Phase-Shift Representation for Nucleon–Nucleon Scattering Above Pion-Production Threshold

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We discuss a number of questions relating to phase-shift representations of nucleon–nucleon scattering in an energy range where inelastic channels are open and may contribute significantly to such representations. While attention will be directed specifically toward nucleon–nucleon scattering above 400 MeV, we see the relevance of such questions being applied to similar strongly interacting systems.

Before discussing the details of partial wave analyses above pion-production thresholds, we should examine the validity of the consequent phase-shift representations. Elastic phase-shift analyses at 350 MeV reveal the importance of $H(l=5)$ waves, which indicate the expected complexity of scattering amplitudes around 1 Bev; that is, we may anticipate the need for extracting 20 or more phase parameters from elastic proton–proton scattering data which is neither too precise nor too abundant. One way to work around the dilemma which results from a sparcity of the data, is to impose certain model constraints on the phase parameters. For instance, we may require that the intermediate and long-range part of the force by given by a one-boson-exchange mechanism, or that the real phases connect smoothly, with appropriately chosen energy dependences, to their well-determined elastic values. Other model predictions may be formulated as constraints on inelastic phase parameters. Perhaps the most fruitful idea in this regard was a suggestion of Mandelstam[1] that, in an energy range where $\Delta(\frac{3}{2},\frac{3}{2})$ production between a final-state pion and proton is important, one can consider the dominant inelastic channels as Δp in relative S, P, or D states. Symmetry considerations combined with reaction cross sections are then used to calculate modulii of elastic scattering elements. This idea has played an important role in the phenomenological analyses attempted so far.[2,3] Recently, Amaldi[4] and co-workers have studied the constraints on the modulii of elastic scattering elements which are implied by a peripheral model for pion production in proton–proton scattering below 1.5 BeV. This idea will be considered in detail momentarily, but let's first examine how such constraints arise and how they can best be built into a phenomenological analysis.

If we exhibit the S matrix in an expanded Hilbert space it has a structure indicated by Eq. (1)

$$S=\begin{pmatrix} S_e & R \\ \tilde{R} & S_i \end{pmatrix}. \tag{1}$$

S_e determines scattering between "elastic" channels; S_i determines scattering between inelastic channels and, of course, R describes transitions between elastic–inelastic channels. Time-reversal invariance gives us symmetry of S_e and S_i, and unitarity tells us that;

$$S_e^*S_e+R^*\tilde{R}=1. \tag{2}$$

If, in an angular momentum representation, S_e is one dimensional, we obtain the familiar result

$$S_e^2=1-\sum_n |R_{ne}|^2. \tag{3}$$

So unitarity, in this instance, prescribes that the modulus of the elastic scattering element be bounded by 1.

Consideration of Eq. (2) when S_e is two dimensional, as is the case in nucleon–nucleon scattering where angular momentum coupling is allowed, produces 3 equations

$$|S_{11}|^2+|S_{12}|^2=1-\sum_n |R_{1n}|^2 \tag{4a}$$

$$|S_{22}|^2+|S_{12}|^2=1-\sum_n |R_{2n}|^2 \tag{4b}$$

$$S_{11}^*S_{12}+S_{12}^*S_{22}=-\sum_n R_{1n}^*R_{2n}, \tag{4c}$$

where

$$S_e=\begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix}.$$

It is important to note that Eq. (4) does not reduce the number of parameters required to define S_e, the number remains at 6 real parameters for a 2 by 2 complex symmetric matrix. It has been a common practice to parameterize for inelastic scattering by simply allowing the phase of the diagonal elements to become complex while keeping the mixing parameter real. This results in a 5-parameter description of a 6-parameter phenomenon.

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[1] S. Mandelstam, Proc. Roy. Soc. (London) **A244,** 491 (1958).
[2] Y. Hama and N. Hoshizaki, Progr. Theoret. Phys. (Kyoto) **31,** 609 (1964).
[3] L. S. Azhgirey, N. P. Klepikov, Yu. P. Kumekin, M. G. Mescheryakov, S. B. Nurushev, and G. D. Stoletov, Phys. Letters **6,** 196 (1963); Zh. Eksperim. i Teor. Fiz. **46,** 1074 (1964) [English transl.: Soviet Phys.—JETP **19,** 728 (1964)].
[4] U. Amaldi, Jr., R. Biancastelli, and S. Francaviglia, Nuovo Cimento **47,** 85 (1967).

TABLE I. Comparison of p–p phase-shift solutions at 660 MeV. A-M Solutions use 107 p–p data from 635–680 MeV.

Parameter	(A-M)^A	A-M	H-H	Azhgirey (Ref. 3)
χ^2	63.2	61.3		
$^1S_0$	-19.6 ± 4.6	-13.1 ± 5.8	-32	-31.9 ± 11.1
$^1D_2$	11.9 ± 2.4	12.6 ± 3.2	10	7.5 ± 6.9
$^1G_4$	7 ± 0.84	6.7 ± 1.3	4.5	5.9 ± 2.1
$^3P_1$	-32.6 ± 7.3	-30.2 ± 7	-37	-35.8 ± 5.7
$^3F_3$	-5 ± 1.8	-3.98 ± 2.1	3.5	1.6 ± 5.3
$^3H_5$	-6.1 ± 0.95	-5.7 ± 1.4	-1.25	(-2.67)
$^3P_2$	25.2 ± 2.6	27.9 ± 3.3	16	18.3 ± 3.3
$^3F_4$	6.1 ± 1.0	5.4 ± 1.5	3	2.3 ± 0.9
$^3H_6$	1.6 ± 0.6	1.7 ± 0.8	0.75	(0.62)
ϵ_2	-2.8 ± 2.6	-4.2 ± 2.8	-3	-2.8 ± 4.6
ϵ_4	-2 ± 1.4	-2.8 ± 1.5	-5.25	-5.7 ± 1.7
$^3P_0$	-41.7 ± 12	-43.7 ± 11.2	-51	-46 ± 18
$^3F_2$	0.3 ± 2.3	-0.8 ± 3.6	-4	-3.6 ± 2.4
$^3H_4$	1.5 ± 1	1.5 ± 1.5	0.75	0.2 ± 0.9
$R(^3P_0)$	0.99	1.00	0.98	0.79 ± 0.18
$R(^3P_1)$	0.89	0.85	0.98	1.1 ± 0.31
$R(^3P_2)$	0.96	0.95	0.90	0.9 ± 0.17
$R(^1D_2)$	0.71	0.75	0.64	0.66 ± 0.04
$R(^3F_2)$	0.97	0.89	0.90	0.89 ± 0.07
$R(^3F_3)$	0.89	0.99	0.68	0.67 ± 0.11
$R(^1G_4)$	0.99	0.95	(1)	(1)

How then should one parameterize the elastic S matrix to accomplish phenomenological analysis? The choice is arbitrary, but should satisfy certain general criterion. First, it should be implicity unitary; equations (3) and (4) should not be violated for any choice of the parameters. Second, it should be capable of describing any situation allowed by unitarity (and it is on this point that a five-parameter description must be criticized). Finally, it would be desirable to have a reasonable connection to the low energy (elastic) parameters. Our choice is given in Eq. (5) for coupled and for uncoupled states

$$S_J^U = \cos\rho_J \exp(2i\delta_J) \qquad \text{for uncoupled states} \tag{5a}$$

$$S_J^C = \begin{bmatrix} \cos\rho_- \cos2\epsilon e^{2i\delta_-} & i\sin2\epsilon e^{i(\delta_+ + \delta_- + \alpha)} \\ i\sin2\epsilon e^{i(\delta_+ + \delta_- + \alpha)} & \cos\rho_+ \cos2\epsilon e^{2i\delta_+} \end{bmatrix}, \tag{5b}$$

for angular momentum coupled states.

As we go below pion-production threshold, all inelastic parameters (ρ's, α's) go to zero and we obtain the familiar Stapp nuclear bar parameterization for elastic scattering.

We now define quantities which are derived from inelastic transition elements and relate them to the parameters of our elastic S matrix

$$X_J \equiv \sum_n |R_{1n}^J|^2 = 1 - \cos^2\rho_J, \tag{6a}$$

for uncoupled states

$$X_\pm \equiv \sum_n |R_{\pm n}^J|^2 = \cos^2 2\epsilon(1-\cos^2\rho_\pm), \tag{6b}$$

for coupled states

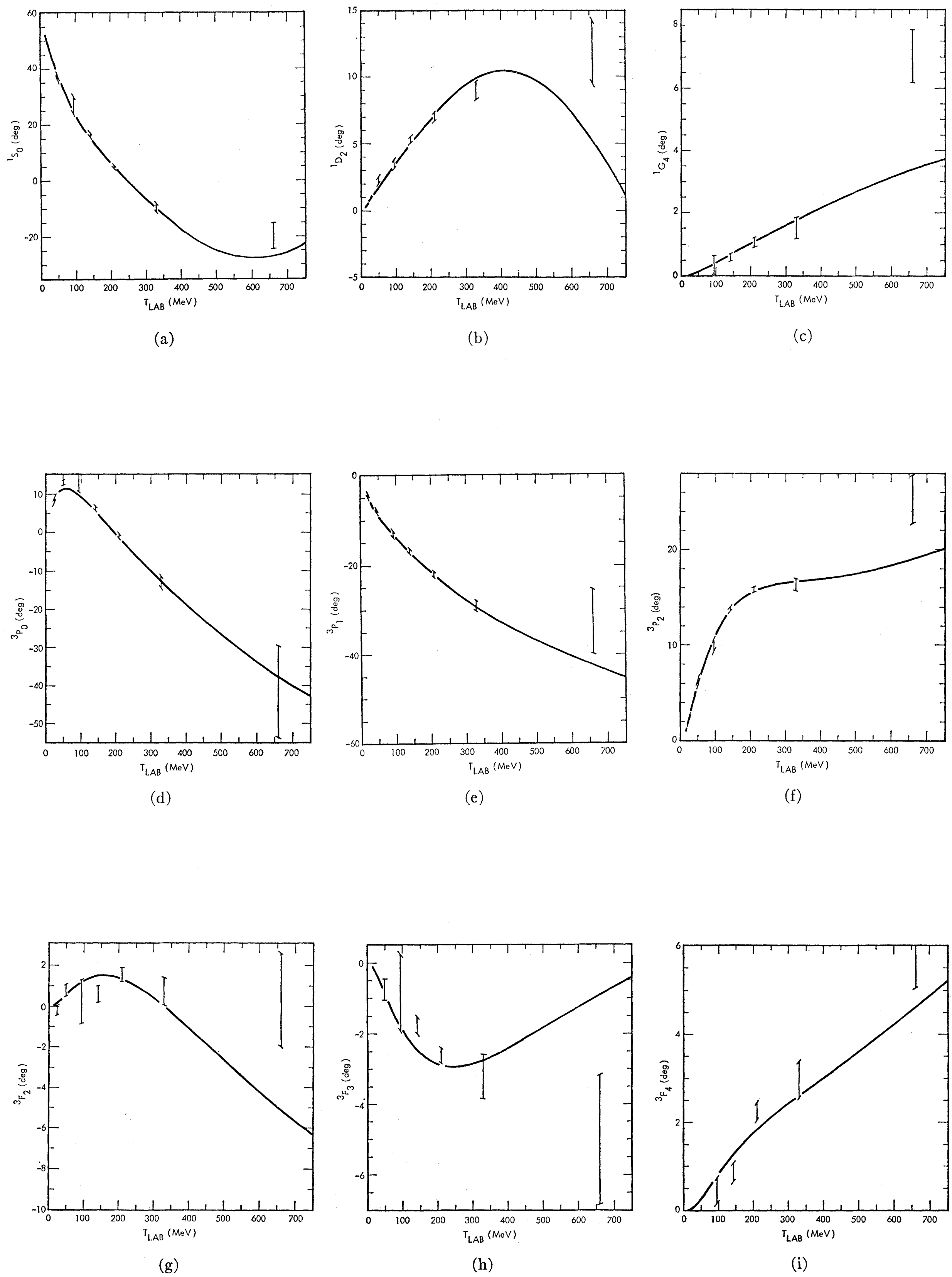
$$X_0 e^{i\beta} \equiv \sum_n R_{-n}^* R_{+n},$$

$$X_0 = \sin 2\epsilon \cos 2\epsilon(\cos^2\rho_+ + \cos^2\rho_- - 2\cos\rho_+\cos\rho_-\cos 2\alpha)^{1/2},$$

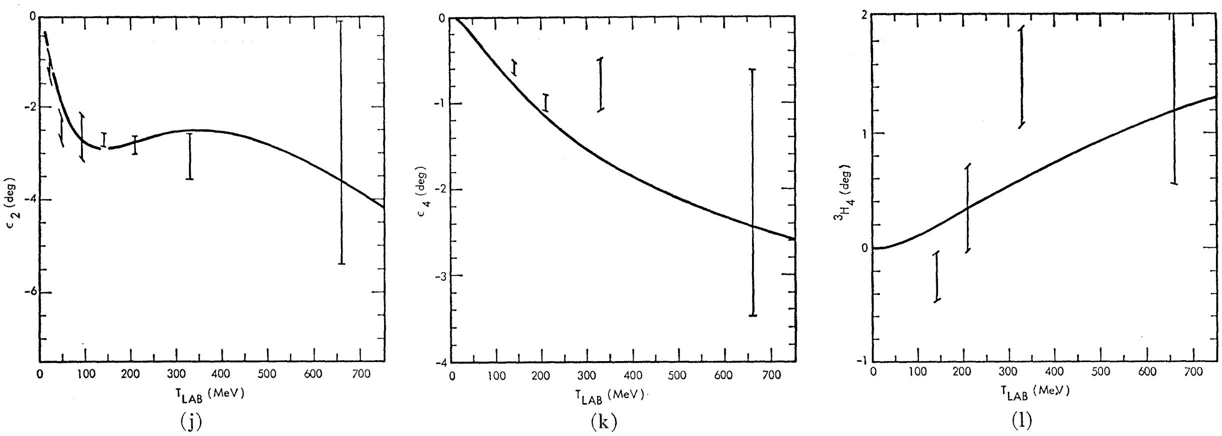
$$\beta = \delta_+ + \delta_- + \tan^{-1}[(\cos\rho_+ - \cos\rho_-)/(\cos\rho_+ + \cos\rho_-)]. \tag{6c}$$

We introduce these parameters as derived, for example, through the Amaldi model, as data with suitably chosen errors. These pseudo-data are then used to complement the existing elastic scattering data. These combination data are then used to extract phase parameters through a phenomenological analysis.

The next question regards the energy dependence of phase parameters. An examination of the existing data reveals that they tend to be somewhat homogeneously distributed in energies above 400 MeV. It seems at least unlikely that sufficient data exist near any one energy to make a single-energy analysis meaningful without being careful to specify the local energy dependence of the phase parameters. One reasonable approach to obtaining local behavior is to first do an energy-dependent phenomenological analysis. The local dependence of the phase parameters is fixed as the derivatives of the energy-dependent curves and the single-energy analysis can then be accomplished by utilizing data in a spread of energies around some central value.



(a)
$^1S_0$ (deg)
T_{LAB} (MeV)
(b)
$^1D_2$ (deg)
T_{LAB} (MeV)
(c)
$^1G_4$ (deg)
T_{LAB} (MeV)
(d)
$^3P_0$ (deg)
T_{LAB} (MeV)
(e)
$^3P_1$ (deg)
T_{LAB} (MeV)
(f)
$^3P_2$ (deg)
T_{LAB} (MeV)
(g)
$^3F_2$ (deg)
T_{LAB} (MeV)
(h)
$^3F_3$ (deg)
T_{LAB} (MeV)
(i)
$^3F_4$ (deg)
T_{LAB} (MeV)



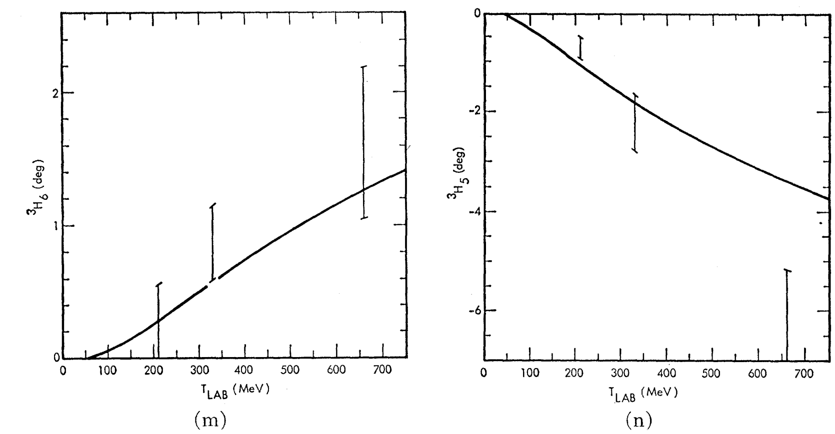


FIG. 1. Energy-dependent and single energy p–p phase shifts to 750 MeV.

The first phase of our program was to do a 36-parameter energy-dependent analysis of p–p data to 750 MeV. The data were composed of most of the available elastic p–p data from 400 to 800 MeV, plus matrix representations[5] for data at 6 elastic scattering energies to 330 MeV. In addition, pseudo-data derived from the Amaldi model at 660 MeV were introduced to serve as a constraint on inelastic phase parameters. The errors on the pseudo-data were obtained from Dr. Amaldi. The analysis is greatly simplified by treating the elastic scattering data through reduced matrix representations, and we suggest that whenever possible, such representations be employed for the development of scattering models.

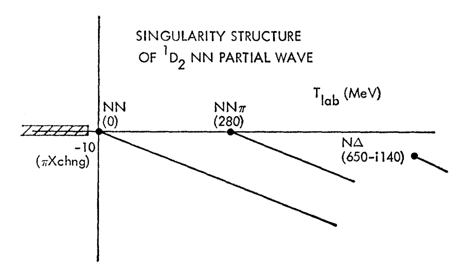


FIG. 2. Singularity structure (near threshold) of p–p $^1D_2$ partial wave amplitude.

The energy-dependent model which we used is a slightly expanded version of our fit to elastic data in which phases through $H(L=5)$ waves were treated as OPEC plus a linear sum over phenomenological basis functions. The functions were essentially Legendre functions of the second kind as would be obtained by partial wave projecting "t-channel" poles of varying masses. The "OPEC" contribution in $^1S_0$ was in fact the value predicted from an effective range expansion, and the phenomenological sum was over "P" wave basis functions; this was to maintain a "proper" low-energy behavior in $^1S_0$. Inelastic parameters were treated as polynomials above 400 MeV and were taken as zero below 400 MeV. We acknowledge that this may be a very crude parameterization of the inelastic phases but it seems a reasonable form to be used as a "first step." Over all χ^2 for the 713 data treated in this analysis came to 722, good enough to make the model "plausible."

Energy derivatives were extracted and used to accomplish a single-energy analysis of the data from 630 to 690 MeV; and some of the characteristic results are given in Fig. 1. Here we have illustrated the conse-

[5] R. A. Arndt and M. H. MacGregor, Phys. Rev. **141**, 873 (1966).

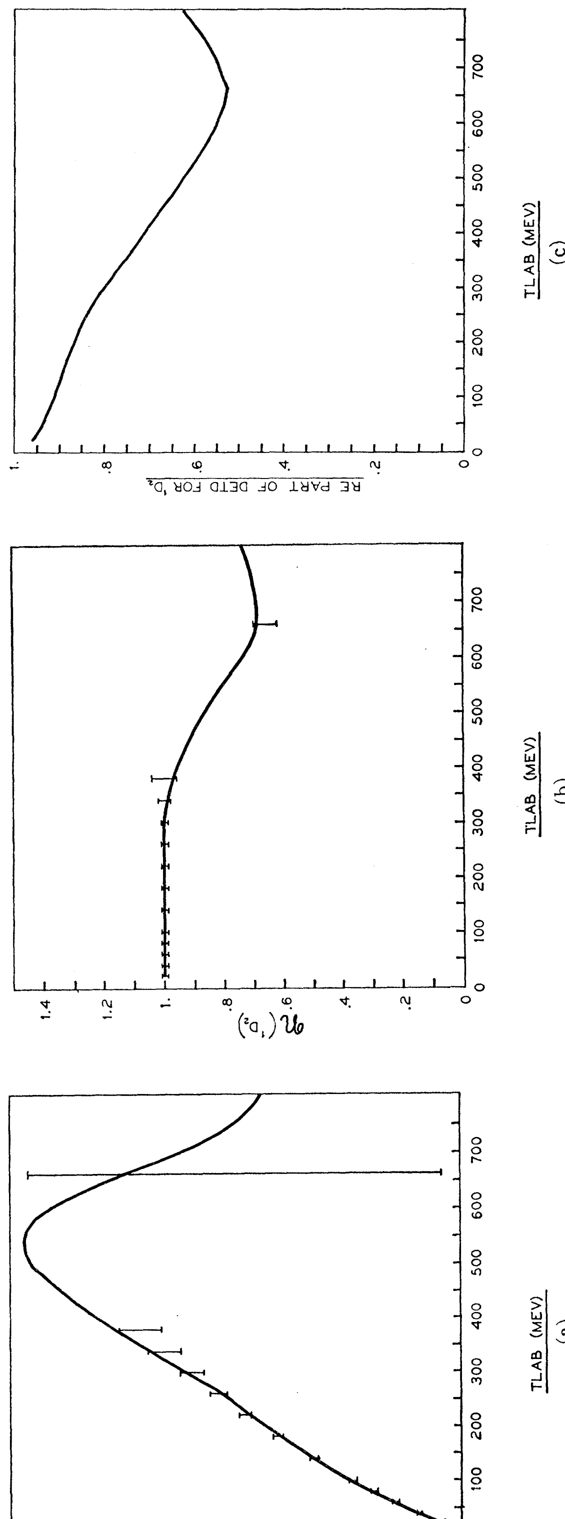


FIG. 3. Coupled channel fit to $^1D_2$ partial wave amplitude. Mass of the Δ is "smeared" out according to an appropriate Breit–Wigner form.

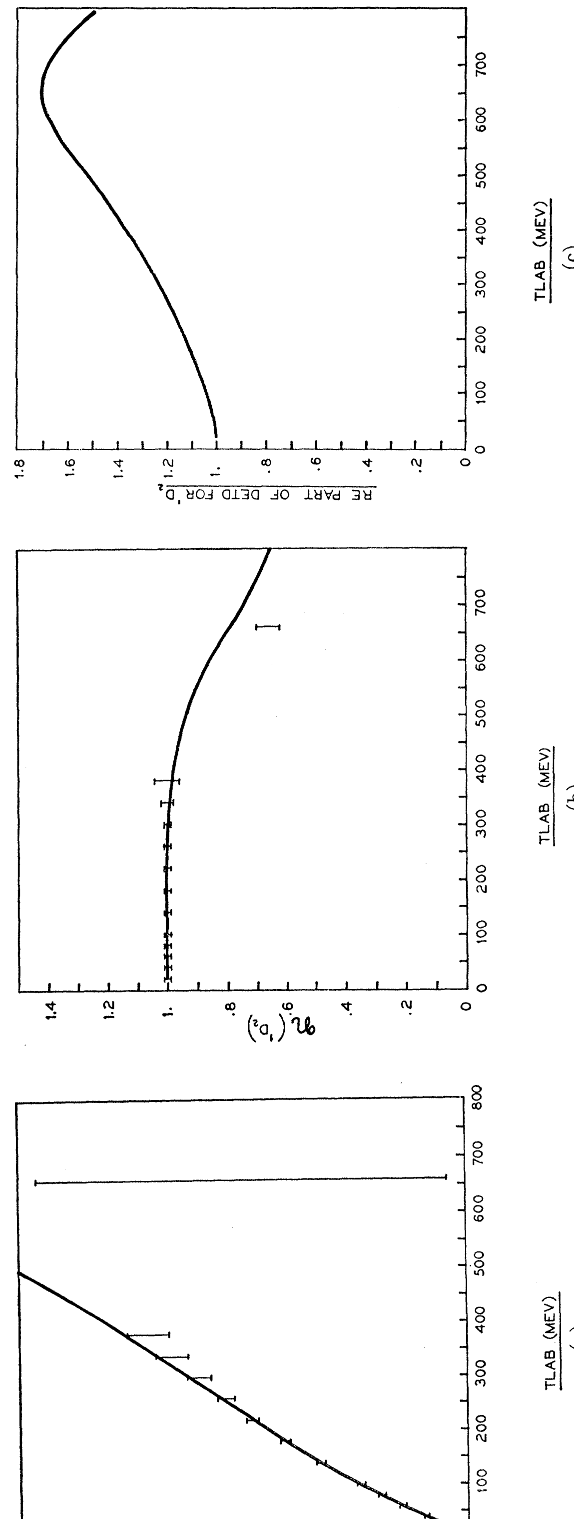


FIG. 4. Coupled channel fit to $^1D_2$ partial wave amplitude. Mass of Δ taken at an appropriate (fixed) complex value.

quences of both types of analysis and the agreement is apparent. The single-energy results at 660 MeV do not differ greatly from results obtained by Hama and Hoshizake,[2] and Azhgirey *et al.* (See Table I).

We can now direct our attention to the question of utility. We have obtained a reasonable, although not too precise, phase-shift representation for p–p scattering to 750 MeV; what theoretical concepts can it be used to test?

It can certainly be applied to the development and testing of the one-boson-exchange hypothesis, and it may prove useful in extending calculations based on partial wave dispersion relations, but it will certainly be of no use in deciding questions of asymptotic behavior.

One model is worth some discussion because it has some interesting theoretical implications and because validity of the model can only be tested through a clear understanding of nucleon–nucleon partial wave structure around 600 or 700 MeV. The idea for the model was suggested by Chew,[6] who pointed out that the high-energy cross-section enhancement in p–p scattering may be due to a pole in the amplitude which would have proper quantum numbers to be the spin-2 Regge recurrence of the $^1S_0$ pole (virtual state). Figure 2 depicts the singularity structure of the $^1D_2$ state; it consists of an elastic unitarity discontinuity starting at threshold ($T=0$), a discontinuity starting at pion production threshold ($T=280$ MeV), and a discontinuity around 650 MeV from the p-$\Delta(\frac{3}{2},\frac{3}{2})$ channel. The complex value of the last branch point is due to the complex mass of the Δ particle.

The singularity structure illustrated in Fig. 2 is similar to that found in the D_{13} pion–nucleon channel where the incident proton is replaced by a pion. In this system as well (πN) there is an enhancement in cross section around 650 MeV which was originally attributed to an S-wave (in the $\pi\Delta$ channel) threshold effect, but subsequent coupled channel unitary calculations revealed the existence of a pole near the 650-MeV threshold. This resonance was verified by subsequent phase-shift analysis and was in fact responsible for the rise in cross section.

Our model for applying these concepts to p–p scattering was a coupled channel N/D equation with the first channel representing elastic p–p scattering in a relative D wave, and the second channel representing p-N^* in a relative S wave with the mass of the N^* particle distributed around the $\Delta(\frac{3}{2},\frac{3}{2})$ according to an appropriate Breit–Wigner form. A single pole was used to represent, phenomenologically, the "potential." The position of this "force" pole was taken to be a few MeV below the pion-exchange threshold (-10 MeV), and we determined subsequently that a "best" fit could be accomplished by keeping this pole between -10 and -30 MeV. The force pole was complemented by a second pole at around -50 MeV. The residue factors at the second pole were not adjustable, but were determined to produce convergent dispersion integrals over the D-wave phase factor.

The results obtained by searching the residue matrix at the force pole are illustrated in Fig. 3. We used as data the precise value of $^1D_2$ obtained from elastic phase-shift analysis to 400 MeV[5] and the single-energy determination of δ and η at 660 MeV.[3] The data which were fit are also depicted in Fig. 3. The existence of a pole is strongly suggested by a dip in the real part of the determinant of the D-function as is illustrated in Fig. 3(c). The exact position of the pole can be obtained by analytic continuation to the second sheet and, for the fit shown in Fig. 3 is found to be at (400–$i\ 300$) MeV.

The speculative nature of this prediction can best be illustrated by considering a second formalism in which the inelastic channel is represented by $p-\Delta$ in a relative S wave and where the Δ mass is taken at its appropriate complex value. Such a model has been tried and we obtain a reasonably good fit to the "data" as is seen in Fig. 4. The formalism does not produce implicitly unitary results below inelastic threshold, and it was necessary to impose "pseudo-data" constraints on the low-energy absorption parameter, η, to guarantee minimal violations of unitarity. As Fig. 4 clearly illustrates, the existing data are adequately accommodated but no pole is predicted. The differences between the two models is in the shape of the scattering parameters at high energy where no reliable determinations yet exist. We have done the calculations taking, as data, the low-energy elastic phase shift and the absorption parameters predicted by the Amaldi model, with the consequence that the second formalism is rejected on its inability to fit this data selection.

In conclusion I would like to say that it seems hopeful that the D-wave pole exists but that its precise position can be determined only through a more complete understanding of the nucleon–nucleon partial wave structure to about 1 BeV.

[6] G. F. Chew (private communication).