Normal Modes of Vibrations of the Universe*

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This paper reviews some aspects of our knowledge of the gravitational theory of fluctuations of density in homogeneous and isotropic models of the universe. Irrotational fluid motions only are considered. All perturbations are assumed to be small and a normal mode analysis is used. The nature and time dependence of the amplitude of the various modes in expanding and contracting models of the universe are considered within the framework of Newtonian and general relativity theories. The origin of celestial structure requires that a uniform universe is unstable against arbitrarily small perturbations. However, the rate of growth of the allowed modes, particularly in an expanding universe, is sufficiently slow to cast doubt on the instability of the models.

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1. INTRODUCTION

Cosmology seeks to account mainly for the global nature of the universe. Instead of a complex and diverse physical universe it deals with a featureless, idealized universe that is everywhere isotropic and homogeneous and contains a uniform fluid of simple properties.[1,2] Out of such sweeping generalizations emerge a variety of elementary models, and one's choice for what it is worth is still largely an act of faith. In spite of its antiquity cosmology is in an immature state; it is hoped that the present idealizations will ultimately be replaced with more realistic representations of the physical universe.

Nobody believes that the universe originated in its present form, complete in all its array of macroscopic detail. Yet if the inchoate universe is featureless, then we must show how differentiation and structure contrive to evolve. The physics of stellar and galactic structure is the concern of astrophysics; but the nature and development of an environment favorable to the formation of astrophysical objects is the concern of cosmology.

The study of the vibrations of the universe is a fascinating subject, apart from any secondary considerations. However, in an attempt to account for the actual universe, as distinct from the uniform models, we might hope to expect that the universe is vibrationally unstable. Thus appropriate modes will grow relatively rapidly and, in a time short compared with the age of the universe, provide the foundations of astrophysical structure. It turns out, as many authors have discovered, that in an expanding universe the growth rates of the various modes of oscillation are generally too small to account for any appreciable irregularity.

In the following treatment all disturbances are assumed to have small amplitude, and the motions are also assumed to be irrotational; long range electromagnetic fields are neglected and only gravitational theory is used. The behavior of the disturbances is studied with both Newtonian and general relativity theory.

2. ORIGIN OF STRUCTURE IN THE UNIVERSE

2.1. Initial Conditions

The provision of both background and initial conditions for the origin and formation of structure is a cosmological problem. The subsequent evolution of structure into its detailed manifestations lies in the provinces of cosmogony, astrophysics, and every other science. At present there are two main hypotheses concerning the initial conditions[3]; these are: the *initial structure hypothesis* and the *instability hypothesis*.

In the initial structure hypothesis it is taken for

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[1] H. P. Robertson, Rev. Mod. Phys. **5**, 62 (1933).

[2] H. Bondi, *Cosmology* (Cambridge University Press, Cambridge, England, 1960).

[3] E. R. Harrison, Mem. Soc. Roy. Sci. Liege **15**, 15 (1967).

granted that structural differentiation in a primitive form originates with the universe and is an indispensable part of its design. The structure is modified and enhanced in the course of time and evolves according to the laws of physics. This hypothesis is as old as cosmology, and elsewhere[3] it is suggested that it should be updated and reconsidered in the light of modern knowledge. If the universe in its earliest stages consists of matter at very high density, then we must inquire whether structural configurations with rotation are a natural state of affairs under these extreme conditions. If by some quirk in the properties of matter,[4] or by some modification of the laws of physics,[5,6] the singular state can be avoided, it is possible that the universe contracts to and then expands from a state of finite density. It is also possible but by no means certain that some irregularity might survive passage through the "bounce" and thus act as the foundation out of which subsequent structure evolves. Although the initial structure hypothesis is interesting and exciting,[3] there is a complete lack of any generally acceptable work in this field, and therefore it will not be considered further in this discussion.

The instability hypothesis on the other hand does not assume any special initial conditions. This hypothesis is as young as the theory of gravitation, and asserts that the universe is unstable against small random perturbations. Jeans[7] points out that in some correspondence Newton remarks: "But if the matter were evenly disposed throughout an infinite space, it could never convene into one mass; but some of it would convene into one mass and some into another, so as to make an infinite number of great masses, scattered great distances from one another throughout all that infinite space. And thus might the sun and fixed stars be formed, supposing that matter were of a lucid nature." In his own work on gravitational instability Jeans writes[8]: "We have found that, as Newton first conjectured, a chaotic mass of gas of approximately uniform density and of very great extent would be dynamically unstable: nuclei would tend to form in it, around which the whole of the matter would ultimately condense." These comments were made with the idea in mind of a static universe. If we consider a universe already fragmented into widely separated "islands," each having a density large compared with the average density, such that the contents of each island do not partake in the expansion between the islands, then Jeans' comments are acceptable for condensations occurring in the islands. But these are the initial conditions of cosmogony: the problem for cosmology is to explain how it is possible in the first place for islands to form.

In view of our limited knowledge in cosmology the principal advantage of the instability hypothesis is that structure is not predetermined but grows naturally from small random disturbances according to the laws of physics. The smallness of such disturbances allows us at least initially to work within the framework of a linearized theory. In the initial structure hypothesis it is doubtful whether a linear theory is valid at any evolutionary stage.

We use conventional gravitation theory and therefore the steady state model is only briefly mentioned. The growth of perturbations in the steady state universe has been considered elsewhere.[9–14]

2.2. Linear Stability Theory

From the simplest of all possible points of view the diversity of the universe consists of variations in the density and motion of matter. Given a cosmological model that is a valid description of the universe in the large, it should be possible to show, with refinements of the cosmic fluid when necessary, that perturbations are capable of evolving in time into configurations of density and motion which resemble the grosser features of the physical world. The first step therefore is to inject some realism into the cosmological models by perturbing their fluid density and motion.

We suppose that all disturbances are small and consist of a superposition of normal modes of a complete set. Within a co-moving system of coordinates the unperturbed uniform fluid of the cosmological models is in a hydrodynamic stationary state. A system in a stationary state is unstable when a small disturbance grows in time and leads to a changed configuration of the system. Thus if one or more modes is time-growing, and the characteristic growth time is short compared with the lifetime of the system, the system is unstable.[15]

Planets, stars, stellar associations and clusters, galaxies, clusters of galaxies, . . . , form a hierarchy of structures in which the density amplitude diminishes as the spatial extent increases. In other words, if ρ is the mean density of the universe and $\rho+\delta\rho$ is the mean density of a system, then for relatively small systems $\delta\rho/\rho\gg 1$, and as the size of the system increases $\delta\rho/\rho$

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[4] E. R. Harrison, Nature **215**, 151 (1967).
[5] F. Hoyle, W. A. Fowler, G. R. Burbidge, and E. M. Burbidge, Astrophys. J. **139**, 909 (1964).
[6] F. Hoyle and J. V. Narlikar, Proc. Roy. Soc. (London) **A278**, 464 (1964).
[7] J. H. Jeans, *Astronomy and Cosmogony* (Cambridge University Press, Cambridge, England, 1929), p. 352.
[8] Reference 7, p. 415.

[9] D. W. Sciama, Monthly Notices Roy. Astron. Soc. **115**, 3 (1955).
[10] W. B. Bonnor, Monthly Notices Roy. Astron. Soc. **117**, 104 (1957).
[11] M. Harwitt, Monthly Notices Roy. Astron. Soc. **122**, 47 (1961).
[12] M. Harwitt, Monthly Notices Roy. Astron. Soc. **123**, 257 (1961).
[13] D. W. Sciama, Quart. J. Roy. Astron. Soc. **5**, 196 (1964).
[14] I. W. Roxburgh and P. G. Saffman, Monthly Notices Roy. Astron. Soc. **129**, 181 (1965).
[15] S. Chandrasekhar, Daedalus **86**, 323 (1957).

diminishes and eventually $\delta\rho/\rho<1$ for systems of large dimensions. Furthermore, as we go back in time the hierarchy of celestial structures progressively dissolves and is submerged in the increasing mean density of the universe. Thus the amplitude of all perturbations relative to the mean density diminishes, that is, $\delta\rho/\rho$ progressively becomes less for any system as we go backward in time. The expression

$$\mu=\delta\rho/\rho \qquad (1)$$

is referred to as the contrast density. It follows therefore that a linear stability theory is essentially cosmological in the sense that it is limited to small contrast densities, either remote in time or extending over cosmic distances. As the universe ages our treatment is therefore restricted to lower and lower modes of vibration of the universe. As the contrast density rises the predictions of the linear theory become less trustworthy, and when $\mu\sim1$, must be viewed with suspicion. To carry through the computation into $\mu>1$ demands more refined and comprehensive techniques.

The success of the instability hypothesis depends on the fulfillment of two conditions.[3] The first condition is morphological: structure must ultimately emerge out of amorphous initial conditions and possess a morphology corresponding with that observed. The second condition is that the rate of growth of the appropriate modes must be adequate. This latter condition is considered briefly in Sec. 6. The question of morphology leads us to two contending points of view: the fragmentation hypothesis and the clustering hypothesis.

In a normal mode analysis all possible wavelengths must not grow at the same rate in order to lay down the foundations of celestial structure. If the universe is initially unstable only for relatively long wavelength perturbations, we might conjecture with Jeans that there is a process of fragmentation[16] of "nebulae out of chaos, of stars out nebulae. . .," and so on. Thus, protogalaxies or larger masses first form and create an environment of enhanced density in which matter no longer expands with the universe, and which favors the formation of smaller condensations. Inhomogeneity, anisotropy, and complex properties of the fluid develop, and we are free to invoke all the cosmogonic paraphenalia of turbulence, magnetic fields, radiation, dust, and so forth, necessary for star formation.[17–21]

Alternatively, if the universe is initially unstable for relatively short wavelengths then we might conjecture that there is a process of clustering[22–25] whereby small scale condensations first form and subsequently interact to create larger and larger gravitationally bound systems. The arguments in favor of this process have not progressed very far nor have they gained wide acceptance.

All this, however, is speculation, and we must wait for cosmology to give a clear account of the origin of a differentiated universe.

From the cosmological point of view the concepts of fragmentation and clustering are by no means mutually exclusive. The universe could be unstable for a large class of modes, or a wide spectrum of wavelengths, and the rates of growth of the different wavelengths determine whether elementary structure evolves by fragmentation or clustering, or by both processes acting simultaneously. Conceivably, very early type stars evolve out of inhomogeneities laid down either prior to or at the same time as those leading to galactic structure.[3]

3. NEWTONIAN COSMOLOGY

3.1. Newtonian Models

In 1934 McCrea and Milne[26,27] used Newtonian theory to derive the equations of a universe obeying the cosmological principle.[2,28,29] It is assumed, as in general relativity, that in the unperturbed state there is everywhere a perfect fluid of uniform mass density ρ and isotropic pressure p. The equations are identical with those derived using general relativity theory, provided the pressure is negligible in comparison with the energy density ρc^2 (c is the speed of light). Several authors[30–36] have discussed the validity and limitations of Newtonian cosmology.

As we shall show, the equations of Newtonian and general relativity theory for the perturbed state are

[16] Reference 7, p. 416.
[17] J. M. Burgers and H. C. van de Hulst, *Gas Dynamics of Cosmic Clouds* (North-Holland Publ. Co., Amsterdam, 1955).
[18] I.A.U. Symposium No. 8, Rev. Mod. Phys. **30**, 905 (1958).
[19] G. R. Burbidge, F. D. Kahn, R. Ebert, S. von Hoerner, and S. Temesevåry, *Die Enstehung von Sternen durch Kondensation diffuses Materie* (Springer-Verlag, Berlin, 1960).
[20] L. Woltjer, *Interstellar Matter in Galaxies* (W. A. Benjamin, Inc., New York, 1962).
[21] L. Spitzer, Jr., "Dynamics of Interstellar Matter and the Formation of Stars," in *Stars and Stellar Systems*, G. P. Kuiper and B. M. Middlehurst, Eds. (University of Chicago Press, Chicago, Ill., to be published), Vol. VII, Chap. 9.
[22] G. Gamow and E. Teller, Phys. Rev. **55**, 654 (1939).
[23] D. Layzer, Astron. J. **59**, 170 (1954).
[24] D. Layzer, Astrophys. J. **137**, 351 (1963).
[25] D. Layzer, Ann. Rev. Astron. Astrophys. **2**, 341 (1964).
[26] E. A. Milne, Quart. J. Math. **5**, 64 (1934).
[27] W. H. McCrea and E. A. Milne, Quart. J. Math. **5**, 73 (1934).
[28] E. A. Milne, *Relativity, Gravitation and World Structure* (Clarendon Press, Oxford, England, 1935), p. 40.
[29] J. D. North, *The Measure of the Universe* (Clarendon Press, Oxford, England, 1965).
[30] D. Layzer, Astron. J. **59**, 268 (1954).
[31] W. H. McCrea, Astron. J. **60**, 271 (1955).
[32] O. Heckmann and E. Shücking, Z. Astrophys. **38**, 95 (1955).
[33] G. C. McVittie, *General Relativity and Cosmology* (Chapman and Hall, London, 1956), Chap. 7, and p. 192.
[34] O. Heckmann and E. Schücking, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1959), Vol. **53**, p. 489.
[35] C. Callan, R. H. Dicke, and P. J. E. Peebles, Am. J. Phys. **33**, 105 (1965).
[36] E. R. Harrison, Ann. Phys. (N. Y.) **35**, 437 (1965).

also identical when the pressure is negligible in comparison with ρc^2. The great advantage of the Newtonian treatment is its simplicity; furthermore, it provides physical insight which helps to reduce the general relativity equations to their simplest form. Before proceeding to the Newtonian equations of a universe in a perturbed state, the treatment for the unperturbed state is presented briefly.

Let $\mathbf{r}$ be the position at time t_0 of any element of fluid. At time t, let the position of the same fluid element be $(R/R_0)\mathbf{r}$, where $R(t)$ is a universal function of time and $R_0=R(t_0)$. This condition ensures that the fluid density ρ remains uniform and is a function only of time. Thus $\mathbf{r}$ is a comoving position vector and r, θ, ϕ are comoving spherical coordinates. The velocity and acceleration of a fluid element are

$$\mathbf{u}=(\dot{R}/R_0)\mathbf{r}, \tag{2}$$

$$d\mathbf{u}/dt=(\ddot{R}/R_0)\mathbf{r}, \tag{3}$$

where dots denote time differentiation. Furthermore, within the comoving coordinate system the ordinary gradient operator ∇' transforms to $(R_0/R)\nabla$.

In a perfect fluid the pressure is a scalar, and the equations of motion, continuity, and Poisson's equation are

$$(R/R_0)(d\mathbf{u}/dt)=-\nabla\psi^*-\rho^{-1}\nabla p, \tag{4}$$

$$(R/R_0)(\partial\rho/\partial t)=-\nabla(\rho\mathbf{u}), \tag{5}$$

$$(R_0/R)^2\nabla^2\psi^*=4\pi G\rho-\Lambda, \tag{6}$$

where p is the pressure, G the gravitational constant, ψ^* the gravitational potential and Λ the cosmological term. In a uniform universe the ∇p term in (4) vanishes. Using (3), and taking the divergence of (4), it follows that

$$3\ddot{R}+(4\pi G\rho-\Lambda)R=0. \tag{7}$$

From the equation of continuity (5),

$$\rho R^3=\text{const}, \tag{8}$$

and hence (7) can be integrated and becomes

$$\dot{R}^2=\tfrac{1}{3}(8\pi G\rho+\Lambda)R^2-\kappa. \tag{9}$$

A universal constant of integration C' is absorbed by the transformation $R\rightarrow R|C'|^{1/2}$, and R has now the dimension of time and κ is dimensionless and has the value of 1, 0, or -1.

Equations (8) and (9) are the Newtonian equations of an isotropic and homogeneous universe, and are identical with those usually derived with general relativity when the pressure is small compared with the energy density ρc^2. (First derived by Friedmann[37,38] for $\kappa=\pm1$, $\Lambda=0$, and Einstein and de Sitter[39] for $\kappa=0$, $\Lambda=0$).

Using the constant

$$C=8\pi G\rho R^3/3, \tag{10}$$

the solutions of (9), when $\Lambda=0$, are

$$\kappa=0\quad\begin{cases}R=C\chi^2,\\ t=\tfrac{2}{3}C\chi^3,\end{cases} \tag{11}$$

$$\kappa=1\quad\begin{cases}R=C\sin^2\chi,\\ t=C(\chi-\sin\chi\cos\chi),\end{cases} \tag{12}$$

$$\kappa=-1\begin{cases}R=C\sinh^2\chi,\\ t=C(\sinh\chi\cosh\chi-\chi),\end{cases} \tag{13}$$

where $t(\chi)$ is found by integrating

$$dt=2Rd\chi. \tag{14}$$

General solutions of (9) are available[40] in terms of elliptic functions and various specific solutions have been given elsewhere.[2,41,42]

The total energy E is

$$E=\Psi+T, \tag{15}$$

where Ψ is the gravitational energy and T is the kinetic energy of the fluid. In a sphere of arbitrary radius r:

$$\Psi=-G\int\frac{\rho^2V}{r}dV=-\frac{16}{15}\left(\frac{R}{R_0}\right)^5\pi^2G\rho^2r^5, \tag{16}$$

$$T=\int\tfrac{1}{2}\rho u^2dV=\frac{2}{5}\frac{\dot{R}^2R^3}{R_0^5}\pi\rho r^5, \tag{17}$$

where $dV=4\pi(R^3/R_0^3)r^2dr$ is an element of volume. From (9) and $E=\Psi+T$, we find

$$E=(\tfrac{1}{3}\Lambda R^2-\kappa)T/\dot{R}^2. \tag{18}$$

When $\Lambda=0$, $E=-\kappa T/\dot{R}^2$, and the energy is positive for $\kappa=-1$, zero for $\kappa=0$, and negative for $\kappa=1$.

The advantage of the Newtonian equations is the ease with which they can be physically interpreted. Thus, assuming $\Lambda=0$ when $\kappa=0$, $E=0$, the fluid elements have velocities equal to their escape velocity and their trajectories are of the parabolic class; and for $\kappa=1(\kappa=-1)$, $E<0(E>0)$, the fluid elements have velocities less (more) than their escape velocity and their trajectories are of the elliptical (hyperbolic)

[37] A. Friedmann, Z. Physik **10,** 377 (1922).
[38] A. Friedmann, Z. Physik **21,** 326 (1924).

[39] A. Einstein and W. de Sitter, Proc. Natl. Acad. Sci. (U. S.) **18,** 213 (1932).
[40] G. Lemaître, Ann. Soc. Sci. Bruxelles **47A,** 49 (1927).
[41] G. Lemaître, Monthly Notices Roy. Astron. Soc. **91,** 483 (1931).
[42] G. C. McVittie, *General Relativity and Cosmology* (University of Illinois Press, Urbana, Ill., 1965).

class. These interpretations are less obvious when (9) is derived from general relativity. In that case κ is the curvature constant and space is flat ($\kappa=0$), spherical or elliptical ($\kappa=1$), and hyperbolic ($\kappa=-1$).

3.2. Perturbed Newtonian Models

Various authors[10,43–55] have used Newtonian gravitational theory to study the time dependence of density fluctuations in a uniform fluid of finite or infinite extension. From the cosmological point of view a Newtonian treatment is scarcely adequate; not only is it limited to low density, but also the long wavelength modes of Euclidean space are inapplicable in curved space. Fluctuations at high density and large-scale fluctuations at low density are the conditions for which the linear theory is most valid but the Newtonian approach is least valid. Nevertheless, the simplicity of the Newtonian approach serves as a valuable guide in the subsequent treatment.

For a collisionless fluid, such as a supergas of stars or galaxies, the formal approach is by way of the Vlasov equations,[56] as in plasma physics. This has been used by Gilbert,[54] and by Sweet[57] for counterstreaming fluids. Particles traveling an appreciable fraction of a wavelength in an oscillation period cause Landau damping. Since we are concerned with an initially structureless fluid we use the fluid approximation; this tends to overestimate the rate of growth of perturbations. In this discussion the velocity components are everywhere single-valued.

Let the disturbed velocity, density, pressure, and gravitational potential be

$$\mathbf{u}\rightarrow\mathbf{u}+\mathbf{v}', \qquad \rho\rightarrow\rho+\delta\rho,$$

$$p\rightarrow p+\delta p, \qquad \psi^*\rightarrow\psi^*+\psi,$$

where the small quantities are functions of $\mathbf{r}$ and t. In the usual coordinates $(t, \mathbf{r}R/R_0)$ the linearized equations of motion and continuity are

$$[(\partial/\partial t)+\mathbf{u}\cdot\nabla']\mathbf{v}'+\mathbf{v}'\nabla'\mathbf{u}+\nabla'\psi+(1/\rho)\nabla'\delta p=0, \quad (19)$$

$$[(\partial/\partial t)+\mathbf{u}\cdot\nabla'+\nabla'\cdot\mathbf{u}]\delta\rho+\rho\nabla'\cdot\mathbf{v}'=0, \quad (20)$$

and terms quadratic in small quantities are neglected. The perturbation of Poisson's equation is

$$\nabla'^2\psi=4\pi G\delta\rho. \quad (21)$$

The transformations to co-moving coordinates are

$$\nabla'\rightarrow(R/R_0)\nabla, \quad (22)$$

$$\partial/\partial t+\mathbf{u}\cdot\nabla'\rightarrow d/dt, \quad (23)$$

where $\mathbf{u}\nabla'=r(\dot{R}/R)\partial/\partial r$, and the time derivative now follows the unperturbed motion of the fluid. In ordinary coordinates $\partial/\partial t$ commutes with ∇', whereas in co-moving coordinates

$$(d/dt)\nabla=\nabla(d/dt). \quad (24)$$

We also express the perturbed velocity in co-moving coordinates:

$$\mathbf{v}'=(R/R_0)\dot{\mathbf{r}}=(R/R_0)\mathbf{v}, \quad (25)$$

[i.e., $dR\mathbf{r}/R_0dt=(\dot{R}\mathbf{r}/R_0)+(R\dot{\mathbf{r}}/R_0)$ and $\mathbf{u}=\dot{R}\mathbf{r}/R_0$, $\mathbf{v}=\dot{\mathbf{r}}$].

Thus, in terms of co-moving coordinates, (19)–(21) become

$$(d/dt)[(R^2/R_0^2)\mathbf{v}]+\nabla\psi+\rho^{-1}\nabla\delta p=0, \quad (26)$$

$$(d/dt)(\delta\rho/\rho)+\nabla\cdot\mathbf{v}=0, \quad (27)$$

$$(R_0^2/R^2)\nabla^2\psi=4\pi G\delta\rho, \quad (28)$$

where $\nabla\cdot\mathbf{u}=3\dot{R}/R$, $\mathbf{v}\cdot\nabla\mathbf{u}=\mathbf{v}\dot{R}/R$, and $\rho R^3=$const.

Taking the curl of (26), we have

$$(d/dt)(R^2\boldsymbol{\zeta})=0, \quad (29)$$

in which $\nabla\wedge\mathbf{v}=\boldsymbol{\zeta}$ is the vorticity in co-moving coordinates. Thus $R^2\boldsymbol{\zeta}$ is a conserved quantity. Equation (29) is identical with the Helmholtz equation[58]

$$(d/dt)(\boldsymbol{\zeta}/\rho)-(\boldsymbol{\zeta}/\rho)\cdot\nabla\mathbf{u}=0,$$

since $\boldsymbol{\zeta}\cdot\nabla\mathbf{u}=\boldsymbol{\zeta}\dot{R}/R$. In the unperturbed state both $\mathbf{v}$ and $\boldsymbol{\zeta}$ are zero; the vorticity is therefore permanently zero and the motion is irrotational. Hence, we can write

$$v=\nabla\varphi. \quad (30)$$

Taking the divergence of (26), and using (27) and (28), we find

$$\left\{\left(\frac{d}{dt}+2\frac{\dot{R}}{R}\right)\frac{d}{dt}-4\pi G\rho\right\}\frac{\delta\rho}{\rho}=\frac{R_0^2p}{R^2\rho}\nabla^2\frac{\delta p}{p}. \quad (31)$$

The variables can be separated, and for any scalar quantity $\psi=\psi(t)\Pi(r,\theta,\phi)$:

$$\nabla^2\Pi+k^2\Pi=0, \quad (32)$$

where k^2 is the separation constant.

Adiabatic fluctuations. For the determination of the four unknowns δp, $\delta\rho$, ψ, and φ there are the three

[43] J. H. Jeans, Phil. Trans. **A199**, 49 (1902); also Ref. 7, p. 345.
[44] G. Gamow, Phys. Rev. **74**, 505 (1948).
[45] F. Hoyle, Astrophys. J. **118**, 513 (1953).
[46] R. Ebert, Z. Astrophys. **37**, 217 (1955).
[47] G. B. van Albada, Bull. Astron. Inst. Netherlands **15**, 165 (1960).
[48] R. Simon, Bull. Acad. Roy. Belg. **47**, 731 (1961).
[49] G. V. van Albada, Astron. J. **66**, 590 (1961).
[50] C. Hunter, Astrophys. J. **136**, 594 (1962).
[51] M. P. Savedoff and S. Vila, Astrophys. J. **136**, 609 (1962).
[52] D. Layzer, Astrophys. J. **137**, 351 (1963).
[53] R. Simon, Ann. Astrophys. **27**, 191 (1964).
[54] I. R. Gilbert, Astrophys. J. **144**, 233 (1966).
[55] T. T. Arny, Astrophys. J. **145**, 572 (1966).
[56] See, for example, I. B. Bernstein, S. K. Trehan, and M. P. H. Weenik, Nucl. Fusion **4**, 61 (1964).
[57] P. A. Sweet, Monthly Notices Roy. Astron. Soc. **125**, 285 (1963).
[58] L. Milne-Thompson, *Theoretical Hydrodynamics* (MacMillan and Co. Ltd., London, 1962), p. 83.

equations (27), (28), and (31), and an equation of state $\delta p(\rho,\ \delta\rho)$ is therefore necessary. We consider briefly some possibilities, including thermal instability.[59–63] Let $L(\rho,\ p)$ be the energy lost by the fluid per unit volume per unit time. Thus if δQ is the energy gained per unit mass, then

$$\rho(dQ/dt) = -L,$$

and therefore from $\delta Q = dU + pdV$,

$$\frac{\rho^\gamma}{\gamma-1}\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right)+L=0. \tag{33}$$

The first-order perturbation of (33) is

$$\frac{1}{\gamma-1}\frac{d}{dt}\left(\frac{\delta p}{p}-\gamma\frac{\delta \rho}{\rho}\right)+\delta\left(\frac{L}{p}\right)=0. \tag{34}$$

In the simplest case of all the fluctuations are adiabatic and $L=\delta L=0$:

$$\delta p/\delta\rho=\gamma(p/\rho)=c_s^2, \tag{35}$$

where c_s is the speed of sound. Thus our three equations, (31), (27), and (28), become

$$\ddot{\mu}+2(\dot{R}/R)\dot{\mu}+[\gamma k^2(p/\rho)(R_0^2/R^2)-4\pi G\rho]\mu=0, \tag{36}$$

$$\dot{\mu}=k^2\varphi, \tag{37}$$

$$4\pi G\rho\mu=-(R_0^2/R^2)k^2\psi, \tag{38}$$

and $\mu=\delta\rho/\rho$. These equations determine the adiabatic fluctuations of density, velocity, and gravitational potential for an inviscid, irrotational fluid. Equation (36) was first derived by Bonnor[10] for the radial modes $\mu(r,\ t)$.

Isothermal fluctuations. We consider next a thermally conducting fluid in which

$$L=-\nabla'\cdot K\nabla' T=-(R_0^2/R^2)\nabla\cdot K\nabla T$$

$$\delta L=-(R_0^2/R^2)K\nabla^2\delta T=(R_0^2/R^2)k^2K\delta T, \tag{39}$$

where $K(\rho,\ p)$ is the thermal conductivity. With the expressions

$$p=\rho \mathrm{k} T/m,$$

$$\delta T/T=(\delta p/p)-(\delta\rho/\rho),$$

(in which m is the mean molecular weight) (34) takes the form

$$\left(\frac{d}{dt}+\alpha\right)\frac{\delta p}{p}=\left(\gamma\frac{d}{dt}+\alpha\right)\frac{\delta\rho}{\rho}, \tag{40}$$

$$\alpha=(\gamma-1)mk^2KR_0^2/R^2\,\mathrm{k}\rho. \tag{41}$$

Equations (31) and (40) combine to give a third-order differential equation in $\delta\rho/\rho$:

$$\left[\left(\frac{d}{dt}+\alpha\right)\frac{\rho R^2}{pR_0^2}\left\{\left(\frac{d}{dt}+2\frac{\dot{R}}{R}\right)\frac{d}{dt}-4\pi G\rho\right\}+k^2\left(\gamma\frac{d}{dt}+\alpha\right)\right]\mu=0. \tag{42}$$

In spite of its appearance this equation is not greatly different from the previous equation (36). For thermal conductivity to play an active role heat must be transported a distance $\lambda\sim k^{-1}$ in a time short compared with the characteristic growth time τ of the disturbance $\delta\rho/\rho$. In other words, δp is influenced mainly by heat conductivity in a time sufficiently short to regard $\delta\rho/\rho$ as constant, and therefore, from (40)

$$\frac{\delta p}{p}-\left(\frac{\delta\rho}{\rho}\right)_{t_0}\simeq\left(\frac{\delta p}{p}-\frac{\delta\rho}{\rho}\right)_{t_0}\exp[-\alpha(t-t_0)]. \tag{43}$$

Hence, for $t-t_0=\tau\gg\alpha^{-1}$, $\delta L=0$, and the fluctuations are isothermal:

$$\delta p/p=\delta\rho/\rho.$$

If, for example, the characteristic growth time is $\tau=(4\pi G\rho)^{-1/2}$, it follows

$$(4\pi G\rho)^{1/2}(\gamma-1)k^2KT \begin{cases}\gg p, & \tilde{\gamma}=1,\\ \ll p, & \tilde{\gamma}=\gamma,\end{cases} \tag{44}$$

where

$$\delta p/\delta\rho=\tilde{\gamma}(p/\rho)=c_s^2. \tag{45}$$

For the unperturbed quantities we have the adiabatic relation

$$p=\text{const}\ \rho^\gamma=(p_0/\rho_0^\gamma)\rho^\gamma,$$

where p_0, ρ_0 refer conveniently to time t_0 when $R=R_0$.

In many instances (42) simplifies therefore to

$$\ddot{\mu}+2(\dot{R}/R)\dot{\mu}+[\tilde{\gamma}k^2(p/\rho)(R_0^2/R^2)-4\pi G\rho]\mu=0, \tag{46}$$

either by using (31) and (45) or by noting that in (42) we obtain (46) when $\alpha\gg d/dt$ with $\tilde{\gamma}=1$, and when $\alpha\ll d/dt$ with $\tilde{\gamma}=\gamma$. In effect, short wavelength fluctuations are isothermal and long wavelength fluctuations are adiabatic.

Radiative cooling. The inclusion of radiation is of particular importance, as first shown by Parker[58] and Weymann.[61] We consider the case where $\epsilon(\rho,\ p)$ is the net energy radiated per unit volume per unit time expressed in the form

$$\epsilon=\text{const}\ \rho^n p^m \tag{47}$$

and neglect all dependence on the energy spectrum.[64] Generally $n>0$, but m may be positive or negative; for bound–free transitions $n=\frac{5}{2}$, $m=-\frac{1}{2}$, and for free–free transitions $n=\frac{3}{2}$, $m=\frac{1}{2}$. The effect of radiation on

[59] E. N. Parker, Astrophys. J. **117**, 431 (1953).
[60] H. Zanstra in *Gas Dynamics of Cosmic Clouds*, J. M. Burgers and H. C. van de Hulst, Eds. (North-Holland Publ. Co., Amsterdam, 1955), Chap. 13.
[61] H. Zanstra in *Vistas in Astronomy* 1, A. Beer, Ed. (Pergamon Press, New York, 1955), p. 256.
[62] R. Weymann, Astrophys. J. **132**, 452 (1960).
[63] G. B. Field, Astrophys. J. **142**, 531 (1965).
[64] R. Weymann, Astrophys. J. **145**, 560 (1965).

the formation of galaxies was first considered by Gamow,[65,66] and has been considered more recently by Peebles[67] following the discovery of blackbody radiation of 3°K by Penzias and Wilson.[68] In a Newtonian treatment of the universe we must assume that the radiation density is low in order that $p \ll \rho c^2$, and furthermore we propose to ignore the radiative drag on the fluid.

Thermal instability is generally discussed in connection with the formation of stars and other such isolated systems from which radiation can escape. From (47) we have $\epsilon \propto \rho^{n+m} T^m$, and hence

$$\delta\epsilon/\epsilon = (n+m)(\delta\rho/\rho) + m(\delta T/T). \qquad (48)$$

Originally Parker[58] considered a static configuration of $\delta\rho = 0$, and clearly when $m < 0$ we have a system which is cooled by radiation, and the more it cools the more it radiates. This runaway state of affairs, possible with bound–bound and bound–free transitions, is referred to as thermal instability.

Retaining the heat conductivity, we have

$$p\delta\left(\frac{L}{p}\right) = \epsilon n \frac{\delta\rho}{\rho} + \epsilon(m-1)\frac{\delta p}{p} + \frac{R_0^2}{R^2} k^2 K \delta T,$$

and hence (34) can be written as

$$\left\{\frac{d}{dt} + \alpha + (m-1)\beta\right\}\frac{\delta p}{p} = \left(\gamma \frac{d}{dt} + \alpha - n\beta\right)\frac{\delta\rho}{\rho}, \qquad (49)$$

$$\beta = (\gamma - 1)\epsilon/p. \qquad (50)$$

As before, from (31), it is found

$$\left[\left\{\frac{d}{dt} + \alpha + (m-1)\beta\right\}\frac{\rho R^2}{p R_0^2} \times \left\{\left(\frac{d}{dt} + 2\frac{\dot{R}}{R}\right)\frac{d}{dt} - 4\pi G\rho\right\} + k^2\left(\gamma\frac{d}{dt} + \alpha - n\beta\right)\right]\mu = 0. \qquad (51)$$

By itself, this equation is inadequate; in a general treatment we should retain the derivatives $\partial\epsilon/\partial\rho$, $\partial\epsilon/\partial p$ and use the radiative transport equations. So far, such a general treatment has not been published. However, in many instances (51) can be simplified. Radiative cooling is effective when $\delta p/p$ changes appreciably in a time short compared with the characteristic time τ for a change in $\delta\rho/\rho$. Therefore, integrating (49)

$$\frac{\delta p}{p} - \left(\tilde{\gamma}\frac{\delta\rho}{\rho}\right)_{t_0} \simeq \left(\frac{\delta p}{p} - \tilde{\gamma}\frac{\delta\rho}{\rho}\right)_{t_0} \times \exp\{-[\alpha + (m-1)\beta](t - t_0)\}, \qquad (52)$$

$$\tilde{\gamma} = (\alpha - n\beta)/[\alpha + (m-1)\beta], \qquad (53)$$

and in time $\tau = t - t_0$ radiative cooling is effective if $t - t_0 \gg [\alpha + (m-1)\beta]^{-1}$. When $m > 1$, the exponential term is small and there is thermal stability[64]; the fluid does not increase its radiative output as it cools. The temperature distribution adjusts itself such that $\delta(L/p) \simeq 0$, and therefore $\delta p/\delta\rho \simeq \tilde{\gamma} p/\rho$. The regions of enhanced density are cooled more rapidly and an equilibrium condition is thus established in which the radiation loss is uniform. Although such a situation is thermally stable, it is not necessarily dynamically stable. Thus, for wavelengths long enough to neglect thermal conduction ($\beta \gg \alpha$), we have $\tilde{\gamma} = -n/(m-1)$, and for $m > 1$ the speed of sound is imaginary and pressure gradients favor the formation of condensations. Presumably this form of thermodynamic instability requires $\lambda\!\!\!^{-2} \ll 3c\tau\lambda_r$, where λ_r is a photon mean free path, in order that radiation diffuses out of a region of enhanced density in a time short compared with τ.

When $m < 1$, then clearly the exponential term in (52) can be large for certain modes and there now exists thermal instability. The value of $\delta p/p$ is positive and increases primarily because the fluid is more or less uniformly cooled by radiation. Thermal instability is of undoubted importance in such subjects as star formation in which a finite isolated region is cooled while in the process of collapse, whereas in cosmology a mechanism of cooling the cosmic fluid has only slight effect while $p \ll \rho c^2$, and merely reduces the pressure gradients without necessarily creating pressure gradients which favor the formation of condensations. In fact, thermal instability, unchecked by other considerations, leads to a condition of zero fluid pressure ($c_s = 0$).

When radiative cooling is effective in the case of thermodynamic instability, then $\delta L = 0$ for an intermediate range of wavelengths of $KT/\epsilon < \lambda\!\!\!^{-2} < 3c\tau\lambda_r$. If the emission and absorption processes balance, and $\epsilon = a\rho^n p^m - b\rho = 0$, then $\delta\epsilon = 0$ gives $\tilde{\gamma} = -(n-1)/m$ for the intermediate wavelengths. In any event we assume that (46) with arbitrary values of γ and $\tilde{\gamma}$ covers many cases of interest. Such a simple single-fluid equation ignores the ubiquitous blackbody radiation, and a completely general multifluid treatment is not yet available. During the time when the radiation density is appreciable[65–67] the situation is more complex and Saslaw[69] has shown that instabilities occur during the helium forming phase and when matter later decouples from the radiation.

3.3. Normal Modes of Vibration

An aribitrary disturbance in a scalar quantity consists of a superposition of normal modes given by (32). These modes can be constructed from plane waves

$$\Pi_k \propto \exp(i\mathbf{k}\cdot\mathbf{r}). \qquad (54)$$

[65] G. Gamow, Rev. Mod. Phys. **21**, 367 (1949).
[66] G. Gamow in *Vistas in Astronomy*, A. Beer, Ed. (Pergamon Press, London, 1956), Vol. 2, p. 1726.
[67] P. J. E. Peebles, Astrophys. J. **142**, 1317 (1965).
[68] A. A. Penzias and R. Wilson, Astrophys. J. **142**, 419 (1965).
[69] W. C. Saslaw, Monthly Notices Roy. Astron. Soc. **136**, 39 (1967).

The modes in spherical coordinates are given for comparison with those derived later for curved space.

Let $\Pi=\Psi(r)\Theta(\theta)\Phi(\phi)$; by separating the variables, Eq. (32) in flat space becomes

$$(d^2\Phi/d\phi^2)+m^2\Phi=0, \tag{55}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(n(n+1)-\frac{m^2}{\sin^2\theta}\right)\Theta=0, \tag{56}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right)+\left(k^2-\frac{n(n+1)}{r^2}\right)\Psi=0. \tag{57}$$

The tesseral harmonics of nth degree and mth order are

$$\Theta\Phi=Y_n^m(\theta,\phi)=(a_{nm}e^{im\phi}+b_{nm}e^{-im\phi})P_n^m(\cos\theta),$$

for integral values of n and m, and the spherical surface harmonics of degree n are

$$Y_n(\theta,\phi)=\sum_{m=0}^{n}Y_n^m(\theta,\phi).$$

The solutions for the radial function $\Psi(r)$, for n integral, are the spherical Bessel functions

$$\Psi_n(\pi/2kr)^{1/2}J_{n+1/2}(kr),\qquad \Psi_{-n}=(\pi/2kr)^{1/2}J_{n-1/2}. \tag{58}$$

There are no boundary or periodic conditions to satisfy, and the only condition is that Ψ must be finite everywhere. For $kr\to 0$,

$$\Psi_n\to(kr)^n/[1\cdot3\cdot5\cdots(2n+1)],$$

$$\Psi_{-n}\to[1\cdot3\cdot5\cdots(2n-1)]/(kr)^{n+1},$$

and $kr\to\infty$,

$$\Psi_n\to(kr)^{-1}\cos[kr-\tfrac{1}{2}(n+1)\pi],$$

$$\Psi_{-n}\to(kr)^{-1}\sin[kr-\tfrac{1}{2}(n+1)\pi],$$

and hence Ψ_{-n} in (58) is rejected since it diverges as $kr\to 0$. The radial functions form a continuous set having all eigenvalues of $k>0$ for each value of n. In particular, for $n=1$, 2, and 3:

$$\Psi_0=(kr)^{-1}\mathcal{S},$$

$$\Psi_1=(kr)^{-1}[(kr)^{-1}\mathcal{S}-\mathcal{C}],$$

$$\Psi_2=(kr)^{-1}[\{3(kr)^{-2}-1\}\mathcal{S}-3(kr)^{-1}\mathcal{C}], \tag{59}$$

$\mathcal{S}=\sin kr$, $\mathcal{C}=\cos kr$. The spatial disturbances are represented by summations and integrations over the complete set of wave functions

$$\Pi=\Psi_n(kr)Y_n^m(\theta,\phi). \tag{60}$$

4. GRAVITATIONAL INSTABILITY

4.1. General Criteria

Let

$$\alpha_m=R^m\delta\rho\propto\delta\rho/\rho^{m/3}. \tag{61}$$

The existence of time-growing modes of an arbitrary quantity such as α_m does not necessarily imply instability in a nonstatic universe. For example, if $\alpha_0=\delta\rho$ grows, an expanding universe is unstable, but a contracting universe is stable if ρ increases more rapidly than $\delta\rho$. The only time-growing quantity that denotes unambiguously a changing configuration is the contrast density $\alpha_3=\mu$. Suppose that α_m grows in time; then in an expanding (contracting) universe $m=3$ is necessary and $m<3(m>3)$ is sufficient for instability. Thus the growth of the gravitational potential $\psi\propto\alpha_2$ is sufficient for instability only in an expanding universe.

When the growth time of a mode is greater than the age of the universe it cannot contribute to a significant change in configuration. For instability we require

$$\dot{\mu}/\mu\gg|\dot{R}/R|, \tag{62}$$

or, if $\mu\propto R^{\pm m}$ (m for $R>0$, $-m$ for $R<0$), then m should be large compared with unity. Even then a clear case of instability requires an adequate amplitude of the initial disturbance.

The Newtonian equations of the unperturbed cosmological models and the linearized Newtonian equations of the perturbed models are identical with the corresponding general relativity equations for small pressure. The Newtonian equations, however, are limited to flat space and cannot be used to determine the behavior of modes in curved space. This is not a severe limitation if the wavelengths are not immoderately large.

From (46), (7)–(9), the general equation for α_m is

$$\ddot{\alpha}_m+(8-2m)\frac{\dot{R}}{R}\dot{\alpha}_m+\left\{Q^2+(6-m)\frac{\ddot{R}}{R}+(4-m)(3-m)\frac{\dot{R}^2}{R^2}-\Lambda\right\}\alpha_m=0, \tag{63}$$

$$Q^2=\tilde{\gamma}k^2pR_0^2/\rho R^2, \tag{64}$$

and $Q=c_skR_0/R$. We assume that $p\propto\rho^\gamma$, and γ, $\tilde{\gamma}$ are arbitrary. For $m=2$, $\alpha_2\propto\psi$ from (28), and therefore

$$\ddot{\psi}+4(\dot{R}/R)\dot{\psi}+[Q^2-2(\kappa/R^2)+\Lambda]\psi=0. \tag{65}$$

For the contrast density, $\alpha_3\propto\mu$, and therefore for $m=3$:

$$\ddot{\mu}+2(\dot{R}/R)\dot{\mu}+(Q^2-4\pi G\rho)\mu=0. \tag{66}$$

This equation, first derived by Bonnor[10] and van Albada[47] for radial type perturbations, is discussed more generally by Savedoff and Vila.[51] Since $\mathbf{v}=d\delta\mathbf{r}/dt$, (27) can be integrated to give

$$\delta\rho/\rho=-\nabla\cdot\delta\mathbf{r}=-i\mathbf{k}\cdot\delta\mathbf{r}, \tag{67}$$

using (54). Introducing the Langrangian displacement $\boldsymbol{\xi}=R\delta\mathbf{r}/R_0$, it follows $\alpha_4\propto\xi$, where ξ is the component of the displacement parallel to $\mathbf{k}$. From (28), with

TABLE I. Time dependence of α_m.[a]

		$m<4$	$m=4$	$m>4$
$\gamma<\frac{4}{3}$	$\dot{R}>0$	$\sim\downarrow$, (?)	$\sim$, ($\uparrow$)	$\sim\uparrow$, ($\uparrow$)
	$\dot{R}<0$	$\sim\uparrow$, ($\uparrow$)	$\sim$, ($\uparrow$)	$\sim\downarrow$, (?)
$\gamma=\frac{4}{3}$	$\dot{R}>0$	?, (?)	$\uparrow$, ($\uparrow$)	$\uparrow$, ($\uparrow$)
	$\dot{R}<0$	$\uparrow$, ($\uparrow$)	$\uparrow$, ($\uparrow$)	?, (?)
$\gamma>\frac{4}{3}$	$\dot{R}>0$	?, ($\sim\uparrow$)	$\uparrow$, ($\sim$)	$\uparrow$, ($\sim\uparrow$)
	$\dot{R}<0$	$\uparrow$, ($\sim\uparrow$)	$\uparrow$, ($\sim$)	?, ($\sim\downarrow$)

[a] The symbols denote: $\dot{R}>0$, expansion; $\dot{R}<0$, contraction; $\uparrow$, growth; $\downarrow$, decay; $\sim$, oscillating; without brackets, $R>R_0$; with brackets, $R<R_0$. For example: ($\sim\downarrow$) means decaying oscillation for $R<R_0$. At $R=R_0$, $\lambda=(3c_0^2/8\pi G\rho_0)^{1/2}$, where c_0 is the speed of sound.

$m=4$, the equation for the displacement is

$$\ddot{\xi}+(Q^2-\tfrac{8}{3}\pi G\rho-\tfrac{1}{3}\Lambda)\xi=0. \tag{68}$$

This last equation shows that when

$$Q^2>\tfrac{1}{3}(8\pi G\rho+\Lambda)=(\dot{R}^2+\kappa)R^{-2}, \tag{69}$$

ξ is oscillatory. This is therefore the condition that $\alpha_m(t)$ is an oscillating function for all values of m. Now, from (64), $Q^2=c_s^2k^2R_0^2/R^2$, and also the velocity of expansion (contraction) is $u=\dot{R}r/R_0$, and therefore from (69) the condition that α_m is an oscillatory function is

$$c_s^2k^2r^2>u^2+(\kappa r^2/R_0^2)=u^2+\kappa(r'^2/R^2), \tag{70}$$

where $r'=rR/R_0$ is the distance from the origin. Thus disturbances are more likely to be periodic in time when when $\kappa=+1$. Suppose that $\kappa=0$; then a mode corresponding to $kr=1$ is periodic in time when

$$c_s^2>u^2. \tag{71}$$

In other words, out to a distance where the expansion (contraction) velocity is equal to the velocity of sound, all modes are oscillatory. The oscillation may of course be time-growing (overstability[70]) or time-decaying, depending on whether the universe is expanding or contracting, and also on the particular variable in the α_m sequence under consideration.

The equation (68) for α_4 is particularly useful for discussing the behavior of all α_m. We have

$$Q^2=Q_0^2(R_0/R)^{3\gamma-1}, \tag{72}$$

$$Q_0^2=\tilde{\gamma}k^2p_0/\rho_0=c_0^2k^2, \tag{73}$$

where c_0 is the speed of sound at $R=R_0$. Thus (68) becomes

$$\ddot{\alpha}_4+\{Q_0^2(R_0/R)^{3\gamma-4}-(C/R_0^3)\}(R_0/R)^3\alpha_4=0, \tag{74}$$

where C is given by (10). Unless stated otherwise we assume that the cosmological term Λ is zero. So far R_0 is arbitrary. For $c_0^2>0$, let us choose $R=R_0$ at that instant when $\ddot{\alpha}_4=0$, and there is marginal stability for a given mode. Or,

$$Q_0^2=c_0^2k^2=8\pi G\rho_0/3, \tag{75}$$

and (74) is now

$$\ddot{\alpha}_4+[(R_0/R)^{3\gamma-4}-1](C/R^3)\alpha_4=0. \tag{76}$$

From this equation we can deduce the results shown in Table I, and it follows that in an expanding (contracting) universe α_m is a growing function of time for $m>4(m<4)$ for all values of γ. More precise conditions are given later with specific solutions of (63). When $\tilde{\gamma}<0$, α_4 and hence all α_m are nonperiodic functions of time.

4.2. Jeans' Criterion

Jeans treatment of gravitational instability resembles Lord Rayleigh's formulation of the problem of oscillations in a fluid of positive and negative charges.[71] Jeans assumes that the uniform and unperturbed gravitating fluid is in a stationary state (which is possible for a neutral plasma) and therefore $\dot{R}=0$. Hence, from (63), for any α_m, and in particular $\alpha_0=\delta\rho$,

$$\ddot{\alpha}_0+(c_s^2k^2-4\pi G\rho)\alpha_0=0, \tag{77}$$

with $R=R_0$. For $\alpha_3\propto\exp(i\omega t)$ this equation gives the dispersion relation

$$\omega^2=c_s^2k^2+\omega_p^2. \tag{78}$$

A similar relation holds for electrostatic oscillations in a plasma, and $\omega_p=(4\pi ne^2/m)^{1/2}$ is the plasma frequency for electrons of number density n and charge to mass ratio e/m. In Jeans' dispersion relation the "gravitational frequency" is imaginary: $\omega_p=(-4\pi G\rho)^{1/2}$. There is thus a marginal state

$$k_J=(4\pi G\rho/c_s^2)^{1/2}, \tag{79}$$

and for $k>k_J$, ω is real and the disturbance oscillates at constant amplitude; and for $k<k_J$, ω is imaginary and the disturbance grows exponentially in time. Jeans' stability criterion is therefore $k>k_J$, or $\lambda<\lambda_J$, where $\lambda=k^{-1}$, $\lambda_J=k_J^{-1}$. The Debye length $\lambda_D=c_s/\omega_p$ plays a similar role in plasma physics and disturbances of $\lambda<\lambda_D$ in many cases tend to be stable. In a Jeans' sphere (as compared with a Debye sphere) of radius λ_J the sum of the thermal and potential energies is of the order $c_s^2\rho\lambda_J^3-4\pi G\rho^2\lambda_J^5=0$. In a sphere of radius $\lambda<\lambda_J$, the thermal energy predominates and collective interactions are of little consequence.

Jeans' analysis suffers from the defect that in general there is no initial stationary state in a uniform nonrotating fluid. When $\lambda\ll\lambda_J$, then $\omega^2\gg\dot{R}^2/R^2$, and the

[70] A. S. Eddington, *The Internal Constitution of the Stars* (Cambridge University Press, Cambridge, England, 1926), p. 201.

[71] Lord Rayleigh, Phil. Mag. **11**, 117 (1906).

dispersion relation (78) is an acceptable approximation. But as λ increases, the oscillation period also increases and is infinite at $\lambda=\lambda_J$; and when $\lambda>\lambda_J$, the e-folding time $i\omega_p^{-1}$ of a disturbance is comparable with the collapse time $i(2/3)^{1/2}\omega_p^{-1}$ of the system. Thus, in the range of interest $\lambda\gtrsim\lambda_J$, the dispersion relation (79) fails and we must fall back on solving (63). It is seen that Jeans' instability criterion $\lambda>\lambda_J$ is necessary for expansion ($\dot{R}>0$) and is sufficient for contraction ($\dot{R}<0$), but in neither case is it both necessary and sufficient.

A marginal state for any value of m is obtained by using

$$q=\int (R/R_0)^{2m-8}dt \qquad (80)$$

as the independent variable in (63):

$$\frac{d^2\alpha_m}{dq^2}+\left\{Q^2+(6-m)\frac{\ddot{R}}{R}+(m-4)(m-3)\frac{\dot{R}^2}{R^2}\right\}\dot{q}^2\alpha_m=0. \qquad (81)$$

For $m=3$, we obtain van Albada's[47] equation

$$(d^2\alpha_3/dq^2)+(Q^2-4\pi G\rho)(R_0/R)^4\dot{q}^2\alpha_3=0, \qquad (82)$$

and therefore Jeans' criterion of marginal stability $Q^2=4\pi G\rho$ holds for $\alpha_3(q)$. However, q is not a linear function of time for $m\neq 4$, and this marginal state does not give an unequivocal stability criterion.

In Einstein's static universe (which has its analogy in Newtonian theory[2] provided $p\ll\rho c^2$) $\dot{R}=0$, $\ddot{R}=0$ at $R=R_0$, and according to (5) and (7) this is possible for $\kappa=1$ and

$$\Lambda=R_0^{-2}=4\pi G\rho_0. \qquad (83)$$

Equation (63) now becomes

$$\ddot{\alpha}_0+(c_s^2k^2-\Lambda)\alpha_0=0, \qquad (84)$$

and as Bonnor[10] has shown, Jeans' dispersion relation (79) holds true without modification in Einstein's static universe. The cosmological term neutralizes the gravitational field and there is similarity with the neutral plasma state. From (84), we have

$$\alpha_0\propto \exp \pm i(c_s^2k^2-\Lambda)^{1/2}t. \qquad (85)$$

If the Einstein universe were static for an indefinite period of time then all wavelengths greater than Jeans' length $(c_s^2/4\pi G\rho)^{1/2}$ would grow exponentially. It is well known,[72–74] however, that the Einstein universe is unstable against perturbations in R. Perturbing (7) and using the equilibrium conditions (83), we find, to a first order,

$$\delta\ddot{R}=\Lambda\delta R, \qquad (86)$$

and therefore

$$\delta R\propto \exp \pm(\Lambda^{1/2}t). \qquad (87)$$

Comparing (85) and (87), we see that the departure from the Einstein equilibrium state grows more rapidly than the condensations, except when $c_s=0$, and the growths are then equal. Thus, contrary to what has been said, the Eddington–Lemaître model[73] appears to offer little advantage over other nonstatic models.

4.3. General Equations

To solve (63) we use the relations (11)–(14) and take χ as the independent variable. For $\kappa=0$ (zero energy)

$$\frac{d^2\alpha_m}{d\chi^2}+\frac{(14-4m)}{\chi}\frac{d\alpha_m}{d\chi}+4\left[L(\chi)+\frac{(m-2)(m-9/2)}{\chi^2}\right]\alpha_m=0,$$

$$L(\chi)=Q_0^2C^2\chi_0^{6\gamma-2}/\chi^{6\gamma-6}, \qquad (88)$$

$\kappa=1$ (negative energy):

$$\frac{d^2\alpha_m}{d\chi^2}+(14-4m)\cot\chi\frac{d\alpha_m}{d\chi}+4\left[L(\sin\chi)-(m-4)(m-3)+\frac{(m-2)(m-9/2)}{\sin^2\chi}\right]\alpha_m=0,$$

$$L(\sinh\chi)=Q_0^2C^2(\sin\chi_0)^{6\gamma-2}/(\sin\chi)^{6\gamma-6}, \qquad (89)$$

and for $\kappa=-1$ (positive energy),

$$\frac{d^2\alpha_m}{d\chi^2}+(14-4m)\coth\chi\frac{d\alpha_m}{d\chi}+4\left[L(\sinh\chi)+(m-4)(m-3)+\frac{(m-2)(m-9/2)}{\sinh^2\chi}\right]\alpha_m=0,$$

$$L(\sinh\chi)=Q_0^2C^2(\sinh\chi_0)^{6\gamma-2}/(\sinh\chi)^{6\gamma-6}, \qquad (90)$$

where $Q_0=c_0k$, $C=8\pi G\rho_0R_0^3/3$, $\Lambda=0$. Of these equations only (88) has a general solution

$$\alpha_m\propto R^{m-13/4}J_{\pm p}\left(\frac{3c_0^2k^2}{2\pi G\rho_0}\left\{\frac{R_0}{R}\right\}^{3\gamma-4}\right)^{1/2}, \qquad (91)$$

$$p=\tfrac{5}{2}(4-3\gamma). \qquad (92)$$

These above equations are now used to consider the time dependence of modes in (i) a cold universe, (ii) an isothermal universe of $\gamma=1$, (iii) $\gamma=\frac{4}{3}$, (iv) arbitrary γ and finally (v) when $\tilde{\gamma}<1$.

[72] A. S. Eddington, Monthly Notices Roy. Astron. Soc. **90,** 668 (1930).
[73] G. Lemaître, *L'hypothese de l'atome primitif* (Griffon, Neufschatel, 1946).
[74] E. R. Harrison, Monthly Notices Roy. Astron. Soc. (to be published).

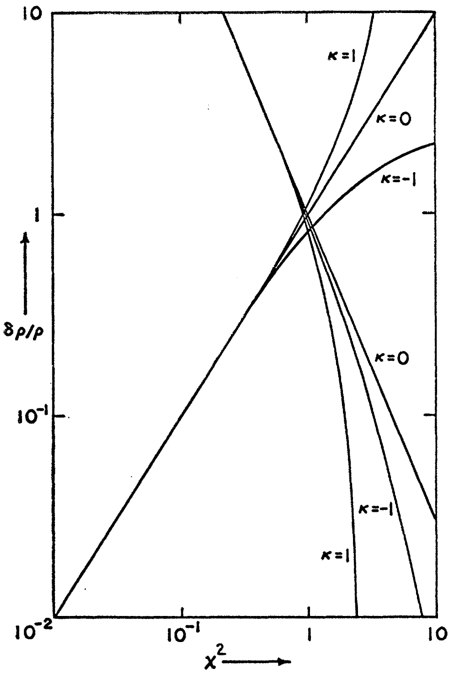

FIG. 1. Curves increasing from left to right show the growth in amplitude of $\delta\rho/\rho$ for $\kappa=0, \pm1$ in an expanding cold universe. Curves increasing from right to left show the corresponding growth in amplitude in a contracting cold universe. The amplitudes are plotted against χ^2. For $\kappa=0$, $\delta\rho/\rho=\text{const}\chi^2$ in an expanding universe, and $\delta\rho/\rho=\text{const}\chi^{-3}$ in a contracting universe. The constants are set equal to unity.

4.4. Cold Universe

A cold universe in which the pressure is zero is particularly interesting because condensation can form unimpeded by random motions. In the cold models Q_0 is zero in (88)–(90). We consider in turn $\kappa=0, +1, -1$.

For $\kappa=0$ the solution of (88) is

$$\alpha_m = A_0\chi^{2m-9}+B_0\chi^{2m-4}, \tag{93}$$

and A_0 and B_0 are constants. Hence, during expansion α_2 is either constant or decays, and during contraction $\alpha_{9/2}$ is either constant or decays. Retaining only the growing term:

Expansion:

$$\mu \propto R \propto \rho^{-1/3}, \tag{94}$$

Contraction:

$$\mu \propto R^{-3/2} \propto \rho^{1/2}, \tag{95}$$

for all wavelengths.

In the $\kappa=1$, or oscillatory model, the solution of (89) is

$$\alpha_m = (\sin\chi)^{2m-7}[A_1P_2^1(i\cot\chi)+B_1Q_2^1(i\cot\chi)], \tag{96}$$

where $0<\chi<\pi$, A_1, B_1 are constants, and P_ν^μ, Q_ν^μ are the associated Legendre functions. Since $P_2^1(-ix)=-P_2^1(ix)$, $Q_2^1(-ix)=Q_2^1(ix)$, we need consider only the range $0<\chi<\frac{1}{2}\pi$. Using the relations[75]

$$P_2^1(ix) = -3(1+x^2)^{1/2}x,$$

$$Q_2^1(ix) = i(1+x^2)^{1/2}[3x\cot^{-1}x-(3x^2+2)/(1+x^2)],$$

(96) becomes

$$\alpha_m = \mathcal{S}^{2m-9}[A_1\mathcal{C}+B_1(3\mathcal{S}-\mathcal{S}^3-3\chi\mathcal{C})], \tag{97}$$

$\mathcal{S}=\sin\chi$, $\mathcal{C}=\cos\chi$.

In the $\kappa=-1$ model the solution of (90) is

$$\alpha_m = (\sinh\chi)^{2m-7}[A_{-1}P_2^1(\coth\chi)+B_{-1}Q_2^1(\coth\chi)], \tag{98}$$

where A_{-1}, B_{-1} are constants and $0\leqslant\chi\leqslant\infty$. From the expressions $(x>1)$

$$P_2^1(x) = 3(x^2-1)^{1/2}x,$$

$$Q_2^1(x) = (x^2-1)^{1/2}\left[\frac{3}{2}x\ln\frac{x+1}{x-1}-\frac{3x^2-2}{x^2-1}\right],$$

it is found

$$\alpha_m = \mathsf{S}^{2m-9}[A_{-1}\mathsf{C}+B_{-1}(3\mathsf{S}+\mathsf{S}^3-3\chi\mathsf{C})], \tag{99}$$

$\mathsf{S}=\sinh\chi$, $\mathsf{C}=\cosh\chi$. When $\chi\gg1$,

$$\alpha_m \simeq AR^{m-4}+BR^{m-3}, \tag{100}$$

(A, B now different constants), and in an expanding (contracting) universe α_m grows for $m>3$ $(m<4)$.

The solutions of (93), (97), and (99) are shown in Fig. 1 with $A_0=A_1=A_{-1}=1$, $B_0=0.4B_1=0.4B_{-1}=1$. In an expanding universe the growth is least as one would expect in the case of positive energy $(\kappa=-1)$, and greatest in the case of negative energy $(\kappa=1)$ but is limited because the universe oscillates.[76]

4.5. Isothermal Universe

In a limited range of density it is possible under certain circumstances for a fluid to expand and contract isothermally.[45] We therefore consider $\gamma=1$, and the L's in (88)–(90) are constant:

$$L = c_0^2k^2R_0^2. \tag{101}$$

Using the Jeans' length $\lambda\!\!\!^-_0=(c_0^2/4\pi G\rho_0)^{1/2}$ at $R=R_0$, we find from (91) for $\kappa=0$,

$$\alpha_m = R^{m-13/4}AJ_{5/2}(x)+BJ_{-5/2}(x),$$

$$x=\lambda\!\!\!^-_0k(6R/R_0)^{1/2}=6^{1/2}\lambda\!\!\!^-_J/\lambda\!\!\!^- \tag{102}$$

where $\lambda\!\!\!^-=R/kR_0$, $\lambda\!\!\!^-_J=(c_0^2/4\pi G\rho)^{1/2}$, and the constants A, B from now on are not necessarily the same in

[75] W. R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill Book Co., Inc., New York, 1950), p. 149.

[76] It was previously thought[3] that the growths could be compared by setting $B_0=B_1=B_{-1}=1$. By letting $\chi\to0$ it can be seen that this assumption was wrong.

different equations. Hence,

$$\lambda^2 \gg 6\lambda_J^2: \qquad \alpha_m = A R^{m-9/2} + B R^{m-2}, \qquad (103)$$

$$\lambda^2 \ll 6\lambda_J^2: \qquad \alpha_m = R^{m-7/2}[A \sin x + B \cos x]. \qquad (104)$$

Wavelengths which are long compared with $6^{1/2}\lambda_J$ behave as in a cold universe, whereas for the short wavelengths $\alpha_{7/2}$ oscillates.

For $\kappa = 1$,

$$\alpha_m = (\sin\chi)^{2m-13/2}[A P_\lambda^{5/2}(\cos\chi) + B Q_\lambda^{5/2}(\cos\chi)], \qquad (105)$$

and $\kappa = -1$,

$$\alpha_m = (\sinh\chi)^{2m-13/2}[A P_\lambda^{5/2}(\cosh\chi) + B Q_\lambda^{5/2}(\cosh\chi)], \qquad (106)$$

where $\lambda = -\frac{1}{2} \pm (1 + 4\kappa c_0^2 k^2 R_0^2)^{1/2}$. These equations are similar to those of the cold model when $2c_0 k R_0 < 1$, and for large χ (106) becomes

$$\alpha_m \propto R^{m-1/2 \pm \lambda/2}.$$

4.6. $\gamma = \frac{4}{3}$

An isotropic photon, neutrino, or relativistic gas has a ratio of specific heats of $\frac{4}{3}$. When their pressure is dominant but their density is small compared with the total density of the fluid, the fluid as a whole has $\gamma = \frac{4}{3}$ and can be treated with the Newtonian approximation.

For $\kappa = 0$,

$$\alpha_m \propto R^{m \pm \epsilon - 13/4}, \qquad (107)$$

where

$$\epsilon = [\tfrac{25}{16} - (3c_0^2 k^2 / 8\pi G \rho_0)]^{1/2} = [\tfrac{25}{16} - \tfrac{3}{2}(\lambda_J^2/\lambda^2)]^{1/2},$$

and c_0 is the speed of sound at density ρ_0. If $\lambda \gg (24/25)^{1/2}\lambda_J$, then α_m behaves as in the cold universe, but when

$$\lambda < (24/25)^{1/2}\lambda_J, \qquad (108)$$

then α_m oscillates:

$$\alpha_m \propto R^{m-13/4} \exp(\pm i\beta \ln R), \qquad (109)$$

where $i\beta = \epsilon$.

When $\kappa = \pm 1$,

$$\alpha_m = R^{m-7/2}[A P_\phi^1(i \cot \kappa\chi) + B Q_\phi^1(i \cot \kappa\chi)], \qquad (110)$$

and

$$\phi = -\tfrac{1}{2} + [\tfrac{25}{4} - 6(\lambda_J^2/\lambda^2)]^{1/2}$$

and P_ϕ^1, Q_ϕ^1 are nonperiodic when the wavelength λ is large compared with λ_J.

4.7. Arbitrary γ

Apparently there are no general solutions for arbitrary γ when $\kappa = \pm 1$. In particular, there are no solutions in terms of simple functions for the important case of $\gamma = \frac{5}{3}$. Savedoff and Vila[51] have discussed the problem using hypergeometric functions and have also given the general solution (91) for $\kappa = 0$. Equation (91) can be written in the form

$$\alpha_m \propto R^{m-13/4} J_{\pm p}(6[R/R_0]^{2-3\gamma/2}), \qquad (111)$$

where $\lambda = \lambda_0$, or

$$\lambda^2 = k^{-2} = c_0^2 / 4\pi G \rho_0, \qquad (112)$$

at $R = R_0$. This method is convenient for it immediately shows the following:

$$\left.\begin{array}{ll} \gamma < \frac{4}{3}, & R \ll R_0 \\ \gamma > \frac{4}{3}, & R \gg R_0 \end{array}\right\} \alpha_m = A R^{m-9/2} + B R^{m-2}, \qquad (113)$$

$$\left.\begin{array}{ll} \gamma < \frac{4}{3}, & R \gg R_0 \\ \gamma > \frac{4}{3}, & R \ll R_0 \end{array}\right\} \alpha_m = R^{m-n-13/4}(A \sin x + B \cos x), \qquad (114)$$

where $n = 1 - 3\gamma/4$, $x = 6(R/R_0)^{2-3\gamma/2}$. Thus for a wavelength which is marginally stable ($\lambda = \lambda_0$) initially, when $R = R_0$, α_m becomes nonperiodic and grows with time in an expanding (contracting) universe for $m > 2$, $\gamma > \frac{4}{3}$ ($m < \frac{9}{2}$, $\gamma < \frac{4}{3}$), and becomes periodic and decays with time in an expanding (contracting) universe for $m < n + \frac{13}{4}$, $\gamma < \frac{4}{3}$ ($m > n + \frac{13}{4}$, $\gamma > \frac{4}{3}$). These results can be partly obtained by physical arguments: at a distance λ the velocity of recession is $\lambda \dot{R}/R = u$, and from (9) with $\Lambda = \kappa = 0$,

$$u^2/c_s^2 = \tfrac{2}{3}(\lambda^2/\lambda_J^2). \qquad (115)$$

Since $\lambda \propto R$, $\lambda_J^2 = \lambda_0^2 (R/R_0)^{6-3\gamma}$, we have $u^2/c_s^2 \simeq (R/R_0)^{3\gamma-4}$, and in an expanding universe this ratio increases for $\gamma > \frac{4}{3}$ and the thermal motions are of diminishing importance.

4.8. $\tilde{\gamma} < 1$

We consider now the important case when $c_s^2 < 0$ for a range of wavelengths, because of radiative transfer. For convenience the discussion is confined to $\kappa = 0$. The argument of $J_{\pm p}$ in (91) is now imaginary, and when large

$$\alpha_m \propto R^{m-(17-3\gamma)/4} \exp \pm \left(-\frac{3c_0^2 k^2}{2\pi G \rho_0} \left\{\frac{R_0}{R}\right\}^{3\gamma-4} \right)^{1/2}. \qquad (116)$$

Thus, if $\gamma = \frac{5}{3}$, then the exponentially growing solution is

$$\mu = \mu_0 \exp[-(3c_0^2 k^2)^{1/2} / 2\pi G \rho_0][1 - (R_0^{1/2}/R^{1/2})], \quad (117)$$

where μ_0 is the initial contrast density. The exponent has a maximum value of $2\lambda c_0^2 / MG$, where $\lambda = k^{-1}$, and $M = 4\pi\rho\lambda^3/3$ is approximately the mass of the eventual condensation.

5. RELATIVISTIC COSMOLOGY

5.1. Unperturbed Models

In its unperturbed state we assume that the universe is homogeneous and isotropic, and the metric is given by the Robertson–Walker line element

$$ds^2=dt^2-[R^2/(1+\tfrac{1}{4}\kappa r^2)^2](dr^2+r^2d\Omega^2), \quad (118)$$

where $d\Omega^2=d\Theta^2+\sin^2\Theta d\phi^2$, r, Θ, ϕ are co-moving coordinates, $R(t)$ has the dimensions of time, and $\kappa=0$, ±1 is the curvature constant. The energy-momentum tensor of a perfect fluid is

$$\mathsf{T}_j{}^i=(\rho c^2+p)g_{kj}u^ku^i-\delta_j{}^ip, \quad (119)$$

in which ρc^2 is the energy density, u^i the four-velocity, and p is the isotropic pressure. For a fluid that is stationary in the co-moving system, $u^0=1$, $u^1=u^2=u^3=0$, and the components of the energy-momentum tensor are

$$\mathsf{T}_0{}^0=\rho c^2, \qquad \mathsf{T}_1{}^1=\mathsf{T}_2{}^2=\mathsf{T}_3{}^3=-p. \quad (120)$$

All that now remains is to determine $R(t)$ in (118) with the Einstein equation

$$\mathsf{R}_j{}^i-\tfrac{1}{2}\delta_j{}^i\mathsf{R}_l{}^l+\delta_j{}^i\Lambda=-(8\pi G/c^2)\mathsf{T}_j{}^i. \quad (121)$$

In this equation, $\mathsf{R}_j{}^i$ is the contracted Riemann–Christoffel or Ricci tensor (its further contraction $\mathsf{R}_l{}^l$ is shown explicitly to avoid confusion with $R(t)$, and the cosmological term Λ is included as in the Newtonian treatment.

Equation (121) is readily solved using the line-element (118) and the components (120) of the energy-momentum tensor.[1,77] The following method however is particularly simple. Transforming to the coordinates:

$$x=r\sin\Theta\cos\phi,$$
$$y=r\sin\Theta\sin\phi,$$
$$z=r\cos\Theta,$$

$r^2=x^2+y^2+z^2$, the line element (118) in the immediate neighborhood of $r=0$ becomes

$$ds^2=dt^2-R^2[1-\tfrac{1}{2}\kappa(x^2+y^2+z^2)](dx^2+dy^2+dz^2). \quad (122)$$

For these coordinates the Christoffel symbols

$$\Gamma_{pq}{}^r=\tfrac{1}{2}g^{rs}\left(\frac{\partial g_{ps}}{\partial x^q}+\frac{\partial g_{qs}}{\partial x^p}-\frac{\partial g_{pq}}{\partial x^s}\right) \quad (123)$$

are[78]

$$\Gamma_{0\nu}{}^\mu=c^2R^{-2}\Gamma_{\mu\nu}{}^0=\delta_\nu{}^\mu(\dot{R}/R), \quad (124)$$

$$\Gamma_{\beta\beta}{}^\alpha=-\Gamma_{\beta\alpha}{}^\beta=-\Gamma_{\alpha\alpha}{}^\alpha=\tfrac{1}{2}\kappa x^\alpha. \quad (125)$$

As $r\to0$, the only surviving Christoffel symbols are $\Gamma_{0\alpha}{}^\alpha$ and $\Gamma_{\alpha\alpha}{}^0$, and the spatial derivatives of (125) are

$$(\partial/\partial x^\alpha)\Gamma_{\beta\beta}{}^\alpha=-(\partial/\partial x^\alpha)\Gamma_{\beta\alpha}{}^\beta$$
$$=-(\partial/\partial x^\alpha)\Gamma_{\alpha\alpha}{}^\alpha=\tfrac{1}{2}\kappa. \quad (126)$$

Since

$$\mathsf{R}_{kj}=(\partial/\partial x^j)\Gamma_{kl}{}^l-(\partial/\partial x^l)\Gamma_{kj}{}^l+\Gamma_{kl}{}^m\Gamma_{jm}{}^l-\Gamma_{kj}{}^l\Gamma_{lm}{}^m, \quad (127)$$

we find from $\mathsf{R}_j{}^i=g^{ik}\mathsf{R}_{kj}$ that

$$R_0{}^0=3\ddot{R}/R, \quad (128)$$

$$R_\alpha{}^\alpha=(\ddot{R}R+2\dot{R}^2+2\kappa c^2)/R^2, \quad (129)$$

and all other components vanish. From (121) we now obtain the well-known equations

$$\dot{R}^2=\tfrac{1}{3}(8\pi G\rho+\Lambda)R^2-\kappa, \quad (130)$$

$$(d/dt)(R^3\rho)+(p/c^2)(dR^3/dt)=0. \quad (131)$$

Every observer can adjust the origin of his co-moving coordinate system to give $r=0$, and therefore (130) and (131) apply to all co-moving observers. Equations (130) and (131) are identical with the Newtonian equations (8) and (9) when p/c^2 is negligibly small compared with ρ. Alternative expressions of (130) and (131) which will be useful are

$$(8\pi G\rho+\Lambda)R^2=3(\dot{R}^2+\kappa), \quad (132)$$

$$(\Lambda-8\pi Gp/c^2)R^2=2R\ddot{R}+\dot{R}^2+\kappa, \quad (133)$$

and therefore

$$\ddot{R}=-\tfrac{1}{3}(4\pi G)[\rho+3(p/c^2)]R+\tfrac{1}{3}\Lambda R. \quad (134)$$

5.2. Equation of State

We have two equations (130) and (131), or (132) and (133), for the determination of R, ρ, and p, and therefore require an equation of state. We use Zel'dovich's equation of state[79]

$$p=(\nu-1)\rho c^2, \quad (135)$$

and in many cases of physical interest in which the pressure is appreciable ν has a constant value. Thus, if ν is constant, then (130) becomes

$$\dot{R}^2=C_\nu R^{2-3\nu}+\tfrac{1}{3}\Lambda R^2-\kappa, \quad (136)$$

$$C_\nu=8\pi G\rho R^{3\nu}/3, \quad (137)$$

and from (131) $C_\nu=$const. Equation (134) is now

$$\ddot{R}=\tfrac{1}{2}(2-3\nu)C_\nu R^{1-3\nu}+\tfrac{1}{3}\Lambda R, \quad (138)$$

and therefore

$$2R\ddot{R}-(2-3\nu)(\dot{R}^2+\kappa)-\nu\Lambda R^2=0. \quad (139)$$

[77] L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields* (Pergamon Press, Oxford, England, 1962), Chap. 12.

[78] The convention adopted is that Latin indices assume all values 0, 1, 2, 3; Greek indices λ, μ, ν assume only the values 1, 2, 3, and α, β, γ are used when there is no summation and $\alpha\neq\beta\neq\gamma$.

[79] Ya. B. Zel'dovich, Zh. Eksperim. i Teor. Fiz. **41**, 1609 (1962) [English transl.: Soviet. Phys.—JETP **14**, 1143 (1962)].

In an isotropic photon or neutrino gas, or in isotropic fluids in which particles and their fields have energies large compared with their rest masses, ν attains a maximum value[80] of $\frac{4}{3}$. (Zel'dovich[79] has proposed increasing the upper limit to 2, but it is possible that this is unrealistic.[81]) The generally accepted physically meaningful values of ν lie in the range $1 \leq \nu \leq \frac{4}{3}$. McCrea[82] has suggested a value of $\nu=0$, or $\rho=$const, as a method of giving sensible meaning to a steady-state universe. As the universe expands stress energy is converted into matter. Whittaker[83] has also proposed a value of $\nu=\frac{1}{3}$ for a model in which $\rho R=$const. The various possible models for different values of the stress constant ν have been classified.[74] In the following treatment we leave the stress constant unspecified except in certain particular instances. McCrea proposes that the pressure in (135) for $\nu<1$ (and presumably also for $\nu>\frac{4}{3}$) can be regarded as a uniform cosmic stress that is not manifest in local and detailed phenomena. Spatial variations in density and pressure are related therefore by a distinctly different equation of state.

For small spatial variations we use

$$\delta p = (\tilde{\nu}-1)\,\delta\rho c^2. \tag{140}$$

Since $dp/d\rho = c_s^2$, we have

$$\tilde{\nu} = 1 + c_s^2/c^2, \tag{141}$$

and $\tilde{\nu}$ is close to unity when $c_s \ll c$, and $\tilde{\nu}=\frac{4}{3}$ for $c_s = c/\sqrt{3}$. Although $\tilde{\nu}$ has an upper limit of $\frac{4}{3}$, and is unity in a fluid consisting of particles with no interactions (other than gravitation), it has in principle no lower limit. That is, $\tilde{\nu}<\frac{4}{3}$, and it is possible that $\tilde{\nu}<1$ for certain mechanisms of transfer. Suppose $dp/d\rho$ is a constant; also ρ_0, p_0, and ρ_1, p_1 are the initial and final values of the density and pressure for an interval of time at a given location. Then

$$\tilde{\nu} = 1 + c^2(\rho_1-\rho_0)/(p_1-p_0).$$

The smallest possible value of p_1 is zero and the maximum possible value of p_0 is $\frac{1}{3}\rho_0 c^2$; in this case

$$\tilde{\nu} = (\rho_1 - \tfrac{4}{3}\rho_0)/(\rho_1-\rho_0),$$

and as $\rho_1 \to \rho_0$, $\nu \to -\infty$. At least in principle we have $\frac{4}{3} \geq \nu \geq -\infty$. The transfer processes themselves, however, will impose physical limitations on this range.

For the Friedmann type models of $\kappa=0$, ± 1, and $\Lambda=0$, it is found

$$\kappa=0, \qquad R=(C_\nu \chi^2)^{1/(3\nu-2)}, \tag{142}$$

$$\kappa=1, \qquad R=(C_\nu \sin^2\chi)^{1/(3\nu-2)}, \tag{143}$$

$$\kappa=-1, \qquad R=(C_\nu \sinh^2\chi)^{1/(3\nu-2)}, \tag{144}$$

where

$$dt/d\chi = [2/(3\nu-2)]R. \tag{145}$$

These equations reduce to the original Friedmann and Einstein–de Sitter solutions (11)–(13) when $\nu=1$.

If ρ_{mat} and ρ_{rad} are the present densities of matter and radiation in the universe, then, neglecting mutual interaction, $\rho_{\text{mat}} \propto R^{-3}$, $\rho_{\text{rad}} \propto R^{-4}$. The two densities are equal when

$$\rho_c = \rho_{\text{mat}} + \rho_{\text{rad}} = 2\rho_{\text{mat}}{}^4 \rho_{\text{rad}}{}^{-3}. \tag{146}$$

Thus if $\rho_{\text{mat}} \sim 10^{-30}$ g cm^{-3} and $\rho_{\text{rad}} \sim 10^{-33}$ g cm^{-3} corresponding to thermal radiation of 3.5°K, the densities are equal when $\rho_c \sim 10^{-21}$. Thus previously, at higher densities, the radiation density was dominant, at least for a period of time, and according to the conventional view ν had a value close to $\frac{4}{3}$. When the mean photon energy is $kT \gtrsim 1$ MeV, pair production populates the accessible particle and antiparticle states, and when $kT > 1$ GeV it is reasonable to suppose as a first approximation[84] that the Fermi energy level of the fermions is comparable with the mean energy kT of the bosons, and the number density of each kind is of the order $(kT/\hbar c)^3$.

For reference the solutions for $\nu=\frac{4}{3}$ are given:

$$\kappa=0, \qquad R=C^{1/2}{}_{4/3}\chi, \tag{147}$$

$$\kappa=1, \qquad R=C^{1/2}{}_{4/3}\sin\chi, \tag{148}$$

$$\kappa=-1, \qquad R=C^{1/2}{}_{4/3}\sinh\chi, \tag{149}$$

$$dt = R\,d\chi. \tag{150}$$

Equation (148) is Tolman's[85] model of a universe containing radiation. Observations are at least consistent with a present value of $\chi_0 \sim 1$, and therefore $\beta_1 \sim R_0$ and at the density ρ_c of (146)

$$\chi_c \sim (R_c/R_0)^{1/2} \sim (\rho_0/\rho_c)^{1/6}.$$

From the present density of $\rho_0 \sim 10^{-30}$ g cm^{-3} it follows that $\chi_c \sim 1/30$. At $\chi < \chi_c$ the models (147)–(149) have negligible difference and for simplicity we can assume $\kappa=0$. This would be true for the high density stage of all models of $\hat{\xi} > \frac{2}{3}$.

5.3. Linearized Equations of Perturbed Models

We consider small departures from the metric (118) as the result of displacements of the fluid. A perturbation treatment of the cosmological models, as distinct from a static and flat metric,[86,87] encounters the slight complication of nonvanishing Christoffel symbols. A

[80] B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (Chicago University Press, Chicago, Ill., 1965).
[81] E. R. Harrison, Astrophys. J. **142,** 1643 (1965).
[82] W. H. McCrea, Proc. Roy. Soc. (London) **A206,** 562 (1951).
[83] J. M. Whittaker, Nature **209,** 491 (1966).
[84] H. Y. Chiu, Ann. Phys. (N. Y.) **26,** 364 (1964).
[85] R. C. Tolman, *Relativity Thermodynamics and Cosmology* (Clarendon Press, Oxford, England, 1934).
[86] L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields* (Pergamon Press, Oxford, England, 1962), p. 349.
[87] J. Weber, *General Relativity and Gravitational Waves* (Interscience Publishers, Inc., New York, 1961).

general treatment was first given by Lifshitz,[88] and more recently contributions have been made by Lifshitz and Khalatnikov,[89] Irvine,[90] Hawking,[91] Silk,[92] Sachs and Wolfe.[93] In general, these solutions include rotational motions and gravitational waves. Simple irrotational motion, which in the elementary Newtonian treatment is of most interest, is often difficult to disentangle owing to conditions imposed on the metric. The following is a simple approach and is analogous to the Newtonian treatment.

Small variations in the metric tensor are expressed as $g_{jk}+\delta g_{jk}$, where g_{jk} is given by (118) and

$$\delta g_{jk}=h_{jk}. \tag{151}$$

We assume h_{jk} and its derivatives are everywhere small, and that quadratic and higher-order terms in small quantities are negligible. Thus the unperturbed tensors g_{jk}, g^{ik} are used for lowering and raising the indices of h^{ik}, h_{jk}: $h_j{}^i=g^{ik}h_{jk}=g_{jk}h^{ik}$, and in effect h_{jk} is a tensor field in the unperturbed g_{jk} space. Since $g_{jk}g^{ik}=\delta_j{}^i$, to a first order

$$\delta(g_{jk}g^{ik})=h_j{}^i+g_{jk}\delta g^{ik}=0,$$

and therefore

$$\delta g^{ik}=-h^{ik}. \tag{152}$$

It is more convenient to use Einstein's equation (121) in the alternative form

$$\mathsf{R}_j{}^i=-(8\pi G/c^2)(\mathsf{T}_j{}^i-\tfrac{1}{2}\delta_j{}^i\mathsf{T})+\delta_j{}^i\Lambda, \tag{153}$$

and its perturbation is

$$\delta\mathsf{R}_j{}^i=-(8\pi G/c^2)\delta(\mathsf{T}_j{}^i-\tfrac{1}{2}\delta_j{}^i\mathsf{T}). \tag{154}$$

The perturbation $\delta\mathsf{R}_j{}^i$ is evaluated in terms of $h_j{}^i$ as follows. From (127),

$$\delta\mathsf{R}_{jk}=[(\partial/\partial x^j)\delta\Gamma_{kl}{}^l-\Gamma_{kj}{}^l\delta\Gamma_{lm}{}^m-\Gamma_{lj}{}^m\delta\Gamma_{km}{}^l+\Gamma_{jm}{}^l\delta\Gamma_{kl}{}^m]$$
$$-[(\partial/\partial x^l)\delta\Gamma_{kj}{}^l-\Gamma_{kl}{}^m\delta\Gamma_{jm}{}^l-\Gamma_{jl}{}^m\delta\Gamma_{km}{}^l+\Gamma_{lm}{}^m\delta\Gamma_{kj}{}^l],$$

thus giving the Palatini equation[94]

$$\delta\mathsf{R}_{jk}=\delta\Gamma_{kl}{}^l{}_{;j}-\delta\Gamma_{kj}{}^l{}_{;l}, \tag{155}$$

where a semicolon denotes covariant differentiation. Also, to a first order,

$$\delta\mathsf{R}_j{}^i=\delta(g^{ik}\mathsf{R}_{jk})=g^{ik}\delta\mathsf{R}_{jk}-h_k{}^i\mathsf{R}_j{}^k, \tag{156}$$

and with (155), we obtain

$$g^{ik}(\delta\Gamma_{kl}{}^l{}_{;j}-\delta\Gamma_{kj}{}^l{}_{;l})=\delta\mathsf{R}_j{}^i+h_k{}^i\mathsf{R}_j{}^k. \tag{157}$$

[88] E. M. Lifshitz, J. Phys. USSR **10**, 116 (1946).
[89] E. M. Lifshitz and I. M. Khalatnicov, Advan. Phys. **12**, 185 (1963).
[90] W. M. Irvine, Ann. Phys. (N. Y.) **32**, 322 (1965).
[91] S. W. Hawking, Astrophys. J. **145**, 544 (1966).
[92] J. Silk, Astrophys. J. **143**, 689 (1966).
[93] R. K. Sachs and A. M. Wolfe, Astrophys. J. **147**, 73 (1967).
[94] C. Møller, *The Theory of Relativity* (Clarendon Press, Oxford, England, 1952), p. 334.

The perturbed Christoffel symbols are

$$\delta\Gamma_{kj}{}^l=\delta(g^{lr}g_{rs}\Gamma_{kj}{}^s)=g^{lr}\delta(g_{rs}\Gamma_{kj}{}^s)-\Gamma_{kj}{}^sh_s{}^l$$
$$=\tfrac{1}{2}g^{lr}\left(\frac{\partial h_{kr}}{\partial x^j}+\frac{\partial h_{jr}}{\partial x^k}-\frac{\partial h_{kj}}{\partial x^r}\right)-\Gamma_{kj}{}^sh_s{}^l, \tag{158}$$

thus giving Lifshitz's[88] equation

$$\delta\Gamma_{kj}{}^l=\tfrac{1}{2}g^{lr}(h_{kr;j}+h_{jr;k}-h_{kj;r}). \tag{159}$$

In particular, $\delta\Gamma_{kl}{}^l=\frac{1}{2}h_{;k}$, where h is the trace $h_l{}^l$.

From (157) and (159) we have a linearized differential equation in $h_j{}^i$:

$$g^{ik}(h_{;kj}-h_k{}^l{}_{;jl}-h_j{}^l{}_{;kl})+g^{lm}h_j{}^i{}_{;kl}=2(\delta\mathsf{R}_j{}^i+h_k{}^i\mathsf{R}_j{}^k). \tag{160}$$

The g_{jk} and $\mathsf{R}_j{}^i$ are known, and $\delta\mathsf{R}_j{}^i$ is given by (154) in terms of the perturbed energy-momentum tensor. A similar equation to (160) was first derived by Lanczos.[95]

The trace of $\mathsf{T}_j{}^i$ is $\mathsf{T}=\rho c^2-3p$, and therefore

$$\delta\mathsf{T}=c^2\delta\rho-3\delta p. \tag{161}$$

Furthermore, $g_{kj}u^ku^j=1$, and from $\delta(g_{kj}u^ku^j)=0$,

$$h_{kj}u^ku^j+g_{kj}\delta u^ku^j+g_{kj}u^k\delta u^j=0,$$

where $u^0=1$, $u^\mu=0$, and therefore

$$h_0{}^0+2u_0\delta u^0=0. \tag{162}$$

Also, $g_{kj}u^ku^i=\delta_0{}^i\delta_j{}^0$, and

$$\delta(g_{kj}u^ku^i)=\delta_0{}^ih_j{}^0+\delta_j{}^0u_0\delta u^i+g_{kj}\delta u^k\delta_0{}^iu^0.$$

We therefore find, from (119)

$$\delta\mathsf{T}_j{}^i=\delta(\rho c^2+p)g_{kj}u^ku^i+(\rho c^2+p)\delta({}_{kj}u^ku^i)-\delta_j{}^i\delta p$$
$$=\delta_0{}^i\delta_j{}^0\delta(\rho c^2+p)-\delta_j{}^i\delta p$$
$$+(\rho c^2+p)(\delta_0{}^ih_j{}^0+\delta_j{}^0u_0\delta u^i+g_{kj}\delta u^k\delta_0{}^iu^0). \tag{163}$$

From (162) and (163) it follows that the components of $\delta\mathsf{T}_j{}^i$ are

$$\delta\mathsf{T}_0{}^0=c^2\delta\rho,$$
$$\delta\mathsf{T}_\alpha{}^0=(\rho c^2+p)(h_\alpha{}^0+g_{\alpha\alpha}\delta u^\alpha),$$
$$\delta\mathsf{T}_\alpha{}^\alpha=-\delta p,$$
$$\delta\mathsf{T}_\beta{}^\alpha=0, \tag{164}$$

where α, $\beta=1$, 2, or 3, $\alpha\neq\beta$, and no summation. From (154) and (164) we now have

$$\delta\mathsf{R}_0{}^0=-4\pi G\delta(\rho+3p/c^2),$$
$$\delta\mathsf{R}_\alpha{}^0=-8\pi G(\rho+p/c^2)(h_\alpha{}^0+g_{\alpha\alpha}\delta u^\alpha),$$
$$\delta\mathsf{R}_\alpha{}^\alpha=4\pi G\delta(\rho-p/c^2), \tag{165}$$

[95] K. Lanczos, Z. Physik **31**, 112 (1925).

and $\delta \mathsf{R}_\beta{}^\alpha = 0$. In addition, the components of $\mathsf{R}_j{}^i$ are

$$\mathsf{R}_0{}^0 = -4\pi G(\rho + 3p/c^2) + \Lambda,$$

$$\mathsf{R}_\alpha{}^\alpha = 4\pi G(\rho - p/c^2) + \Lambda. \quad (166)$$

Collecting together the equations (160), (165), and (166), we have, for $i=j=0$,

$$h_{;00} - 2h_0{}^l{}_{;0l} + g^{lm} h_0{}^0{}_{;lm} = -8\pi G(h_0{}^0 + \delta)(\rho + 3p/c^2) + 2h_0{}^0\Lambda; \quad (167)$$

for $i=0$, $j=\alpha$, ($\alpha = 1$, 2, or 3; no summation over α);

$$h_{;0\alpha} - h_0{}^l{}_{;\alpha l} - h_\alpha{}^l{}_{;0l} + g^{lm} h_\alpha{}^0{}_{;lm} = -8\pi G[h_\alpha{}^0(\rho + 3p/c^2) + 2(\rho + p/c^2) g_{\alpha\alpha}\delta u^\alpha] + 2h_\alpha{}^0\Lambda; \quad (168)$$

for $i=\alpha$, $j=\alpha$:

$$g^{\alpha\alpha}(h_{;\alpha\alpha} - 2h_\alpha{}^l{}_{;\alpha l}) + g^{lm} h_\alpha{}^\alpha{}_{;lm} = 8\pi F(h_\alpha{}^\alpha + \delta)(\rho - p/c^2) + 2h_\alpha{}^\alpha\Lambda; \quad (169)$$

and for $i=\alpha$, $j=\beta$, ($\alpha \neq \beta$):

$$g^{\alpha\alpha}(h_{;\alpha\beta} - h_\alpha{}^l{}_{;\beta l} - h_\beta{}^l{}_{;\alpha l}) + g^{lm} h_\beta{}^\alpha{}_{;lm} = 0. \quad (170)$$

Altogether, (167)–(170) provide ten equations for the determination of the ten unknowns: h_{kj} (four components can be discarded by coordinate transformations), δu^α (since $\delta u^0 = -\frac{1}{2}h_0{}^0$), and $\delta\rho$ (δp is given by an equation of state).

5.4. Irrotational Motion

For irrotational motion it can be shown that (167)–(170) reduce to three equations determining ψ, φ, $\delta\rho$, as in the Newtonian treatment. A sufficient condition for irrotational motion is that all nondiagonal components of $h_j{}^i$ are zero, and $h_0{}^0 = -h_\alpha{}^\alpha$.

The simplest procedure is to use local coordinates in which the Christoffel symbols are given by (124) and (125). We also adopt a system of coordinates in which

$$h_{0\alpha} = 0, \quad (171)$$

$$h_\alpha{}^i{}_{;0} - \tfrac{1}{2}\delta_\alpha{}^i h_{;0} = \theta, \quad (172)$$

and the perturbed velocity is

$$\delta u^\alpha = \partial\varphi/\partial x^\alpha. \quad (173)$$

Then (168) becomes

$$\tfrac{1}{2}(\partial h/\partial t) - (1/R^2)(\partial/\partial t)(R^2 h_0{}^0) - \theta = 16\pi G(\rho + p/c^2) R^2\varphi, \quad (174)$$

and since θ can be absorbed into φ, we assume $\theta = 0$, and therefore

$$h_0{}^0 = -h_1{}^1 = -h_2{}^2 = -h_3{}^3 = 2\psi/c^2. \quad (175)$$

The $2c^{-2}$ is included so that ψ is equivalent to the gravitational potential of Newtonian theory. We are thus left with six equations: (167), (168), (174), and the three equations (170), for the determination of the six quantities $h_\beta{}^\alpha$, $\delta\rho$, ψ and φ. However, the $h_\beta{}^\alpha$ according to (170) are propagated independently of the fluid disturbance and therefore we can set them equal to zero. For small irrotational fluid disturbances the line element is therefore

$$ds^2 = \left(1 + 2\frac{\psi}{c^2}\right) dt^2 - \frac{R^2(1 - 2\psi c^{-2})}{(1 + \frac{1}{4}\kappa r^2)^2}(dr^2 + r^2 d\Omega^2). \quad (176)$$

McVittie[96] has used a similar line element for an Einstein universe containing discrete condensations.

A transformation of the coordinates $x^i \to x'^i = x^i + \epsilon^i$, where ϵ^i are small quantities, leads to $g_{jk} + \gamma_{jk}$, where

$$\gamma_{jk} = h_{jk} - \epsilon_{j;k} - \epsilon_{k;j}.$$

The line element (176) is unchanged with the Killing equation[97]

$$\epsilon_{j;k} + \epsilon_{k;j} = 0.$$

From

$$(g_{jk} + h_{jk})\frac{dx^j}{ds}\frac{dx^k}{ds} = (g_{jk} + \gamma_{jk})\frac{dx'^j}{ds}\frac{dx'^k}{ds} = 1,$$

it is found $d\epsilon^0/ds = d\epsilon^0/dt = 0$. Also, for δu^α to remain unchanged, $d\epsilon^\alpha/dt = 0$. Clearly, these conditions are not limited to infinitesimal transformations, and any $x^i \to x'^i$ leading to $\psi(x^i) \to \psi'(x'^i)$ is admissible.

5.5. Equations of Irrotational Motion

With the line element (176) the only surviving equations of (167)–(170) are

$$4\ddot{\psi} + 12\dot{R}R^{-1}\dot{\psi} - \Box^2\psi = 4\pi G(2\psi + c^2\delta)(\rho + 3p/c^2) - 2\psi\Lambda, \quad (177)$$

$$4\dot{R}R^{-1}\dot{\psi} + 4(\ddot{R}R^{-1} + 2\dot{R}^2R^{-2})\psi + \Box^2\psi = 4\pi G(2\psi - c^2\delta)(\rho - p/c^2) + 2\psi\Lambda, \quad (178)$$

$$(d/dt)(R\psi) = -4\pi G(\rho + p/c^2) R^3\varphi c^2. \quad (179)$$

These equations are obtained either by working through the covariant differentiations, or more simply using local coordinates. The results have been checked with Dingle's[98] formulas for an orthogonal line element. With

$$\Box^2\psi = \left(\frac{\partial^2}{\partial t^2} + 3\frac{\dot{R}}{R}\frac{\partial}{\partial t} + \frac{k^2}{R^2}\right)\psi,$$

$$\nabla^2\psi = -k^2\psi,$$

(177) and (178) become

$$4\pi G c^2 \delta\rho = -3\dot{R}R^{-1}\dot{\psi} - (k^2 + 3\dot{R}^2 - 3\kappa)R^{-2}\psi, \quad (180)$$

$$4\pi G\delta p = \ddot{\psi} + 4\dot{R}R^{-1}\dot{\psi} + (2\ddot{R}R + \dot{R}^2 - \kappa)R^{-2}\psi. \quad (181)$$

[96] G. C. McVittie, Monthly Notices Roy. Astron. Soc. **92,** 274 (1931).

[97] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, N. J., 1949), p. 233.

[98] H. Dingle, Proc. Natl. Acad. Sci. (U. S.) **19,** 559 (1933).

The fluid motions are therefore parallel to the pressure and density gradients.

Let $p=(\nu-1)\rho c^2$, $\delta p=(\tilde{\nu}-1)\delta\rho c^2$, then using (136), (139), the above set of equations (179)–(181) become

$$\ddot{\psi}+(1+3\tilde{\nu})\dot{R}R^{-1}\dot{\psi}+\{(\tilde{\nu}-1)k^2+\tilde{\nu}\Lambda R^2+8\pi G\rho(\tilde{\nu}-\nu)R^2-(6\tilde{\nu}-4)\kappa\}\times R^{-2}\psi=0, \quad (182)$$

$$(d/dt)(R\dot{\psi})+4\pi G\nu\rho c^2R^3\varphi=0, \quad (183)$$

$$\dot{\mu}+3(\tilde{\nu}-\nu)(\dot{R}/R)\mu-3(\nu/c^2)\dot{\psi}-k^2\nu\varphi=0, \quad (184)$$

where $\mu=\delta\rho/\rho$. The last two equations can also be obtained from $\delta T_{0;i}{}^{i}=0$ and $\delta T_{\mu;i}{}^{i}=0$. Equations (182)–(184) are the general equations with arbitrary ν, $\tilde{\nu}$ for the determination of ψ, φ, μ.

At low pressure we have $p\ll\rho c^2$ and ν is close to unity; also $\tilde{\nu}=1+c_s^2/c^2$, with $c_s^2\ll c^2$; therefore (182) becomes

$$\ddot{\psi}+4\dot{R}R^{-1}\dot{\psi}+\{(c_sk/c)^2+\Lambda R^2-2\kappa\}R^{-2}\psi=0. \quad (185)$$

This is identical with the Newtonian result (65) if k is replaced with kR_0c (k is now dimensionless), or if R is replaced with R/R_0c. Similarly, (184) is identical with (37); and since $c^2\rightarrow\infty$, (183) becomes (38).

For small scale irregularities of $\lambda\ll Rc$, $\lambda\dot{R}/R\ll c$, in which the fluid velocity is small compared with the velocity of light, an approximate form of (160) is

$$\Box^2h_0{}^0=-8\pi G\delta(\rho+3p/c^2). \quad (186)$$

which is Irvine's[90] equation. In this approximation $\delta\rho/\rho$, $\delta p/p$, need not be small quantities. The results are equivalent to the Newtonian results for small scale irregularities and $p\ll\rho c^2$. Lifshitz's basic equations are similar to (167)–(170). Coordinate transformations are possible which allow four of the $h_j{}^i$ to be zero; which of the $h_j{}^i$ are made zero determines to some extent the simplicity of the equations for a given physical problem. Lifshitz makes the choice $h_j{}^0=0$ and derives solutions in terms of scalar, vector, and tensorial harmonics.[99]

5.6. Normal Modes of Vibration in Curved Space

Using r, θ, ϕ coordinates in

$$\nabla^2\psi+k^2\psi=0,$$

and separating the variables $\psi=\psi(t)\Psi(r)Y_n{}^m(\theta,\phi)$, we find that only the radial function $\Psi(r)$ is different in the three cases $\kappa=0,\pm1$.

[99] The author is indebted to R. K. Sachs for pointing out that Lifshitz's solutions do not exclude irrotational motions.

For spherical harmonics of degree n the radial equation is

$$\frac{1}{q^{3/2}r^2}\frac{d}{dr}\left(q^{1/2}r^2\frac{d\Psi}{dr}\right)+\left[k^2-\frac{n(n+1)}{qr^2}\right]\Psi=0, \quad (187)$$

where $q=(1+\frac{1}{4}\kappa r^2)^{-2}$. We consider briefly the wave functions and eigenvalues in a space of negative and positive curvature.[89,100–102]

For $\kappa=1$ let

$$\sin\alpha=r/(1+\tfrac{1}{4}r^2);$$

hence,

$$d\alpha=dr/(1+\tfrac{1}{4}r^2), \quad (188)$$

and (187) becomes

$$(\sin^2\alpha)^{-1}\frac{d}{d\alpha}\left(\sin^2\alpha\frac{d\Psi}{d\alpha}\right)+\left[k^2-\frac{n(n+1)}{\sin^2\alpha}\right]\Psi=0. \quad (189)$$

With $\Psi=\Pi\sin^{1/2}\alpha$, this equation is

$$(\sin\alpha)^{-1}\frac{d}{d\alpha}\left(\sin\alpha\frac{d\Pi}{d\alpha}\right)+\left[\lambda(\lambda+1)-\frac{(n+\frac{1}{2})^2}{\sin^2\alpha}\right]\Pi=0, \quad (190)$$

and $\lambda(\lambda+1)=k^2+\frac{3}{4}$.

For $\kappa=-1$ let

$$\sinh\alpha=r/(1-\tfrac{1}{4}r^2);$$

hence,

$$d\alpha=dr/(1-\tfrac{1}{4}r^2), \quad (191)$$

and $\Psi=\Pi\sinh^{-1/2}\alpha$; then (187) becomes

$$(\sinh\alpha)^{-1}\frac{d}{d\alpha}\left(\sinh\alpha\frac{d\Pi}{d\alpha}\right)+\left[\lambda(\lambda+1)-\frac{(n+\frac{1}{2})^2}{\sinh^2\alpha}\right]\Pi=0, \quad (192)$$

and $\lambda(\lambda+1)=k^2-\frac{3}{4}$.

Let $\xi=\kappa^{1/2}\alpha$, then both (190) and (192) can be expressed in the one equation

$$(\sin\xi)^{-1}\frac{d}{d\xi}\left(\sin\xi\frac{d\Pi}{d\xi}\right)+\left[\lambda(\lambda+1)-\frac{(n+\frac{1}{2})^2}{\sin^2\xi}\right]\Pi=0, \quad (193)$$

where now $\lambda(\lambda+1)=\kappa k^2+\frac{3}{4}$, or

$$\lambda_{1,2}=-\tfrac{1}{2}\pm(1+\kappa k^2)^{1/2}. \quad (194)$$

The solutions of (193) are the associated Legendre functions $P_\nu{}^\mu(\cos\xi)$, $Q_\nu{}^\mu(\cos\xi)$, and $\mu=\frac{1}{2}+n$, $\nu=\lambda$.

Positive curvature ($\kappa=1$). In this case it is more convenient to set

$$k^2=\gamma(\gamma+2). \quad (195)$$

[100] V. Fock, Z. Physik **98,** 148 (1935).
[101] E. Schrödinger, Physica **6,** 899 (1939).
[102] E. Schrödinger, *Expanding Universes* (Cambridge University Press, Cambridge, England, 1957).

Since $P_\nu^\mu = P_{-\nu-1}^\mu$, or $P_{\lambda_1}^\mu = P_{\lambda_2}^\mu$, we use $\nu=\lambda_1$. Because $\mu\pm\nu$ is an integer, but μ is not an integer, we can use P_ν^μ and $P_\nu^{-\mu}$ as linearly independent solutions.[103] These are

$$\Psi_\gamma^n = (\pi/2\sin\alpha)^{1/2} P_{1/2+\gamma}^{\;1/2+n}(\cos\alpha), \qquad (196)$$

$$\Psi_\gamma^{-n} = (\pi/2\sin\alpha)^{1/2} P_{1/2+\gamma}^{\;-1/2-n}(\cos\alpha). \qquad (197)$$

From the definition

$$P_{1/2+\gamma}^{\;\pm(1/2+n)}(\cos\alpha) = \frac{(\cot\frac{1}{2}\alpha)^{\pm(1/2+n)}}{\Gamma(1\mp(\frac{1}{2}+n))} \times F(\tfrac{1}{2}-\gamma, \tfrac{1}{2}+\gamma; 1\mp(\tfrac{1}{2}+n); \sin^2\tfrac{1}{2}\alpha), \qquad (198)$$

it is seen that as $\alpha\rightarrow 0$ (or $r\rightarrow 0$)

$$\Psi_\gamma^{\pm n} \rightarrow \sim \frac{(\cot\frac{1}{2}\alpha)^{\pm(1/2+n)}}{(\sin\alpha)^{1/2}\Gamma(1\mp(\frac{1}{2}+n))},$$

and as n is a positive integer, only Ψ_γ^{-n} is regular at the origin. Therefore

$$\Psi = \Psi_\gamma^{-n} = (\pi/2\sin\alpha)^{1/2} P_{1/2+\gamma}^{\;-1/2-n}(\cos\alpha). \qquad (199)$$

We have assumed that γ is integral; it can be shown that this is necessary in order that Ψ is periodic or single-valued:

$$\Psi_\gamma^{-n}(-\cos\alpha) = \cos(\gamma-n)\pi\Psi_\gamma^{-n}(\cos\alpha). \qquad (200)$$

Thus the wave function is symmetric (antisymmetric) about $\alpha=\frac{1}{2}\pi$, or $r=2$, when $\gamma-n$ is an even (odd) integer. This has interesting consequences for elliptical space $0<r<2$ and spherical space $0<r<\infty$. The transformation $r\rightarrow 4/r$ leaves the metric (118) unchanged. It is therefore said[104] that elliptical and spherical space are indistinguishable because $2<r<\infty$ is merely a remapping of elliptical space. In a perturbed universe, however, elliptical space is not a mirror image of $2<r<\infty$ for the antisymmetric wave functions, and therefore it would seem to be an inadequate model of the universe.

For $n=0, 1, \cdots, \gamma$,

$$\Psi_\gamma^{-n} = \frac{\sin^n\alpha}{M_n}\frac{d^{n+1}[\cos(1+\gamma)\alpha]}{d(\cos\alpha)^{n+1}},$$

$$M_n = (1+\gamma)^2[(1+\gamma)^2-1]\cdots[(1+\gamma)^2-n^2]. \qquad (201)$$

It follows, for $n=0$,

$$\Psi_\gamma^0 = \sin(1+\gamma)\alpha/(1+\gamma)\sin\alpha, \qquad (202)$$

and for $n=\gamma$,

$$\Psi_\gamma^{-\gamma} = (\sin\alpha)^\gamma/(1.3.5\cdots 2\gamma+1). \qquad (203)$$

Also

$$\Psi_{\gamma+1}^{-\gamma} = \cos\alpha\Psi_\gamma^{-\gamma}, \qquad (204)$$

$$\Psi_{\gamma+2}^{-\gamma} = \left(1-\frac{2\gamma+4}{2\gamma+3}\sin^2\alpha\right)\Psi_\gamma^{-\gamma}. \qquad (205)$$

Hence, the radial functions of the lowest modes are

$$\Psi_1^0 = \cos\alpha, \qquad \Psi_1^{-1} = \tfrac{1}{3}\sin\alpha, \qquad (206)$$

$$\Psi_2^0 = 1-\tfrac{4}{3}\sin^2\alpha, \qquad \Psi_2^{-1} = \tfrac{1}{3}\sin\alpha\cos\alpha,$$

$$\Psi_2^{-2} = \tfrac{1}{15}\sin^2\alpha, \qquad (207)$$

$$\Psi_3^0 = \cos\alpha(1-2\sin^2\alpha), \qquad \Psi_3^{-1} = \tfrac{1}{3}\sin\alpha(1-\tfrac{6}{5}\sin^2\alpha), \qquad (208)$$

$$\Psi_3^{-2} = \tfrac{1}{15}\cos\alpha\sin^2\alpha, \qquad \Psi_3^{-3} = \tfrac{1}{105}\sin^3\alpha. \qquad (209)$$

In a space of uniform positive curvature the eigenvalues are

$$k^2 = \gamma(\gamma+2), \qquad \gamma = 1, 2, 3, \cdots,$$

and $m\leq n\leq\gamma$. The fundamental mode has an eigenvalue of $k^2=3$.

Negative curvature ($\kappa=-1$). For the eigenvalues in this case we use

$$k^2 = \gamma^2+1, \qquad (210)$$

and from (194), $\lambda_{1,2} = -\frac{1}{2}\pm i\gamma$. $P_\nu^{-\mu}$ and Q_ν^μ are linearly dependent because $\mu=\frac{1}{2}+n$ is half-integral, and for the linearly independent solutions P_ν^μ and $P_\nu^{-\mu}$ are again chosen. Since $P_{\lambda_1}^\mu = P_{\lambda_2}^\mu$, we chose $\nu=\lambda_1$, and therefore

$$\Psi_\gamma^n = (\pi/2\sinh\alpha)^{1/2} P_{-1/2+i\gamma}^{\;1/2+n}(\cosh\alpha), \qquad (211)$$

$$\Psi_\gamma^{-n} = (\pi/2\sinh\alpha)^{1/2} P_{-1/2+i\gamma}^{\;-1/2-n}(\cosh\alpha). \qquad (212)$$

The hypergeometric expression

$$P_{-1/2+i\gamma}^{\;\pm(1/2+n)}(\cosh\alpha) = \frac{(\coth\frac{1}{2}\alpha)^{\pm(1/2+n)}}{\Gamma(1\mp(\frac{1}{2}+n))} \times F(\tfrac{1}{2}-i\gamma, \tfrac{1}{2}+i\gamma; 1\mp(\tfrac{1}{2}+n); -\sinh^2\tfrac{1}{2}\alpha) \qquad (213)$$

is real, and as $\alpha\rightarrow 0$,

$$\Psi_\gamma^{\pm n} \rightarrow \sim \frac{(\coth\frac{1}{2}\alpha)^{\pm(1/2+n)}}{(\sinh\alpha)^{1/2}\Gamma(1\mp(\frac{1}{2}+n))},$$

and therefore only Ψ_γ^{-n} is regular at the origin. Thus,

$$\Psi = \Psi_\gamma^{-n} = (\pi/2\sinh\alpha)^{1/2} P_{-1/2+i\gamma}^{\;-1/2-n}(\cosh\alpha), \qquad (214)$$

and because space is open, there are no periodic conditions to satisfy, and γ can have any real value.

For integral values of n,

$$\Psi_\gamma^{-n} = \frac{\sinh^n\alpha}{N_n}\frac{d^{n+1}(\cos\gamma\alpha)}{d(\cosh\alpha)^{n+1}},$$

$$N_n = -\gamma^2(\gamma^2+1)\cdots(\gamma^2+n^2), \qquad (215)$$

[103] W. Magnus and F. Oberhettinger, *Functions of Mathematical Physics* (Chelsea Publ. Co., New York, 1943).

[104] A. S. Eddington, *Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, England, 1950), p. 157.

and for $n=0$,

$$\Psi_\gamma^0=\sin\gamma\alpha/\gamma\sinh\alpha, \qquad (216)$$

$n=1$,

$$\Psi_\gamma^{-1}=(\gamma^2+1)^{-1}(\gamma\cot\gamma\alpha-\coth\alpha)\Psi_\gamma^0, \quad (217)$$

and so forth.

We have found that the eigenvalue spectra for $\kappa=0$, 1, and -1 are

$$\begin{aligned}&\kappa=0, && k^2=\gamma^2, && \gamma^2\geq 0,\\ &\kappa=1, && k^2=\gamma(\gamma+2), && \gamma=1,2,3\cdots,\\ &\kappa=-1, && k^2=\gamma^2+1, && \gamma^2\geq 0,\end{aligned} \qquad (218)$$

and the lowest eigenvalues are $k^2=0$, 3, and 1, respectively. The eigenvalues form continuous spectra for zero and negative curvatures, and a discrete spectrum for positive curvature.

5.7. Time Dependence of Modes in a Static Universe

In the static models $\dot{R}=\ddot{R}=0$ and therefore, from (132) and (133),

$$\Lambda=4\pi(3\nu-2)G\rho=(3\nu-2)\kappa/\nu R^2. \qquad (219)$$

Equations (182)–(184) now become

$$\ddot{\psi}+[(\tilde{\nu}-1)k^2-(3\tilde{\nu}-2)\kappa]R^{-2}\psi=0, \qquad (220)$$

$$\dot{\psi}-4\pi G\nu\rho c^2R^2\varphi=0, \qquad (221)$$

$$\kappa\mu+(\nu\psi/c^2)(k^2-3\kappa)=0. \qquad (222)$$

It follows that

$$\psi\propto\delta\rho\propto\exp\pm i\{(\tilde{\nu}-1)k^2-(3\tilde{\nu}-2)\kappa\}^{1/2}R^{-1}t. \quad (223)$$

With ν arbitrary there are an infinite number of both unstable and stable static models. Let $R\rightarrow R+\delta R$, then from (130), (134), to a first order

$$\delta\ddot{R}-\nu\Lambda\delta R=0, \qquad (224)$$

or

$$\delta R\propto\exp\pm\{(3\nu-2)\kappa\}^{1/2}R^{-1}t. \qquad (225)$$

It can be shown[74] that unstable models occur when $\kappa=1$, $\nu>\frac{2}{3}$, and when $\kappa=-1$, $\nu<0$; and stable models occur when $\kappa=-1$, $\nu<0$.

Consider first $\kappa=1$. Then, from (223)

$$\delta\rho\propto\exp\pm i\{(\tilde{\nu}-1)\gamma(\gamma+2)-(3\tilde{\nu}-2)\}^{1/2}R^{-1}t, \quad (226)$$

where $k^2=\gamma(\gamma+2)$. For $\nu=\tilde{\nu}=\frac{4}{3}$, which is of particular interest,

$$\delta\rho\propto\exp\pm i\{\gamma(\gamma+2)-6\}^{1/2}t/3^{1/2}R, \qquad (227)$$

and all modes $\gamma>2$ are oscillatory. The fundamental model $\gamma=1$ grows as $\exp(t/R)$, whereas the universe departs from its equilibrium state as $\exp 2^{1/2}t/R$, from (225). The mode $\gamma=2$ varies $At+Bt^{-1}$. The case of $\nu=1$ has been previously considered in (4.2); when $\tilde{\nu}<1$, all modes tend to be unstable.

In the case of $\kappa=-1$, $k^2=\gamma^2+1$

$$\delta\rho\propto\exp i\{(\tilde{\nu}-1)(\gamma^2+1)-(3\tilde{\nu}-2)\}^{1/2}R^{-1}t, \qquad (228)$$

with $\gamma^2>0$, and a given mode grows exponentially when $\tilde{\nu}<(\gamma^2-1)/(\gamma^2-2)$.

It was hoped[96] that the formation of condensations in an unstable static universe would increase the volume to $V+\delta V$, thus launching the universe on a career of expansion rather than contraction. Using the line element (176), it can be seen

$$\delta V=3R^2\int_V\sin^2\alpha\sin\theta\psi(\alpha,\theta,\phi)\,d\alpha d\theta d\phi, \quad (229)$$

and in the linear approximation, $\delta V=0$, as was subsequently discovered.[105] It is interesting to notice that the solution of Lapace's equation in spherical space is

$$\Psi_0^0=q\cot\alpha=q(r^{-1}-\tfrac{1}{4}r), \qquad (230)$$

where q is a constant, as shown by Whittaker[106] and Copson.[107] The transformation $\alpha\rightarrow\pi-\alpha$ or $r\rightarrow 4/r$ reverses the sign of Ψ_0^0. Thus, if the universe consists of a large number of condensations, then by taking conjugate pairs at (α,θ,ϕ), $(\pi-\alpha,\pi-\theta,\phi+\pi)$ the net gravitational potential vanishes, as it must, since the effect of gravity is already included in the assumed curvature.

Spherical space—not necessarily static—has the intriguing property of providing a form of "CPT" invariance. If $\kappa=1$ and $r^2=x^2+y^2+z^2$ in the line element (118), then

$$\begin{aligned}x&=2\tan\tfrac{1}{2}\alpha\sin\theta\cos\phi,\\ y&=2\tan\tfrac{1}{2}\alpha\sin\theta\sin\phi,\\ z&=2\tan\tfrac{1}{2}\alpha\cos\theta.\end{aligned} \qquad (231)$$

The operation $0(r\rightarrow 4/r)$ maps $r<2$ onto $r>2$ with reversed parity: $x\rightarrow -x$, $y\rightarrow -y$, $z\rightarrow -z$. Similarly, if q is an electric charge, then $q\rightarrow -q$. Thus if the charge conjugation operator C is limited to electric charge, then 0 conserves CP. If now we consider a perturbed spherical space, it can easily be shown from the geodesic equations that CP invariance is violated whereas CPT is conserved, where T is the time (or momenta) reversal operator.

5.8. Time Dependence of Modes in Nonstatic Models

We assume that $\Lambda=0$, and take first the conventional view that $1\leq\nu\leq\frac{4}{3}$. The case of $p\ll\rho c^2$ has been previously considered and we therefore consider the extremely

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interesting case of $\nu=\frac{4}{3}$. In the range $10^{-21}<\rho<10^{-2}$ g cm^{-3}, where the upper limit corresponds to a photon energy of the order of 1 MeV, radiation is the dominant constituent in the universe and both ν and $\tilde{\nu}$ have values close to $\frac{4}{3}$. At higher density the composition of the cosmic fluid becomes progressively more complex; nevertheless, it seems likely that ν retains a value close to $\frac{4}{3}$.

Defining β_m as

$$\beta_m=\psi R^m, \tag{232}$$

we have from (182) with $\Lambda=0$, $\nu=\tilde{\nu}=\frac{4}{3}$,

$$\ddot{\beta}_{5/2}+(\tfrac{1}{3}k^2-\tfrac{10}{3}\pi G\rho R^2-\tfrac{1}{4}\kappa)R^{-2}\beta_{5/2}=0. \tag{233}$$

Thus $\beta_{5/2}$, and therefore all β_m, are periodic in time when

$$\lambda\!\!\!^{-}<(c^2/10\pi G\rho)^{1/2}, \tag{234}$$

where $\lambda\!\!\!^{-}=cRk^{-1}$, and the curvature term is negligible compared with k^2.

Because χ is generally a small quantity at high density (182) transforms to (for $\kappa=0, \pm1$)

$$\psi''+4\chi^{-1}\psi'+(\tfrac{1}{3}k^2-4\kappa)\psi=0, \tag{235}$$

using (147), where primes denote differentiation with respect to χ. Either from (180) or (184), the contrast density is

$$\tfrac{1}{2}\mu=-\chi\psi'-(\tfrac{1}{3}k^2\chi^2+1)\psi. \tag{236}$$

The solution of (235) is

$$\psi\propto\chi^{-3/2}J_{\pm3/2}(a\chi),$$

$$a=(\tfrac{1}{3}k^2-4\kappa)^{1/2}. \tag{237}$$

For the low-order modes $a\chi\ll1$, and therefore $\mu\propto\psi$ and

$$\mu=A+BR^{-3}, \tag{238}$$

where $\chi\propto R\propto t^{1/2}$. Thus $\delta\rho/\rho$ is either constant or diminishes in an expanding universe. For the high-order modes of $a\chi\gg1$, we have $\mu\propto\chi^2\psi$, and therefore

$$\mu=A\sin a\chi+B\cos a\chi. \tag{239}$$

The condition $a\chi\gg1$ is simply $\lambda\!\!\!^{-}\ll(c^2/8\pi G\rho)^{1/2}$. We can summarize by saying that the contrast density oscillates at constant amplitude at a frequency of $2c/3^{1/2}\lambda$ for short wavelengths, and is either constant or decays in an expanding universe for long wavelengths.

The possibilities occurring when $\nu<1$ are not particularly interesting, except perhaps when $\nu=0$, as in the steady-state model. Let $\tilde{\nu}=1+c_s^2/c^2$, $\kappa=0$, then for $c_s^2\ll c^2$ (182) becomes

$$\psi''+\frac{6}{3\nu-2}\frac{\psi'}{\chi}+\left(\frac{2}{3\nu-2}\right)^2\left(\frac{c_s^2}{c^2}k^2+(1-\nu)\frac{3}{\chi^2}\right)\psi=0. \tag{240}$$

For $\nu=0$, and maximum growth of $c_s=0$, we find

$$\delta\rho\propto\psi R^{-2}=AR^{-3}+BR^{-4}, \tag{241}$$

and in the steady-state theory condensations cannot form as a result of perturbations in the average density, as shown by Bonnor.[10]

6. DISCUSSION

The vibrations of the universe are moderately well behaved and show no signs of a catastrophic growth in amplitude. This conclusion has been reached by several authors in various ways. Within the limitations of simple linearized theory the expanding universe is reasonably stable when subject only to gravitational interactions. This is illustrated in an approximate manner by Jeans' theory. The gravitational frequency is of the order $i(\rho G)^{1/2}$, and the age of the universe is of the order $(\rho G)^{-1/2}$, and hence perturbations cannot grow significantly in the time available.

In an expanding cold universe of zero curvature we have

$$\mu=\mu_0(\rho_0/\rho)^{1/3}, \tag{242}$$

where μ_0, ρ_0 denote initial conditions. If the disturbances originate from thermal fluctuations, the mean-square fluctuation in N particles is

$$\langle(\Delta N)^2\rangle=kT(\partial N/\partial\mu)_{T,V}, \tag{243}$$

where μ is the chemical potential. The right-hand side of this equation is equal to N for a Maxwellian distribution, and is of the same order for a relativistic gas of bosons, and for fermions in which the Fermi energy is equal to kT. Therefore

$$\mu_0\sim N^{-1/2},$$

and $N=M/m$, where m is the mass of the constituent particles and M is the eventual mass of the condensation. Hence, (242) becomes

$$\mu\sim(m/M)^{1/2}(\rho_0/\rho)^{1/3}, \tag{244}$$

where ρ is the mean density of the universe when the condensation has formed, and $\rho_0\sim M/\lambda_0^3$ is the initial mean density where λ_0 is the characteristic size. Now μ cannot be made as large as we please by increasing ρ_0, for then the assumption of a cold universe breaks down and (239) replaces (242). There is thus a limited range of ρ_0/ρ which is not large enough to compensate for the extreme smallness of m/M.

A redeeming feature of gravitational theory is that the universe in the large tends to remain homogeneous and isotropic. If gravitational theory alone adequately accounted for the growth of irregularity, then the low-order modes would also develop large amplitudes and the universe would possess pronounced macroscopic anisotropy. What is required is that the universe is unstable for an intermediate range of wavelengths at some stage in the expansion, and the density in various regions thereafter ceases to diminish with time. Such a concept demands that the gravitational potential of the disturbance increases with time and attains a value of $|\psi|\sim GM/\lambda$. But in the cold universe, and in the radiation universe, ψ is not an increasing function of time.

A linearized theory limited to irrotational motions and gravitational interactions is open to several criticisms. The neglect of all forms of rotation is a gross simplification, since angular momentum is a common and indispensible feature of galactic and stellar systems. It seems plausible that at subnuclear density, at least, a treatment based on rotational motions will lead to even slower rates of growth owing to the presence of inertial forces. Gamow[108,109] has proposed a primordial turbulent state of large amplitude fluctuations for the initial conditions. Similarly, Weizsäcker[110] assumes an initial state of turbulent gas clouds. Bonnor[111] points out, however, that turbulence is more likely to be the result rather than the cause of condensations. Furthermore, initial conditions of this nature add to the mystery rather than clarifying it, and their postulation falls within the province of the initial structure hypothesis.

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[110] C. F. von Weizsäcker, Astrophys. J. **14,** 165 (1951).
[111] W. B. Bonnor, Z. Astrophys. **39,** 143 (1956).

Tolman,[112] Bonnor,[111] and Peebles[113] have used nonlinear theory to study the growth of single condensations, and Bonnor has shown that the formation of the nebulae is an improbable occurrence if $\mu_0 \sim N^{-1/2}$, where N is the number of atoms.

The assumption that initial disturbances are small demands that rapid growth is possible at some stage in the expansion of the universe. This would appear to be impossible unless we abandon the rudimentary fluid prescription. More complex fluids with mechanisms of radiative transfer are an attractive possibility. However, the difficulty is that radiation is the dominant constituent of the universe over an extremely wide range of density. In this period the motion of matter is impeded by radiative drag,[67] and thereafter ρ_0/ρ in (244) increases approximately by only three orders of magnitude for galaxies. Either condensations evolve in spite of radiative drag or they must evolve rapidly after the radiation deluge has subsided.[114]

A study of the modes of vibrations of the universe shows that the origin of basic structure is an intriguing and challenging problem.

[112] R. C. Tolman, Proc. Natl. Acad. Sci. (U. S.) **20,** 169 (1934).
[113] P. J. E. Peebles, Astrophys. J. **147,** 859 (1967).
[114] *Note added in proof.* It has since been proposed [E. R. Harrison, Phys. Rev. Letters **18,** 1011 (1967)] that the early rudimentary structure of the universe consists of compositional fluctuations in a fluid of uniform density, rather than density fluctuations in a fluid of uniform composition. Thus fluctuations in the *b* argon number are amplified and an expanded universe is left in a fragmented state consisting of separate regions of matter and antimatter.