SESSION D—MESON THEORY OF *N–N* INTERACTION

CHAIRMAN: *W. Watari, Osaka City University, Osaka, Japan*

Meson Theoretic N–N Interactions for Nuclear Physics*

A. E. S. GREEN, T. SAWADA
Department of Physics and Astronomy, University of Florida, Gainesville, Florida

One-boson-exchange interactions are derived for pseudoscalar (P), vector (V), scalar (S), axial vector (A), and antisymmetric tensor (T) mesons. The Schrödinger–Pauli–Breit limit is obtained for a combination of such Diracian OBE interactions. We calculate the phase shifts for purely relativistic $\pi-5\omega$ model and find it possible to fit the S-wave phase shifts by adjusting one coupling constant and one regularization parameter. Then by breaking this model slightly we develop several accurate representations of the N–N interaction for all waves with a moderate number of adjustable parameters, several of which are fixed by external considerations. We present a physical interpretation of the results particularly the velocity-dependent interactions. Finally we discuss the results in relationship to other PVS models.

1. INTRODUCTION

A number of groups have recently found theoretical models involving pseudoscalar (P), vector (V), and scalar (S) mesons which fit nucleon–nucleon scattering data quantitatively using only 5–10 adjustable constants. All of these models use (1) The one-boson-exchange contribution (OBEC) or potential (OBEP) of the π meson which dominates the long-range interaction; (2) The OBEC of the ω and ρ mesons which play important roles in the middle range; (3) The OBEC of a scalar meson (s) which cancels the major static effects of the omega meson.

There are now a large number of variations of the pseudoscalar, vector, scalar (PVS) theme embodying approximately the $\omega-s$ cancellation feature which achieve for p and higher waves fits which are comparable to the best purely phenomenological models. These works represent a vast advance with respect to pre-1961 models of the N–N interaction not only in the reduction of free parameters used (from about 20–50 to 5–10) but also in the goodness of the fits to the data. Unfortunately, it is likely that there are so many variations of the PVS theme that it may be difficult in the near future to make a rational choice between them. In view of this, it would seem appropriate that we improve upon the current accomplishments within the PVS theme to the point that meson theoretic N–N interactions provide a good description of S waves which is essential for applications to nuclear physics. To place this study in perspective, it may be helpful to give a brief account of how it fits into the context of earlier work in field theory.

Viewed from the standpoint of the boson field Lagrangian, quantum electrodynamics, the original form of quantum field theory as developed in the early 1930's by Heisenberg, Pauli, Dirac, Rosenfeld, Fock, Fermi, Podolsky, Breit, Moller, etc.[1–9] may be characterized as one in which the photon field Lagrangian depends quadratically upon first derivatives of the field potentials. The extension of field theory to allow for the dependence (quadratic) upon the meson field coordinates themselves was made in 1935 by Yukawa.[10] This explains the short range character of the NN interaction; i.e., the $\mathcal{C}=r^{-1}$ Green's function of electrodynamics goes over to $Y=r^{-1}e^{-r/a}$ of Yukawa's theory where $a=\hbar/mc$ with m the mass of the field bosons. Yukawa's original suggestion which was made in conjunction with a scalar (S) field coordinate was extended by Proca[11] in 1936 to embrace vector (V) fields. Then Kemmer,[12] in 1938, extended the theory further by including pseudoscalar (P), axial vector (A),

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[3] P. A. M. Dirac, V. A. Fock, and B. Podolsky, Z. Phys. Sowjetunion **2**, 473 (1932).
[4] E. Fermi, Rev. Mod. Phys. **4**, 487 (1932).
[5] V. A. Fock, Z. Phys. Sowjetunion **6**, 449 (1934).
[6] V. A. Fock and B. Podolsky, Z. Phys. Sowjetunion **1**, 801 (1932).
[7] L. Rosenfeld, Z. Physik **76**, 729 (1932).
[8] G. Breit, Phys. Rev. **34**, 553 (1929); **36**, 383 (1930); **39**, 616 (1932).
[9] C. Møller, Z. Physik **70**, 786 (1931).
[10] H. Yukawa, Proc. Phys.-Math. Soc. Japan **3**, 17, 18 (1935).
[11] A. Proca, J. Phys. Radium (VII) **7**, 347 (1936).
[12] N. Kemmer, Proc. Roy. Soc. (London) **A166**, 127 (1938).

and antisymmetric tensor (T) meson fields, allowing for what would now be referred to as the isotopic spin dependence of these fields and permitting both direct and derivative-type couplings.

With the broad realm of possible interactions derivable from such fields, the major problem became that of coping with the highly singular nature of many of the interactions and that of selecting particular interactions which fit the experimental data. Møller and Rosenfeld[13] and Schwinger[14] made the first serious effort in these directions by using mixtures of interactions. We refer the reader to the original papers and early texts[15] for the details of these developments.

While this work was underway in the early 1940's, the treatment of electrodynamics itself was extended by Podolsky[16] and by Bopp[17] who permitted the field Lagrangian to depend quadratically upon second derivatives. Still further extensions of Lagrangian theory to deal with meson fields by including both the potentials as well as arbitrarily high derivatives were made by Green[18] and others in the late 1940's. The strength of these generalized field theories was their freedom from infinite self-energies and short-range singularities which plague ordinary electrodynamics and meson theory. Their weakness was the problem of giving physical interpretation to the subtractive components of the boson field which arise naturally in generalized field theories. Quite apart from this interpretive problem there were the more immediate problems of choosing the tensorial character and isotopic character of the meson field variable and the nature of the field–particle coupling.

In 1948, Green[19] became involved with the PVS theme in an attempt to deal with the auxiliary condition problem in vector meson theory which, in certain treatments, requires the introduction of a scalar field. While examining various variations of vector–scalar combinations, it became apparent that some lead to cancellation of large static N–N interaction terms. This then leads in a simple way to a very complex nuclear force which is largely or entirely relativistic in nature. Then the fact that the pseudoscalar–pseudoscalar interaction is also purely relativistic was the motivation for considering this particular interaction.

The singularities associated with relativistic interactions arising in such a PVS theory would normally present great difficulties. However when used in conjunction with a generalized meson field with higher derivatives in the Lagrangian, these singularity difficulties do not arise.

It would be too great a digression to summarize the trends of the main streams[20] of investigations on the NN interaction during the intervening years other than to point out that in the early 1960's the only theoretical models which could deal reasonably well with experimental observations required some 30 to 50 adjustable constants.

Then the advances referred to in the opening paragraph came in independent works by Bryan and Scott,[21] Scotti and Wong,[22] McKean,[23] and Hoshizaki, Otsuki, Sawada, Ueda, Watari, and Yonezawa[24] who found combinations of OBEP which required only about ten adjustable constants to fit the experimental data. While the formalisms used by these groups are somewhat diverse, it seems clear now that the primary reasons for their success are the common ingredients described at the beginning of the section.

The work of Bryan and Scott who solved Schrödinger equation with OBE potentials associated with these combinations of pseudoscalar, vector, and scalar fields prompted Green and Sharma[25] to reinvestigate the relativistic interaction model. They found, without availing themselves of any adjustable constants, that both the isoscalar and isovector sets of spin–spin, tensor, and spin–orbit interactions implicit in Green's earlier work agreed quite well with those of Bryan and Scott. However the relativistic model displayed important velocity-dependent terms not contained in the Bryan and Scott study.

Green, Sawada, and Sharma,[26] using a modified version of the Bryan–Scott code, then carried out a preliminary study of phase shifts generated by a purely relativistic model consisting of a generalized π meson interaction and isoscalar, scalar, and vector interactions. They obtained encouragingly good results using only one adjustable coupling constant and two "cutoff" parameters.

At this time the efforts of the two groups, although they use entirely different formalisms in deriving their Schrödinger potentials, have almost merged into one. On the one hand, in a recent study, Green and Sawada[27] follow Bryan and Scott by allowing for simultaneous direct and derivative coupling to ρ vector mesons. On the other hand, Bryan and Scott[28] now include the velocity-dependent terms in the treatment of the vector

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[13] C. Møller and L. Rosenfeld, Proc. Cop. **17**, No. 8 (1940).
[14] J. Schwinger, Phys. Rev. **61**, 381 (1942).
[15] L. Rosenfeld, *Nuclear Forces* (North-Holland Publ. Co., Amsterdam, The Netherlands, 1949), p. 22.
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[23] R. S. McKean, Jr., Phys. Rev. **125**, 1399 (1962).
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[25] A. E. S. Green and R. D. Sharma, Phys. Rev. Letters **14**, 380 (1965).
[26] A. E. S. Green, T. Sawada, and R. D. Sharma, in *Proceedings of the Conference on Isobaric Spin in Nuclear Physics*, J. D. Fox and D. Robson, Eds. (Academic Press Inc., New York, 1966).
[27] A. E. S. Green and T. Sawada, Nucl. Phys. **B2**, 267 (1967).
[28] B. L. Scott and R. A. Bryan, Bull. Am. Phys. Soc. **10**, 736 (1965).

and scalar interactions. The one remaining difference relates to the use of regularized fields by Green *et al.*, which may also be interpreted as equivalent to the use of form factors for the nucleons. In practical fitting of data this difference plays an important role, particularly for S waves which are of greatest importance in applications to nuclear physics. Accordingly, let us begin by describing how regularized fields entered our work.

2. THE GENERALIZED MESON FIELD

In this study to deal with the well-known singularity difficulties of meson theory we use a generalized meson field Lagrangian which embodies "regularization" in an essential way. Thus we assume all component fields are characterized by a quadratic Lagrangian density of the form

$$L=-(1/2a^2)[C_0Q^2+C_1a^2(\partial_{\lambda_1}Q)^2+\cdots+C_Na^{2N}(\partial_{\lambda_1}\partial_{\lambda_2}\cdots\partial_{\lambda_N}Q)^2], \quad (2.1)$$

where $a=\hbar/Mc$, the nucleon Compton wavelength, is our unit length, the C's are dimensionless natural constants. $Q(x_\nu)$ the field coordinate is a real or imaginary function

$$x_\nu=(\mathbf{r}, ict), \qquad \partial_\lambda=\partial/\partial x_\lambda,$$

and

$$\partial_{\lambda_1}\cdots\partial_{\lambda_N}=\partial^N/\partial x_{\lambda_1}\cdots\partial x_{\lambda_N}.$$

Here Q represents a scalar field. For other tensorial types we append an appropriate subscript. If we identify $C_0=(m/M)^2$ with m the meson mass, $C_1=1$ and $C_2=C_3\cdots=C_N=0$, then Eq. (2.1) becomes the usual meson field Lagrangian.

Applying the variational equation $\delta\int Ld\Omega=0$ and assuming that the potential and all derivatives up to the order $N-1$ are held constant on the boundaries of the four-dimensional space, we obtain the equation of motion

$$\sum_{\eta=0}^{N}[C_\eta(-a^2\Box)^\eta]Q=0. \quad (2.2)$$

From our Lagrangian we can derive the energy momentum four-vector P_μ and the energy momentum tensor $t_{\mu\nu}$. It can be shown that $\partial_\nu t_{\mu\nu}=0$ thus ensuring conservation of energy and momentum and also that the antisymmetric part of $t_{\mu\nu}$ is a four-divergence.

The equation of motion of the field may be written as

$$\left[\prod_{\sigma=1}^{N}\left(\Box-\frac{\xi_\sigma^2}{a^2}\right)\right]Q(x_\nu)=0, \quad (2.3)$$

where ξ_σ^2 are the roots (dimensionless) of

$$F(\xi^2)=-\sum_\eta C_\eta(-\xi^2)^\eta=0. \quad (2.4)$$

A generalized Fourier solution of Eq. (2.3) is given by

$$Q(x_\nu)=(1/2\pi)^{3/2}\sum_\sigma\int[Q_\sigma(\mathbf{k})\exp(ik_{\sigma\nu}x_\nu)+Q_\sigma^*(\mathbf{k})\exp(-ik_{\sigma\nu}x_\nu)]d\mathbf{k}, \quad (2.5)$$

where

$$k_{\sigma\nu}=(\mathbf{k}, i\omega_\sigma/c) \quad \text{and} \quad \omega_\sigma^2=c^2(\mathbf{k}\cdot\mathbf{k}+\xi_\sigma^2/a^2). \quad (2.6)$$

We may identify $\xi_\sigma=m_\sigma/M$, where m_σ are masses associated with the field. If the masses are to be real, then the C's in the Lagrangian must all have the same sign. Using Eq. (2.5) we may express the energy momentum four-vector and hence the field Hamiltonian in terms of Fourier amplitudes. For the field Hamiltonian we obtain

$$H=\sum_\sigma(\gamma_\sigma/c^2)\int\omega_\sigma^2[Q_\sigma(\mathbf{k})Q_\sigma^*(\mathbf{k})+Q_\sigma^*(\mathbf{k})Q_\sigma(\mathbf{k})]d\mathbf{k}, \quad (2.7)$$

where

$$\gamma_\sigma=\sum_\eta \eta C_\eta(-\xi_\sigma^2)^{\eta-1}=[dF/d\xi^2]_{\xi=\xi_\sigma}. \quad (2.8)$$

The γ_σ which arise naturally during this last calculation are factors whose reciprocals weight the fields associated with the different bosons. Thus, if we define $B_\sigma=1/\gamma_\sigma C_N$ it follows that

$$B_\sigma=\prod_{\tau=1}^{N}{}'(\xi_\tau^2-\xi_\sigma^2)^{-1}, \quad (2.9)$$

where we omit $\tau=\sigma$ in forming the product and that

$$\sum B_\sigma=0 \qquad \sum B_\sigma\xi_\sigma^2=0 \quad \cdots \quad \sum B_\sigma\xi_\sigma^{2(N-2)}=0. \quad (2.10)$$

The weight factors B_σ alternate in sign so that if a realistic interpretation is given to these mesons, every other one would act subtractively with respect to its neighbor.

The subtractive terms which play a vital role in the success of regularization have their direct counterparts in other current studies of the NN interaction. This may be significant in view of the fact that in a quantum field treatment the annihilation and creation operators associated with such subtractive fields satisfy "wrong sign" commutation relations. While there are ways to calculate with such quantities, unfortunately, they present serious interpretive problems and they have varyingly been taken to imply a nonunitary theory, negative probabilities, an indefinite metric, or anti-Hermitian interactions. Nevertheless, despite many critical discussions[29] such entities keep reappearing in various forms, e.g., Feynman's smearing the source

[29] A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145 (1950).

function,[30] Pauli–Villars regularization method,[31] Lee's peritization method,[32] Heisenberg's ghost state,[33] and Frautschi's superposition of pole terms.[34] Since no one has resolved the fundamental difficulties of field theory without embodying a subtractive device, there currently appears to be an open mindedness with respect to equations which may be successful in relation to experiment even though "a simple physical view by which all of the contents of the equation can be seen is still lacking".[35] In this spirit we consider applications of such fields.

3. QUANTIZING THE MESON FIELD

We may now quantize the meson field by accepting Eq. (2.7) as a quantum-mechanical Hamiltonian and requiring that the field obey the generalized Heisenberg equation of motion

$$\dot{Q}(\mathbf{r},t)=(i/\hbar)[H,Q(\mathbf{r},t)]. \tag{3.1}$$

Expressing both sides in terms of Fourier integrals and equating coefficients of corresponding exponentials, we find that the Fourier amplitudes must satisfy

$$[Q_\sigma^*(\mathbf{k}),Q_\tau(\mathbf{k}')]=(-1)^\sigma\delta_{\sigma\tau}\delta(\mathbf{k}-\mathbf{k}')\hbar c^2/2\omega_\sigma|\gamma_\sigma|. \tag{3.2}$$

It is convenient now to introduce the operators

$$b_\sigma(\mathbf{k})=Q_\sigma(\mathbf{k})\,(2\omega_\sigma\gamma_\sigma/\hbar c^2)^{1/2}$$
$$b_\sigma^*(\mathbf{k})=Q_\sigma^*(\mathbf{k})\,(2\omega_\sigma\gamma_\sigma/\hbar C^2)^{1/2} \tag{3.3}$$

which then have the commutation properties

$$[b_\sigma^*(\mathbf{k}),b_\tau(\mathbf{k}')]=(-1)^\sigma\delta_{\sigma\tau}\delta(\mathbf{k}-\mathbf{k}'). \tag{3.4}$$

For odd values of σ these are just the familiar commutation rules for a neutral positive definite scalar field. Following Fock we represent the operator $b_\sigma^*(\mathbf{k})$ for $\sigma=1, 3$, etc. by functional multiplication of the function $\bar{b}_\sigma(\mathbf{k})$ and the operator $b_\sigma(\mathbf{k})$ by a functional derivative $\delta/\delta\bar{b}_\sigma(\mathbf{k})$. Then it follows that the number operator

$$N_\sigma=\iiint b_\sigma^*(\mathbf{k})\,b_\sigma(\mathbf{k})\,d\mathbf{k} \tag{3.5}$$

has an eigenfunctional belonging to an n particle state

$$\Omega_{n,\sigma}=(n!)^{-1/2}\int\cdots\int\Psi_{n,\sigma}(\mathbf{k}_1\cdots\mathbf{k}_n)\times\bar{b}_\sigma(\mathbf{k}_1)\cdots\bar{b}_\sigma(\mathbf{k}_n)\,d\mathbf{k}_1\cdots d\mathbf{k}_n, \tag{3.6}$$

where $\Psi_{n,\sigma}(\mathbf{k}_1\cdots\mathbf{k}_n)$ is the wave function in momentum space characterizing an n-meson state of the σ component. When nucleons are present this wave function also depends upon the coordinates and times of the nucleons as well.

For even values of σ the operators b_σ^* and b_σ satisfy "wrong sign" commutation relations. From our present standpoint we handle them pragmatically by identifying for these components

$$b_\sigma^*(\mathbf{k})\to\delta/\delta\bar{b}_\sigma(\mathbf{k})\qquad b_\sigma(\mathbf{k})\to\bar{b}_\sigma(\mathbf{k}). \tag{3.7}$$

In relationship to the present problem, a possible insight as to the significance of the higher derivative field theories may be obtained by considering the static terms in Eq. (2.3) when a delta function source exists in the field. The generalization of Laplace equation becomes

$$\left[\prod_{\sigma=1}^{N}(\nabla^2-\kappa_\sigma^2)\right]\phi=-4\pi g\delta(\mathbf{r}), \tag{3.8}$$

where $\kappa_\sigma=\xi_\sigma/a$. The static spherically symmetric solution of this equation is

$$\phi=g\sum_\sigma B_\sigma Y_\sigma,$$

where

$$Y_\sigma=R^{-1}\exp(-\kappa_\sigma R). \tag{3.9}$$

The nonsingular nature of ϕ as compared to Y_1 is a major factor in the success of current work using one boson exchange potentials.

Now it is possible to look upon Eq. (3.8) and Eq. (3.9) as equivalent to

$$(\nabla^2-\kappa_1^2)\phi=gf(R), \tag{3.10}$$

where

$$f(R)=\sum B_\sigma\kappa_\sigma^2Y_\sigma-\kappa_1^2\sum B_\sigma Y_\sigma. \tag{3.11}$$

Thus, in effect, we can get the same generalized potential by assuming that the source of the first meson field is a smeared out function with form given by the Yukawa functions of the heavier mesons. Since nucleons are known to have form factors characteristic of meson Compton wavelengths, this suggests the equations might eventually find a physical interpretation in terms of intrinsically nonlocal fields.[36]

It is important to keep in mind that the generalized theory automatically eliminates ultraviolet divergences in meson momentum space. For example, where an ordinary meson theory would simply give the Yukawa interaction between two particles, a generalized theory will give

$$J=\frac{g^2}{(2\pi)^3C_N}\sum_\sigma B_\sigma\int\frac{d\mathbf{k}[\exp(ik_{\nu\sigma}X_\nu)]}{2(k^2+\kappa_\sigma^2)}=\frac{g^2}{(2\pi)^2C_N}\sum_\sigma B_\sigma\int\frac{k\sin kR}{R(k^2+\kappa_\sigma^2)}\,dk=\frac{g^2}{4\pi C_N}\sum B_\sigma Y_\sigma, \tag{3.12}$$

[30] R. P. Feynman, Phys. Rev. **76**, 769 (1949).
[31] W. Pauli and F. Villars, Rev. Mod. Phys. **21**, 434 (1949).
[32] T. D. Lee, Phys. Rev. **95**, 1329 (1954).
[33] W. Heisenberg, Nucl. Phys. **4**, 532 (1957).
[34] S. C. Frautschi, *Regge Poles and S-Matrix Theory* (W. A. Benjamin, Inc., New York, 1963), p. 131.
[35] R. P. Feynman, Nobel Lecture "The Development of the Space Time View of Quantum Electrodynamics," Phys. Today **19**, 31 (August 1966).
[36] E. C. G. Sudarshan *et al.* Phys. Rev. **123**, 2183, 2193 (1961); **137**, B1085 (1965).

where $X_\nu = x_{\nu 1} - x_{\nu 2}$ and the times are set equal. If we expand

$$\sum B_\sigma [1+(\kappa_\sigma/k)^2]^{-1} = \sum B_\sigma - k^{-2} \sum B_\sigma \kappa_\sigma^2 - k^{-4} \sum B_\sigma \kappa_\sigma^4 \cdots \quad (3.13)$$

we see how the regularization conditions suppress the initial terms in the series. Now if J occurs in relativistic interaction terms in combinations such as J'/R, $R(J'/R)'$, $R^{-2}(R^2J')'$, where prime denotes differentiation with respect to R, additional k's will appear in the numerator of the integrand. These would normally lead to difficulties at $R=0$. However, here the particular regularization conditions which appear automatically in a generalized meson theory prevent such ultraviolet catastrophies.

Let us view this matter from the standpoint of the behavior of the scattering amplitude at large energies and momentum transfer as is done in discussions of matrix theory and dispersion relations. The elastic scattering amplitude for a weighted superposition of Yukawas is[34]

$$\pi^{-1} \sum \frac{B_\sigma}{m_\sigma^2 - t} = \pi^{-1} \frac{\sum B_\sigma}{t} + \frac{\sum B_\sigma m_\sigma^2}{t^2} + \frac{\sum B_\sigma m_\sigma^4}{t^3} + \cdots. \quad (3.14)$$

At large energies and momentum transfers this would behave like t^{-1}. However if $\sum B_\sigma = 0$, it behaves as t^{-2} and if $\sum B_\sigma m_\sigma^2 = 0$ it behaves as t^{-3} etc.

In the following sections we do not work explicitly with a generalized meson field. Instead we simply replace the ordinary Yukawa function where it would appear in the usual form of meson theory by the well-regulated combination of Yukawa functions given by

$$J(r) = \frac{g^2}{r}\left[e^{-\kappa r} - \frac{U^2-\kappa^2}{U^2-\Lambda^2} e^{-\Lambda r} + \frac{\Lambda^2-\kappa^2}{U^2-\Lambda^2} e^{-Ur}\right], \quad (3.15)$$

where g^2 is the intrinsic coupling constant κ is the inverse Compton wavelength of the meson and Λ and U are regularization parameters assigned to this meson. This potential originally arose in our work as the static solution of a sixth-order wave equation for a delta-function source density. However, the corresponding potential arises as the static solution of the usual Klein Gordon equation with a source function whose form factor has the characteristic outer length Λ^{-1} and inner cutoff at U^{-1}.

We note that when $U \rightarrow \infty$ or when $U \rightarrow \Lambda$ this goes over to $g^2 r^{-1}[e^{-\kappa r} - e^{-\Lambda r}]$ a weakly regulated combination of Yukawa functions. This then goes over to the ordinary Yukawa function when $\Lambda \rightarrow \infty$. In applications of these well-regularized functions at this time we do not use U as an adjustable parameter. Instead we fix it at a large value using plausibility arguments. Thus for most purposes we set U to $20M$ which correspond to the nominal mass of q and $\bar{q}$ when unbound where q is the "ultimate" system; i.e., the urbaryon, the quark, or ace. With this prescription, Λ will sometimes be set at $2M$ which will be referred to as our "standard" regularization. This regularization prescription which scarcely affects the region from 0.2 F outward eliminates the singularities of all the relativistic interactions which we will encounter, even those which, in a purely mathematical sense, are manageable singularities. This prescription is very useful in practical numerical calculations since it minimizes the integration errors caused by taking finite steps near the origin.

In most of our studies we have not fixed Λ but have treated it as negotiable or adjustable. By treating Λ as negotiable, we mean that we use some known external considerations as a basis for estimating its value rather than search on the phase shifts themselves. For example, since the ρ meson is a two-pion resonance and there are indications for a 90° $I=0$ and $J=0$ phase shift in this same mass neighborhood of the two-pion system it is not unreasonable to set $\Lambda_\pi \approx m_\rho$. The prescription, $\Lambda = 2M$, is in the spirit of the nucleon and antinucleon as ingredients of mesons as in the original Fermi–Yang model[37] or the Sakata model.[38] However, for most purposes we have treated Λ as adjustable, in which case we are in effect determining a scale length for the form factor of the meson–nucleon vertex.

4. DERIVATION OF ONE-MESON-EXCHANGE INTERACTIONS

In this section we consider the broad picture of meson interactions and concentrate our attention on direct couplings between various tensorial types of meson fields and two nucleons. Such a study was first made by Kemmer.[12] Since we are not quantizing the nucleon field, rather than duplicate Kemmer's formalism, we use the multitime formalism. Here the basic wave equation for each Dirac particle interacting with the field at its location is

$$[c\boldsymbol{\alpha}_i \cdot \mathbf{p}_i + \beta_i Mc^2 + I(\mathbf{r}_i, t_i)]\Psi(\mathbf{r}_1, \mathbf{r}_2, t_1 t_2, t, Q) = 0, \quad (4.1)$$

where I is the interaction between the meson field and the particle and Ψ is the state functional which depends upon the coordinates and times of both particles as well as the field coordinates. A similar equation may be written for the second particle. In the case of direct-coupling interactions associated with a generalized meson field we have

$$I(\mathbf{r}_i t_i) = -g\vartheta Q(\mathbf{r}_i, t_i), \quad (4.2)$$

where ϑ is the appropriate Dirac matrix associated with the component of the meson field. For example, for scalar field, ϑ is β. We now add the Dirac equations for the two particles and set the times equal, then we use Rosenfeld's transformation[7] to convert the Dirac–

[37] E. Fermi and C. N. Yang, Phys. Rev. **76**, 1739 (1949).
[38] S. Sakata, Progr. Theoret. Phys. (Kyoto) **16**, 686 (1956).

Fock–Podolsky multitime wave equation into the Heisenberg–Pauli single-time equation.

$$(H_p+H_f+H_I-i\hbar\partial/\partial t)\Psi(Q,\ \mathbf{r}_1,\ \mathbf{r}_2,\ t)=0, \quad (4.3)$$

where

$$H_p=H_1+H_2 \quad \text{and} \quad H_I=H_{I,1}+H_{I,2}. \quad (4.4)$$

We next assume that our wave functional has the form given by Eq. (3.6) where each Ψ_n is a function of the coordinates of two nucleons. Using the properties of Fock operators we obtain an infinite series of coupled integrodifferential equations the first three members of which are

$$D\Psi_0-\int d\mathbf{k}G^*(\mathbf{k}_1)\Psi_1(\mathbf{k}_1)=0, \quad (4.5)$$

$$D\Psi_1+\hbar\omega_1\Psi_1-\sqrt{2}\int d\mathbf{k}G^*(\mathbf{k})\Psi_2(\mathbf{k},\mathbf{k}_1)-G(\mathbf{k}_1)\Psi_0=0, \quad (4.6)$$

$$D\Psi_2+\hbar(\omega_1+\omega_2)\Psi_2-\sqrt{3}\int d\mathbf{k}G(\mathbf{k})\Psi_3(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)$$
$$-\sqrt{2}^{-1}[G(\mathbf{k}_2)\Psi(\mathbf{k}_1)+G(\mathbf{k}_2)\Psi(\mathbf{k}_1)]=0, \quad (4.7)$$

where

$$D=H_p-i\hbar\partial/\partial t$$

and

$$G(\mathbf{k})=\sum_i g(2\pi)^{-3/2}\vartheta_i c(\hbar/2\omega)^{1/2}\exp(-i\mathbf{k}\cdot\mathbf{x}_i). \quad (4.8)$$

To derive the OBEP we assume in the second equation that $\hbar\omega\Psi_1$ is large compared to $D\Psi_1$, and we also drop the amplitudes Ψ_2. When these steps are taken it follows that

$$\Psi_1(\mathbf{x}_1,\mathbf{x}_2,t,\mathbf{k}_1)\approx G(\mathbf{k})\Psi_0(\mathbf{x}_1,\mathbf{x}_2,t)/\hbar\omega. \quad (4.9)$$

If we insert this into Eq. (4.5) we find

$$(H_1+H_2+V^D-i\hbar\partial/\partial t)\Psi_0(\mathbf{x}_1,\mathbf{x}_2,t)=0, \quad (4.10)$$

where V^D, the Dirac one-boson-exchange potential is given by

$$V^D=-\vartheta_1\vartheta_2\int d\mathbf{k}G^*(\mathbf{k})G(\mathbf{k})/\hbar\omega$$
$$=-\vartheta_1\vartheta_2\frac{g^2}{(2\pi)^3}\sum_{i,j}\int d\mathbf{k}\frac{\exp i\mathbf{k}\cdot(\mathbf{x}_j-\mathbf{x}_i)}{2(k^2+\kappa^2)}$$
$$=-\vartheta_1\vartheta_2 J, \quad (4.11)$$

where

$$J=(g^2/4\pi)(e^{-\kappa r}/r).$$

As we indicated earlier, we shall in our detailed calculations replace this by the well regulated J given by Eq. (3.15).

Equation (4.11) provides the basis of our derivation of several OBEP. The same result follows from the second-order term in usual power series perturbation procedure and indeed from a classical treatment of the field.

The assumptions made in this derivation of OBEP are (1) two-boson exchange and higher-order processes are omitted; (2) nucleons are treated as point source functions; (3) direct couplings only are treated although later in the calculations of phase shifts we consider the derivative coupling terms as well; and (4) it was assumed that $D\Psi_1\ll\hbar\omega\Psi_1$.

An attempt to go beyond this one-boson-exchange approximation by attempting to solve Eqs. (4.5), (4.6), (4.7) and corresponding higher-order Fock equations by successive approximations was made earlier by one of the writers (AESG).[39] Efforts in this direction have been recently renewed[40] giving considerations to closely related studies which bear upon this problem, such as the renormalization of the wave functional, the virtual nucleon pair productions and the suppressions, the existence of nucleon isobars, the correlated meson exchanges, and so on.[41,42]

5. RELATIVISTIC OBEP

To apply our technique for deriving OBEP, let us first consider the case of an isoscalar scalar field which is directly coupled to nucleons. To do this, we simply identify ϑ with β to obtain the scalar–scalar Dirac potential

$$V_S=-\beta_1\beta_2 J. \quad (5.1)$$

To treat the case of isovector scalar fields we would simply identify ϑ with $\boldsymbol{\tau}\beta$ to obtain the corresponding interaction.

This procedure may now be extended to deal with vector–meson fields. Here we represent the meson field by a four-vector $A_\alpha=(\mathbf{A},iA_0)$. The timelike component of the four-vector field leads to a Hamiltonian which is negative definite. We discuss possible ways of overcoming this problem in a subsequent work. Here we simply note that the formal consequence of such a negative definite Hamiltonian is that the odd operators $b_{\sigma,0}$ and $b_{\sigma,0}^*$ now satisfy the "wrong-sign" commutation relations and the even $b_{\sigma,0}$ and $b_{\sigma,0}^*$ satisfy right-sign commutation relations. Formally we treat this case simply by reversing the roles of b_σ and b_σ^*.

For isoscalar vector field we take the interaction

$$I=-g\vartheta_\alpha A_\alpha(x_\nu), \quad (5.2)$$

where $\vartheta_\alpha=(\boldsymbol{\alpha},\ i)$. The final result due to the four fields is simply

$$V_V=(1-\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2)J. \quad (5.3)$$

This particular vector–meson interaction is implicit in many prior studies.[43,44]

[39] A. E. S. Green, Phys. Rev. **77**, 719 (1950).
[40] B. Chern, W. A. Wilson, and A. E. S. Green (unpublished).
[41] D. Feldman, Phys. Rev. **98**, 1456 (1955).
[42] K. A. Brueckner, M. Gell-Mann, and M. L. Goldberger, Phys. Rev. **90**, 476 (1953).
[43] W. Pauli, *Handbuch der Physik* (Julius Springer-Verlag, Berlin, 1933), 2nd ed., Vol. 24:1.
[44] H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 191 (1936).

In the nonrelativistic limit Eq. (5.1) is dominated by the large attraction $-J$. On the other hand, Eq. (5.3) is dominated by the large repulsion J. Accordingly, if we take a direct synthesis of interactions given by Eq. (5.1) and Eq. (5.3) using the same coupling constant and mass we obtain the purely relativistic interaction

$$V_{5V}=(1-\beta_1\beta_2-\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2)J. \tag{5.4}$$

This purely relativistic interaction proposed some time ago[19] has been reexamined recently,[25–27] and when used in conjunction with pseudoscalar π meson has been found to give a semiquantitative account of nucleon–nucleon scattering. In addition to spin–spin and tensor forces it accounts in a very simple way for the spin orbit as well as velocity-dependent terms which have been found in phenomenological studies including some which do not explicitly state the detailed assumptions as to the nature of the meson field.

Let us consider next the pseudoscalar field with direct coupling. Here the field is intrinsically a fourth-rank antisymmetric tensor with only one nonvanishing component, A_{1234}. For consistency we should, in view of the presence of the fourth index, represent this field by a purely imaginary potential, i.e., $A_{1234}=i\phi_P$. To ensure a Hermitian Hamiltonian the Dirac matrix associated with the interaction is now $\vartheta_P=i\beta\gamma_5$. From this it follows that the pseudoscalar direct coupling interaction is

$$V_P=\beta_1\gamma_{5,1}\beta_2\gamma_{52}J. \tag{5.5}$$

In a similar way we may treat the axial vector field which intrinsically is a third rank antisymmetric tensor. The four nonvanishing components of such a field are A_{123}, A_{124}, A_{134}, A_{234}. Thus we may represent such a field by the pseudovector $\phi_\alpha=(\phi_0, i\mathbf{A})$ which has three purely imaginary components. The corresponding Dirac matrices needed to ensure a Hermitian Hamiltonian are $\vartheta_\alpha=(\gamma_5, i\boldsymbol{\sigma})$. It follows immediately that the direct coupling axial vector interaction is

$$V_A=(\gamma_{5,1}\gamma_{5,2}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)J. \tag{5.6}$$

Note that the part of the OBEP corresponding to the real field has a negative sign, whereas the parts arising out of the imaginary field give rise to a positive sign.

We may construct one more direct coupling OBEP interaction which in a certain sense was ignored by Kemmer. Here we assume the field to be an intrinsic second rank antisymmetric tensor. The six nonvanishing components of such a tensor may be identified by $\phi_{\alpha\beta}=(\phi_{12}, \phi_{13}, \phi_{23}; \phi_{14}, \phi_{24}, \phi_{34})=(\mathbf{A}, i\mathbf{B})$. The associated set of Dirac matrices which produce a Hermitian Hamiltonian are $\vartheta_{\alpha\beta}=(\beta\boldsymbol{\sigma}, \beta\boldsymbol{\alpha})$. Thus the Dirac OBEP is

$$V_T=-(\beta_1\beta_2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+\beta_1\beta_2\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2)J. \tag{5.7}$$

There are several important aspects of the OBEP interactions derived in this simple way. In the first place they are consistent with possible interaction forms between nucleons inferred from invariance considerations prior to the discovery of meson theory as discussed by Pauli[42] and Bethe and Bacher[43] in connection with beta decay and by Fermi and Yang[44] in connection with the nucleon–antinucleon interaction.

In the second place they correspond in their static limits to very simple interactions. Thus in the static limit we have

$$V_S=-J \qquad V_V=J \qquad V_P=0$$
$$V_A=-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2J \qquad V_T=-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2J, \tag{5.8}$$

where here the $\boldsymbol{\sigma}$'s now correspond to the usual Pauli matrices. It is intriguing to note that from these five direct coupling interactions one may, by choosing masses and coupling constants identical, construct three purely relativistic interactions, i.e., V_P, V_S+V_V, and V_A-V_T. The first two purely relativistic interactions have been considered previously. In view of the discovery of 1^+ mesons it appears important to consider now the V_A and V_T interactions. The fact that the OBEP is well represented by V_P, a purely relativistic interaction and the fact that the purely relativistic combination V_S+V_V also seems to characterize quite well components now identified in the nucleon–nucleon force suggests that serious attention be given to the third purely relativistic interaction V_A-V_T. It is interesting to note that these three purely relativistic interactions would follow from the three possible five dimensionally invariant combinations of tensorial and spinorial fields explored by Watanabe[45] providing one uses the same masses and coupling constants in the scalar and vector interaction and the same pair in the axial vector and tensor interactions.

The direct coupling Diracian OBEP interactions derived by Kemmer[12] for the corresponding scalar, vector, pseudovector, and pseudoscalar fields are somewhat different particularly in the relativistic terms. The differences reflect the difference in treatment in the use of auxiliary conditions and in the selection of the field Lagrangian. We shall consider some of these differences in a subsequent work.

In concluding this section we might note that in 1937 Breit,[46] from considerations of approximate relativistic invariance, inferred interactions due to scalar and vector fields which are more complicated than Eq. (5.1) or Eq. (5.3). However the combination $V_{BS}-V_{BV}$ is precisely the purely relativistic interaction given by Eq. (5.4) when Breit J is identified with the Yukawa function.

6. TWO-PARTICLE DIRAC EQUATION WITH OBEP

The Dirac equation for two nucleons 1 and 2 interacting via OBEP may be written as:

$$\{-c\boldsymbol{\alpha}_1\cdot\mathbf{p}_1-c\boldsymbol{\alpha}_2\cdot\mathbf{p}_2-(\beta_1+\beta_2)Mc^2+V^D\}\Omega=E\Omega, \tag{6.1}$$

[45] S. Watanabe, Phys. Rev. **74**, 1864 (1948).
[46] G. Breit, Phys. Rev. **51**, 248 (1937).

where $E=W+2Mc^2$ is the total energy of the system, $M=M_1=M_2$ is the mass of the nucleons, and $\mathbf{p}_1$ and $\mathbf{p}_2$ are their momenta. In the reductions to follow we assume that the Diracian interaction V^D is given as a sum of the OBEP arising from scalar (S), pseudoscalar (P), unconstrained vector (V), axial vector (A), antisymmetric tensor (T). Other interactions can be treated in a similar way.

Thus far we have only calculated interactions mediated by isoscalar mesons. If isovector mesons are also involved we simply assume that each of the potentials J_n is given by the sum of an isoscalar and an isovector part

$$J_n=J_n^0(r)+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 J_n^1(r), \tag{6.2}$$

where $\boldsymbol{\tau}_i$ is the isospin matrix for the ith nucleon and r is the interparticle distance, and where $J_n^0(r)$ and $J_n^1(r)$ are given by Eq. (3.15).

The explicit representation of the Dirac matrices $\boldsymbol{\alpha}$, β, $\boldsymbol{\sigma}_D$, and γ_5 are

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\qquad \beta=\begin{pmatrix}I_\sigma & 0\\ 0 & -I_\sigma\end{pmatrix},$$

$$\boldsymbol{\sigma}_D=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix},\qquad \gamma_5=i\alpha_x\alpha_y\alpha_z=\begin{pmatrix}0 & -I_\sigma\\ -I_\sigma & 0\end{pmatrix}, \tag{6.3}$$

where $\boldsymbol{\sigma}$ is the Pauli spin matrix and I_σ is the 2×2 unit matrix. The wave function Ω is a 16-component two-particle spinor which is the direct product of 2 one-particle spinors,

$$\Omega=\Omega_1\otimes\Omega_2$$

$$=\begin{pmatrix}u_1\\ w_1\end{pmatrix}\otimes\begin{pmatrix}u_2\\ w_2\end{pmatrix}=\begin{pmatrix}u_1u_2, & u_1w_2\\ w_1u_2, & w_1w_2\end{pmatrix}=\begin{pmatrix}\zeta, & \chi_1\\ \chi_2, & \psi\end{pmatrix}, \tag{6.4}$$

where u and w are the small and the large components of the spinor Ω_1 for the first nucleon etc. In Eq. (6.4) the functions ζ, χ_1, χ_2, and ψ are introduced in place of u_1u_2, u_1w_2, w_1u_2, and w_1w_2, each of which is a 4×4 matrix. The decomposition of Eq. (6.1) into a set of four coupled equations for ζ, χ_1, χ_2, and ψ is most conveniently done by introducing the 2×2 matrices[47]

$$\rho_{1,i}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad \rho_{2,i}=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$\rho_{3,i}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad I_{\rho,i}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{6.5}$$

($i=1$ and 2) which operate between w_i and u_i taking them as single units. Thus, for example, the effect of operating $\rho_{1,i}$ on Ω_1 is to switch u_1 and w_1. The usual Pauli spin matrices $\boldsymbol{\sigma}_i$ and $I_{\sigma,i}$ operate within w_i or u_i; therefore $\boldsymbol{\rho}_i$ and $\boldsymbol{\sigma}_i$ commute with each other, and a product of two Dirac matrices can be carried out in ρ matrices and in σ matrices independently. From Eqs. (6.5) and (6.7) one finds

$$\boldsymbol{\alpha}_i=\rho_{1,i}\otimes\boldsymbol{\sigma}_i,\qquad \beta_i=\rho_{3,i}\otimes I_{\sigma,i}$$

$$\boldsymbol{\sigma}_{D,i}=I_{\rho,i}\otimes\boldsymbol{\sigma}_i,\qquad \gamma_{5,i}=-\rho_{1,i}\otimes I_{\sigma,i}. \tag{6.6}$$

For example,

$$\beta_i\gamma_{5,i}=(\rho_{3,i}\otimes I_{\sigma,i})(-\rho_{1,i}\otimes I_{\sigma,i})$$

$$=-(\rho_{3,i}\rho_{1,i})\otimes(I_{\sigma,i}I_{\sigma,i})=-i\rho_{2,i}\otimes I_{\sigma,i}. \tag{6.7}$$

Similarly,

$$\beta_i\boldsymbol{\sigma}_{D,i}=\rho_{3,i}\otimes\boldsymbol{\sigma}_i,\qquad \beta_i\boldsymbol{\alpha}_i=i\rho_{2,i}\otimes\boldsymbol{\sigma}_i. \tag{6.7b}$$

The effects of operating $\boldsymbol{\rho}$ matrices on Ω are given by

$$\rho_{1,1}I_{\rho,2}\Omega=\begin{pmatrix}\chi_2, & \psi\\ \zeta, & \chi_1\end{pmatrix},\qquad I_{\rho,1}\rho_{1,2}\Omega=\begin{pmatrix}\chi_1, & \zeta\\ \psi, & \chi_2\end{pmatrix},$$

$$\rho_{3,1}I_{\rho,2}\Omega=\begin{pmatrix}\zeta, & \chi_1\\ -\chi_2, & -\psi\end{pmatrix},\qquad I_{\rho,1}\rho_{3,2}\Omega=\begin{pmatrix}\zeta, & -\chi_1\\ \chi_2, & -\psi\end{pmatrix},$$

$$\rho_{1,1}\rho_{1,2}\Omega=\begin{pmatrix}\psi, & \chi_2\\ \chi_1, & \zeta\end{pmatrix},\qquad \rho_{2,1}\rho_{2,2}\Omega=\begin{pmatrix}-\psi, & \chi_2\\ \chi_1, & -\zeta\end{pmatrix},$$

$$\rho_{3,1}\rho_{3,2}\Omega=\begin{pmatrix}\zeta, & -\chi_1\\ -\chi_2, & \psi\end{pmatrix}. \tag{6.8}$$

Expressing Eqs. (6.1) and (6.2) in terms of ρ matrices and σ matrices as per Eqs. (6.6) and (6.7) and applying Eq. (6.8) to the resulting expression, we find the following coupled equations:

$$(E-2Mc^2-V_a)\psi+P_1\chi_1+P_2\chi_2-V_b\zeta=0, \tag{6.9a}$$

$$P_1\psi+(E-V_c)\chi_1-V_d\chi_2+P_2\zeta=0, \tag{6.9b}$$

$$P_2\psi-V_d\chi_1+(E-V_c)\chi_2+P_1\zeta=0, \tag{6.9c}$$

$$-V_b\psi+P_2\chi_1+P_1\chi_2+(E+2Mc^2-V_a)\zeta=0, \tag{6.9d}$$

where

$$P_i=c(\boldsymbol{\sigma}_i\cdot\mathbf{p}_i) \tag{6.10}$$

and

$$V_a=-J_S+J_V-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(J_A-J_T),$$

$$V_b=J_P+J_A-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(J_V-J_T),$$

$$V_c=J_S+J_V-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(J_A+J_T),$$

$$V_d=-J_P+J_A-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(J_V+J_T). \tag{6.11}$$

Eliminating χ_1 and χ_2 from Eqs. (6.9) one finds

$$\mathcal{L}_1(\psi+\zeta)=0,\qquad \mathcal{L}_2(\psi-\zeta)=0, \tag{6.12}$$

[47] A. E. S. Green, unpublished notes (1948–1950).

with

$$\mathcal{L}_1=[(E+h_1)(E+h_2)/4Mc^2]-Mc^2-(4Mc^2)^{-1}\{(E+h_2)P_S(E+h_3)^{-1}P_S+P_D(E+h_4)^{-1}P_D(E+h_1)\},$$

$$\mathcal{L}_2=[(E+h_1)(E+h_2)/4Mc^2]-Mc^2-(4Mc^2)^{-1}\{P_S(E+h_3)^{-1}P_S(E+h_2)+(E+h_1)P_D(E+h_4)^{-1}P_D\}, \quad (6.13)$$

where

$$P_S=c(\boldsymbol{\sigma}_1\cdot\mathbf{p}_1)+c(\boldsymbol{\sigma}_2\cdot\mathbf{p}_2), \qquad P_D=c(\boldsymbol{\sigma}_1\cdot\mathbf{p}_1)-c(\boldsymbol{\sigma}_2\cdot\mathbf{p}_2), \quad (6.14)$$

and

$$h_1=-V_a-V_b=J_S-J_V-J_P-J_A+(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(J_A+J_V-2J_T),$$
$$h_2=-V_a+V_b=J_S-J_V+J_P+J_A+(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(J_A-J_V),$$
$$h_3=-V_c-V_d=-J_S-J_V+J_P-J_A+(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(J_A+J_V+2J_T),$$
$$h_4=-V_c+V_d=-J_S-J_V-J_P+J_A+(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(J_A-J_V). \quad (6.15)$$

Equations (6.12) and (6.13) have been derived by Breit for the case of π meson field.[48] One could solve Eq. (6.12) formally for ψ. However, for our purpose, it is sufficient to insert the identities

$$(E+h_i)^{-1}=E^{-1}\{1-[h_i/(E+h_i)]\} \qquad (i=1,2,3,4)$$

into Eq. (6.13) and to add the two equations (6.12) together. We obtain

$$\left\{\frac{(E+h_1)(E+h_2)}{4Mc^2}-Mc^2-\frac{p_1^2+p_2^2}{2M}-(8Mc^2)^{-1}\left(P_D^2\frac{h_1}{E}+\frac{h_1}{E}P_D^2+\frac{h_2}{E}P_S^2+P_S^2\frac{h_2}{E}\right)\right.$$
$$\left.+(4Mc^2)^{-1}\left(P_S\frac{h_3}{E}P_S+P_D\frac{h_4}{E}P_D\right)\right\}\psi$$
$$+\left\{(8Mc^2)^{-1}\left(P_D\frac{h_4}{E+h_4}P_D\frac{h_1}{E}+\frac{h_1}{E}P_D\frac{h_4}{E+h_4}P_D+\frac{h_2}{E}P_S\frac{h_3}{E+h_3}P_S+P_S\frac{h_3}{E+h_3}P_S\frac{h_2}{E}\right)\right.$$
$$\left.-(4Mc^2)^{-1}\left(P_S\frac{h_3^2}{E(E+h_3)}P_S+P_D\frac{h_4^2}{E(E+h_4)}P_D\right)\right\}\psi$$
$$-(8Mc^2)^{-1}\left(P_D^2\frac{h_1}{E}-\frac{h_1}{E}P_D^2+\frac{h_2}{E}P_S^2-P_S^2\frac{h_2}{E}-P_D\frac{h_4}{E+h_4}P_D\frac{h_1}{E}+\frac{h_1}{E}P_D\frac{h_4}{E+h_4}P_D\right.$$
$$\left.-\frac{h_2}{E}P_S\frac{h_3}{E+h_3}P_S+P_S\frac{h_3}{E+h_3}P_S\frac{h_2}{E}\right)\zeta=0. \quad (6.16)$$

This is still an exact expression. The separation of the terms has been done in such a way that the first curly bracket contains all the terms up to $1/M^2c^4$ and the remainder to $1/M^3c^6$ or higher.

7. THE SCHRÖDINGER–PAULI–BREIT EQUATION

The advantage of Eq. (6.16) is that it allows us to introduce the Breit–Pauli limit in one simple step. Within the energy regions of interest here, that is $E_{\text{lab}}<300$ MeV, the condition $W^2/M^2c^4=(v/c)^2\ll1$ is well satisfied. We shall make approximations which apply at the interparticle distances where terms like

$$J_nJ_m/M^2c^4, \qquad WJ_n/M^2c^4, \quad \text{and} \quad J_np^2/M^3c^4$$

are much smaller than unity, where J_n and J_m stand for any of the J in Eq. (6.15). Thus we can neglect the second and the third terms in Eq. (6.16) compared to the first term in which we substitute $1/2Mc^2$ for $1/E$. Remembering that in the lowest order

$$\zeta\approx\{(-(\boldsymbol{\sigma}_1\cdot\mathbf{p})(\boldsymbol{\sigma}_2\cdot\mathbf{p})+V_b)/4Mc^2\}\psi$$

we obtain

$$\left\{\frac{(E+h_1)(E+h_2)}{4Mc^2}-Mc^2-\frac{p^2}{M}-(16M^2c^4)^{-1}(P_D^2h_1+h_1P_D^2+P_S^2h_2+h_2P_S^2)+(8M^2c^4)^{-1}(P_Sh_3P_S+P_Dh_4P_D)\right\}\psi=0, \quad (7.1)$$

[48] G. Breit, Phys. Rev. **111**, 652 (1958).

where we have chosen the center-of-mass system so that

$$(\mathbf{p}_1+\mathbf{p}_2)\psi=0 \quad \text{and} \quad \mathbf{p}=\mathbf{p}_1=-\mathbf{p}_2.$$

Using $E=W+2Mc^2$ we find

$$\frac{(E+h_1)(E+h_2)}{4Mc^2}-Mc^2=w\left(1+\frac{h_1+h_2}{4Mc^2}\right)+\tfrac{1}{2}(h_1+h_2)+\frac{W^2}{4Mc^2}+\frac{h_1\cdot h_2}{4Mc^2}. \tag{7.2}$$

Thus the lowest-order approximation to Eq. (7.1) becomes

$$(W+\tfrac{1}{2}(h_1+h_2))\psi=(p^2/M)\psi. \tag{7.3}$$

This gives the lowest-order expression for W,

$$W=p^2/M-\tfrac{1}{2}(h_1+h_2), \tag{7.4}$$

which is substituted into $W^2/4Mc^2$ appearing in Eq. (7.2). The result is given by

$$W\psi=\left\{\frac{p^2}{M}-\frac{p^4}{4M^3c^2}-\frac{h_1+h_2}{2}-\frac{h_1\cdot h_2}{4Mc^2}+\frac{(h_1+h_2)^2}{16Mc^2}+(8M^2c^2)^{-1}(p^2(h_1+h_2)-(h_1+h_2)p^2)\right.$$
$$\left.+(16M^2c^4)^{-1}(P_D{}^2h_1+h_1P_D{}^2+P_S{}^2h_2+h_2P_S{}^2)-(8M^2c^4)^{-1}(P_Sh_3P_S+P_Dh_4P_D)\right\}\psi. \tag{7.5}$$

Substituting V_a, V_b, V_c, and V_d back in place of h_i $(i=1, 2, 3, 4)$ and using Eq. (6.15) we find for the following terms:

$$-\frac{h_1\cdot h_2}{4Mc^2}+\frac{(h_1+h_2)^2}{16Mc^2}=\frac{V_b{}^2}{4Mc^2}, \tag{7.6a}$$

$$P_D{}^2h_1+h_1P_D{}^2+P_S{}^2h_2+h_2P_S{}^2=-4c^2\{p^2V_a+V_ap^2+(\boldsymbol{\sigma}_1\cdot\mathbf{p})(\boldsymbol{\sigma}_2\cdot\mathbf{p})V_b+V_b(\boldsymbol{\sigma}_1\cdot\mathbf{p})(\boldsymbol{\sigma}_2\cdot\mathbf{p})\}, \tag{7.6b}$$

$$P_Sh_3P_S+P_Dh_4P_D=2c^2\{-(\boldsymbol{\sigma}_1\cdot\mathbf{p})V_c(\boldsymbol{\sigma}_1\cdot\mathbf{p})-(\boldsymbol{\sigma}_2\cdot\mathbf{p})V_c(\boldsymbol{\sigma}_2\cdot\mathbf{p})+(\boldsymbol{\sigma}_1\cdot\mathbf{p})V_d(\boldsymbol{\sigma}_2\cdot\mathbf{p})+(\boldsymbol{\sigma}_2\cdot\mathbf{p})V_d(\boldsymbol{\sigma}_1\cdot\mathbf{p})\}. \tag{7.6c}$$

Using Eq. (3.6) into Eq. (3.5) we obtain

$$W\psi=\left[\frac{p^2}{M}-\frac{p^4}{4M^3c^2}+V_a+\frac{V_b{}^2}{4Mc^2}-\frac{p^2}{2M^2c^2}V_a-(4M^2c^4)^{-1}\mathcal{V}\right]\psi \tag{7.7}$$

where we have defined

$$\mathcal{V}=(\boldsymbol{\sigma}_1\cdot\mathbf{p})(\boldsymbol{\sigma}_2\cdot\mathbf{p})V_b+V_b(\boldsymbol{\sigma}_1\cdot\mathbf{p})(\boldsymbol{\sigma}_2\cdot\mathbf{p})-(\boldsymbol{\sigma}_1\cdot\mathbf{p})V_c(\boldsymbol{\sigma}_1\cdot\mathbf{p})-(\boldsymbol{\sigma}_2\cdot\mathbf{p})V_c(\boldsymbol{\sigma}_2\cdot\mathbf{p})+(\boldsymbol{\sigma}_1\cdot\mathbf{p})V_d(\boldsymbol{\sigma}_2\cdot\mathbf{p})+(\boldsymbol{\sigma}_2\cdot\mathbf{p})V_d(\boldsymbol{\sigma}_1\cdot\mathbf{p}). \tag{7.8}$$

The second term on the right-hand side of Eq. (7.7) is the relativistic correction to the kinetic energy.

Due to the appearance of $-(p^2/2M^2c^2)V_a$ in Eq. (7.7), the effective Hamiltonian is non-Hermitian. This corresponds to the fact that ψ is not normalized properly and hence probability is not conserved. This is remedied by applying a "renormalization" to ψ which takes advantage of the fact that the original equation (6.1) conserves probability. Within the approximation mentioned at the beginning of this section we have

$$\Omega^\dagger\Omega\approx\psi^\dagger[1+p^2/2M^2c^2]\psi. \tag{7.9}$$

Therefore the "renormalized" wave function

$$\Psi=(1+p^2/4M^2c^2)\psi \tag{7.10}$$

conserves the probability within our approximation. Expressing ψ in terms of Ψ using Eq. (7.10) and inserting the expression into Eq. (7.7) we obtain after a little arrangement

$$W\Psi=\left[\frac{p^2}{M}-\frac{p^4}{4M^3c^2}+V_a+\frac{V_b{}^2}{4Mc^2}\right.$$
$$\left.-(4M^2c^2)^{-1}(p^2V_a+V_ap^2)-(4M^2c^4)^{-1}\mathcal{V}\right]\Psi. \tag{7.11}$$

The Hamiltonian corresponding to Eq. (7.11) is Hermitian as expected.

We note in Eq. (7.11) the term $V_b{}^2/4Mc^2$ which appears due to the Diracian interactions (6.2) with odd Dirac matrices. This term is similar to terms which might be expected out of the second order of perturbation theory. We omit this term throughout the remaining calculations postponing the related discussions for Sec. 10.

All the remaining terms in Eq. (7.11) are of first order in the interactions. Ordinarily, in an inner region where interactions due to heavier mesons such as ρ and

TABLE I. Contribution to V_{Tot} $[a^2=(\hbar/Mc)^2,\ J_1=r^{-1}(d/dr)J,\ J_2=r^{-1}(d/dr)J_1]$.

	5V				
	$S(J=J_S)$	$V(J=J_V)$	$P(J=J_P)$	$A(J=J_A)$	$T(J=J_T)$
V_c	$-J-\frac{1}{4}a^2\langle\nabla^2J\rangle$	J			$\frac{1}{4}a^2\langle\nabla^2J\rangle$
V_σ		$\frac{1}{6}a^2\langle\nabla^2J\rangle$	$(1/12)a^2\langle\nabla^2J\rangle$	$-J-\frac{1}{3}a^2\langle\nabla^2J\rangle$	$-J-(5/12)a^2\langle\nabla^2J\rangle$
V_{LS}	$\frac{1}{2}a^2J_1$	$\frac{3}{2}a^2J_1$		$-\frac{1}{2}a^2J_1$	$\frac{3}{2}a^2J_1$
V_T		$-(1/12)a^2r^2J_2$	$(1/12)a^2r^2J_2$	$-a^2[J+(1/12)r^2J_2]$	$a^2[J_1+(1/12)r^2J_2]$
V_Δ	$-a^2J$	$-a^2J$			$-2a^2(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)J$
V_∇	$-a^2J_1$	$-a^2J_1$			$-2a^2(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)J_1$
V_R				$-a^2J_1$	a^2J_1
V_Q				$2(a/r)^2J$	$-2(a/r)^2J$

ω are present, V_a is the major term and the rest of the interactions is of the order v^2/c^2 smaller than V_a. However, as can be seen from the definition of V_a given by Eq. (6.11), if we have chosen the masses and the coupling constants of the scalar and the vector meson interactions to be the same, and similarly for the axial vector and the antisymmetric tensor meson interactions, then V_a becomes identically zero. Since the major term cancels, the remaining terms which normally would be of the order $(v/c)^2$ smaller than V_a become the zeroth-order interaction. In such highly relativistic interactions, the effects of the velocity dependence, spin–orbit forces, tensor forces, etc., become very prominent.

One can rewrite Eq. (7.11) in the form

$$[p^2/M+V_{\mathrm{Tot}}]\Psi=(\hbar^2k^2/M)\Psi, \qquad (7.12)$$

where

$$E_{\mathrm{c.m.}}=(E^2/4Mc^2)-Mc^2=W+(W^2/4Mc^2) \qquad (7.13)$$

is the center-of-mass energy which is related to the lab energy E_{lab} by $E_{\mathrm{c.m.}}=E_{\mathrm{lab}}/2$ and $k^2=ME_{\mathrm{c.m.}}/\hbar^2$ a relationship which is a relativistic as well as a nonrelativistic expression. The correction to the kinetic energy included in Eq. (7.11) is taken care this way to the order $(v/c)^2$. Omitting the term $V_b^2/4Mc^2$, the total interaction is given by

$$V_{\mathrm{Tot}}=V_a-(4Mc^2)^{-1}(p^2V_a+V_ap^2)-(4M^2c^4)^{-1}\mathcal{V}, \qquad (7.14)$$

where V_a and $\mathcal{V}$ are defined by Eqs. (6.11) and (7.8), respectively.

After carrying out somewhat tedious but straightforward algebra involving the Pauli spin matrices and the gradient operator appearing in V_{Tot}, Eq. (7.14) can be expressed in the following way:

$$V_{\mathrm{Tot}}=V_c+V_\sigma(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)+V_{LS}(\mathbf{l}\cdot\mathbf{S})+V_TS_{12}$$
$$+V_\Delta(r)\nabla^2+V_\nabla(r)(\mathbf{r}\cdot\boldsymbol{\nabla})+V_RR_{12}+V_QQ_{12}, \qquad (7.15)$$

where

$$\mathbf{l}=(\mathbf{r}\times\mathbf{p})/\hbar \qquad \mathbf{S}=\tfrac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2),$$

and

$$S_{12}=(3/r^2)(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) \qquad (7.16)$$

is the usual tensor operator. The terms ∇^2 and $\mathbf{r}\cdot\boldsymbol{\nabla}$ are the velocity dependence. There appear two additional tensor operators R_{12} and Q_{12} which are defined by

$$R_{12}=(i/\hbar)\{(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{p})+(\boldsymbol{\sigma}_2\cdot\mathbf{r})(\boldsymbol{\sigma}_1\cdot\mathbf{p})\}-S_{12} \qquad (7.17)$$

$$Q_{12}=(r^2/\hbar^2)(\boldsymbol{\sigma}_1\cdot\mathbf{p})(\boldsymbol{\sigma}_2\cdot\mathbf{p}) \qquad (7.18)$$

and which arise due to the inclusion of the axial vector and antisymmetric tensor interactions. The tensor operator R_{12} is related to the operator $T_{ab}{}^\dagger$ introduced by Breit[46] in the following manner.

$$R_{12}=(i/\hbar)[\tfrac{2}{3}T_{ab}{}^\dagger+(\tfrac{4}{3}\mathbf{S}^2-2)(\mathbf{r}\cdot\mathbf{p})]-S_{12}. \qquad (7.19)$$

The choice for R_{12} is made here so that it has no diagonal matrix elements. The operator Q_{12} is a new operator. Both R_{12} and Q_{12} contain the gradient operator $\mathbf{p}$ which acts on the angular part of Ψ as well as the radial part. Thus these operators yield additional velocity dependence, spin–spin forces, and tensor forces.

Contributions to each term in V_{Tot} from different meson interactions are given in Table I.

8. NUCLEON–NUCLEON PHASE SHIFTS

The Schrödinger equation is solved numerically utilizing a program similar in most respects to that developed by Bryan and Scott but differing in a number of important details. The numerical integrations of the Schrödinger equations have been performed on an IBM 709 digital computer in University of Florida. The partial wave equations for all the uncoupled states and the coupled states at higher energies are solved by Noumerov method, and the coupled equations at lower energies ($E_{\mathrm{lab}}\leq142$ MeV) are solved by a Runge–Kutta method where the Noumerov method shows some instability for the case of no cutoff and one cutoff. The use of Noumerov method speeds up the integrations approximately three times compared to the Runge–Kutta method. The phase shifts are obtained by matching the numerical solutions to Coulomb functions for $I=1$ states and to spherical Bessel functions for $I=0$ states at a radius $r=8$ F where all the OBEP contributions become negligibly small. For the coupled states, the reaction matrices are calculated from the numerical solutions and by diagonalizing the matrices the Blatt–Biedenharn phase shifts are obtained. The nuclear bar phase shifts are found from the Blatt–Biedenharn phase shifts by a usual transformation.

The automatic search for the minimum χ^2 can be carried out for most of the parameters.

The nucleon–nucleon scattering phase shifts are calculated using the isoscalar-scalar (s) and vector (ω) mesons and π meson interactions given by columns 2, 3, and 4 in Table I. The Schrödinger equation (7.12) is solved numerically utilizing the well-regulated potentials given by Eq. (3.15). The results obtained from the purely relativistic $\pi-5\omega$ model for the parameters $g_\pi{}^2=14.7$ and $g_{5\omega}{}^2=41.25$ with $\Lambda_{5\omega}=2M$ and $\Lambda_\pi=720$ are shown by smooth curves in Fig. 1.

These theoretical curves are shown in relationship to the energy-dependent phase shifts of Arndt and MacGregor.[49] The S-wave phase shifts below 25 MeV are based upon the effective range approximation. We see that the purely relativistic model despite its simplicity of structure is realistic enough to reproduce most of the characteristic features of the experimental phase shifts. In this model the large central repulsion due to the ω is largely canceled by the large central attraction due to s, leaving relativistic effects only, including a weak attractive residual as well as the spin–spin, tensor, spin–orbit force and the other velocity dependent components.

Our studies indicate that $g_{5\omega}{}^2$ can be varied from about 15 to 150 and that the good fit to the $^1S_0$ and $^3S_1$ phases will be preserved providing both Λ_π and $\Lambda_{5\omega}$ are appropriately readjusted. We have made detailed studies in neighborhoods characterized by $g_{5\omega}{}^2=15.22$, 34.25, and 137. These correspond to $eg_{5\omega}/\hbar c=\frac{1}{3}$, $\frac{1}{2}$, and 1. The values $\frac{1}{2}$ and 1 lead to the coupling constants proposed for magnetic monopoles by Dirac[50] and Schwinger.[51] One might speculate that these bear some relationship to coupling constants for neutral vector mesons. Figure 2 shows the S, P, and D phase shifts for $\frac{1}{3}$, $\frac{1}{2}$, and 1 and all phase shifts for $\frac{1}{2}$. It should be clear that the $^3S_1$ phase shifts are quite good throughout the large range of vector meson coupling constants. However the $^1S_0$ phase shifts are good only for $g_{5\omega}{}^2\geq 50$. On the other hand, the P wave phase shifts prefer the range $30\geq g_{5\omega}{}^2\geq 15$. Our exploratory searches suggest that this range provides many good points of departure for "broken" relativistic models which require only small additions to the pure $\pi-5\omega$ model to achieve good fits to all the phase shifts.

From the standpoint of applications to nuclear physics it would be helpful to have a simple model which gives good S-wave phase shifts, moderately good P- and D-wave phase shifts and embodies the long range pion interactions accurately. To do this we have

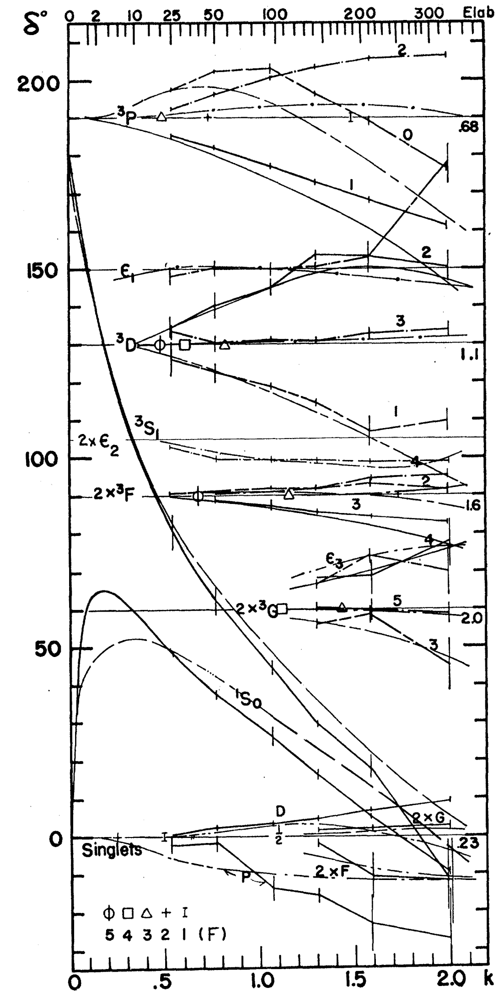

Fig. 1. Relativistic π-5ω model. The smooth curves represent the results of the purely relativistic π-5ω model with $g_\pi{}^2=14.7$, $m_\pi=138$ MeV, $m_s=m_\omega=782$ MeV, $\Lambda_{s\omega}=2M$ *ab initio* and with the adjusted constants $g_{5\omega}{}^2=41.2$ and $\Lambda_\pi=720$. The Arndt–MacGregor energy-dependent phase-shift solutions are shown as experimental points joined by straight-line segments. Assigned errors less than or equal to $\frac{1}{2}$ degree are shown by the same sized bars. The lower horizontal scale is the propagation constant in inverse fermis. The $^3S_1$ and all singlet curves are referred to the ordinates as shown. The other curves are referred to the horizontal lines indicated. The phase shifts for F and G waves and ϵ_2 and ϵ_3 have been doubled. The symbol codes in the lower left corner indicate impact parameter distances according to the rule $b=(l+\frac{1}{2})/k$. Impact distances for S waves are shown along the 0° ordinate. The corresponding distances for P, D, F, and G waves along the triplet reference axes. The numbers on the extreme left represent impact parameter distances at 400 MeV. The theoretical singlet curves are distinguished using the international Morse code: $S\cdot\cdot\cdot$, $P\cdot$—$\cdot$, D—$\cdot\cdot$, $F\cdot\cdot$—$\cdot$, and G—$\cdot$.

been guided by a separate study[52] on how to maintain S-waves phase shifts in the neighborhood of that required by effective range theory at low energies and how to lower the phase shifts at high energies by various combinations of soft cores with velocity dependence. This study has guided us in our meson

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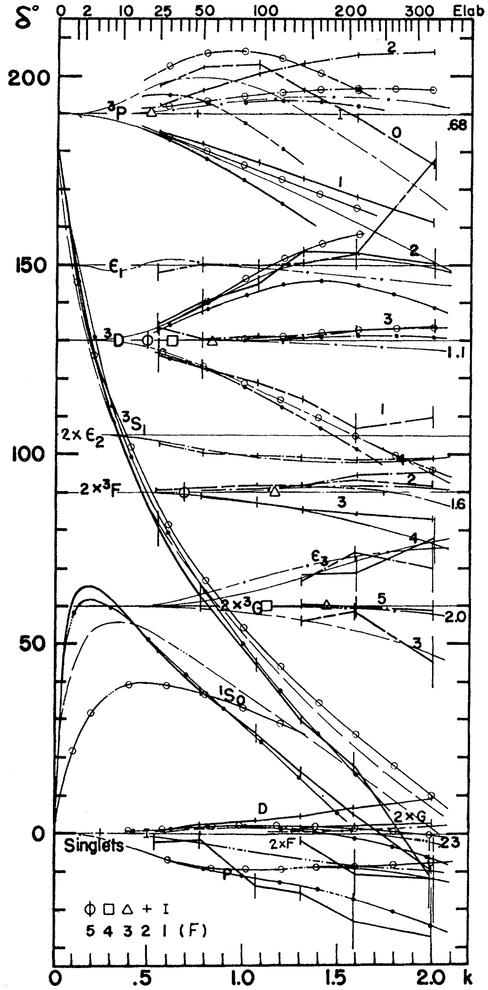

FIG. 2. Relativistic π-5ω models with $g_\pi^2=15.22$ ($eg_\pi/\hbar c=\frac{1}{3}$). The curves without circles represent $g_{5\omega}^2=34.25$ ($eg_{5\omega}/\hbar c=\frac{1}{2}$), $\Lambda_{5\omega}=2390$, $\Lambda_\pi=675$. The curves with closed circles represent $g_{5\omega}^2=137$ ($eg_{5\omega}/\hbar c=1$), $\Lambda_{5\omega}=1038$, $\Lambda_\pi=825$. The curves with open circles represent $g_{5\omega}^2=g_\pi^2$, $\Lambda_{5\omega}=20M$, $\Lambda_\pi=782$.

theoretic efforts. From this study it became clear that to fit the rapid rise of the singlet S state we basically must adjust the potential for that state so as to be very near the critical region for binding an S wave at low energies. To accommodate the rapid falloff of the $^3S_1$ state the potential must achieve the slightly overbound condition. To do this without going to the extremely large coupling constants represented in Fig. 2 whose high-velocity dependence disturbs the p waves, we should seek solutions at lower values of $g_{5\omega}^2$ (15–30). In this region p waves tend to be reasonable and $^3S_1$ can readily be fitted. The main problem is then to bring the $^1S_0$ phase shift closer to critical binding at low energies. At the same time we wish to bring down the high-energy phase shifts somewhat more rapidly. The solution to these problems is quite transparent.[52] In the first place we must strengthen the well by extending outward the region of attraction sufficiently to almost bind the $^1S_0$ state at zero energy. In the second place we must embody a somewhat stronger effective repulsion or core to bring down the high-energy phase shifts. These objectives can be very simply implemented. First we can reduce the scalar mass which we regard as a two-pion effect and modify the one-pion-exchange potential to have as a form factor the same mass as occasioned by the two-pion effect. Thus if we let $\Lambda_\pi=m_s$ and furthermore we strengthen the vector-meson coupling constant to provide an extra short-range repulsion, we can accomplish a gross improvement upon the π–5ω model. To preserve an average cancellation of the vector and scalar static interactions we should impose the relation

$$g_v^2/m_v^2=g_s^2/m_s^2. \tag{8.1}$$

The combination g^2/m^2 corresponds to the VR^2 "well strength" familiar in shell and optical model studies.

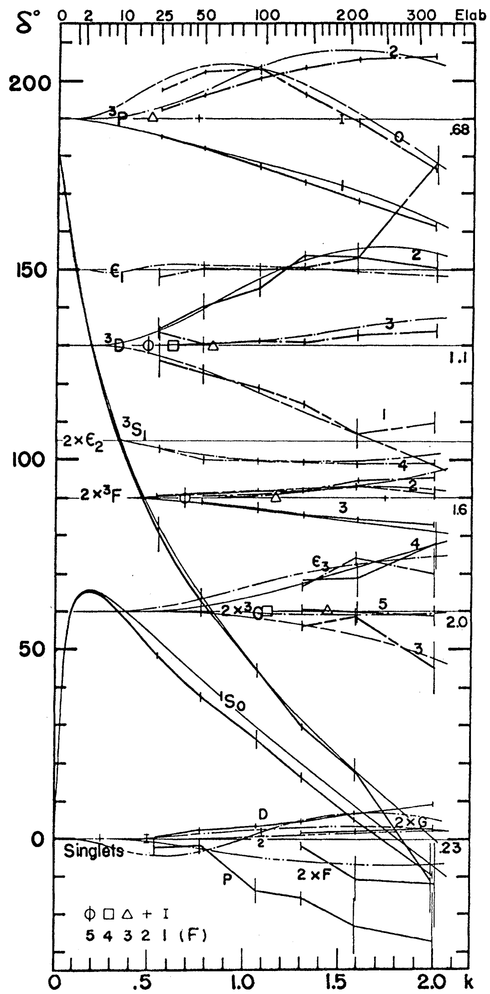

FIG. 3. One-parameter modified model. Here we fix $m_\pi=138.7$, $m_\omega=782.8$, $\Lambda_\omega=\Lambda_s=2M_p$, $U=20\ M_p$, adjust $m_s=\Lambda_\pi$ to 600 MeV and take the corresponding $g_\omega^2=g_s^2(m_\omega^2/m_s^2)=25.0$, where $g_s^2=g_\pi^2=14.7$.

Figure 3 gives the results of a one-parameter modified relativistic model embodying these ideas. Here $m_s = \Lambda_\pi = 600$ MeV is the only parameter fixed on the basis of N–N phases. The parameter $g_v^2 = 25.0$ follows from Eq. (8.1) when $g_s^2 = g_\pi^2 = 14.7$ is taken *ab-initio* as are the remaining parameters in the model. It is seen that this one-parameter model provides a good over-all representation of all the partial wave shifts with the exception of the $^1P_1$ and, to a lesser extent, the $^1D_2$ wave. Some of the discrepancies between theory and experiment, particularly at 330 MeV are probably due to experimental errors.

It is the opinion of the authors that, pending the clarification of very many fundamental theoretical questions, the N–N interaction represented in Fig. 3 should serve as a reasonable meson theoretic N–N interaction for nuclear physics.

Let us next take a particle physicist point of view and try to embody all the known effects which we believe

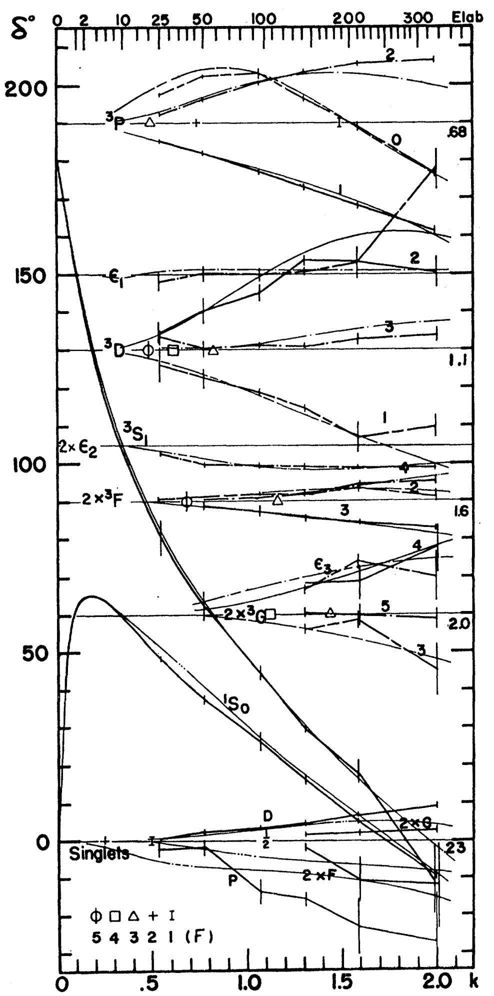

FIG. 4. Broken model with weakly coupled long-range scalar. $g_\pi^2 = 14.42$, $g_\omega^2 = 24.25$, $g_s^2 = 15.60$, $g_{s'}^2 = 2.5$, $m_s = 782$, $m_{s'} = 421$, $g_\rho^2 = 0.65$, $f_\rho/g_\rho = 3.75$, $g_{\rho s}^2 = 0.816$, $\Lambda_\pi = 817$, $\Lambda_R = 1470$.

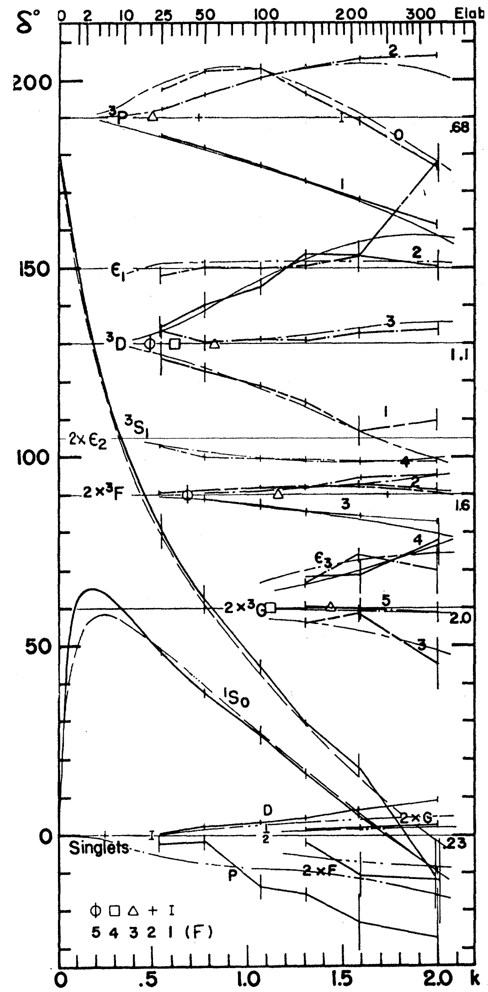

FIG. 5. Broken model. Here we fix $m_\pi = 138.7$ MeV, $m_\omega = 782.8$, $m_\eta = 548.7$, $g_\rho^2 = 0.65$, $m_\rho = 763$, and $f_\rho/g_\rho = 3.75$ but search on m_s, g_π^2, g_s^2, g_ω^2, g_η^2, $g_{\rho s}^2$, Λ_π, and Λ_R (a common cutoff for the remaining mesons). The values obtained are $g_\pi^2 = 13.5$ which is within the range allowed by pion nucleon experiments, $g_s^2 = 13.6$, $m_s = 599$, $g_\omega^2 = 28.1$, $\Lambda_\pi = 1006$, $\Lambda_R = 1189$, $g_\eta^2 = 3.83$, $g_{\rho s}^2 = 1.32$.

are represented in the N–N interaction. Thus we "break" our model to encompass both direct and derivative coupling to the vector mesons. Here we follow closely the work of Bryan and Scott, but embody our well regulated potentials along with the velocity-dependent terms (which they also now include). We accept, however, as constraints various auxiliary pieces of information deduced from experiment which do not depend directly on nucleon–nucleon scattering. For example, electromagnetic form factor analysis[53] and Sakurai's studies[54] of universal ρ couplings suggest that we take $g_\rho^2 = 0.65$ and $f_\rho/g_\rho \simeq 3.75$. Electromagnetic form factor studies also indicate that the f/g for the ω meson is approximately 0. The question of the scalar mass is a critical one in this problem.

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[54] J. J. Sakurai, Phys. Rev. Letters **17**, 1021 (1966).

Here first we maintain m_s at 782 MeV (the ω meson mass) but break the degeneracy of the coupling constants. Furthermore, we incorporate a low mass scalar s' in the spirit of the study of Amati *et al.*,[55] of Furuchi and Watari,[56] and of Durso[57] who show that the N^* resonance implies an effect in the $I=0$, $J=0$ channel. Figure 4 represents the results of a search on $g_{s'}^2$, g_s^2, g_ω^2, Λ_π, and Λ_V (all scalar and vector mesons are given the same form factor). We see that this simple modification leads to a very good set of phase shifts for S, P, and D waves. Hence the corresponding potential should be adequate for most nuclear physics applications. We note that again the vector meson has reached to a higher coupling constant. This has the effect of introducing an extra repulsion at short ranges which helps the velocity dependence of our potential in bringing down the high-energy phase shifts. On the other hand, the weakly coupled long-range scalar (s') helps bring the $^1S_0$ state closer to binding at zero energy which helps bring up its phase shift at low energies. The gross feature of cancellation of large effects is still preserved on the average.

We have also searched the same five parameters with m_s fixed at 900 MeV and at 700 MeV. We obtained very similar phase shifts as in the case of $M_s=782$, the effect of the slight changes in mass being simply taken up by rather small changes in the coupling constant of s. Thus we see that a weak long-range two-pion effect would provide just the perturbation needed to account for the observed phase shifts providing a strong scalar dipion is present in the mass neighborhood of the ω.

We have also investigated the possibility that the only one scalar effect exists by simply breaking our π-5ω model to let m_s and g_s and g_v search out their optimum values. While doing so, we have also included the η meson letting its coupling constant search. In this respect we have come very close to the model of Bryan and Scott except that we have embodied regularized potentials and have constrained the ρ meson coupling constants. The results of the study are shown in Fig. 5. The scalar meson in this case searches out a lower mass value and lower coupling constant. What seems to be preserved in these variations is the approximate equality described by Eq. (8.1) just as in our one-parameter model. This might be expected from the well-known VR^2 ambiguity of scattering studies. One might expect that a scalar particle at 600 MeV with so large a coupling constant would be easy to find experimentally since it is sufficiently far from the ρ meson peak. Accordingly we feel the evidence tends to be somewhat more favorable to a heavier scalar meson in conjunction with some residual long-range effects.

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9. A PHYSICAL DESCRIPTION OF THE NUCLEON–NUCLEON FORCE

As it has evolved, the nucleon–nucleon interaction as derived from field theory and inferred from experiment has turned out to be such an exceedingly complex interaction that it may appear impossible to picture it in simple terms. Of course, we can get some help by pictorial aids such as Feynman diagrams or pictures in the complex momentum plane or complex angular momentum plane. However, such diagrams primarily have appeal to mathematically sophisticated theoreticians. What the average physicist needs is a way of visualizing the meson field in terms similar to the electromagnetic field which all physicists have studied.

The fact that the ω meson is a neutral vector meson, similar in many respects to photons, plays a dominant role in the middle interaction range (0.5 to 1 F) suggests this possibility may exist. Thus on the basis of the close correspondence of the relativistic effects which appear in electrodynamics and in VS meson theory, it may be possible to adapt the Faraday electromagnetic flux concept as a means of visualizing the nucleon–nucleon force.

Let us neglect the long-range pi meson interactions and assume that the two nucleons are coupled only by the exchange of the ω and s mesons with approximately the same coupling constants ($g^2 \sim 10^3 e^2$) and masses.

We now recall from studies of electrodynamics that two Dirac particles are automatically endowed with spin. If we assume we are dealing with positive nucleonic particles, it follows that this spin generates a "meso-

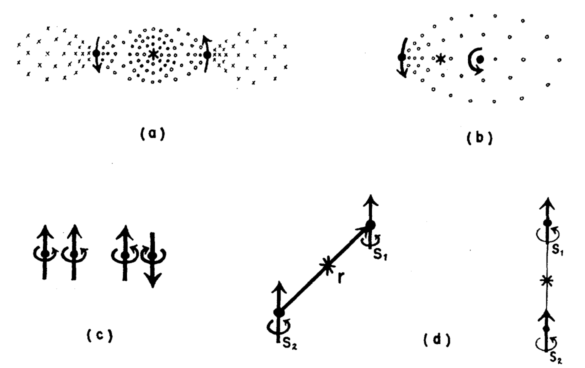

FIG. 6. Faraday-like visualization of components of relativistic nucleon–nucleon interactions. (a) flux representation of velocity-dependent interaction of two nucleons propagated by ω and s mesons. Note that flux lines concentrate in center giving repulsive force. (b) flux representation of spin–orbit force. Note flux lines associated with the left particle at the position of right particle are upward (by the right-hand rule). Thus the most stable orientation for the right particle is as shown, in which the parallel spin and orbital arrangement is the most stable configuration. (c) representation of the spin–spin force propagated by ω meson. By analogy with magnetism, the two mesomagnets prefer the right configuration to the left. (d) magnetic analogy of tensor interaction propagated by ω meson showing general position and the most stable position since the interaction favors the cigar-like configuration.

magnetic" dipole moment along the axis of spin. Further, the orbital motion of the nucleon about a center of force, generates a "meso-magnetic" field parallel to the direction of the orbital angular momentum. In Fig. 6 we illustrate the "meso-magnetic" flux lines associated with two "positive" particles. Flux lines going into the paper are denoted by crosses. Flux lines coming out are denoted by circles. The orbital vector for such a situation is coming upwards out of the paper. We assume that the gv flux line density is proportional to the effective current (gv), and that the flux lines have the properties of the electromagnetic flux lines. It is thus obvious that two particles circulating in the fashion shown in Fig. 6 would repel each other. The repulsion which is proportional to v^2 goes up very rapidly as the distance between mesons becomes small. These are precisely the physical consequences of the velocity dependent terms in any of the vector plus scalar models when large cancellation has occurred. It would appear that this effect, which is the analogue of current–current interaction has been attributed to a hard core of a radially dependent static potential.

Let us next consider the spin–orbit force. This arises because the particles are moving in the upward mesomagnetic field occasioned by their orbital motion. We consider the case in which both spins are parallel and are rigidly coupled into an $S=1$ state. It is then clear by analogy with electromagnetics that both dipoles wish to orient along the field lines. Thus the stable position, i.e., the minimum energy position, corresponds to spin and orbital motion parallel in agreement with experiment. This agrees with the nature of the spin orbit potential generated by the ω and s mesons as evidenced by the separation of the phase shifts at large energies of the $J=L+1$ and $J=L-1$ triplet states.

Let us next consider the question of two spinning mesomagnets close together and ask what orientation they would prefer. Here, it should be obvious that the parallel position would concentrate the flux lines in the center and yield a repulsive force. Accordingly, the antiparallel position is the stable position exactly as it would be in the case of two ordinary magnets in coincidence.

Finally, let us consider the tensor force propagated by the ω meson. Here, we picture two magnets constrained to parallelism, but separated by a radial vector $\boldsymbol{r}$. It should be immediately obvious that the stable position is the cigar-like position in which L, S, and $\boldsymbol{r}$ are parallel. This conforms to the fact that the coefficient of the tensor force propagated by the ω meson is a negative potential.

The over-all effects of the spin–spin and tensor force operators are complicated by the contributions of the π meson which is dominant at long ranges. It is simple to extend the mesomagnetic flux line picture for isotopic spin 1 vector fields by invoking analog of charges which reverse sign for certain states. However for pseudoscalar fields the analog becomes more complicated and loses the beauty of its close relationship to electrodynamics.

10. SUMMARY AND CONCLUSIONS

We have presented in some detail the background for the two major aspects of our work. (1) The use of generalized meson fields which are intrinsically regularized; and (2) The use of relativistic interactions. We have shown that purely relativistic models predict the phase shifts moderately well and slightly broken models do very well. The fact that this work with OBEP in Schrödinger's equation bears a close relationship to other recent approaches based upon substantially different formalisms tends to support this direction of attack.

There are some differences in detail between our studies to date and these other studies. For example, the scalar meson used in the other works tends to be in the 400–600-MeV range whereas our original speculations involved scalar mesons with the same mass and coupling constant as the vector meson. As of this moment, the experimental evidence for scalar mesons tends to be more favorable towards a scalar meson in the 700–900-MeV range[58,59] rather than the 400–600-MeV range. However, the final disposition of this point must await further meson spectral analyses. Actually a difference in masses would not seriously alter the main aspect of the concept of relativistic or almost relativistic interactions providing there is an "average" type of cancellation of the vector repulsion by the attraction due to a scalar meson or the attractions caused by a strongly coupled heavy scalar particle in conjunction with a weak long-range effect which simulates a scalar particle.

Because of the complexity of the velocity-dependent tensor interactions we have omitted axial vector and antisymmetric tensor mesons from our calculations. Since 1^{++} and 1^{+-} mesons have been found,[60] the future more complete analysis should include these meson interactions as well. The occurrence of the velocity-dependent tensor operators R_{12} and Q_{12} as defined by Eqs. (7.17) and (7.18) complicates the situation substantially for the coupled states.

We note that the quadratic term $V_b^2/4Mc^2$ which appears in Eq. (7.11) and which is neglected throughout the calculation arises from Diracian interactions with odd Dirac matrices. A similar quadratic term also appears when one treats the equation (6.12) exactly. A similar quadratic term was discussed in the early studies[61–63] of fine structures of two-electron atoms with

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[63] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One-and-Two-Electron Atoms* (Academic Press Inc., New York, 1957).

the interaction introduced by Breit. In the case of π-meson interactions, if one performs the Dyson transformation to the *ps–ps* coupling and retains the OPE term only neglecting the meson pair term, the quadratic term does not appear since then the interaction contains even Dirac matrices only. More generally, if one retains only the equation for the Fourier components of Ω in which the two nucleons are both in positive energy states throughout the OBE processes, then no quadratic term occurs and the resulting interactions in the Pauli limit agree with those given in Sec. 7. The results of Gupta[64] and Bryan and Scott[21] also support the conclusion although that of Breit[48,65] and Charaplyvy[66] do not. It should be clear therefore that this question along with other theoretical questions will have to be resolved by further study.

A question requiring immediate attention is the problem of the forms of vector meson interaction given by Eq. (5.3). This is somewhat different from other forms arising from vector field theory.[11,12] If the scalar meson cancels the main vector–meson interaction term then one might surmise that the relativistic terms which survive might act quite differently on the phase shifts. This consideration is the motivation for further studies which are underway.

In conclusion, while it should be clear that much remains to be done, nevertheless, we have come quite far as compared to the state of understanding as of 3 or 4 years ago when 20–50 adjustable parameters were required in models which could be made reasonably consistent with experiment. Indeed it would appear that we are approaching the point where our understanding of the *NN* interaction might be adequate for the purposes of treating the nuclear many-body problem.

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