Determination of the p–p Elastic Scattering Matrix

MALCOLM H. MACGREGOR
Lawrence Radiation Laboratory, University of California, Livermore, California

A phase-shift analysis of p–p scattering data from 10 to 400 MeV is presented. Phase-shift energy-dependent forms having a plausible analytic structure are used, and matrix techniques are employed for the search procedure. An interation process was used to arrive at a self-consistent p–p data set. This analysis is being extended to higher energies and to the n–p system.

1. INTRODUCTION

Many groups have carried out phase-shift analyses of (p, p) scattering data in recent years. There is general agreement now as to the existence of a single solution in the elastic energy region, which we may define for practical purposes as extending from zero to 400 MeV. In the research work reported on here, we have attempted to combine our laboriously acquired knowledge of how to carry out phase-shift analyses, together with a complete updating of the data selection, to obtain the best isotopic spin $I=1$ phase shifts that can be obtained from the existing data.

The results described here are in a sense preliminary, since we expect to obtain additional data and, perhaps more importantly, comments on the data we already have, from other participants at this conference, which will necessitate further small changes in the analysis. These results will be published in detail, so that the discussion here is a general description of what we have done, with no attempt to document the work in detail. Only occasional references to other workers and to the data sources are provided.

Since we are primarily interested in the $I=1$ scattering matrix that spans the elastic energy region, our analysis is an energy-dependent one. Results of energy-independent analyses are presented only for purposes of comparison.

2. PHASE-SHIFT ENERGY-DEPENDENT PARAMETERIZATION

There are two common approaches to a phase-shift energy parameterization. The first approach, as used at Yale,[1] is to select pure mathematical forms that have just enough freedom to provide the wiggles demanded by the data. The second approach, as used at Livermore,[2] is to select forms that are plausible from the standpoint of theory. The first method has the advantage that the results do not depend on the choice of any particular model. The second method has the advantage that one can, at least in principle, get by with fewer parameters. And if the results do become somewhat model-dependent, then the results may be expected to give some indication of the validity of the model.

The two approaches just described are somewhat idealized limits. The Yale workers do use some aspects of theory—the high phase shifts are calculated from one-pion exchange. And in the Livermore work we supply a sufficient number of parameters so that we feel the results are not being appreciably dictated by the model.

In a dispersion-theory context, a (p, p) partial wave amplitude in the complex energy plane is characterized by a right-hand unitarity cut and a left-hand series of discontinuities due to "driving forces" originating from resonances in crossed channels. By our use of phase shifts, we automatically satisfy unitarity. The phase-shift energy dependence is thus constructed from a sum of functions having singularities along the left-hand energy axis. In early Livermore calculations,[3] these singularities were represented by a sum of poles. However we find a significant improvement[2] by including the one-pion-exchange cut exactly, and by taking a sum of terms having cuts on the left that can be adjusted to match the 2π-exchange cut, the 3π-exchange cut, and so on. The improvement consists of the facts that we get by with about half the number of free parameters, and that our solution can be extrapolated with some confidence above the region in which the data are included. This extrapolation was not possible in the older work.[4]

The phase-shift forms used are of the following type:

$$\delta_l=\delta_l^{\mathrm{OPE}}+f\sum_j a_j Q_l\{1+[(j\mu)^2/MT]\}. \qquad (1)$$

M is the nucleon mass and T is the nucleon laboratory kinetic energy. The Q_l, Legendre functions of the second kind, have discontinuities on the left in the complex T plane that correspond to the exchange of a system of mass $(j\mu)$, where μ is the pion mass. The a_j are the coefficients searched on, and f is a factor put in to ensure the proper T^l threshold behavior. We used the values $j=2, 3, 5, 9, 20$ in the sum. An s phase or p phase requires four or five values of j in the sum, while an h phase requires in general only one j value plus the one-pion-exchange term. The $^1S_0$ phase shift was further modified to have the correct extrapolation to the scattering length and effective range at low energies.

[1] G. Breit *et al.*, Phys. Rev. **128**, 826 (1962); M. H. Hull, Jr., *et al.*, Phys. Rev. **128**, 830 (1962).
[2] R. A. Arndt and M. H. MacGregor, Phys. Rev. **141**, 873 (1966).
[3] M. J. Moravcsik, H. P. Noyes, H. P. Stapp, and R. M. Wright (unpublished).
[4] M. J. Moravcsik, *The Two-Nucleon Interaction* (Clarendon Press, Oxford, England, 1963), Figs. 34–37.

3. SEARCH PROCEDURE

The (p, p) data set from 0–400 MeV included 758 data points. 33 free parameters were used to represent the phase shifts, and the data set included 56 normalization parameters, which were also included in the search. Thus the mathematical search problem consisted of fitting 814 pieces of data by adjusting 89 free parameters to minimize χ^2. The best way to carry out the minimization is to use a matrix search. Since the phase parameters to be searched on are highly correlated, an efficient search must utilize these correlations. We can write

$$\chi^2(p)=\chi^2(p_0)+\sum_j(\partial\chi^2/\partial p_j)\,dp_j +\tfrac{1}{2}\sum_{j,k}(\partial^2\chi^2/\partial p_j\partial p_k)\,dp_j\,dp_k+\cdots \quad (2)$$

or, in a rough vector notation,

$$\chi^2(p)=\chi^2(p_0)+\beta\cdot d\mathbf{p}+d\mathbf{p}\alpha d\mathbf{p}+\cdots, \quad (3)$$

where p is the parameter set and p_0 represents values in the neighborhood of the solution. Then the solution is to be found where

$$d\chi^2/dp=0=\beta+\alpha d\mathbf{p}. \quad (4)$$

Thus a step

$$d\mathbf{p}=-\alpha^{-1}\beta \quad (5)$$

will take the vector $\mathbf{p}_0$ very close to the value at the minimum. α is $\frac{1}{2}$ the second derivative matrix, and the elements of α^{-1} give the fully correlated error matrix. $\alpha^{-1}=1$ in Eq. (5) would give the gradient search.

The matrix search has been used for years in many applications. It was first applied to nucleon–nucleon phase-shift analyses by Signell. As modified by Arndt,[5] the matrix can be subdivided by separating the phase shift parameters from the normalization constants and searching the two sequentially. This roughly halves the size of the matrix that one must handle. Since the second derivative matrix for the normalization constants is diagonal, this separation is a particularly convenient one.

4. (p, p) DATA SELECTION

In view of the lack of any quantitative theory of nuclear forces, we can in general judge the accuracy of a set of nucleon–nucleon scattering data only by comparing it to other data. This comparison requires two factors: (1) a theoretical model for the phase shifts to allow for the fact that the data occur at different energies; and (2) an accepted set of data against which to make the comparison. For the theoretical model, we chose the energy-dependent parameterization described above, with 33 free phase-shift parameters. For the standard data set, we chose the selection contained in our published papers,[2,6] which is about two years old. In particular, we used the matrix representation of the parameters that gave a fit to that data selection. Thus armed, we proceeded to reexamine all of the available (p, p) data from 0–400 MeV. The data in our standard data set [2,6] included only the data that were grouped in narrow energy bands near 25, 50, 95, 142, 210, and 310 MeV, although this in fact includes the majority of the available data.

(p, p) data below 10 MeV, namely the very accurate Wisconsin cross-section data[7] at 1.4–3 MeV, were excluded from the analysis, since treatment of these data requires precise corrections for vacuum polarization, including S-wave corrections. A few other older data sets that are known to be inaccurate were also deleted. The data examined included 758 data points extending from 9.68 to 400 MeV. Using the parameter matrix, we did not have to search on each set of data. The matrix could be used directly to predict the χ^2 sum when each set of data in turn was compared to the matrix.

As a criterion for selection of data, we arbitrarily deleted any individual data points that deviated more than 3 standard deviations from the predicted values. The statistical probability for such a deviation is only about 1 chance in 300. We also eliminated a few small-angle cross-section points with obvious systematic errors. With these exceptions, all data sets were either accepted or rejected *in toto*.

The selection of a data set is of necessity a sort of bootstrap process, with refinements being made on the basis of self-consistency. Thus the starting basis set used here[2,6] was the result of a long series of investigations of the data at various energies. Using this set, we checked the available data from 9.68 to 400 MeV, as noted above, and selected all data sets having an average χ^2 value of less than 1 per data point. This set, consisting of 439 data points, was then used as a new basic data set, and all of the data were reexamined. This time all data sets having an average χ^2 value of less than 1.55 per data point were selected, and another basic data set was made up having 656 data points. This set was used to make a final selection of data having an average χ^2 value of less than 1.6 per data point. The final data set contained 632 data points. Since 33 free phase-shift parameters were used, the "expected" χ^2 value is 632–33=599. The value we actually obtained was $\chi^2=573$. The cutoff point of 1.6 was not chosen to make the χ^2 value come out as expected, but rather was chosen because it empirically represented a reasonable line of demarcation between

[5] R. A. Arndt and M. H. MacGregor, *Methods in Computational Physics* (Academic Press Inc., New York, 1966), Vol. 6.

[6] M. H. MacGregor, R. A. Arndt, and A. A. Dubow, Phys. Rev. **135**, B628 (1964); M. H. MacGregor and R. A. Arndt, *ibid.* **139**, B362 (1965); H. P. Noyes, D. S. Bailey, R. A. Arndt, and M. H. MacGregor, *ibid.* **139**, B380 (1965).

[7] D. J. Knecht, P. F. Dahl, and S. Messelt, Phys. Rev. **148**, 1031 (1966).

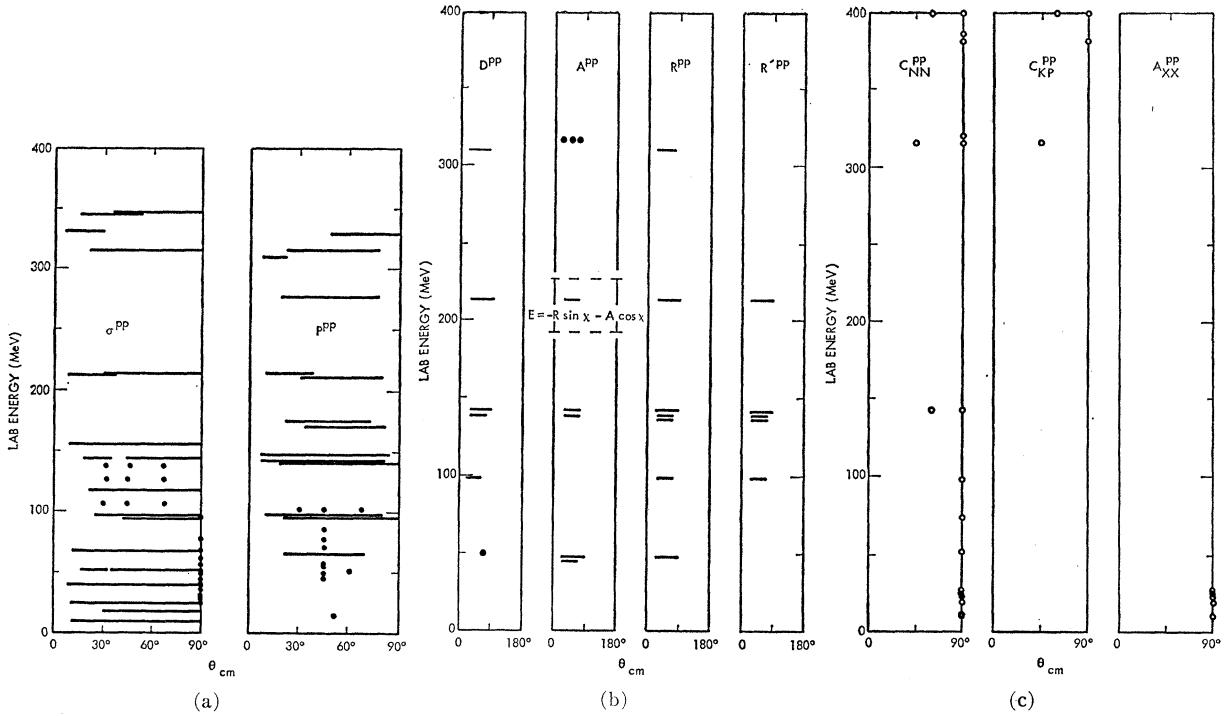

FIG. 1. (p, p) scattering data from 10 to 400 MeV. Only the data that were selected for our final data compilation, as discussed in the text, are included.

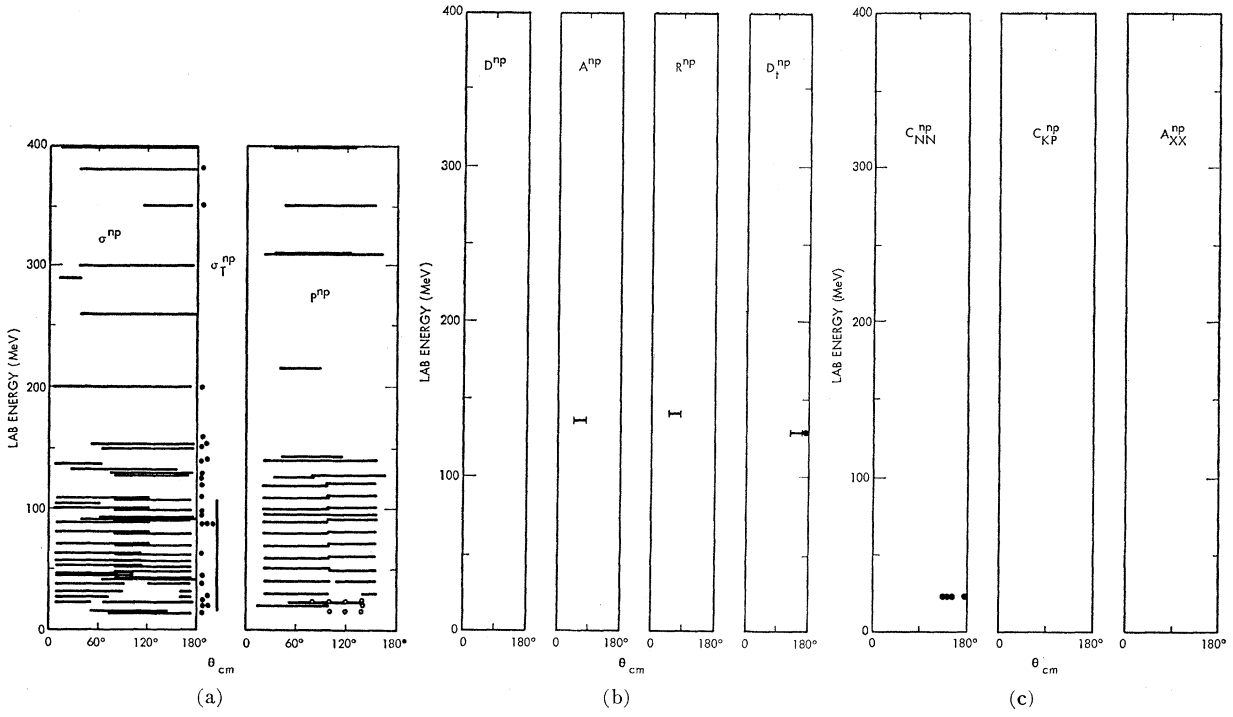

FIG. 2. (n, p) scattering data from 10 to 400 MeV. All available data are included, as the selection for compatibility has not yet been completed.

the "good" and the "bad" data sets. Since we examined altogether 758 data points, only 126 points were deleted as being incompatible with the rest. The most difficult energy region to assess was that above 300 MeV, since this was at the end of the energy range under consideration, and since the data in this region are older and not as complete as at the other energies. By adding data above 400 MeV, it was concluded that the shape of the 380-MeV differential cross section[8] is not consistent with the shape of other cross section data above and below this energy, so this data set was deleted. This set could not be deleted solely by a χ^2 test, since the solution can choose either to fit it or else to fit other nearby, conflicting data.

The (p, p) data set that we finally selected is given schematically in Fig. 1. We are in the process of repeating this analysis for the (n, p) elastic scattering data. These data are given in Fig. 2. Since the (n, p) data collection is manifestly incomplete, the analysis can be carried out only by invoking charge independence and using the (p, p) data to determine the $I=1$ amplitudes. Even with this assumption, the determination of the $I=0$ amplitudes over the 0–400 MeV range will be only qualitative.

5. RESULTS OF THE (p, p) ELASTIC PHASE-SHIFT ANALYSIS

The $I=1$ phase shifts obtained from the present analysis are shown in Figs. 1 and 2 of the summary article at the end of this issue. Also shown are error flags for our single-energy solutions at 25,[9] 50,[10] 95,[6] 142,[6] 210,[6] and 310[6] MeV. The agreement with other workers doing (p, p) elastic phase-shift analyses is quite good. It seems fair to conclude that the $I=1$ nucleon–nucleon scattering matrix is quantitatively well-determined from 0–300 MeV, and to a somewhat lesser extent also to 400 MeV.

As an extension of this work, we added 126 (p, p) data points between 400 and 500 MeV and made a fit to the entire data set of 758 points, assuming zero inelasticity throughout. The 33-parameter solution was capable of giving a good fit over this entire range, with little readjustment being required in the phases below 400 MeV. Of course inelasticity effects begin to become important in this energy region, so that the solution thus obtained is only qualitative above 400 MeV. A continuation made all the way to 750 MeV, with inelasticity effects included, showed that for a highly inelastic phase such as $^1D_2$, inelasticity effects are important above 400 MeV, while for an elastic phase such as $^1S_0$ (which does not couple to a final-state (3, 3) resonance at energies just above the pion production threshold), the inelasticity effects seem to be small at energies below 600 MeV.

A detailed summary of the inelastic analysis is given by Richard Arndt in another paper presented at this conference.

ACKNOWLEDGMENTS

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[8] J. R. Holt, as quoted by W. N. Hess, Rev. Mod. Phys. **30,** 368 (1958).

[9] R. A. Arndt and M. H. MacGregor, Phys. Rev. **154,** 1549 (1967).

[10] R. M. Wright, M. H. MacGregor, and R. A. Arndt, Phys. Rev. (to be published).