History of the Lenz–Ising Model

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Many physico-chemical systems can be represented more or less accurately by a lattice arrangement of molecules with nearest-neighbor interactions. The simplest and most popular version of this theory is the so-called "Ising model," discussed by Ernst Ising in 1925 but suggested earlier (1920) by Wilhelm Lenz.

Major events in the subsequent history of the Lenz–Ising model are reviewed, including early approximate methods of solution, Onsager's exact result for the two-dimensional model, the use of the mathematically equivalent "lattice gas" model to study gas–liquid and liquid–solid phase transitions, and recent progress in determining the singularities of thermodynamic and magnetic properties at the critical point. Not only is there a wide range of possible physical applications of the model, there is also an urgent need for the application of advanced mathematical techniques in order to establish its exact properties, especially in the neighborhood of phase transitions where approximate methods are unreliable.

After many years of being scorned or ignored by most scientists, the so-called "Ising model" has recently enjoyed increased popularity and may, if present trends continue, take its place as the preferred basic theory of all cooperative phenomena. Whereas previously it appeared that the greatly over-simplified representation of intermolecular forces on which this model is based would make it inapplicable to any real systems, it is now being claimed that the essential features of cooperative phenomena (especially at the critical point) depend not on the details of intermolecular forces but on the mechanism of propagation of long-range order, and the Ising model is the only one which offers much hope of an accurate study of this mechanism. Whether or not it does eventually turn out that gas–liquid critical phenomena, magnetic Curie points, order–disorder transitions in alloys, and phase separation in liquid mixtures can all be described, to a good first approximation, by the same model, the problem of a generalized description of cooperative phenomena now deserves serious attention. We are just beginning to realize some of the implications of such a generalized description: specialists in the properties of gases and liquids could not afford to ignore progress being made in research on phase transitions in solids, and conversely. While some scientists might not appreciate the burden imposed by the need for keeping up with the literature in unfamiliar fields which now suddenly appear to be related to their own, students should benefit by the prospective unification of different subjects. No longer would it be necessary to learn a different theory for each kind of cooperative phenomenon.

The historical development of the "Ising model" also shows the same disregard for traditional boundaries between disciplines. Physics, chemistry, metallurgy, and mathematics have all been involved, and some of the most recent applications have been in biology. The most striking success in the history of the Ising model—the exact solution of the two-dimensional problem—involved such difficult mathematics that it stumped all the physicists who attempted it, and was finally accomplished by...a chemist. (Just as ironical is the fact that the supposed inventor of the model, Ernst Ising, gave up research in physics after thinking he had proved that his model had no physical usefulness, and only discovered twenty years later that he had become famous as a result of work on his model by other scientists.)

THE MODEL

We assume that the physical system can be represented by a regular lattice arrangement of molecules in space. We are interested in three kinds of physical systems: (1) magnets, in which each molecule has a "spin" that can be oriented either up or down relative to the direction of an externally applied field; (2) mixtures of two kinds of molecules; (3) mixtures of molecules and "holes" (empty spaces). All three kinds can be represented abstractly by the same model, if we simply say that each node of a regular space lattice is assigned a two-valued variable. Depending on whether this variable has the value $+1$ or -1, we say that the molecule at that node (1) has spin up or down, or (2) belongs to one or the other of two species, (3) is present or absent. Usually the two-valued variable is called the *spin* σ_i associated with node i of the lattice.

A *configuration* of the lattice is a particular set of values of all the spins; if there are N nodes, there will be 2^N different configurations. A typical configuration is shown in Fig. 1.

We assume that the molecules exert only short-range forces on each other; in particular, we assume that the interaction energy depends only on the configurations of neighboring nodes of the lattice. For example, we could say that the forces are such that when two neighboring spins are the same (both $+1$ or both -1) the energy is $-U$, and when two neighboring spins are different (one is $+1$, the other -1) the energy is $+U$. In other words, the interaction tends to make neighboring spins the same. In the three types of physical systems mentioned above, such an interaction could lead to (1) spontaneous magnetization, with all or
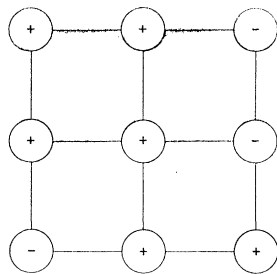
FIG. 1. A possible configuration of a finite square lattice. The energy of this configuration is $E = -2U + 3\mu H$.

most spins in the same direction even in the absence of an external field, (2) phase separation, with molecules of the same kind clustering together, (3) condensation of molecules into one region of space, leaving empty space in the rest of the container. Whether the interaction does in fact lead to such phenomena remains to be seen, of course; one must first work out the statistical mechanical theory of the model.

If we simply reverse the sign of U, so that the energy is positive when two neighboring spins are the same and negative when they are different, then the interaction tends to produce a regular alternation of up and down spins. Physically, this type of configuration could correspond to (1) antiferromagnetic ordering, (2) superlattice structure in an alloy, (3) a solid-like arrangement of molecules with repulsive forces.

We have assumed that each nearest-neighbor pair contributes an interaction energy which can be written (as one easily sees) in the form $-U\sigma_i\sigma_j$, where U is either positive or negative. In addition, we assume that the total energy of a configuration also includes a term of the form μH_i for each spin. The notation suggests that μ is a magnetic moment and H is an external magnetic field, and this would indeed be a reasonable interpretation when we are applying the model to magnetic systems. In general, μH may be any parameter which plays the role of a "chemical potential" in determining the average number of up and down spins, or average composition of a mixture, or average density of a molecule–hole system.

The mathematical problem associated with our model is the following: Find a closed-form analytic expression for the statistical mechanical partition function

$$Q = \sum_{\text{conf}} \exp\left(-E/\kappa T\right) \quad \text{(sum over all configurations)},$$

where

$$E = -\underset{\substack{\text{(sum over all} \\ \text{nearest-neighbor} \\ \text{pairs)}}}{\sum} U\sigma_i\sigma_j + \mu H \underset{\text{(sum over } i)}{\sum} \sigma_i. \quad .$$

From the partition function one can then derive all the thermodynamic functions of the system by the usual procedures of statistical mechanics, and in particular one can find out whether the system undergoes a phase transition.

LENZ AND ISING

This model is commonly referred to as the "Ising model" although one has only to read Ising's original paper[1] to learn that it was previously proposed by Ising's research director, Wilhelm Lenz, in 1920.[2] It is rather curious that Lenz's priority has never been recognized by later writers. Of three biographical notes on Lenz,[3-5] only one[4] even mentions this paper, and then simply summarizes it in one sentence without mentioning its connection with the "Ising model." Lenz himself apparently never made any attempt later on to claim credit for suggesting the model, and even his colleagues at Hamburg University were not aware of his contribution.[6]

Lenz suggested that dipolar atoms in a crystal may be free to rotate around a fixed rest position in a lattice. To understand the physical basis of his assumptions, it would be necessary to go into some of the forgotten details of the old quantum theory, which we will not do here. Instead, we will simply quote Lenz's own formulation (my translation):

"In a quantum treatment certain angles α will be distinguished, among them in any case $\alpha = 0$ and $\alpha = \pi$. If the potential energy W has large values in the intermediate positions, as one must assume taking account of the crystal structure, then these positions will be very seldom occupied, Umklapp processes will therefore occur very rarely, and the magnet will find itself almost exclusively in the two distinguished positions, and indeed on the average will be in each one equally long. In the presence of an external magnetic field, which we assume to be in the direction of the null position for the sake of simplicity, this equivalence of the two positions will disappear, and one has, according to the Boltzmann principle, a resulting magnetic moment of the bar magnet at temperature T:

$$\bar{\mu} = \mu(e^a - e^{-a})/(e^a + e^{-a}); \qquad a = \mu H/kT.$$

For sufficiently small values of a, this reduces to

$$\bar{\mu} = \mu^2 H/kT,$$

i.e., we obtain the Curie law....

"For ferromagnetic bodies, in addition to the temperature dependence of the susceptibility, one has to explain first of all the fact of spontaneous magnetization, as is observed in magnetite and pyrites....

"If one assumes that in ferromagnetic bodies the potential energy of an atom (elementary magnet)

[1] E. Ising, Z. Physik 31, 253 (1925).
[2] W. Lenz, Physik. Z. 21, 613 (1920).
[3] Ortwein, Physik. Blatter 4, 30 (1948).
[4] A. Sommerfeld, Z. Naturforsch. 3a, 186 (1948).
[5] P. Jordan, Physik. Blatter 13, 269 (1957).
[6] Professor Dr. H. Raether (private communication).

with respect to its neighbors is different in the null position and in the π position, then there arises a natural directedness of the atom corresponding to the crystal state, and hence a spontaneous magnetization. The magnetic properties of ferromagnetics would then be explained in terms of nonmagnetic forces, in agreement with the viewpoint of Weiss, who has by calculation and experiment established that the internal field, which he introduced and which generally gives a good representation of the situation, is of a nonmagnetic nature. It is to be hoped that one can succeed in explaining the properties of ferromagnetics in the manner indicated."

Lenz was at Rostock University in 1920, but the following year he was appointed Ordinary Professor at Hamburg. One of his first students was Ernst Ising. Since no biographical information has ever been published about Ising (aside from a brief entry in *American Men of Science*), we shall quote here the "Lebenslauf" from his Dissertation (my translation)[7]:

"I, Ernst Ising, was born on May 10, 1900, the son of the merchant Gustav Ising and his wife Thekla, at Löwe, Köln. Shortly thereafter my parents moved to Bochum, Westfall, where I started school in Easter 1907. I received my diploma at the Gymnasium there, in 1918. After brief military training I began my studies of mathematics and physics at Göttingen University in Easter 1919. After an absence of one semester, I con-

FIG. 2. Wilhelm Lenz (1888–1957). Photograph courtesy of Professor Dr. H. Raether.

[7] A copy of the Lebenslauf was kindly sent to me by Professor Dr. H. Raether; the dissertation itself is hard to obtain outside of Germany. A copy is now available at the archives of the Center for History and Philosophy of Physics of the American Institute of Physics in New York.
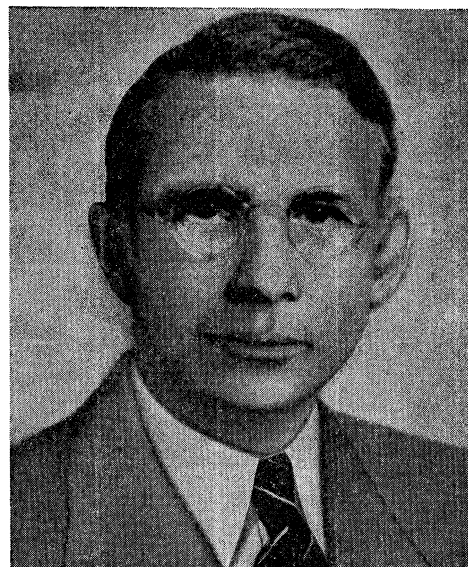

FIG. 3. Ernst Ising. Photograph courtesy of Professor Ising.

tinued my studies in Bonn, where I studied astronomy among other things. Two semesters later I went to Hamburg. There I turned especially to the study of theoretical physics at the suggestion of Professor Dr. W. Lenz, and at the end of 1922 I began under his guidance the investigation of ferromagnetism, which led to the results presented here."

In his dissertation, summarized in a short paper published in 1925,[1] Ising carried out an exact calculation of the partition function for the model described above, for the special case of a one-dimensional lattice. His analysis showed that there was no phase transition to a ferromagnetic ordered state at any temperature. This result can be understood by a qualitative argument: suppose one had an ordered state in which, say, all the spins were up. Then if any random thermal fluctuation should cause spins in the middle of the lattice to flip to the down position, the ordering would be destroyed, because there would be nothing to prevent all the spins on one side from flipping simultaneously. In other words, the ordered state is unstable at any finite temperature, because the "communication" between any two parts of the lattice can be broken by a single defect. However, Ising did not realize that this argument is valid only in one dimension, and he gave some approximate calculations purporting to show that his model could not exhibit a phase transition in three dimensions either.

According to a letter from Ising to the author:

"At the time I wrote my doctor thesis Stern and Gerlach were working in the same institute on their famous experiment on space quantization. The ideas

we had at that time were that atoms or molecules of magnets had magnetic dipoles and that these dipoles had a limited number of orientations. We assumed that the field of these dipoles would die down fast enough so that only interactions should be taken into account, at least in the first order.... I discussed the result of my paper widely with Professor Lenz and with Dr. Wolfgang Pauli, who at that time was teaching in Hamburg. There was some disappointment that the linear model did not show the expected ferromagnetic properties.... I do not know of any reaction to my paper while I was in Europe, except that Heisenberg mentioned it in one of his publications. Only after I had come to this country [U.S.A.] in 1947 did I learn that the idea had been expanded. I have tried to extend my model to more complicated forms, but have not published anything yet.

"After I got my doctor's degree I worked in the research department of the Allgemeine Elektrizitäts-Gesellschaft (General Electric Company) in Berlin. I was not satisfied, returned to the University and became a teacher. When Hitler came to power in 1933 I was dismissed from public schools, and for four years I was the head of a private Jewish school near Potsdam. I left Germany in 1939, but was not able to come to the U.S.A. immediately. We survived the war in a small town in Luxembourg, but I was there completely shut off from scientific and social life. After coming to the U.S.A. I taught for one year at the State Teachers College in Minot, N.D., and since 1948 I have been teaching physics at Bradley University [Peoria, Illinois]."

Ising remarks that the only contemporary citation of his paper was by Heisenberg. Heisenberg, when he proposed his own theory of ferromagnetism in 1928,[8] said:

"Ising succeeded in showing that also the assumption of directed sufficiently great forces between two neighboring atoms of a chain is not sufficient to explain ferromagnetism."

Thus Heisenberg used the supposed failure of the Lenz–Ising model to explain ferromagnetism as one justification for developing his own theory based on a more complicated interaction between spins. In this way the natural order of development of theories of ferromagnetism was inverted; the more sophisticated Heisenberg model was exploited first, and only later did theoreticians return to investigate the properties of the simpler Lenz–Ising model.

Indeed, as a result of Ising's own rejection of the model, we might never have heard any more about it, if it had not been for developments in a different area of physics: order–disorder transformations in alloys. Tamman had proposed in 1919 that the atoms in alloys may be in a definite ordered arrangement, and a number of workers had developed the idea that order–disorder transformations result from the opposing effects of temperature and the lower potential energy of order.[9] In 1928, Gorsky tried to construct a statistical theory on this basis by assuming that the work needed to move an atom from an "ordered" position to a "disordered" position is proportional to the degree of order already existing.[10] Bragg and Williams developed Gorsky's theory further in 1934, and the assumption has subsequently become known as the "Bragg–Williams approximation."[11] It differs from the Lenz–Ising model in that the energy of each configuration of an individual atom is assumed to depend only on the average degree of order throughout the entire system, rather than on the configurations of neighboring atoms.

In 1935, Hans Bethe showed how the Bragg–Williams theory could be improved by taking account of the short-range ordering produced by interactions between neighboring atoms.[12] He did this by constructing an approximate combinatorial factor based on configurations of the first shell of lattice sites around a central one. In the same year, E. A. Guggenheim developed a theory of liquid solutions in which nearest-neighbor interactions were taken into account by what is known as the "quasi-chemical" (QC) method.[13] In the QC method one constructs an approximate combinatorial factor by counting configurations of neighboring pairs or larger groups of atoms, assuming that these groups can be treated as independent statistical entities. (This is of course not strictly true since each atom belongs to several groups, so that the configurations of these groups cannot really be independent.) The QC method was improved and extended by Rushbrooke[14] by means of Bethe's method. Guggenheim[15] then showed that the

[8] W. Heisenberg, Z. Physik **49**, 619 (1928). Ising's result is also cited by L. Nordheim in the article "Quantentheorie des Magnetismus" in Mueller-Pouillet's *Lehrbuch der Physik* (Frederick Vieweg und Sohn, Braunschwieg, Germany, 1934), 11th ed., Vol. IV, Teil 4, p. 859. (I am indebted to Professor Ising for this reference.)

[9] G. Tammann, Z. Anorg. Chem. **107**, 1 (1919); U. Dehlinger, *Röntgenforschung in der Metallkunde* (Julius Springer-Verlag, Berlin, 1930); Z. Physik **74**, 267 (1932); **79**, 550 (1932); **83**, 832 (1933); Z. Physik. Chem. **B26**, 343 (1934); G. Borelius, Ann. Physik **20**, 57 (1934).

[10] W. Gorsky, Z. Physik **50**, 64 (1928).

[11] W. L. Bragg and E. J. Williams, Proc. Roy. Soc. (London) **A145**, 699 (1934); **A151**, 540 (1935); E. J. Williams, Proc. Roy. Soc. (London) **A152**, 231 (1935). Bragg and Williams acknowledged the priority of Gorsky but claimed that he had made an error in his formulation; but according to R. H. Fowler and E. A. Guggenheim [*Statistical Thermodynamics* (Cambridge University Press, Cambridge, England, 1939), p. 574], Gorsky's formula is equivalent to that of Bragg and Williams. (I am indebted to Professor Guggenheim for pointing this out to me.)

[12] H. A. Bethe, Proc. Roy. Soc. (London) **A150**, 552 (1935).

[13] E. A. Guggenheim, Proc. Roy. Soc. (London) **A148**, 304 (1935).

[14] G. S. Rushbrooke, Proc. Roy. Soc. (London) **A166**, 296 (1938).

[15] E. A. Guggenheim, Proc. Roy. Soc. (London) **A169**, 134 (1938).

QC method is really equivalent to the Bethe method, and more convenient to use in many cases such as those involving complicated lattices. In 1940, Fowler and Guggenheim[16] published a general formulation of the QC method and applied it to alloys with long-range order.[17]

ATTEMPTS TO FIND AN EXACT SOLUTION

The concept of the "Ising model" as a mathematical object existing independently of any particular physical approximation seems to have been developed by the Cambridge group led by R. H. Fowler in the 1930's. Fowler discussed rotations of molecules in solids in a paper published in 1935,[18] where he stated that the need for a quantitative theory of such phenomena

". . . was first brought clearly to my notice at a conference on the solid state held at Leningrad in 1932. As will appear, however, an essential feature of the theory is an application of the ideas of order and disorder in metallic alloys, where the ordered state is typically cooperative, recently put forward by Bragg and Williams. As soon as their ideas are incorporated the theory 'goes'."

In 1936, R. Peierls published a paper with the title "On Ising's Model of Ferromagnetism;"[19] in which he recognized the equivalence of the Ising theory of ferromagnetism, the Bethe theory of order–disorder transformations in alloys, and the work of Fowler and Peierls[20] on adsorption isotherms. Peierls gave a simple argument purporting to show that (contrary to Ising's statement) the Lenz–Ising model in two or three dimensions *does* exhibit spontaneous magnetization at sufficiently low temperatures. He pointed out that each possible configuration of ($+$) and ($-$) spins on the lattice corresponds to a set of "boundaries" between regions of ($+$) spins and regions of ($-$) spins. If one can show that, at sufficiently low temperatures, the average ($+$) area or volume enclosed by boundaries is only a small fraction of the total area or volume, then it will follow that the majority of spins must be ($-$), which corresponds to a system with net magnetization. Unfortunately Peierls' proof is not rigorous because of an incorrect step, which was discovered by M. E. Fisher and S. Sherman.[21] The basic idea is still of interest—the analysis of boundaries between magnetized regions plays an important role in some of the more recent combinatorial methods.

The next advance was made by J. G. Kirkwood, who developed in 1938 a systematic method for expanding the partition function in inverse powers of the temperature.[22] His method was based on the semi-invariant expansion of T. N. Thiele (1838–1910), used in statistics to characterize a distribution function by its moments.[23] Since only a small number of terms in the expansion could actually be computed, the result was "essentially equivalent to Bethe's in its degree of approximation, but somewhat less unwieldy in form." In 1939 Bethe and Kirkwood, then both at Cornell, published a joint paper giving a comparison of their methods, and including a calculation of the next term in Kirkwood's expansion.[24] Chang (1941) evaluated two more terms, and Wakefield (1951) determined three more.[25] The present status of this expansion, and its use in determining the critical point, is reviewed in survey papers by Domb and Fisher.[26]

The first exact quantitative result for the two-dimensional Ising model was obtained by Kramers and Wannier in 1941[27]; they located the transition temperature by using the symmetry of the lattice to relate the high- and low-temperature expansion of the partition function. They showed that the partition function can be written as the largest eigenvalue of a certain matrix; they attribute some of the ideas used in their analysis to Montroll, who subsequently published a similar method.[28] Kramers and Wannier developed a method that yields the largest eigenvalue of a sequence of finite matrices and should in principle converge to the exact solution if sufficiently large matrices could be analyzed. They did not succeed in obtaining an exact solution in closed form, but they did develop a variational method which is fairly accurate, and which has been used occasionally in later studies.[29]

Some other exact but incomplete results were obtained by other workers at about the same time. Zernike[30] had used the Bethe method to derive a nonlinear finite difference equation for the "correlation function" (correlation of spin variables at various dis-

[16] R. H. Fowler and E. A. Guggenheim, Proc. Roy. Soc. (London) **A174**, 189 (1940).

[17] See also: E. A. Guggenheim, *Mixtures* (Clarendon Press, Oxford, England, 1952); J. A. Barker, Proc. Roy. Soc. (London) **A216**, 45 (1953); S. G. Brush, Trans. Faraday Soc. **54**, 1781 (1958). (Extension of QC method to higher approximations.)

[18] R. H. Fowler, Proc. Roy. Soc. (London) **A149**, 1 (1935).

[19] R. Peierls, Proc. Cambridge Phil. Soc. **32**, 477 (1936).

[20] R. Peierls, Proc. Cambridge Phil. Soc. **32**, 471 (1936).

[21] S. Sherman (private communication); see R. B. Griffiths, Phys. Rev. **136**, A437 (1964).

[22] J. G. Kirkwood, J. Chem. Phys. **6**, 70 (1938).

[23] See, for example, M. G. Kendall, *Advanced Theory of Statistics* (Griffin, London, 1947), 3rd ed., Vol. I, Chap. 3. Kendall prefers to call these quantities "cumulants" but the name "semi-invariant" has survived in statistical mechanics. The history of semi-invariants is discussed briefly by H. M. Walker in *Studies in the History of Statistical Method* (The Williams and Wilkins Co., Baltimore, 1929), p. 81.

[24] H. A. Bethe and J. G. Kirkwood, J. Chem. Phys. **7**, 578 (1939).

[25] T. S. Chang, J. Chem. Phys. **9**, 169 (1941); A. J. Wakefield, Proc. Cambridge Phil. Soc. **47**, 419, 799 (1951).

[26] C. Domb, Advan. Phys. **9**, 150 (1960); M. E. Fisher, J. Math. Phys. **4**, 278 (1963): *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, Colo., 1965), Vol. VIIC, p. 1.

[27] H. A. Kramers and G. H. Wannier, Phys. Rev. **60**, 252, 263 (1941).

[28] E. Montroll, J. Chem. Phys. **9**, 706 (1941); **10**, 61 (1942).

[29] D. ter Haar and B. Martin, Phys. Rev. **77**, 721 (1950); B. Martin and D. ter Haar, Physica **18**, 569 (1952); T. Oguchi, Phys. Rev. **76**, 1001 (1949); J. Phys. Soc. Japan **5**, 75 (1950).

[30] F. Zernike, Physica **7**, 565 (1940).

tances), Ashkin and Lamb[31] derived an exact low-temperature series for the correlation function of the two-dimensional lattice using the Kramers–Wannier method. They compared the results of Zernike's approximation with their own method, and with van der Waerden's exact series expansion[32] for a three-dimensional lattice. They also proved that the existence of long-range order implies degeneracy of the largest eigenvalue of the Kramers–Wannier matrix, as previously remarked by Lassettre and Howe.[33] Ashkin and Teller[34] located the transition temperature in four-component two-dimensional systems by symmetry methods.

A contribution by R. Kubo[35] deserves especial mention because it was published only in Japanese and was therefore unknown to Western physicists. (An English translation is now available.) Kubo developed the matrix formulation and showed how a possible phase transition in the two- or three-dimensional system would be related to the degeneracy of the largest eigenvalue of a matrix, but he did not give detailed calculations except for the one-dimensional case.

ONSAGER'S EXACT SOLUTION

At a meeting of the New York Academy of Science, 28 February 1942, Lars Onsager announced his solution of the two-dimensional Lenz–Ising problem in zero magnetic field.[36] The details were published two years later.[37] The method is similar to that of Kramers and Wannier, and of Montroll, except that Onsager emphasized "the abstract properties of relatively simple operators rather than their explicit representation by unwieldy matrices." We quote Onsager's summary of his method:

"The special properties of the operators involved in this problem allow their expansion as linear combinations of the generating basis elements of an algebra which can be decomposed into direct products of quaternion algebras. The representation of the operators in question can be reduced accordingly to a sum of direct products of two-dimensional representations, and the roots of the secular equation for the problem in hand are obtained as products of the roots of certain quadratic equations. To find all the roots requires complete reduction, which is best performed by the explicit construction of a transforming matrix, with valuable

by-products of identities useful for the computation of averages pertaining to the crystal. It so happens that the representations of maximal dimension, which contain the two largest roots, are identified with ease from simple general properties of the operators and their representative matrices. The largest roots whose eigenvectors satisfy certain special conditions can be found by a moderate elaboration of the procedure; these results will suffice for a qualitative investigation of the spectrum. To determine the thermodynamic properties of the model it suffices to compute the largest root of the secular equation as a function of temperature.

"The passage to the limiting case of an infinite base involves merely the substitution of integrals for sums. The integrals are simplified by elliptic substitutions, whereby the symmetrical parameter of Kramers and Wannier appears in the modulus of the elliptic functions. The modulus equals unity at the 'Curie point'; the consequent logarithmic infinity of the specific heat confirms a conjecture made by Kramers and Wannier."

Onsager's method was subsequently simplified by Kaufman[38] and by Newell and Montroll,[39] using ideas from the theory of spinors and Lie algebras. A good exposition of the method is given in the review by Newell and Montroll[39]; shorter summaries appear elsewhere.[40] The result is that the partition function for a rectangular lattice of $n \times m$ points can be written in the form

$$\lim_{n,m\to\infty} \frac{\log Q}{mn} = \log 2 + (1/2\pi^2) \int_0^\pi d\omega \int_0^\pi d\omega$$

$$\times \log \left(\cosh 2K \cosh 2K' - \sinh 2K \cos \omega \right.$$

$$\left. - \sinh 2K' \cos \omega \right),$$

where $K = U/kT$, $K' = U'/kT$ (there can be different interaction energies U and U' in horizontal and vertical directions).

Onsager's formula for the spontaneous magnetization of the square lattice was

"... first exposed to the public on 23 August 1948 on a blackboard at Cornell University on the occasion of a conference on phase transitions.... To tease a wider audience, the formula was again exhibited during the discussion which followed a paper by Rushbrooke at the first postwar IUPAP statistical mechanics meeting in Florence in 1948; it finally appeared in print as a dis-

[31] J. Ashkin and W. E. Lamb, Jr., Phys. Rev. **64**, 159 (1943).
[32] B. L. van der Waerden, Z. Physik **118**, 473 (1941).
[33] E. N. Lassettre and J. P. Howe, J. Chem. Phys. **9**, 747 (1941).
[34] J. Ashkin and E. Teller, Phys. Rev. **64**, 178 (1943); see also R. B. Potts, Proc. Cambridge Phil. Soc. **48**, 106 (1952).
[35] R. Kubo, Busseiron-kenkyu **1**, 1 (1943) [English transl.: UCRL-Trans. 1030(L), available from Lawrence Radiation Laboratory, Livermore, California].
[36] See T. Shedlovsky and E. Montroll, J. Math. Phys. **4**, 145 (1963).
[37] L. Onsager, Phys. Rev. **65**, 117 (1944).

[38] B. Kaufman, Phys. Rev. **76**, 1232 (1949).
[39] G. F. Newell and E. W. Montroll, Rev. Mod. Phys. **25**, 353 (1953).
[40] C. Domb, Advan. Phys. **9**, 150 (1960); D. ter Haar, *Elements of Statistical Mechanics* (Rinehart and Co., New York, 1954); K. Huang, *Statistical Mechanics* (John Wiley & Sons, Inc., New York, 1963); H. S. Green and C. A. Hurst, *Order-Disorder Phenomena* (Interscience Publishers, Inc., New York, 1964).

cussion remark. However, Onsager never published his derivation. The puzzle was finally solved by C. N. Yang.[41]"

The exact partition functions and other properties of several other two-dimensional lattices have been deduced from the Onsager result by various workers. In a few cases it has been possible to obtain information about behavior in a magnetic field for models somewhat similar to the Ising model by transforming the variables in Onsager's partition function.[42] However, although Onsager's method has been applied to some closely related problems, no further really new results have been obtained with it. All the two-dimensional lattices appear to have qualitatively similar properties, at least for the case of nearest-neighbor ferromagnetic interactions. The problem of calculating the exact partition function in a finite magnetic field remains unsolved, though significant results for the spontaneous magnetization and susceptibility in a vanishingly small field have been obtained. And, 25 years after the announcement of Onsager's original result, no one has yet succeeded in solving the problem in three dimensions. Probably this is because the abstract algebraic approach of Onsager and Kaufman involves techniques that are as yet too difficult and unfamiliar to most of the physicists who have been interested in the Lenz–Ising model; and the mathematicians who are competent in this area have not shown much interest in physical applications. This is regrettable even from the point of view of the progress of pure mathematics, for research stimulated by the Lenz–Ising model has revealed a relationship between two separate areas of mathematics whose significance is apparently not yet understood by mathematicians themselves (see below).

In an attempt to provide a heuristic explanation of Onsager's result, Kac and Ward[43] constructed a matrix whose determinant provided a generating function for a certain combinatorial problem related to the Lenz–Ising model partition function. Van der Waerden[32] had shown how the Lenz–Ising problem could be reduced to the problem of counting the number of lattice graphs containing closed polygons with various total perimeters. These polygons may be thought of as boundaries between regions of $(+)$ and $(-)$ spins, as suggested by Peierls,[19] but van der Waerden and Kac and Ward investigated the approach more systematically. The validity of the Kac–Ward method depends on the assumption that their determinant is the correct generating function for $n(L)$, defined as the number of graphs of L bonds that can be constructed on the lattice

subject to certain conditions. Since "most" graphs (in some asymptotic sense) are indeed counted correctly, and since the determinant does reduce to just the same function that occurs in the integrand of the Onsager partition function, it was tempting to conclude that the method could be made rigorous. It turned out, however, that some graphs are not counted correctly (see the counterexample given by Sherman[44]) and that a somewhat different though closely related method is required. The clue to the correct approach was found by Feynman,[45] who conjectured a relation between functions of graphs and of closed paths (random walks) on a lattice. If one accepts the validity of Feynman's relation, he can then make use of methods previously developed[46] to count random walks on a lattice, and thereby get to the Onsager partition function by a route somewhat analogous to that suggested by Kac and Ward.[47] The difficult part is to prove the Feynman relation between graphs and paths. This was first done by Sherman[44] and a simplified proof was devised by Burgoyne.[48] The proof uses a result on crossings of curves in a plane, originally derived by Whitney.[49]

What began as an attempt to provide a more comprehensive derivation of a known result has turned up some unsuspected insights and conjectures regarding the borderline of modern algebra and combinatorial analysis. This is why mathematicians should pay more attention to the Lenz–Ising model, though it is impossible to give any more than a brief hint here of some of the technical aspects involved. Here is a morsel for the experts to chew on: Sherman[50] remarks that he was informed by M. P. Schutzenberger that his (Sherman's) combinatorial theorem, of which Feynman's conjecture is a special case, involves an identity used to establish a formula of W. E. Witt[51] on "the dimension of the linear space of Lie elements of degree r in a free Lie algebra with k generators over a field of characteristic zero." Sherman's theorem is a generalization of Witt's identity "to any planar 1-cycle with sufficient smoothness so that winding numbers can be defined. It constitutes a relation between the fundamental group and the first homology group over the integers mod 2 of this planar 1-cycle. An analogous relation between the two groups for 1-cycles in 3-space might very well crack the long attacked Ising problem." Sherman mentions some other mathematical problems whose

[41] E. W. Montroll, R. B. Potts, and J. C. Ward, J. Math. Phys. **4**, 308 (1963); L. Onsager, Nuovo Cimento (Suppl.) **6**, 261 (1949); C. N. Yang, Phys. Rev. **85**, 809 (1952); C. H. Chang, *ibid.* **88**, 1422 (1952).

[42] M. E. Fisher, Phys. Rev. **113**, 969 (1959); Proc. Roy. Soc. (London) **A254**, 66 (1960); **A256**, 502 (1960).

[43] M. Kac and J. C. Ward, Phys. Rev. **88**, 1332 (1952).

[44] S. Sherman, J. Math. Phys. **1**, 202 (1960).

[45] Unpublished. An account is given by Harary in the draft of a chapter of his book on graph theory, dated 1958 but not yet published. See also Feynman's lecture notes, Hughes Research Laboratory, Malibu, California (1960).

[46] See, for example, H. N. V. Temperley, Phys. Rev. **103**, 1 (1956).

[47] Harary, Ref. 45; Sherman, Ref. 44; N. Burgoyne, J. Math. Phys. **4**, 1230 (1963).

[48] N. Burgoyne, Ref. 47.

[49] H. Whitney, Compos. Math. **4**, 276 (1937).

[50] S. Sherman, Bull. Am. Math. Soc. **68**, 225 (1962).

[51] W. E. Witt, J. Reine Angew. Math. **177**, 152 (1937).

solution might be of use in tackling the three-dimensional problem.

Domb's address to the Royal Statistical Society of London in 1964 stimulated an interesting discussion on the relations between professional statisticians and theoretical physicists, apropos of "Some statistical problems connected with crystal lattices."[52] Among other pertinent remarks was the following by Professor H. E. Daniels: "whereas nowadays nearly every difficult problem in applied mathematics can be adequately solved on a computer, the three-dimensional Ising lattice problem must be one of the rare examples where ultimately only a complete mathematical solution will really do." But if mathematicians are to participate effectively in solving problems in theoretical physics, some psychological or at least linguistic barriers must be broken down (the varying usages of the term "ergodic" in physics and mathematics is a good example from a neighboring field of statistical mechanics).

Specialists in quantum field theory should also be interested in the Lenz–Ising model, in view of the mathematical similarities between these two topics.[53] One particular approach, proposed by Hurst and Green,[54] uses field theory techniques to arrive at expressions similar to those that occur in the Kac–Ward method. Each lattice point in a graph in the expansion of the partition function is associated with a set of noncommuting operators, chosen in such a way that a product of certain combinations of these operators, when expanded as a sum over lattice graphs, will be identically equal to the partition function. This is sometimes known as the "Pfaffian" method; it involves triangular arrays of quantities related to antisymmetrical determinants.[55] Fisher,[56] Temperley and Fisher,[57] and Kasteleyn[58] applied the Pfaffian method to the problem of the configurations of dimer molecules on a lattice, and Kasteleyn[59] showed the connection between the dimer problem and the Ising model. The theory of Toeplitz matrices turned out to be useful in much of this research, and has since found applications in other areas of statistical mechanics.[60]

THE LATTICE GAS

Though it had been recognized earlier[61] that formulas derived for the Lenz–Ising model could apply equally well to systems of atoms and holes in a lattice, it was Lee and Yang who first used the term "lattice gas" in a published paper.[62] For this reason, and because they did give a systematic account of the transcription of variables and formulas, as well as deriving some new results, their paper is usually cited as the origin of work on the lattice gas.

Whereas the set of all possible spin values of N lattice sites corresponds to a canonical ensemble for a magnet, it corresponds to a grand canonical ensemble for a lattice gas. The total volume is fixed but the total number of atoms can vary. To calculate the partition function one sums over all possible values of the number of atoms, from zero up to the total number of lattice sites. Thus the magnetization of the magnet (difference between number of up and down spins) is directly related to the density of the lattice gas (fraction of occupied sites). The role of the external magnetic field, which is a controllable parameter for the magnetic model, is now played by the fugacity for the lattice gas. The case of zero external field—the only one that can be treated exactly by Onsager's method—now corresponds to the case in which half of the lattice sites are filled on the average. Below the transition temperature the system may split into two phases of different densities; these correspond to the two possible states of spontaneous magnetization of a magnet in zero field (whether the magnetization is $+M$ or $-M$ depends on the direction of the field before it was turned off).

The lattice gas model with attractive forces between neighboring atoms has frequently been studied as a possible model for the liquid–gas transition and the critical point (see below). However, if the interaction is repulsive, so that configurations with alternating filled and vacant sites are favored, one has a model which is of interest in connection with the theory of solidification. Such a model involving only repulsive forces was suggested by the results of computer experiments on systems of a few hundred hard spheres. These experiments showed that, at least for finite systems, repulsive forces can produce an ordered phase at high densities which is reached by a first-order phase transition from the disordered phase at medium and

[52] C. Domb, J. Roy. Stat. Soc., Ser. B26, 367 (1964) with discussion remarks by J. M. Hammersley, H. E. Daniels, D. C. Handscomb, J. F. C. Kingman, D. J. A. Welsh, and E. S. Page.

[53] Y. Nambu, Progr. Theoret. Phys. (Kyoto) 5, 1 (1950); R. L. Ingraham, Nuovo Cimento 21, 29 (1961); T. D. Schultz, D. C. Mattis, and E. H. Lieb, Rev. Mod. Phys. 36, 856 (1964); R. Abe, Progr. Theoret. Phys. 33, 600 (1965).

[54] C. A. Hurst and H. S. Green, J. Chem. Phys. 33, 1059 (1960); A. M. Dykhne and Y. B. Rumer, Usp. Fiz. Nauk 75, 101 (1961) [English transl.: Soviet Phys. Usp. 4, 698 (1962)]; C. A. Hurst, J. Chem. Phys. 38, 2558 (1963); H. S. Green, Z. Physik 171, 129 (1962); E. W. Montroll, in Applied Combinatorial Mathematics, E. F. Beckenbach, Ed. (John Wiley & Sons, Inc., New York, 1964); H. S. Green and C. A. Hurst, Order-Disorder Phenomena (Interscience Publishers, Inc., New York, 1964).

[55] See, for example, G. Brunel, Mem. Soc. Sci. Bordeaux 5, 165 (1895); W. T. Tutte, J. London Math. Soc. 22, 107 (1947).

[56] M. E. Fisher, Phys. Rev. 124, 1664 (1961).

[57] H. N. V. Temperley and M. E. Fisher, Phil. Mag. 6, 1061 (1961).

[58] P. W. Kasteleyn, Physica 27, 1209 (1961).

[59] P. W. Kasteleyn, J. Math. Phys. 4, 287 (1963).

[60] E. W. Montroll, R. B. Potts, and J. C. Ward, J. Math. Phys. 4, 308 (1963); A. Lenard, J. Math. Phys. 5, 930 (1964).

[61] Y. Muto, J. Chem. Phys. 16, 519, 524, 1176 (1948); S. Ono, Mem. Fac. Eng. Kyushu Univ. 10, (No. 4), 196 (1947); T. Tanaka, H. Katsumori, and S. Toshima, Progr. Theoret. Phys. (Kyoto) 6, 17 (1951).

[62] T. D. Lee and C. N. Yang, Phys. Rev. 87, 410 (1952).

low densities.[63] Temperley and others have therefore pursued the study of antiferromagnetic versions of the Lenz–Ising model in the hope of learning more about the liquid–solid transition.[64]

THE CRITICAL POINT

Until recently, the lack of an exact solution for the three-dimensional Lenz–Ising model had thwarted attempts to reach definite conclusions about the nature of the phase transition. Many approximate methods had been developed, but comparison of their predictions with Onsager's exact two-dimensional solution showed that the approximations generally failed to reproduce the nature of the singularity in the specific heat and other thermodynamic functions at the transition point. It seemed that nothing short of an exact solution could offer any convincing evidence about the singularity in three dimensions.

During the early 1960's, Domb and his colleagues at London had been continuing the earlier work[22,24,25] of computing further terms in the series expansions of the partition functions and thermodynamic properties. These expansions are either in positive or negative powers of the temperature, starting from zero or infinite temperature, respectively. Although the expansions were not expected to give reliable values in the neighborhood of the transition point, it was thought possible to estimate the location of the critical point by various extrapolation methods.[26,65] Then, in 1961, Baker introduced the method of Padé approximants[66] for extrapolating series expansions and estimating the location and nature of singularities of the functions

represented by those expansions.[67] His results for the high-temperature susceptibility confirmed the estimates of Domb and Sykes[65]: for temperatures near the critical temperature T_c, the susceptibility behaves as

$$\chi(T) \approx C^+/[1-(T_c/T)]^\gamma,$$

where $\gamma = \frac{7}{4}$ in two dimensions, and $= \frac{5}{4}$ in three dimensions. Baker also showed that the Padé approximant method could be applied successfully to low-temperature series, which previous methods had failed to handle. For example, Baker's calculations suggested that the spontaneous magnetization goes to zero at the transition temperature in such a way that

$$I_0(T) \approx D[1-(T/T_c)]^\beta,$$

where $\beta = 0.30 \pm 0.01$ for three-dimensional lattices. It was already known[41] that $\beta = \frac{1}{8}$ in two dimensions.

The Padé approximant method was quickly taken up by the London group and elsewhere; the possibility of obtaining significant and reliable results for the critical point of three-dimensional systems provided a strong motivation for grinding out more terms in the series expansions.[68] The results have been summarized in papers by Fisher.[69]

This recent progress in determining the critical-point behavior of the Lenz–Ising model has revealed that the model may have much greater applicability to real physical systems than was previously thought. Fisher[70] pointed out that according to Van der Waals theory of the liquid–gas transition, the difference between liquid and gas phase densities should go to zero as the square root of the difference between temperature and critical temperature, as the critical point is approached from below:

$$\rho_L - \rho_G \sim A(T_c-T)^{1/2} \quad \text{as} \quad T \rightarrow T_c^-.$$

The compressibility along the critical isochore should behave as

$$K_T \sim B/\mid T-T_c \mid \qquad (\rho=\rho_c, \ T \rightarrow T_c^+).$$

The specific heat at constant volume along the critical isochore should rise to a maximum and then fall discontinuously as T increases through T_c:

$$c_v(T) \sim C_c^\pm - D^\pm(\mid T-T_c \mid), \qquad T \lessgtr T_c$$

with

$$C_c^- - C_c^+ = \Delta C > 0.$$

The compressibility of the gas and the liquid along the coexistence curve should also diverge as simple poles.

[63] B. J. Alder and T. E. Wainwright, J. Chem. Phys. **27**, 1208 (1957); the possibility of such a transition had previously been suggested by J. G. Kirkwood and E. Monroe, J. Chem. Phys. **10**, 394 (1942) and by I. Z. Fisher, Zh. Eksperim. i Teor. Fiz. **28**, 437 (1955) [English transl.: Soviet Phys.—JETP **1**, 273 (1955)]. For reasons why there should *not* be a phase transition, see J. E. Mayer, Phys. Today **11**, 22 (January 1958).

[64] H. N. V. Temperley, Proc. Phys. Soc. London **74**, 183, 432, 444 (1959); **84**, 339 (1964); **86**, 180 (1965). For earlier work on the antiferromagnetic model, see J. E. Brooks and C. Domb, **A207**, 343 (1951); Y. Y. Li, Phys. Rev. **84**, 721 (1951); J. M. Ziman, Proc. Phys. Soc. (London) **64**, 1108 (1951); other recent work: M. F. Sykes and M. E. Fisher, Phys. Rev. Letters **1**, 321 (1959); Physica **28**, 919 (1962); M. E. Fisher and M. F. Sykes, Physica **28**, 939 (1962); D. M. Burley, Proc. Phys. Soc. (London) **75**, 262 (1960); **77**, 451 (1961); Physica **27**, 768 (1961); C. Domb, Nuovo Cimento Suppl. **9**, 1 (1958); B. Jancovici, Physica **31**, 1017 (1965); D. S. Gaunt and M. E. Fisher, J. Chem. Phys. **43**, 2840 (1965); L. K. Runnels, Phys. Rev. Letters **15**, 581 (1965).

[65] C. Domb and M. F. Sykes, J. Math. Phys. **2**, 63 (1961); Proc. Roy. Soc. (London) **A240**, 214 (1957); M. E. Fisher and M. F. Sykes, Physica **28**, 939 (1962); M. F. Sykes, J. W. Essam, and D. S. Gaunt, J. Math. Phys. **6**, 283 (1965). There has also been some work on series expansions for models in nonzero magnetic fields: C. Domb, Proc. Roy. Soc. (London) **A196**, 36 (1949); **A199**, 199 (1949) and later papers.

[66] H. Padé, Ann. Sci. Ecole Normale Superieure (Paris) **9**, Suppl. 1–92 (1892); G. A. Baker, Jr., J. L. Gammel, and J. G. Wills, J. Math. Anal. Appl. **2**, 405 (1961); G. A. Baker, Jr., in *Advances in Theoretical Physics*, K. A. Brueckner, Ed. (Academic Press Inc., New York, 1965).

[67] G. A. Baker, Jr., Phys. Rev. **124**, 768 (1961).

[68] J. W. Essam and M. E. Fisher, J. Chem. Phys. **38**, 802 (1963); A. V. Ferris-Prabhu, Phys. Letters **15**, 127 (1965).

[69] M. E. Fisher, J. Math. Phys. **4**, 278 (1963).

[70] M. E. Fisher, J. Math. Phys. **5**, 944 (1964); *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, Colo., 1965), Vol. VIIC, p. 1; see also C. N. Yang and C. P. Yang, Phys. Rev. Letters **13**, 303 (1964).

These predictions are also made by most other approximate theories, and in fact they are "essentially a direct consequence of the implicit or explicit assumption that the free energy and the pressure can be expanded in a Taylor series in density and temperature at the critical point."

Experimental results on gas-liquid systems near the critical point are definitely in disagreement with the predictions of the van der Waals theory, however. According to older data, as analyzed by Guggenheim in 1945,[71] the exponent is about $\frac{1}{3}$ rather than $\frac{1}{2}$. More recent experiments also show that the exponent (usually denoted by β, corresponding to the exponent for spontaneous magnetization) is between 0.33 and 0.36 for most systems. Thus the experimental coexistence curve is much flatter than the van der Waals curve. On the other hand, the three-dimensional lattice gas model leads to a value of β between 0.303 and 0.318, possibly just $\frac{5}{16} = 0.3125$. While this is somewhat outside the range for real systems, it is certainly much closer than the van der Waals theory, and creates the presumption that the lattice gas provides an extremely good first approximation for the behavior of fluids at the critical point.

It will be observed, incidentally, that theoreticians have a strong prejudice towards finding simple fractions for these exponents, whatever may be the raw data. This is probably justified by the results of the two-dimensional exact calculations, but as J. F. C. Kingman has pointed out, "there must have been a time when considerable plausibility attached to the conjecture $\pi = \frac{22}{7}$."[52]

For the singularity in the compressibility (corresponding to the susceptibility for a magnetic system), the exponent γ mentioned above was thought for some time to have the theoretical value $\frac{5}{4}$, but expert opinion has recently tended toward the value $\frac{21}{16}$.[72] Experimental data, according to Fisher,[70] lead to a value somewhat larger than 1.1, though there is not sufficient evidence for a precise estimate. Furthermore, most of the evidence is relevant only below the critical temperature, and γ may have a different value (γ') above the critical temperature. Nevertheless, it seems likely that the lattice-gas value is better than the van der Waals value.

The nature of the specific heat singularity has not been settled. Fisher assumes a relation of the form

$$c_v(T) \approx (A^{\pm}/\alpha)[\, |1 - (T/T_c)\,|^{-\alpha} - 1] + B^{\pm},$$

where A and B are constants that may have different values above and below T_c; there may also be a different exponent above (α) and below (α'). For the two-dimensional lattice gas, we know that $\alpha = 0$ (representing, in this formula, a logarithmic singularity).

For the three-dimensional lattice gas, α is positive but less than 0.2; the best guess at present[72] is $\alpha' = \frac{1}{16}$.

At this point there is an opportunity for the use of thermodynamic, as opposed to statistical mechanical reasoning, in deducing relations among the three exponents. Essam and Fisher,[68] on the basis of "heuristic arguments related to the Frenkel–Bijl–Band picture of condensation," conjectured the relation

$$\alpha' + 2\beta + \gamma' = 2.$$

This is valid in the two-dimensional Lenz–Ising model, where $\alpha = 0$, $\beta = \frac{1}{8}$, and $\gamma = \frac{7}{4}$. Rushbrooke[73] showed that the relation is valid as an inequality,

$$\alpha' + 2\beta + \gamma' \geq 2$$

for ferromagnets, and Fisher applied the same reasoning to fluids. Griffiths[74] derived this and several similar inequalities by thermodynamic arguments. Widom, Kadanoff, and others have recently investigated general equations of state for lattice gases and ferromagnets in nonzero magnetic field, in order to discover what properties a system must have in order to be consistent with information now available about the critical point behavior.[75] A new pattern of relations between critical indices now seems to be emerging from this research, but it would perhaps be premature to report here on the tentative conclusions that have been reached.[76]

For magnets, the Heisenberg model is generally considered to be more accurate than the Lenz–Ising model. Attempts have been made to apply the Padé approximant method to exact series expansions for the Heisenberg model, but the calculations appear to be more difficult and the conclusions less certain. It was thought for awhile that the exact value of γ for the Heisenberg model may be $\frac{4}{3}$, which would agree very well with some of the experimental results though not all of them.[77] For example, Kouvel and Fisher analyzed the old data of Weiss and Forrer (1926) and concluded that $\gamma = 1.35 \pm 0.02$ for nickel.[78] But more recent theoretical work suggests a value of about 1.42 for γ.[79]

[71] E. A. Guggenheim, J. Chem. Phys. **13**, 253 (1945).
[72] M. E. Fisher (private communication).

[73] G. S. Rushbrooke, J. Chem. Phys. **39**, 842 (1963).
[74] R. B. Griffiths, Phys. Rev. Letters **14**, 623 (1965); J. Chem. Phys. **43**, 1958 (1965).
[75] B. Widom, J. Chem. Phys. **43**, 3892, 3898 (1965); L. Kadanoff, Physics **2**, 263 (1966); A. Z. Patashinskii and V. L. Pokrovskii, Zh. Eksperim. i Teor. Fiz. **50**, 439 (1966) [English transl.: Soviet Phys.—JETP **23**, 292 (1966)].
[76] C. Domb and D. L. Hunter, Proc. Phys. Soc. (London) **86**, 1147 (1965); C. Domb, Ann. Acad. Sci. Fennicae A. VI. **210**, 167 (1966); L. P. Kadanoff et al., Rev. Mod. Phys. **39**, 395 (1967); M. E. Fisher, J. Appl. Phys. **38**, 981 (1967).
[77] C. Domb and M. F. Sykes, Phys. Rev. **128**, 168 (1962); J. Gammel, W. Marshall, and L. Morgan, Proc. Roy. Soc. (London) **A275**, 257 (1963).
[78] J. S. Kouvel and M. E. Fisher, Phys. Rev. **136**, A1626 (1964).
[79] G. A. Baker, H. F. Gilbert, J. Eve, and G. S. Rushbrooke, Phys. Letters **20**, 146 (1966).

In this brief account we have not been able to review the applications of the Lenz–Ising model to liquid mixtures, alloys, polymers, and random walk problems, and many other areas of science; we have not even mentioned all the theoretical work on the model itself.[80] Research on critical exponents is currently one of the fastest-moving fields in science, and many of the results mentioned here may well be obsolete by the time this article appears in print. But there does seem to be one general feature of the impact of the Lenz–Ising model on science which is well-enough established to be worth noting.[81] In constructing a theory to interpret a complex physical phenomenon, a scientist frequently has to choose between two approaches. On the one hand, he may want to make his theory contain all the possible factors which he knows must influence the effects he observes in the laboratory; the theory must be as "realistic" as possible. But this approach usually leads to formulations that are mathematically so complicated that the consequences of the hypotheses cannot be deduced from the theory without gross approximations; if the predictions of the theory disagree with the experimental facts, it is often difficult to know whether to attribute this to defects in the original hypotheses, or to errors incurred by approximations in the calculations. On the other hand, one may intentionally sacrifice some of the more realistic features of a model in order to obtain a simpler model that is exactly, or almost exactly, soluble. Such oversimplified theories are often scorned by empirically minded scientists, since they may be completely irrelevant to the problem of interpreting the subtle effects that are important in practical problems. It is encouraging, therefore, to those who favor the second approach, to have at least one example in which insistence on getting an exact solution of a simple model has really paid off. (And of course it is also extremely useful to be able to test other theoretical methods, such as the currently popular integral equations for pair distribution functions, on a model where the exact solution is known.[82]) The Lenz–Ising model provides this example: a many-body problem in which the interactions between particles cannot be ignored, or even treated successfully by perturbation theory; and yet it *is* possible, if one works hard and cleverly enough, to get an exact solution. Here is a modern paradigm for the fruitful application of mathematics to physical science.

ACKNOWLEDGMENTS

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[80] A bibliography of papers published up to 1964 is given in my report "History of the Lenz–Ising model," University of California, Lawrence Radiation Laboratory, Livermore, UCRL-7940 (19 June 1964). A supplement covering more recent papers is now in preparation.

[81] Cf. the opening remarks and views of Frenkel quoted in Fisher's Boulder lectures (cited in Ref. 26).

[82] F. H. Stillinger, Jr., and H. L. Frisch, Physica 27, 751 (1961).

FIG. 2. Wilhelm Lenz (1888–1957). Photograph courtesy of Professor Dr. H. Raether.

Fig. 3. Ernst Ising. Photograph courtesy of Professor Ising.