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Physical Axiomatics*

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The peculiarities of physical axiom systems, by contrast to the mathematical ones, are examined. In particular, the problem of attaching a physical meaning to a formalism is handled. The main views concerning meaning—formalism, operationalism, the double language doctrine, and realism—are analyzed and arguments for realism are advanced. The realistic view is illustrated by analyzing typical physical quantities and by axiomatizing a theory—special relativistic kinematics. It is argued that all the components of a physical theory—the formalism as well as the correspondence or semantic hypotheses—contribute to sketching the meaning of the theory, and that this meaning is best found out upon displaying the basic assumptions in an axiomatic fashion. The advantages and scope of the axiomatic approach are finally discussed.

The axiomatic approach has seldom been tried in physics, partly because the term 'axiomatic' is still widely mistaken for 'self-evident' or for '*a priori*,' partly because physical theories are often regarded as mere data processing devices in no need of logical organization, and partly because of a fear of rigor and clarity. As a result, between Newton's naive axiomatization of point mechanics (1687) and the birth of modern axiomatics (Hilbert, 1899), no significant effort in the logical organization of physical thought seems to have been made. And even though mathematical logic, metamathematics, and semantics have vigorously developed during our century, only a few essays in physical axiomatics have been influenced by these developments—namely those of Hilbert[1] (phenomenological radiation theory), McKinsey *et al.*[2] (classical particle mechanics), Noll[3] (classical continuum mechanics), Wightman[4] (quantum field theory), and Edelen[5] (general classical field theory). Most other attempts have failed to pinpoint and characterize the basic (undefined) concepts and/or to give a sufficient set of postulates entailing the typical theorems of the theory concerned. In particular, the works of Carathéodory[6] (thermostatics) and von Neumann[7] (quantum mechanics) fall short of the requirements of modern axiomatics. In short, physical axiomatics is having a protracted infancy. It would therefore be unfair to judge it by its fruits.

Not only are there too few physical theories organized in a logically satisfactory fashion, but the existing axiomatizations have either or both of the following shortcomings: (a) an inadequate characterization of the physical meaning of the symbolism, and (b) an insufficient metamathematical analysis (of consistency, independence, etc.). To the writer's knowledge only two works[1,2] pay adequate attention to the metatheoretical aspect. This is understandable, for it is more rewarding and far easier to reconstruct a theory than to perform consistency and independence tests. What is not so easily excusable is the first shortcoming, namely the weakness of most of the existing axiomatizations on the semantical side. This weakness is particularly interesting to the philosopher because it can be traced to certain views concerning meaning—a typically philosophical subject. We therefore concentrate in this paper on the semantical aspect of physical axiomatics and on the philosophical issues related to it. And, rather than just preaching a given doctrine concerning physical meaning without caring to check whether it is viable, we exhibit a specimen of a physical theory axiomatized in accordance with our philosophy. But before doing this we must examine the prevailing views.

I. FOUR DOCTRINES CONCERNING PHYSICAL MEANING

A. Formalism

When it comes to fixing the physical content of a set of formulas, most physicists take a happy-go-lucky attitude: while recognizing that the formulas must mean something, they trust that the context in which they occur will make that content clear. This informal attitude, which is hardly a doctrine, can be contrasted to the formalist one found among many mathematicians working in physics. The mathematician will naturally tend to approach the axiomatization of a physical theory as if it were one more mathematical theory. That is, he will focus on the formalism, with neglect of the physical content.

Two species of formalists can be distinguished: the radical and the moderate ones. The uncompromising

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[1] D. Hilbert, Physik. Z. **13**, 1056 (1912); **14**, 592 (1913); **15**, 878 (1914).

[2] J. C. C. McKinsey, A. C. Sugar, and P. Suppes, J. Ratl. Mech. Anal. **2**, 253 (1953).

[3] W. Noll, "The Foundations of Classical Mechanics in the Light of Recent Advances in Continuum Mechanics," in L. Henkin, P. Suppes, and A. Tarski, Eds., *The Axiomatic Method with Special Reference to Geometry and Physics* (North-Holland Publ. Co., Amsterdam, 1959).

[4] R. F. Streater and A. S. Wightman, *PCT, Spin & Statistics, And All That* (W. A. Benjamin, Inc., New York, 1964).

[5] D. G. B. Edelen, *The Structure of Field Space* (University of California Press, Berkeley and Los Angeles, 1962).

[6] C. Carathéodory, Math. Ann. **67**, 355 (1909).

[7] J. V. Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, N.J., 1955).

formalist will say, for instance, that the electromagnetic field *is* a tensor field F over a certain manifold and satisfying certain equations: he will regard electromagnetism as a branch of differential geometry. And, since most of mathematics can be reconstructed on the basis of set theory, he may go as far as claiming that the proper axiomatization of a physical theory turns it into a part of set theory. The moderate formalist will abstain from identifying things (e.g., physical fields) with ideas (e.g., tensor fields) and correspondingly theoretical physics with mathematics: he will grant that certain mathematical symbols have special names in physics, but he will not bother to inquire what these names name. In any case, when reconstructing an electromagnetic field theory, the moderate formalist will be willing to add *designation rules* such as

> *DR 'F' designates* [or *names* or *is called*] *an electromagnetic field.*

The physicist cannot be satisfied with this concession—nor can the philosopher, who will object that naming assigns no meaning. Names are indeed conventional tags while meaning assumptions are either true or false. Thus the hypothesis that the components of the electromagnetic tensor F *stand for* (represent, symbolize) the elongations of oscillating aether particles is now regarded as untestable and leading to contradiction: we think of F as *representing* the basic trait of a peculiar substance, namely an electromagnetic field in vacuum.

The physicist and the philosopher, then, will probably agree on the following points: (a) a physical theory includes a mathematical formalism but it is more than this; (b) this something more is the physical meaning, which is not assigned by laying down arbitrary designation rules; (c) physical meanings either take care of themselves (informal attitude) or they are assigned by adding *correspondence* or "dictionary" statements linking theoretical symbols to extralinguistic items—as first emphasized by Campbell.[8] So far so good: as soon as this agreement is reached, a violent argument is likely to start concerning the way symbols acquire a meaning, i.e., the nature of the correspondence statements. The disagreement bears mainly on the physical partner of the sign–physical object correspondence: it is a philosophical quarrel concerning what physics is about. There are two main views on this issue: (a) physical objects are items of human experience, in particular observations (empiricism), and (b) physical objects are components of an autonomously existing external world (realism). If empiricism is adopted, then the correspondence statements are said to consist in symbol–experience relations; on a realistic philosophy they will be symbol–objective item relations. Let us steal a glance at these two conflicting views.

[8] N. R. Campbell, *Physics: The Elements* (Cambridge University Press, Cambridge, England, 1920).

B. Empiricism

The dominant view concerning physical theory seems to be this: Every physical theory is a mathematical formalism or calculus that is assigned a physical meaning by reference to experience and, in particular, to laboratory operations. This assignment of meaning is a term by term affair except for the purely formal symbols such as 'and' and '+'. There are two variants of this doctrine: an extreme and a mild one. According to the radical view all physical concepts must be *reduced* to observational concepts by way of identities of some sort, preferably the so-called operational definitions. The moderate view is that there exist irreducibly theoretical terms, but they must all be *related* within the theory to observational concepts via correspondence postulates; furthermore the theoretical items must dwell on the higher regions of the theory while the lower-level statements of it (the weakest theorems) must contain only observational concepts. Let us handle these two versions in succession.

1. Operationalism

The view that physics must contain only observables can be traced back to Ptolemy, Berkeley, d'Alembert, Kirchhoff, and Mach. This doctrine seems to have entered physical axiomatics via Carathéodory's famed axiomatization of termostatics,[6] where he claimed that the meaning (not only the test) of his initial assumptions had to be "defined" by establishing experimentally the conditions (e.g., the equations) describing the properties of the entities concerned. Thus, dealing with thermally transparent enclosures and the like, he wrote: "What is meant by these various expressions must be exactly *defined* by *experimentally* establishing the conditions [. . .] describing the thermodynamic properties of the wall under investigation." Fifteen years later, in his unsuccessful axiomatization of special relativity,[9] Carathéodory claimed that this theory could be based on time readings alone—a view recently revived by J. Synge. Of course in neither case was he *defining* symbols: he was laying down truth conditions for whole statements; nor was he attaching them a *meaning*: he was stipulating testability conditions. Yet the confusion spread and worsened with the backing of what was quickly becoming the official philosophy of physics, namely logical positivism—the most advanced philosophical school of the 1920's. From then on, to assign physical meanings came to be regarded as identical with giving "operational definitions."

This doctrine—operationalism—was first explicitly stated by Dingler,[10] whose writings were very influential in the German-speaking world. Operationalism was

[9] C. Carathéodory, Sitzber. Preuss. Akad. Wiss. Phys.-Math. Kl. **1924**, 12.

[10] H. Dingler, *Grundlinien einer Kritik und exakten Theorie der Wissenschaften, insbesondere der Mathematik* (Ackermann, Munich, 1907).

independently reinvented by Eddington,[11] who introduced it to the English-speaking world and who ruled that the starting point for any physical theory consisted of "physical quantities defined by operations of measurement." One year later, in his pseudoaxiomatization of special relativity, Reichenbach[12] attempted to "define" time sequences in terms of operations. Bridgman's popular book of 1927[13] was a systematic exploration of the same idea. Even though he subsequently corrected the doctrine to encompass pencil and paper (i.e., mental) operations,[14] many scientists still regard *The Logic of Modern Physics* as the scripture of philosophical wisdom.

This extreme version of empiricism is not only widespread but it exerts a powerful influence on the valuation and even the construction of physical theories. Thus Heisenberg's founding papers on the S-matrix theory[15] were prompted by his complaint that the standard quantum theories teem with unobservables, and by the requirement that physical theories should contain only observables.[16] Some go as far as holding that the ideal physical theory is the one whose basic symbols are either definable or interpretable in terms of direct elementary human experiences—a natural requirement for an empiricist to make, since laboratory operations are soaked in theory.[17] Moreover such an ideal theory should—granting only a modest inductive leap—be inferable from coarse experiential (not experimental) items alone. Furthermore, the poorer the powers of observation of the subject—i.e., the less refined and the less use they make of instruments and consequently of theoretical formulas—the better from such a standpoint. The ideal empiricist theory is indeed the one that could have been evolved by a "primitive observer," i.e., a subject endowed with meager powers of observation.[18] While this view does not explain why physical science did not emerge 10^6 years ago, it does suggest that radical empiricism is not an adequate philosophy of physics.

Operationalism can be criticized on several counts.[19] First, no existing physical theory complies with the operationalist program, for every such theory contains concepts with no counterpart in sensory experience—such as those of potential, Lagrangian, plane wave, and mass point. Yet such concepts are physically meaningful in the sense that they concern, if only in a sketchy and roundabout way, things and properties of things supposed to be out there. Second and consequently, if the operationalist strictures were admitted, then all our present theories would have to go, leaving a dreadful conceptual vacuum. Third and consequently, no significant laboratory operation would then be possible, for every such operation is backed up and guided by numerous fragments of theories. Fourth and more to the point: there *are* no operational definitions proper, and this for the following reasons: (a) a fact, such as a measurement operation, can be described by a set of statements, never by a single concept, e.g., the one of length; (b) while measuring is a theory-backed empirical fact, defining is a conceptual operation made in a theoretical context; consequently (c) the structure and content of a theoretical symbol can only be disclosed by a theoretical analysis, never by a laboratory operation: what a measurement does is to sample numerical values of magnitudes or, rather, estimates of such. In summary, there are no and there can be no operational definitions.

What we do have is empirical *tests* of some physical statements and empirical *interpretations* of some physical symbols. These two operations have been consistently mixed up by operationalism. Moreover, it has never been proved that all the basic symbols occurring in fundamental theories, such as electromagnetism and quantum mechanics, do have an empirical interpretation and, in particular, an interpretation in terms of possible laboratory operations. True, it is sometimes said that the proper time in a (or relative to) a point particle is the time read by an observer riding the particle—but this is just a didactic prop, since clocks as well as observers are complex systems that could not possibly be carried by a particle. A genuine interpretation must be literal not metaphorical if it is to belong to science rather than to science fiction. One may also feel tempted to say that a current intensity is what an ammeter measures—but this is again a prop, it is unilluminating, and it is doubly misleading, for ammeters can also be used to measure differences of potential, and because a current intensity is a function, and a function should not be mistaken for its values. As to the trick of calling 'observable' any dynamical variable that cannot be measured without the assistance of whole theories, it is as effective as christening every body Leo to make sure he will be valiant.[20]

Most philosophers of science initially adopted operationalism.[21] In later years they have gone some way in

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[11] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, England, 1923).

[12] H. Reichenbach, *Axiomatik der relativistischen Raum-Zeit-Lehre* (Frederick Vieweg und Sohn, Braunschweig, Germany, 1924).

[13] P. W. Bridgman, *The Logic of Modern Physics* (The Macmillan Co., New York, 1927).

[14] P. W. Bridgman, Daedalus **88,** 518 (1959).

[15] W. Heisenberg, Z. Physik **120,** 513, 673 (1943).

[16] For a criticism of the claim that S-matrix theory does comply with operationalism see the author's "Phenomenological Theories," in *The Critical Approach,* M. Bunge, Ed. (Free Press, New York, 1964).

[17] P. Duhem, *The Aim and Structure of Physical Theory* (Atheneum, New York, 1962).

[18] R. Giles, *Mathematical Foundations of Thermodynamics* (Pergamon Press, Oxford, England, 1964).

[19] For details, see the author's *Scientific Research* (Springer-Verlag, Berlin, 1967), Vol. I, Chap. 3.

[20] For a criticism of phony observational languages, see the author's "A Ghost-Free Axiomatization of Quantum Mechanics," in *Quantum Theory and Reality,* M. Bunge, Ed. (Springer-Verlag, Berlin, 1967).

[21] For example, R. Carnap, *Foundations of Logic and Mathematics* (University of Chicago Press, Chicago, Ill., 1939).

criticizing it.[22] Still, operationalism keeps recurring in milder forms. A slightly sophisticated version of empiricist reductionism is Braithwaite's.[23] According to this view, every correspondence hypothesis or "dictionary axiom" has the form of an identity, namely: $o=(\cdots t_1\cdots t_2\cdots)$, where o is an "observable" (observational) term while the t_i are theoretical terms. Consequently whenever a theoretical expression of the form '$(\cdots t_1\cdots t_2\cdots)$' occurs in a theorem it can be replaced by the corresponding observational term. In this way theoretical terms become harmless euphemisms for clusters of experiences: their function is not semantical but syntactical. The trouble with this thesis is that it takes for granted the reducibility it sets out to prove: indeed, it assumes that the correspondence axioms are actually in the form of identities of the above kind. No actual example of a scientific theory is offered in support of this thesis—nor is it known to the writer.

A somewhat more refined version of this thesis is Carnap's.[24] On this view, every scientific theory contains both observational terms o and theoretical terms t, but the latter can be defined in terms of the former via the postulates of the theory. This reduction is performed with the help of Hilbert's ϵ operator, namely so: "$t=\epsilon_u\Phi(u, o)$," read 't is an object satisfying the logical predicate Φ that summarizes the postulates of the theory.' There are three objections to this view: (a) "$\epsilon_u\Phi$" designates *an* object u that satisfies the condition $\Phi(u, o)$: unlike the definite description operator ι, ϵ is an indefinite description operator and therefore unsuitable to frame definitions proper; (b) before the conjunction of postulates $\Phi(u, o)$ can be written out, the theoretical terms u must be at hand either as primitives or as defined concepts—whence if "$t=\epsilon_u\Phi(u, o)$" were a definition (which it is not), it would be circular; (c) there are no purely observational terms, such as "blue" and "rough," in a physical theory—but this borders on the next section. Ramsey's influential proposal,[25] of "eliminating" the t's by reformulating the theory in the form "There exists at least one u such that $\Phi(u, o)$" is open to the last two objections.

In conclusion, empiricist reductionism in its various versions is scientifically and philosophically inadequate: we need a more tolerant doctrine of meaning.

2. *The Double Vocabulary View*

The dominant view among philosophers[26] is no longer that every nonlogical term of a scientific theory must be operationally "defined" (interpreted), but that every scientific theory contains, alongside genuine observational terms such as 'hot,' others which are not reducible to sense experience, such as 'temperature.' In other words, the specific vocabulary of every scientific theory can be partitioned into two sets: a collection of strictly observational terms and another set of strictly theoretical terms. Consequently the sentences of a scientific theory fall into three jointly exhaustive classes: observational, theoretical, and mixed. Among the mixed sentences the correspondence rules or postulates stand out.

On this view, while the observational sentences are fully meaningful because they are directly testable, the theoretical ones are by themselves deprived of meaning because they can be subjected to no direct empirical tests. It is only the mixed sentences, and particularly the basic ones—i.e., the correspondence sentences—that confer a (partial) empirical meaning upon the theory. For, while the observational terms are fully meaningful because they are directly anchored to experience, the theoretical symbols have no such interpretation: they are partially meaningful since they obtain their meaning indirectly, via the correspondence rules and empirically testable theorems containing no theoretical terms at all: "The calculus is thus interpreted from the bottom upwards."[27] Scientific theories are then regarded as semi-interpreted hypothetico-deductive systems rather than as fully interpreted ones, and this because meaning is equated with empirical meaning and the latter in turn with testability—in keeping with the Vienna Circle verifiability doctrine of meaning. We shall reach a similar conclusion, from different premises, concerning the semantical incompleteness of theories (see Sec. IC).

This version of empiricism is an advance over operationalism insofar as it acknowledges the occurrence of nonobservational concepts in scientific theories. It is superior also in that it does not wish to be normative but just descriptive: in fact it assumes modestly that theories are in fact that way, instead of legislating the permissible kind of theory. Unfortunately physical theories do not happen to be that way: they fail to contain observational concepts *stricto sensu*, such as 'hot' and 'blue' (Carnap's favorite examples). These terms occur only in psychological theories (for they refer to sensations), in the language of the experimental physicist, and in didactic presentations of physical theories

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[22] See, e.g., C. G. Hempel, "The Theoretician's Dilemma," in *Minnesota Studies in the Philosophy of Science*, H. Feigl, M. Scriven and G. Maxwell, Eds. (Minnesota University Press: Minneapolis, 1958), Vol. II; A. Pap, "Are Physical Magnitudes Operationally Definable?" in *Measurement: Definitions and Theories*, C. W. Churchman and P. Ratoosh, Eds. (John Wiley & Sons, Inc., New York, 1959); and K. R. Popper, *The Logic of Scientific Discovery* (Basic Books, Inc., New York, 1959).

[23] R. B. Braitwaite, "Axiomatizing a Scientific System by Axioms in the Form of Identifications," in the collective volume cited in Ref. 3.

[24] R. Carnap, "On the Use of Hilbert's ϵ-operator in Scientific Theories," in *Essays on the Foundations of Mathematics*, Y. Bar-Hillel, Ed. (Magnes Press, Jerusalem, 1961).

[25] F. P. Ramsey, *The Foundations of Mathematics* (Routledge and Kegan Paul, London, 1931), Chap. IX.

[26] See R. B. Braithwaite, *Scientific Explanation* (Cambridge University Press, Cambridge, 1953); R. Carnap, "The Methodological Character of Theoretical Concepts," in *Minnesota Studies in the Philosophy of Science*, H. Feigl, M. Scriven, and G. Maxwell, Eds. (University of Minnesota Press, Minneapolis, 1956), Vol. I; and *Philosophical Foundations of Physics* (Basic Books, Inc., New York, 1966); and C. G. Hempel's paper cited in Ref. 22.

[27] Braithwaite in Ref. 23.

(so far the major source of inspiration of the philosophy of science). They do not and ought not occur in theoretical physics, however important they may be elsewhere. In particular, they should not occur in the semantical statements or "correspondence rules." Thus the statement that the electromagnetic radiation of a given wavelength elicits a certain color sensation—one of Carnap's examples of a "correspondence rule"[28]—is a statement in psychophysiological optics not in physical optics. Moreover it is not a rule or prescription but a full-fledged hypothesis and, more precisely, a corrigible statement serving as cause-symptom relation and therefore important in experimental physics. But it does not and it should not occur in theoretical physics, which is observer-invariant (objective). In short, physical theories are free from strictly observational or phenomenal concepts.

Basic physical theories do not even contain observational terms *lato sensu*, i.e., symbols standing for objective aspects of real experimental situations. In fact, a basic physical theory is an idealized model or sketch of a physical system (electron, field, fluid, etc.), not a literal description of complex experimental situations such as the measurement of electric charges by means of electrometers, or the determination of collision cross sections by means of scintillation counters. It is rather the other way around: the accurate description and *a fortiori* the explanation of an experimental situation calls for ideas belonging to a number of scientific theories. Thus a length measurement, even if direct and therefore coarse, involves a set of assumptions concerning the geometry of physical space, the behavior of bodies under transport, and the propagation of light. Accurate and therefore indirect length measurements involve much more than that—usually whole pieces of mechanics, electromagnetic theory, and also quantum electronics if they employ lasers. And every measurement in nuclear and atomic physics uses both microphysical and macrophysical theories. On the other hand, none of these basic theories is couched in observational terms and none contain descriptions of the construction of instruments and rules for operating and reading them—contrary to what some philosophers believe.[28]

True, some theories—even high-brow ones such as general relativity and quantum mechanics—are often worded as if they did concern experimental situations alone. But even a summary analysis shows that this empirical interpretation is metaphorical not literal (recall Sec. IB1). Indeed, none of their basic formulas contains parameters concerning pieces of apparatus—much less sentient observers. Thus Einstein's gravitational field equations are about fields and matter but they do not even hint at the ways the curvature tensor might be measured—this being one reason why it took forty years to design such a measurement. The theory itself is needed in order to devise ways in which the components of that magnitude could be measured, just as classical mechanics was needed in order to measure mass values. Similarly the Schrödinger equation is fairly general and it contains no macrovariables describing traits of measuring instruments. Genuine accounts of experimental situations (not just of gedankenexperiments) are specific because instruments happen to be specific. Moreover such accounts involve macrophysical and particularly classical ideas, because what we manipulate and observe are only macrofacts—not either microfacts or megafacts. Therefore the formulation of those high-level theories in operational terms is phony: it involves metaphorical not literal interpretations. Furthermore those theories can be reformulated without using the fiction of the ever-present observer who is ever ready to take direct and exact measurements of any magnitude.[29]

In conclusion, we have as little use for the dual vocabulary doctrine as we had for the operationalist tenet. We must look elsewhere for a more realistic view on physical meaning.

C. Objectivism

The empiricist doctrines examined in Sec. IB rest on a narrow interpretation of the expressions 'factual meaning' and 'content,' namely as being identical with 'empirical meaning.' We shall drop this restriction for it does not square with scientific practice. Thus the components of the energy-stress tensor of a body, or even of an atom, are regarded as physically meaningful symbols, for they are supposed to correspond to an objective state of the innards of the thing they refer to, even though they are not directly measurable, much less observable in a strict sense. Similarly in electromagnetic field theory the field tensor F mentioned in Sec. IA makes sense even in the absence of charged bodies, when there is no possibility of measuring F. In such a theory one would have, in addition to the postulates determining the mathematical structure of F (and of the underlying space), a set of field equations (the basic law statements of the theory), and one or more interpretive hypotheses, or *semantic assumptions*, sketching the meaning of F. One such semantic assumptions could be

SA *F represents [models, mirrors] an electromagnetic field* φ,

or

SA $F \hat{=} \varphi$

for short, where "$\hat{=}$" symbolizes the reference relation.[30] This formula is a correspondence statement in the sense of Sec. IA, for it establishes a correspondence

[28] Carnap in the second of his works cited in Ref. 26.

[29] M. Bunge, *Foundations of Physics* (Springer-Verlag, Berlin, 1967).

[30] M. Bunge, Ref. 29, Chap. 1.

between the symbol 'F' (or the concept F designated by the sign 'F') and the thing (field) named φ.

Our SA above complies with none of the philosophies examined so far. In the first place, it is not a rule but a full-blown hypothesis, both because it involves a trans-observational or hypothesized entity (the electromagnetic field) and because it might be falsified and moreover it is pointless in any action at a distance theory. (Every change in the semantical assumptions of a scientific theory yields a different theory, even though the formal structure of it is kept unchanged.) Nor is our SA either an operational "definition" (interpretation) or a correspondence rule *à la* Carnap or Braithwaite. In fact, φ does not name an empirical item but a physical one, and SA does not state that its conceptual partner F is a set of experimentally found values: no confusion between a function and some of its values is made herein.

SA is an *objective interpretation hypothesis*, i.e., an assumption conferring a factual meaning onto a theoretical symbol. Moreover it is literal not metaphoric. In particular, our SA does not state that the components F_{0i} of F are the components of the acceleration felt by an observer riding a particle of unit mass and unit charge. The reason for our preference for objective and literal over empirical and metaphorical interpretation assumptions is clear: physics is supposed to be about what is or may be the case not about what may appear to a subject, much less to a fictitious observer. If the reference to observers is meant seriously then it must be supplemented and tested with the science of observers, namely human psychophysiology; but if the reference to observers is phony, then the observer concept is out of place in theoretical physics.

Let us now find out to what extent does an objective interpretation hypothesis (=semantic assumption) specify the meaning of a theoretical symbol. To this end we must recall that every concept has a connotation or intension, and a denotation or extension. Now the *meaning* of a symbol s that stands for a concept c may be defined[31] as the ordered pair constituted by the intension (set of properties) and the extension (domain of applicability) of c. (In short:

$$\mathfrak{D}sc \Rightarrow [\text{Mean } s \overset{\text{df}}{=} \langle \mathcal{I}(c), \mathcal{E}(c) \rangle.)$$

Clearly, SA hints at the intended denotation or extension of the concept F, for it says that F applies to an arbitrary member $\varphi \in \{\varphi\}$ of the set $\{\varphi\}$ of (actual and possible) electromagnetic fields. (The extension of the whole family F of tensor fields on which the theory focuses is of course the aggregate $\{\varphi\}$.) This takes care of only a part of the extension of F. The other part is specified by the field equations (zero four-divergence and zero four-curl of F for the free space). Indeed, not every F but just those F that happen to satisfy the basic law statements embodied in the theory will (hopefully) represent electromagnetic fields. In short, the law statements in conjunction with SA (in psychological terms: the former read in terms of the latter) fix the extension of F. As to the connotation or intension of this concept, i.e., the collection of its properties, it is specified in the theory by both the basic law statements and the mathematical assumptions (real valuedness, continuity, etc.) underlying those law statements. Other theories and experimental physics will supply further properties of F.

We see then that the meaning of the symbol 'F' is specified jointly by the following components of electromagnetic theory: (a) the formal or structural hypotheses (without which the field equations would make no sense); (b) the initial physical assumptions or basic law statements (variational principle or field equations and boundary conditions), and (c) the objective interpretation assumptions. Now a basic theory like the one we are considering has no other components; in particular it contains no prescriptions for building measuring instruments and reading them. Therefore we feel justified in jumping to the general conclusion that the content or meaning of a physical symbolism is determined jointly by its formalism and by its semantic assumptions as long as the latter are objective. Equivalently: the meaning of a set of symbols is specified by the whole theory in which they occur.

So far we have dealt with objective or factual meaning: what about empirical meaning or meaning for an observer? This cannot be found out by examining the foundations of the theory, for the description of empirical situations involves some of the logical consequences of the basic principles—e.g., some solutions of the field equations. Yet this too is insufficient, for a theory has empirical implications only if conjoined with other theories and with special assumptions. Thus the electromagnetic theory remains untestable unless conjoined with some fragments of dynamics—e.g., a law of motion for charged particles. The funny thing is that the theories functioning as auxiliaries in the task of finding the empirical implications of a given theory may be inconsistent with the latter. For example, classical mechanics, which is necessary to design and operate any instrument, is inconsistent with electromagnetic theory. In any case by itself no theory has an empirical meaning. This vindicates Quine's thesis[32] that the unit of empirical meaning is neither the term nor the sentence but the whole of science.

In conclusion, we nail down the following heresies on the temple of the orthodox philosophy of physics:

(*a*) *The realist thesis*: A physical theory has an objective or factual content. No physical theory has an empirical *content*, for it contains no observational concepts. But it does acquire an empirical *import* provided it allies to further theories.

[31] M. Bunge, *Scientific Research* (Springer-Verlag, Berlin, 1967), Vol. I, Chap. 2.

[32] W. V. Quine, *From a Logical Point of View* (Harvard University Press, Cambridge, Mass., 1953).

(*b*) *The downward flow thesis*: In an axiomatized theory meanings flow downwards, from basic (primitive) to defined symbols and from axioms to theorems. It is only in the process of theory construction that the meaning of some symbols are found by examining low-level statements.

(*c*) *The wholeness thesis*: All the basic components of a theory (formal assumptions, basic law statements, and semantic assumptions) contribute to determine the meaning of its symbolism. Even so, they just sketch meanings: the interpretation of a physical symbolism remains always incomplete, hence in the making (see Sec. IIC).

Let us now perform a more thorough exploration of the same area.

II. THE SEMANTIC ASPECT OF A PHYSICAL THEORY

A. An Analysis of Physical Concepts

In order to uncover the semantic side of physical theories it is convenient to analyze their ultimate components, i.e., their undefined (primitive) concepts. Mathematically, these are either sets or maps—e.g., a set of bodies, and the electric charge. Among maps, physical magnitudes stand out. Every physical magnitude is a function on a set of individuals every one of which represents a physical system. (This holds for single-sorted theories like mechanics; for many-sorted or pluralistic theories like electrodynamics, the basic set is a set of n-tuples of concepts, every one of which represents a physical system of a kind.) Thus every electric charge value is the charge *of* a body of a certain kind. The usual notations, e.g., 'Q_σ,' for the charge of the σth system, suggest that Q is a function on a set Σ of systems σ. Similarly the force $F_{\sigma\sigma'}$ exerted by a body σ on another individual σ' of the same kind Σ, is a function on $\Sigma\times\Sigma$. On the other hand, the refractive index of light beams of a certain wavelength in a given medium is given ("defined") on the set $L\times M$ of pairs light beam medium. Every magnitude is a family indexed by some set or, more generally, by the Cartesian product $\Sigma_1\times\Sigma_2\times\cdots\Sigma_n$ of n classes of physical systems. (We are here ignoring the other eventual arguments of the functions concerned, such as position and time, because we are focusing on the referents of our concepts.) The sets Σ_i on which physical magnitudes are given are or represent collections of individuals, i.e., objects that have no members (in the set-theoretical sense) although they may have parts (in the mereological sense). For example, the reference class of classical electrodynamics is the set of triples: charged-body σ-electromagnetic field φ-physical reference frame k.

Now an axiomatized physical theory complying with the realistic doctrine of meaning outlined in Sec. IC will have to characterize the mathematical status and the physical meaning of the basic (undefined) symbols concerned, and it will have to interrelate them. Thus, if "electric charge" is one of our basic concepts and if we wish to elucidate it in the context of elementary electrostatics, then we must do something like this. We start by borrowing from mathematics the real line R and the concepts of real function and integral. We then introduce, as if it were out of the blue (i.e., forgetting all about the history of the subject), four basic sets—call them M, Σ, B, and $\{\varphi\}$—a basic family of functions $\{D\}$, and a basic function Q. Finally we lay down the following conditions on Q:

FA Q is a real valued function on Σ.

PA In any [region of space] $V\subset M$, for every [body] $\sigma\in\Sigma$, every [field] $\varphi\in\{\varphi\}$, every [body image] $b\in B$, and every [field image] $D\in\{D\}$: if $b\subseteq V$ and if ∂V is the boundary of V and if n is the outer normal of V, then: if

$$b\mathrel{\widehat{=}}\sigma \quad\text{and}\quad D\mathrel{\widehat{=}}\varphi,$$

then

$$Q(\sigma)=q=(1/4\pi)\int_{\partial V} d^2x(D\cdot n).$$

SA For any $\sigma\in\Sigma$, $Q(\sigma)$ represents the total electric charge of σ.

FA is of course a formal or mathematical axiom while *PA* is a physical axiom (Gauss' law in integral form) and *SA* is a semantic assumption. The three together determine the form and content of Q, while further axioms will take care of the remaining primitive concepts, in particular of D. But even completing the axiom system in such a way that every primitive be characterized both mathematically and physically, the whole thing must be embedded in a wider context if it is to make any sense. This wider context will include, in particular, all the presuppositions specifying the structure and content of the generic primitives M and $\widehat{=}$, i.e., manifold geometry and physical geometry for M, and semantics for $\widehat{=}$. In short, a fuller specification of the structure and content of the theory requires exhibiting or at least mentioning its *background*, which is a motley collection of theories.[33]

Even so, the meaning of 'Q' will not be *fully* specified by the above axioms. Indeed, so far *SA* is only slightly more informative than a conventional designation rule (see Sec. IA). Psychological consequence: to the novice the key symbol 'Q' remains obscure if the expression 'electric charge' occurring in *SA* fails to evoke in his brain a previously acquired notion or a past experience. But it will evoke a more or less definite idea in the experienced physicist, who has met Q before in a number of formulas and possibly also in the description of certain experiments. The mere description of charge measuring procedures, though, will be just as un-

[33] See the author's "The Structure and Content of a Physical Theory" in *Delaware Seminar in the Foundations of Physics*, M. Bunge, Ed. (Springer-Verlag, Berlin, 1967).

illuminating to the novice: such descriptions will make sense only in the light of a set of physical theories involving at the very least Coulomb's law in the form "$V=q/r$" (entailed by the PA above in the case of a point charge). That is, the several ways in which electric charges can be measured do not determine the meaning of 'Q.' If they sometimes do seem to do so it is only because such procedures are embedded in a theoretical context including, among other things, a theory specifying the form and content of Q if only in outline.

In any case the basic and specific concepts of a scientific theory are not defined but are introduced as primitives in it. Of course 'primitive' does not mean "unanalyzed," "irrational" or "obscure": the basic concepts are analyzed both formally and semantically within the theory itself, i.e., by way of postulates of the kinds FA, PA, and SA. How fine is this analysis? The formal analysis can be as complete as one cares to, for it is done in terms of concepts already elucidated in the mathematical theories presupposed by the given scientific theory: one can always refer to the pertinent branches of mathematics lying in the formal background of the theory. On the other hand the semantical analysis of the specific primitives is bound to remain incomplete. Not because every primitive, in order to acquire a factual content, should point to some item of ordinary experience—which is not a well analyzed item anyhow—but because the meaning of a theoretical symbol is specified both within the theory concerned and by all the other theories in which it occurs if only secondarily. For example, the meaning of 'Q' is specified not only by electrostatics but also by dynamics, which offers the proper setting for the elementary law "$V=q/r$" derived from Gauss' law.

In conclusion, the basic concepts of an axiomatized scientific theory are introduced formally by means of axioms of three kinds: formal, physical, and semantical. These initial hypotheses jointly determine in outline the connotation or intension of the various basic concepts. A fuller specification of the intension requires working out the theory and relating it to other fields of research. The intended denotation or extension is sketched in like manner. On the other hand, the actual extension or "domain of validity" of a given theoretical concept must be explored by experimental science. The whole actual extension remains unknown except when empty: by empirical means we can only spot a limited number of individuals satisfying the theory and therefore making up the known subset of the actual extension of the theory. Needless to say, by other means no individuals whatever will be identified.

We are now in a better position to understand the differences between the physical and the mathematical interpretation of an axiomatic framework.

B. Models in Mathematics and in Physics

As is well known, there are two kinds of mathematical axiom systems: the purely syntactical or abstract ones and the semantical or interpreted ones. While the former are single-level structures, the latter are usually formulated by using two different languages: one is the object language of the basic abstract theory, the other is a metalanguage in which something is said concerning what the basic symbols of the object language are. Thus if we write '$x\in R$' without specifying what 'R' stands for, we make a purely syntactical statement: we just say that x (nondescript) is in R (nondescript). But if we write down 'x is a real number,' then we assign 'R' a definite interpretation and we do it by talking about x; i.e., we make a metalinguistic statement. Clearly, physics has no use for purely syntactic systems: in our science we must know what we are talking about even if what we are saying is false. Let us then cast a glance at interpreted mathematical theories—the tools of the theoretical physicist.

The metalanguage used in interpreting an abstract mathematical theory in mathematical terms is, of course, yet another mathematical language. Equivalently: when subjected to interpretation, an abstract theory is interpreted within mathematics. More precisely, the foundation of an abstract theory T is a set of conditions (axioms) on an n-tuple

$$B(T)=\langle S_0, S_1, \cdots, S_{n-1}\rangle,$$

where S_0 is a nondescript or abstract set and the S_i for $i>0$ are predicates of certain degrees (monadic, two-place, etc.). These basic symbols are characterized only by the axioms of T. By contrast, any of the corresponding *mathematical models* T_m of T is about a specific n-tuple

$$B(T_m)=\langle P_0, P_1, \cdots, P_{n-1}\rangle$$

such that (a) P_0 is a fixed ("concrete") set, e.g., the real line, and the P_i for $i>0$ are specific predicates or functors with the same structure as the corresponding S_i (e.g., if S_2 is a binary relation so is P_2), and (b) every axiom (and consequently every formula) of the original abstract formalism T is satisfied when every S_i is specified or interpreted to be the corresponding P_i. Thus while the truth concept is pointless with regard to the abstract framework T, a mathematical model (the mth model) of T is true under the given interpretation. Every interpretation consists of a mapping of the set $S=\{S_0, S_1, \cdots, S_{n-1}\}$ of symbols characterized only by T, into a set $P=\{P_0, P_1, \cdots, P_{n-1}\}$ of concepts known outside T. Such an assignment of fully meaningful concepts to the basic symbols of an abstract theory, i.e.,

$$\text{Int}: S\rightarrow P,$$

is in mathematics an *internal* affair, in the sense that both the domain S and the range P of Int are concepts.

By contrast, a *physical model* or interpretation of a mathematical framework is a mapping of the set S of basic symbols of a theory into a set P of physical objects—things, properties of things, or relations among things. In short, while the domain of Int is conceptual, its range is extraconceptual: in physics the map Int is with one foot in the mind and the other in the external world. Thus in the simplest case of a single-sorted (monistic) physical theory, P_0 will be a set of things, e.g., fluids, while the P_i for $i>0$ will be either properties of those concrete individuals or relations among them—hence their conceptual representatives will be functions on S_0 (in general on S_0^p, where p is the number of places of the predicate concerned). In the case of a many-sorted (pluralistic) physical theory, P_0 will be analyzable into pairs, triples, or in general n-tuples of physical things.

In any case, the peculiarity of a nonmathematical interpretation is then that it is partly an *external* affair in the sense that it is made by reference to things supposed, rightly or wrongly, to be out there. Thus while the set B in Sec. IIA was a set all right and therefore a concept characterizable in mathematics (set theory and measure theory), the set Σ which B was supposed to deputize is the collection of all actual and possible bodies. In short, Int: $B \rightarrow \Sigma$. The reference relation $\hat{=}$ occurring in our SA's—e.g., in the statement "$b \hat{=} \sigma$"—is a subrelation of Int and it relates a theoretical object to a physical object. (When the physical object is not observable, it is often called a theoretical entity—which is of course a contradiction in terms.) However, the physical interpretations are not wholly external: they are bridges between the mind and the world. Thus the range of most magnitudes is interpreted as an interval of the real line. The point, though, is that physical interpretation is partly external just because physical theories are supposed to concern extratheoretical entities. This being so, current model theory,[34] restricted as it is to mathematics, is necessary but insufficient to handle physical axiomatics.

In any case the display of a formalism is necessary but insufficient to have a nonabstract (interpreted) theory in mathematics and *a fortiori* in physics. In addition to all the formal axioms needed to characterize mathematically every primitive, and to all the physical statements that link the given primitive to other specific concepts, we need one semantical axiom per primitive. Any theory complying with this requirement will be said to be *primitive complete*. Unless this requirement is fulfilled, no definite interpretation can be secured, for one and the same set of formal and physical axioms may be shared by different theories, that is, by theories concerning different sets of physical entities and/or different physical properties. Thus one and the same scalar field theory, with a given wave equation at the center of it, can be interpreted either as describing certain features of a fluid, or as describing a nonmaterial field. Or take a far simpler example. For most materials, the electric resistance increases with the temperature. Therefore on the set Σ of materials of this kind, the relation $<$ may be interpreted either as "less resistive than" or as "at a lower temperature than." In other words, the content of the relational structure $B(T)=\langle D, <\rangle$ is not at all uniquely determined by fixing the domain D of individuals, i.e., by adding the interpretation assumption "Int $(D)=\Sigma$." We need, in addition, an interpretation assumption for $<$, e.g., "$\mathrm{Int}_1(<)=$ less resistive than" or "$\mathrm{Int}_2(<)=$ at a lower temperature than." A mere mention of "the intended interpretation" of a set of symbols is much too ambiguous.

Having outlined certain traits of physical axiomatics, let us see how far it can take us.

C. Scope of Physical Axiomatics

An axiomatized physical theory should satisfy the basic metamathematical requirements usually imposed on mathematical theories,[35] notably consistency and independence, both at the level of concepts (mutual independence of the primitive concepts) and at the propositional level (mutual independence of the axioms). The fact that consistency is hard to prove does not render it the less desirable: indeed, consistency is the supreme desideratum concerning the organization of a body of knowledge. In addition to internal consistency, what may be called external consistency[36] should be satisfied: any given physical axiom system should tally with other (not with all) accepted theories, chiefly with its own presuppositions. If on top of all this the theory is also factually true to a reasonable extent, so much the better. But even a false physical theory is a physical theory and, if bold and deep, it may be heuristically valuable; on the other hand a shallow phenomenological theory, if false, has no value whatsoever.[37]

What about completeness? Of the various senses of 'completeness' the following two are of interest in this connection: primitive completeness and deductive completeness. Let us start with the former, which was quickly characterized in Sec. IIB. Suppose a certain constant C—which may well stand for the velocity of light in vacuum—is among the primitives of a certain theory—which may well be special relativity. Clearly, it is not the business of the theory to fix the numerical value of C: this is the concern of experimental physics. Consequently the axiom(s) specifying the nature of C should state only that its value is a real number, its dimension that of a velocity, and its referent the speed of propagation of an electromagnetic signal in (flat) free space. Generally speaking, we want our theories to be as nearly primitive complete as is consistent with the function of experiment. Exhaustive p-completeness would end up in Eddingtonian subjectivism.

Something similar holds for deductive completeness, or the power to yield every formula in a given field.

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[34] See A. Robinson, *Introduction to Model Theory and to the Metamathematics of Algebra* (North-Holland Publ. Co., Amsterdam, 1963).

[35] See R. R. Stoll, *Set Theory and Logic* (W. H. Freeman and Co., San Francisco, Calif., 1963).

[36] See Ref. 31, Chap. 8.

[37] See Ref. 16, Ref. 36, and the author's "The Maturation of Science," in *Problems in the Philosophy of Science*, I. Lakatos and A. Musgrave, Eds. (North-Holland Publ. Co., Amsterdam, 1967).

Since the work of K. Gödel we know that no consistent theory involving a modicum of number theory can be complete; consequently no consistent physical theory using analysis can be complete, notwithstanding the unproven claim that quantum mechanics is a complete theory. Yet we certainly would like every one of our theories to include all the standard theorems and particularly the essential formulas of the field it covers. But even if Gödel's theorem did not hold, we should not want a physical theory such that every formula must be either provable or refutable within the theory. Indeed, we want to be able to introduce new special assumptions, such as constraints and initial conditions, as well as special hypotheses—e.g., that σ is a point charge—which are neither axioms nor theorems of the theory. Equivalently: we wish to be left the freedom to add nonfundamental formulas to the theory without thereby falling into contradictions. But this is just a way of saying that we desire deductive *in*completeness as far as the nonfundamental formulas are concerned.

In short, our theories should be both p-complete and d-complete—but not quite so. In other words, we should axiomatize the *core* of a physical theory, i.e., the nucleus made up of its central hypotheses, avoiding specifications that would render it too narrow, or make it usurp the functions of experiment. This desirable amount of incompleteness or openness to fresh data and fresh assumptions should reassure those who mistrust axiomatics thinking that it brings perfection and therefore the end of research. This should be no source of worry, not only because we want to axiomatize only the core of every theory but also because of the following reasons. First, every physical statement, even if mathematically unobjectionable, is at best partially true concerning facts.[38] Second, the best available organization of a field of factual knowledge is not the best possible one, if only because the formal (logical and mathematical) tools are being sharpened unceasingly. Third, even assuming that the foundations of a theory could be laid down once and for all—a strange claim often made in connection with quantum mechanics—one still could work them out for all eternity, by adding special assumptions and deriving the corresponding new theorems. In short axiomatization does not lead to stagnation. Quite the contrary, it facilitates the growth and maturation of science.[39]

III. AN APPLICATION: RELATIVISTIC KINEMATICS

A. Background and Basic Concepts

Let us apply the preceding ideas to the logical organization of a mathematically simple but semantically tricky theory: special relativistic kinematics or SRK for short.[40] This theory is the basis of relativistic physics, for it does not specify the nature of the entities on which it focuses its attention. Historically and logically SRK presupposes Maxwell's electromagnetic theory for the vacuum and consequently it shares the latter's background. This consists of the following theories: (a) ordinary logic (predicate calculus with identity), a fragment of semantics (the theory concerning '$\hat{=}$'), algebra, topology, analysis, manifold geometry, and whatever else these theories presuppose in turn—in particular naive set theory; (b) a theory of local or frame-dependent time (the concern of which is to elucidate "T"),[41] physical geometry (which elucidates the concept of space structure relative to a frame and the concept of physical reference frame), and a general systems theory—which elucidates the concepts of juxtaposition or physical addition of physical systems, and of being the part of a whole.[42] Maxwell's theory supplies three primitives of SRK: the concept S of electromagnetic signal (or source-independent radiation field), the concept C of velocity of an $s\in S$, and the electromagnetic concept Λ of inertial (Lorentz) frame, defined in Maxwell's theory[43] as that physical frame relative to which Maxwell's equations hold. This third concept occurs in the formulations of the metastatement known as the principle of relativity—e.g., "The basic laws of physics ought to hold relative to any inertial reference frame." This is not a law of nature but a metalaw statement[44] and should therefore not occur among the axioms of SRK although the theory should of course comply with it: it is a heuristic or theory-construction principle, not a constitutive one. This is not easily recognizable before axiomatizing SRK.

We choose the following septuple as the *primitive base* (set of primitive concepts) for SRK:

$$B(\mathrm{SRK})=\langle \Sigma, S, \Lambda, E^3, T, \{X\}, C\rangle.$$

Σ will be interpreted as the set of physical systems of any kind—material, field-like, or quantum mechanical. S will be interpreted as the set of electromagnetic signals; Λ as the set of inertial (Lorentz) frames; E^3 as the Euclidean three-space; T as the range of the time function studied in chronology[41]; an element of the family $\{X\}$ as the instantaneous position of a $\sigma\in\Sigma$ relative to a frame $\lambda\in\Lambda$; and $C(s,\lambda)$ as the speed of an electromagnetic signal in vacuum, relative to a $\lambda\in\Lambda$.

On the basis of the preceding primitives and of certain formal concepts borrowed from logic and mathematics we build the following derived concepts:

Df. 1. $E^{3+1}\overset{\mathrm{df}}{=}E^3\times T$ (not to be mistaken for the space–time of events, which is pseudoeuclidean).

Df. 2.

$$X^0\overset{\mathrm{df}}{=}ct \text{ (cotime)}.$$

Df. 3. If $\sigma\in\Sigma$, $\lambda\in\Lambda$, and $t\in T$, then:

$$V(\sigma,\lambda,t)\overset{\mathrm{df}}{=}dX(\sigma,\lambda,t)/dt.$$

[38] For an axiomatic theory concerning partial truth, see the author's *The Myth of Simplicity* (Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963).

[39] See the paper cited in Ref. 37.

[40] For a similar system and more details see Chap. 3, Sec. 2 of the book cited in Ref. 29.

[41] For a relational theory of local time, see Chap. 2, Sec. 3 of the book cited in Ref. 29.

[42] M. Bunge, Ref. 29, Ch. 2, Sec. 5.

[43] M. Bunge, Ref. 29, Ch. 3, Sec. 1.

[44] M. Bunge, Am. J. Phys. **29**, 518 (1961).

Df. 4. If $\sigma \in \Sigma$, $\lambda \in \Lambda$, and $t \in T$, then: σ is in uniform rectilinear motion w.r.t. $\lambda \overset{\text{df}}{=} V(\sigma, \lambda, t) = \text{const}$.

B. The Basic Statements

The above basic and defined concepts are tied up in the following axioms, every one of which focuses on one primitive. Its formal, physical or semantical status is indicated by *FA*, *PA*, and *SA*, respectively (see Sec. IIA).

A1: *Systems*

(a) $\Sigma \neq \varnothing$. [*FA*]
(b) Every $\sigma \in \Sigma$ represents a physical system. [*SA*]

A2: *Signals*

(a) $S \neq \varnothing \wedge S \subset \Sigma$. [*FA*]
(b) Every $s \in S$ represents an electromagnetic signal. [*SA*]

A3: *Frames*

(a) $\Lambda \neq \varnothing \wedge \Lambda \subset \Sigma - S$. [*FA*]
(b) Every $\lambda \in \Lambda$ represents an inertial [Lorentz] reference frame. [*SA*]
(c) For every $\lambda \in \Lambda$ there is a basis $e = \langle e_0, e_1, e_2, e_3 \rangle$ in E^{3+1} such that $e \widehat{=} \lambda$. [*SA*]

A4: *Local space*

(a) E^3 is a tridimensional Euclidean space endowed with inner product. [*FA*]
(b) E^3 represents ordinary space relative to [metaphorically: "as seen from"] any given $\lambda \in \Lambda$. [*SA*]

A5: *Local time*

(a) T is an interval of the real line. [*FA*]
(b) T is the range of the time function in the local time theory. [Equivalently: a $t \in T$ represents an instant of λ-time.] [*SA*]
(c) For every λ, $\lambda' \in \Lambda$, the associated cotimes X^0 and $X^{0'}$ are such that $\partial X^0 / \partial X^{0'}$ exists and is positive. [Equivalently: no over-all time reversal.] [*PA*]

A6: *Localization*

(a) $\{X\}$ is a nonempty family of functions. [*FA*]
(b) Every $X \in \{X\}$ is a function from $\Sigma \times \Lambda \times T$ to R^3. [*FA*]
(c) $X(\sigma, \lambda, t)$ represents the position of a point of the system σ, referred to the frame λ, at the instant t relative to [metaphorically: "measured by"] λ. [*SA*]
(d) For every point event there exists a sextuple $\langle \sigma, s, \lambda, X^0, X^1, X^2, X^3 \rangle \widehat{=}$ event. [*SA*]

A7: *Constancy of light velocity*

(a) C is a real valued function on $S \times \Lambda$. [*FA*]
(b) Every $s \in S$ propagates in vacuum, relative to any $\lambda \in \Lambda$, with uniform rectilinear motion at the speed c—i.e., $C(s, \lambda) = c$. [*PA*]

C. Comments

The preceding axiom system is *p*-complete and *d*-complete in the weak sense described in Sec. IIC. Indeed, it characterizes all the primitives both formally and semantically—provided the background of the theory is recalled as was done in Sec. IIIA—and it entails all the typical formulas of SRK[45]: the relativity of simultaneity (a law not a convention), the Lorentz transformation formulas, the invariance of the line element in the space of events $\Sigma \times S \times \Lambda \times E^{3+1}$ (not in E^{3+1}) and, of course, whatever these formulas entail in turn, particularly the length "contraction" and the time "dilatation." Unlike other axiomatizations—notably those of Carathéodory[9] and Reichenbach[12]—ours is observer free—as it should be, since the gist of covariance is precisely the (numerical) invariance of law statements under certain substitutions of frames, in particular observers. This is not a merit of axiomatics but a peculiarity of the realistic philosophy underlying our particular brand of physical axiomatics (recall Sec. IC).

Our axiomatization shows that SRK is neither about the empty space–time (Minkowski's formalistic construal) nor about material points (mechanistic construal) nor about rods and clocks (operationalist interpretation), much less about a set of intercommunicating and qualified observers (subjectivist interpretation). It shows that SRK is about any ordered triples $\langle \sigma, s, \lambda \rangle$, since Σ, S, and Λ are the reference sets or domains of physical entities of the theory. It also shows that SRK employs two different spaces: the Euclidean space E^{3+1} tied to every single frame, and the over-all (interframe) pseudo-Euclidean space of events $\Sigma \times S \times \Lambda \times E^{3+1}$ whose line element is Lorentz invariant. (The usual identification of E^{3+1} as the space of events or the world is objectionable because there are no events without physical entities. It is correct only if E^{3+1} is constructed out of events, as proposed by Noll.[46]) Consequently the Lorentz transformation formulas are given a physical not just a mathematical interpretation: they are seen to hold for the physical coordinates X not for the points x of E^{3+1}. Whether this particular interpretation of SRK is accepted or not, one thing is clear: such a clarity concerning the referents of a physical theory is gained only through its axiomatization. This possibility of identifying the actual referent of a theory becomes particularly valuable in the case of elementary quantum mechanics, the referent of which

[45] See Ref. 40.
[46] W. Noll, "Space-Time Structures in Classical Mechanics," in the volume cited in Ref. 33.

is still a matter of controversy: some take it to be an observer, others an experimental setup, still others a statistical aggregate of microsystems, or else a potential (Gibbs) ensemble of microsystems, and finally others an individual microsystem. The axiomatization of the theory makes it clear which of these interpretations is allowed by the formalism.[47]

It will be noticed that most of our axioms for SRK are either *FA*'s or *SA*'s: there are only two law statements proper, namely *A5(c)* (no over-all time reversal) and *A7(b)* (constancy of light velocity). The former is never explicitly stated yet it is used in deriving the Lorentz formulas as it fixes the sign of the coefficient L_{00} of the transformation matrix. Hidden assumptions like this one are bound to be unearthed upon axiomatization. As to the relativity principle, as mentioned before it is obeyed by our axiom system but is not included in it because it is a heuristic metastatement. Although the foundations of SRK contain only two *PA*'s, every *FA* becomes a *PA* when translated into physical terms with the help of the accompanying *SA*. Thus *A1* says that Σ is nonempty (*FA*) and that Σ represents the aggregate of physical systems (*SA*)—which is tantamount to the assertion that there are physical systems. And the latter is a physical statement even though it is not a law statement. Something similar holds for the theorems: they are physical statements because the *SA*'s say that the basic symbols have a physical import. Notice finally that the concepts of reference frame, electromagnetic signal, and local time, are used but not analyzed in SRK: their analysis is entrusted to the underlying theories.

This somewhat dry discussion of the foundations of SRK must suffice here for it was not meant to enliven the controversy over the meaning of SRK but to illustrate our objectivistic approach to physical axiomatics.[48]

IV. CONCLUDING REMARKS

While many physical theories are known, comparatively little is known about their precise nature and, particularly, about their content. In this respect physics is hardly more advanced than mathematics in 1900, when modern axiomatics and metamathematics were just being born. In a situation like this, the philosopher is expected to be of some help for he is supposed to be the analyst of ideas *par excellence*. Now scientific ideas are not properly analyzed by examining them out of their context but by bringing out their systemicity—their relation to other ideas—as well as their relation to the things they deal with. And so far, axiomatization is the most effective way of systematizing and therefore elucidating a body of ideas. Granted, physical axiomatics cannot be perfect, if only because it must leave some windows through which new knowledge can enter, and because it consists in the organization of a body of statements many of which are at best partially true (recall Sec. IIC). Also, axiomatization does not replace the creation of original theories. Yet, however limited in scope, axiomatization is a stage in maturation.

Here go some reasons for maintaining that axiomatization does promote the maturation of science:

(a) it discloses many tacit presuppositions (the background of the theory) and many hidden assumptions, putting them therefore under control;

(b) it displays the structure of the theory, facilitating thereby the control of derivations;

(c) it sketches the interpretation of the theory, preventing thereby *ad hoc* (particularly metaphorical) interpretations;

(d) it allows us to check whether we have overdrawn from our fund of assumptions, thereby helping us in spotting invalid deductions and in suggesting how to enrich the set of assumptions;

(e) the key concepts and hypotheses are identified, so that one does not succumb to the temptation of defining and proving everything;

(f) the consequences of possible changes in the foundations are better realized;

(g) strangers—in particular the psychological intruders inflamed with philosophical or pedagogical zeal—are kept out;

(h) the shortcomings of the theory can be better spotted and corrected;

(i) philosophical vagaries are cut out and philosophical analyses are facilitated.

In short, axiomatization enhances clarity and rigor and it also facilitates criticism and therefore growth. Therefore to oppose axiomatics is to foster woolliness and dogmatism.

[47] See Chap. 5 of the book cited in Ref. 29, and the author's "Quanta and Philosophy," in *7e. Congrès interamèricain de philosophie* (Presses de l'Université Laval, Québec, 1967).

[48] For more comments on the nature of relativity, both special and general, see Ref. 29, Chap. 3, Secs. 2 and 3.