chromism is identical to the first-order term of the present theory [Eq. (35)]. Hoffmann's results are contained, as an approximation, in the present theory. For completeness, we point out discussions of the hypochromism problem by McLachlan and Ball,[19] Fowler,[20] and Bullough,[21] whose conclusions are in general agreement with ours.

The theory, therefore, contains nothing which is basically new, but does serve to coordinate previous theories under one "theoretical roof". We feel that, because of the generality, directness, and completeness of the approach, the use of linear response (retarded) Green's functions will point the way to better approximation methods.

The same method should be readily applicable to other optical phenomena, besides absorption, such as rotatory dispersion and molecular crystal reflection.

[19] A. D. McLachlan and M. A. Ball, Mol. Phys. **8**, 581 (1964).
[20] G. N. Fowler, Mol. Phys. **8**, 383 (1964).
[21] R. G. Bullough, J. Chem. Phys. **43**, 1927 (1965).

REVIEWS OF MODERN PHYSICS VOLUME 39, NUMBER 2 APRIL 1967

Electrodynamics of a Semiclassical Free-Electron Gas*

M. G. CALKIN, P. J. NICHOLSON
Dalhousie University, Halifax, Nova Scotia, Canada

This article presents a simplified treatment of the high density, collisionless, free-electron gas, based on the ideas of a wave number and frequency-dependent conductivity and dielectric constant. The formalism is applied to solve a number of problems: the screening of the electrostatic potential of a foreign point charge placed in the electron gas, the rate of energy loss of a charged particle moving through the electron gas, plasma oscillations, the reflection of electromagnetic waves from the electron gas, and ultrasonic attenuation in metals due to the interaction of the sound waves with the conduction electrons. In a final section it is indicated how the methods may be generalized. Explicit expressions for the conductivity of the electron gas are obtained in an appendix.

INTRODUCTION

In recent years the quantum theory of the electron gas has attracted the attention of a large number of investigators, with the result that many features of such a system are now well understood. Although the power and generality of the quantum-mechanical approach cannot be denied, there are times when a simplified analysis is desired, if only to introduce concepts and aid the intuition. Thus motivated, this article presents a semiclassical treatment of a series of problems, which can be classified under the heading: the electrodynamics of an electron gas. Most of the results have been obtained previously, and are available, for example, in the works of Lindhard,[1] Rukhadze and Silin,[2] Pines,[3] Kittel,[4] Ziman,[5] and Pippard.[6] It is hoped, however, that the following treatment will prove a useful introduction, serving to bridge the gap between the old-fashioned and more modern pictures.

Although the ideas can be applied to, and in some cases derive from, the study of a low-density electron gas at high temperatures, this article is primarily concerned with the high-density electron gas at low temperatures. That is, in the absence of any perturbing electric field, the system considered consists of a gas of electrons, moving without collisions through a uniform, smeared out distribution of positive charge, with charge density equal and opposite to that of the electrons. It is further assumed that the electrons obey Fermi statistics, and that the temperature is zero. The average number of electrons in a volume $d^3\mathbf{x}$ with velocities in the range $d^3\mathbf{v}$ is thus $(3n_0/4\pi v_F^3)\,d^3\mathbf{x}d^3\mathbf{v}$ for $v<v_F$ and 0 for $v>v_F$. The Fermi velocity v_F is given by $(\hbar/m)(3\pi^2 n_0)^{1/3}$.

The results obtained are used to describe the behavior of the conduction electrons in a metal. In most cases this application should be regarded as a crude approximation to the true state of affairs. Firstly, the periodicity of the lattice changes the unperturbed electron states, leading, for example, to nonspherical and in some cases multiply-connected Fermi surfaces. The modifications to the theory presented here to allow for these effects are described by Pippard.[7] Secondly, the theory is valid only if the mean Coulomb inter-

* This research was supported by the National Research Council.

[1] J. Lindhard, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **28**, No. 8 (1954).
[2] A. A. Rukhadze and V. P. Silin, Usp. Fiz. Nauk **74**, 223 (1961); **76**, 79 (1962) [English transl.: Soviet Phys.—Uspekhi **4**, 459 (1961); **5**, 37 (1962)].
[3] D. Pines, *The Many-Body Problem* (W. A. Benjamin, Inc., New York, 1961); *Elementary Excitations in Solids* (W. A. Benjamin, Inc., New York, 1963).
[4] C. Kittel, *Quantum Theory of Solids* (John Wiley & Sons, Inc., New York, 1963).
[5] J. M. Ziman, *Electrons and Phonons* (Oxford University Press, Oxford, England, 1960); *Principles of the Theory of Solids* (Cambridge University Press, Cambridge, England, 1964).
[6] A. B. Pippard, Rept. Progr. Phys. **23**, 176 (1960); in *Low Temperature Physics, Les Houches*, 1961, C. DeWitt, B. Dreyfus, and P. G. DeGennes, Eds. (Gordon and Breach, Science Publishers, Inc., New York, 1962).
[7] A. B. Pippard, Ref. 6.

action energy between electrons, e^2/r, where r is some average distance between electrons, is small compared to the kinetic energy of an electron, $\frac{1}{2}mv^2$, where v is some typical velocity. Taking for r the radius of a sphere with volume equal to the volume per electron, $r=(3/4\pi n_0)^{1/3}$, and for v the Fermi velocity v_F, this condition gives $r_s\ll 1$, where $r_s=r/a_0$ is r measured in units of the Bohr radius $a_0=\hbar^2/me^2$. For metals, however, r_s ranges from 1.8 to 5.7. Procedures for interpolating between the high-density ($r_s\ll 1$) and low-density ($r_s\gg 1$) regions are discussed by Hubbard[8] and by Nozieres and Pines.[9]

With these limitations in mind let us now turn our attention to the basic problem discussed here, the response of an electron gas to an electric field.

I. CONDUCTIVITY TENSOR

Consider then the effects caused by the application of an electric field $\mathbf{E}(\mathbf{x}, t)$ to the electron gas. This field generates an electric current $\mathbf{J}_{(e)}(\mathbf{x}, t)$ which, if the field is weak, is assumed to be linearly related to the field. This, however, does not mean that one can simply set $J_{(e)}{}^i(\mathbf{x}, t)=\sigma_j{}^iE^j(\mathbf{x}, t)$, for such relationship is not only linear but local. That is, the electric field at a particular space–time point determines the current density at the same space–time point. This is not true if the mean free path of the electrons is long compared to distances over which the field changes appreciably, or if the mean time between collisions is long compared to times over which the field changes appreciably. Under these conditions an electric field at $(\mathbf{x}', t')$ can generate a current density at different space–time points $(\mathbf{x}, t)$. To allow for such effects, one postulates the following linear but nonlocal relationship between $\mathbf{J}_{(e)}$ and $\mathbf{E}$

$$J_{(e)}{}^i(\mathbf{x}, t)=\int d^3\mathbf{x}'\, dt' K_j{}^i(\mathbf{x}, t; \mathbf{x}', t')E^j(\mathbf{x}', t'). \quad (1)$$

The "kernel," $K_j{}^i(\mathbf{x}, t; \mathbf{x}', t')$, of this integral relation is the current in the "i" direction at the space–time point $(\mathbf{x}, t)$ caused by a unit impulse of electric field in the "j" direction at the space–time point $(\mathbf{x}', t')$.

In general such a nonlocal relationship between the electric field and current density is difficult to handle. If, however, one makes use of translational invariance considerable simplification occurs. Firstly, the kernel $K_j{}^i$ becomes a function only of the differences of the space–time coordinates,

$$K_j{}^i=K_j{}^i(\mathbf{x}-\mathbf{x}'; t-t').$$

Secondly, if one introduces the space–time Fourier transforms of the current density and electric field

$$J_{(e)}{}^i(\mathbf{k}, \omega)=\int d^3\mathbf{x}\, dt J_{(e)}{}^i(\mathbf{x}, t) \exp(-i\mathbf{k}\cdot\mathbf{x}+i\omega t)$$

$$E^j(\mathbf{k}, \omega)=\int d^3\mathbf{x}\, dt E^j(\mathbf{x}, t) \exp(-i\mathbf{k}\cdot\mathbf{x}+i\omega t) \quad (2)$$

one finds that $J_{(e)}{}^i(\mathbf{k}, \omega)$ and $E^j(\mathbf{k}, \omega)$ are related by

$$J_{(e)}{}^i(\mathbf{k}, \omega)=\sigma_j{}^i(\mathbf{k}, \omega)E^j(\mathbf{k}, \omega), \quad (3)$$

where

$$\sigma_j{}^i(\mathbf{k}, \omega)=\int d^3\mathbf{x}\, dt K_j{}^i(\mathbf{x}, t) \exp(-i\mathbf{k}\cdot\mathbf{x}+i\omega t) \quad (4)$$

is the space–time Fourier transform of the kernel $K_j{}^i$. Because of the formal similarity between Eq. (3) and the more elementary relation $J_{(e)}{}^i(\mathbf{x}, t)=\sigma_j{}^iE^j(\mathbf{x}, t)$, one refers to Eq. (3) as "Ohm's law," and calls $\sigma_j{}^i(\mathbf{k}, \omega)$ the conductivity tensor at wave vector $\mathbf{k}$ and angular frequency ω.

Ohm's law can be still further simplified if one makes use of spatial isotropy. The conductivity tensor can be written

$$\sigma_j{}^i(\mathbf{k}, \omega)=\sigma^L(k, \omega)k^ik_j/k^2+\sigma^T(k, \omega)(\delta_j{}^i-k^ik_j/k^2), \quad (5)$$

where σ^L and σ^T are the longitudinal and transverse conductivities, and are functions only of the magnitude of $\mathbf{k}$, k, and the angular frequency. The terminology "longitudinal" and "transverse" is appropriate, since a purely longitudinal electric field $\mathbf{E}^L$, with $\mathbf{k}\times\mathbf{E}^L=0$, generates a longitudinal current density

$$\mathbf{J}_{(e)}{}^L=\sigma^L\mathbf{E}^L, \quad (6)$$

whereas a purely transverse electric field $\mathbf{E}^T$, with $\mathbf{k}\cdot\mathbf{E}=0$, generates a transverse current density

$$\mathbf{J}_{(e)}{}^T=\sigma^T\mathbf{E}^T. \quad (7)$$

For the highly degenerate free electron gas, these conductivities can be calculated explicitly,[10] with the results

$$\sigma^L=\frac{3}{8}\frac{\omega_p{}^2}{v_Fk}\frac{\omega^2}{v_F{}^2k^2}\theta\left(1-\frac{\omega}{v_Fk}\right)-i\frac{3}{4\pi}\frac{\omega_p{}^2}{v_Fk}\frac{\omega}{v_Fk}\times\left[1+\frac{1}{2}\frac{\omega}{v_Fk}\ln\left|\frac{1-\omega/v_Fk}{1+\omega/v_Fk}\right|\right]$$

$$\sigma^T=\frac{3}{16}\frac{\omega_p{}^2}{v_Fk}\left(1-\frac{\omega^2}{v_F{}^2k^2}\right)\theta\left(1-\frac{\omega}{v_Fk}\right)+i\frac{3}{8\pi}\frac{\omega_p{}^2}{v_Fk}\frac{\omega}{v_Fk}\times\left[1+\frac{1}{2}\frac{\omega^2/v_F{}^2k^2-1}{\omega/v_Fk}\ln\left|\frac{1-\omega/v_Fk}{1+\omega/v_Fk}\right|\right]. \quad (8)$$

In these expressions $\omega_p=(4\pi n_0e^2/m)^{1/2}$ is the plasma frequency, and $\theta(x)$ is the unit step function, $\theta(x)=0$ for $x<0$ and $\theta(x)=1$ for $x>0$. For low frequencies

[8] J. Hubbard, Proc. Roy. Soc. (London) **A243**, 336 (1957).
[9] P. Nozieres and D. Pines, Phys. Rev. **111**, 442 (1958).
[10] See the appendix for details.

($\omega/v_Fk \ll 1$) these expressions reduce to

$$\sigma^L = -i\frac{3}{4\pi}\frac{\omega_p^2}{v_Fk}\frac{\omega}{v_Fk}\left[1+i\tfrac{1}{2}\pi\frac{\omega}{v_Fk}+\cdots\right]$$

$$\sigma^T = \frac{3}{16}\frac{\omega_p^2}{v_Fk}\left[1+i\frac{4}{\pi}\frac{\omega}{v_Fk}+\cdots\right], \tag{9}$$

whereas for high frequencies ($\omega/v_Fk \gg 1$) they become

$$\sigma^L = \frac{i}{4\pi}\frac{\omega_p^2}{\omega}\left[1+\frac{3}{5}\frac{v_F^2k^2}{\omega^2}+\cdots\right]$$

$$\sigma^T = \frac{i}{4\pi}\frac{\omega_p^2}{\omega}\left[1+\frac{1}{5}\frac{v_F^2k^2}{\omega^2}+\cdots\right]. \tag{10}$$

II. DIELECTRIC CONSTANT

A complete description of the average electromagnetic behavior of the electron gas is obtained by combining Ohm's law with Maxwell's equations. In Gaussian units, these latter equations for the total electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ read

$$\begin{aligned} i\mathbf{k}\times\mathbf{E} &= i(\omega/c)\mathbf{B},\\ i\mathbf{k}\cdot\mathbf{B} &= 0,\\ i\mathbf{k}\times\mathbf{B} &= (4\pi/c)\mathbf{J}-i(\omega/c)\mathbf{E},\\ i\mathbf{k}\cdot\mathbf{E} &= 4\pi\rho. \end{aligned} \tag{11}$$

The variables $\mathbf{J}$ and ρ denote the total current and charge densities. They are the sum of the current and charge densities $\mathbf{J}_{(f)}$ and $\rho_{(f)}$ due to foreign charges, or charges which are not part of the system consisting of the electron gas and uniform positive charge background, and the current and charge densities $\mathbf{J}_{(e)}$ and $\rho_{(e)}$ due to deviations of the electron gas from equilibrium,[11]

$$\begin{aligned} \mathbf{J} &= \mathbf{J}_{(f)}+\mathbf{J}_{(e)}\\ \rho &= \rho_{(f)}+\rho_{(e)}. \end{aligned} \tag{12}$$

The variables $\mathbf{J}_{(e)}$ and $\rho_{(e)}$ can be eliminated from the field equations by using the equation of continuity to express $\rho_{(e)}$ in terms of $\mathbf{J}_{(e)}$, and then using Ohm's law to express $\mathbf{J}_{(e)}$ in terms of the total electric field $\mathbf{E}$. The two Maxwell equations which relate the fields to the sources become[12]

$$\begin{aligned} (i\mathbf{k}\times\mathbf{B})^i &= (4\pi/c)J_{(f)}{}^i - i(\omega/c)(\epsilon_j{}^iE^j)\\ ik_i(\epsilon_j{}^iE^j) &= 4\pi\rho_{(f)}, \end{aligned} \tag{13}$$

where

$$\epsilon_j{}^i = \delta_j{}^i + (4\pi i\sigma_j{}^i/\omega). \tag{14}$$

The resulting system is formally identical to that used to describe the electromagnetic field in a medium with dielectric constant $\epsilon_j{}^i$. Because of this one calls $\epsilon_j{}^i$ the "dielectric constant" of the electron gas. Note, however, that in the elementary theory $\epsilon_j{}^i$ is a constant. Here $\epsilon_j{}^i$ is a function of $\mathbf{k}$ and ω.

Using spatial isotropy one can write the dielectric constant in the form

$$\epsilon_j{}^i(\mathbf{k},\omega) = \epsilon^L(k,\omega)k^ik_j/k^2 + \epsilon^T(k,\omega)(\delta_j{}^i - k^ik_j/k^2), \tag{15}$$

where

$$\epsilon^{(L,T)} = 1+(4\pi i\sigma^{(L,T)}/\omega) \tag{16}$$

are the longitudinal and transverse dielectric constants. The physical significance of these parameters can be appreciated by splitting Maxwell's equations into two sets, one containing only longitudinal variables, the other only transverse variables. For this purpose one sets

$$\begin{aligned} \mathbf{E} &= \mathbf{E}^L+\mathbf{E}^T,\\ \mathbf{J}_{(f)} &= \mathbf{J}_{(f)}{}^L+\mathbf{J}_{(f)}{}^T, \end{aligned} \tag{17}$$

where $\mathbf{E}^L = \mathbf{k}\mathbf{k}\cdot\mathbf{E}/k^2$ and $\mathbf{J}_{(f)}{}^L = \mathbf{k}\mathbf{k}\cdot\mathbf{J}/k^2$ are the longitudinal parts of $\mathbf{E}$ and $\mathbf{J}_{(f)}$, and $\mathbf{E}^T = \mathbf{k}\times(\mathbf{E}\times\mathbf{k})/k^2$ and $\mathbf{J}_{(f)}{}^T = \mathbf{k}\times(\mathbf{J}_{(f)}\times\mathbf{k})/k^2$ are the transverse parts of $\mathbf{E}$ and $\mathbf{J}_{(f)}$. The longitudinal electric field and current density then satisfy

$$\begin{aligned} i\omega(\epsilon^L\mathbf{E}^L) &= 4\pi\mathbf{J}_{(f)}{}^L\\ i\mathbf{k}\cdot(\epsilon^L\mathbf{E}^L) &= 4\pi\rho_{(f)}. \end{aligned} \tag{18}$$

These equations show that $\epsilon^L\mathbf{E}^L$ is the longitudinal electric field $\mathbf{E}_{(f)}{}^L$ generated by the foreign charges,

$$\mathbf{E}_{(f)}{}^L = \epsilon^L\mathbf{E}^L. \tag{19}$$

The transverse electric field and current density satisfy

$$\begin{aligned} i\mathbf{k}\times\mathbf{E}^T &= i(\omega/c)\mathbf{B}\\ i\mathbf{k}\times\mathbf{B} &= (4\pi/c)\mathbf{J}_{(f)}{}^T - i(\omega/c)(\epsilon^T\mathbf{E}^T) \end{aligned} \tag{20}$$

showing that for transverse fields ϵ^T plays the role of dielectric constant. The magnetic field $\mathbf{B}$ can be eliminated from Eqs. (20) to give the following relation between the total transverse electric field $\mathbf{E}^T$ and the transverse current $\mathbf{J}_{(f)}{}^T$ due to the foreign charges

$$[k^2-\epsilon^T(\omega/c)^2]\mathbf{E}^T = (4\pi i\omega/c^2)\mathbf{J}_{(f)}{}^T. \tag{21}$$

In free space a transverse current $\mathbf{J}_{(f)}{}^T$ generates a transverse electric field $\mathbf{E}_{(f)}{}^T$, which is given by an equation analogous to Eq. (21), the only difference being that ϵ^T is replaced by 1. Thus one obtains the following relation between the transverse electric field $\mathbf{E}_{(f)}{}^T$ generated by the foreign charges and the total

[11] There is also a current and charge density due to the polarizability of the ion cores. This is neglected here.

[12] This method of eliminating the electron current and charge densities from the field equations is not unique. One could equally well write

$$\begin{aligned} i\mathbf{k}\times(\mu^{-1}\mathbf{B}) &= (4\pi/c)\mathbf{J}_{(f)} - i(\omega/c)(\epsilon^L\mathbf{E}),\\ i\mathbf{k}\cdot(\epsilon^L\mathbf{E}) &= 4\pi\rho_{(f)}, \end{aligned}$$

where $\mu^{-1} = 1+(\omega/ck)^2(\epsilon^L-\epsilon^T)$ in place of Eq. (13). In this picture the influence of the electron gas on the fields is described with the aid of a dielectric constant ϵ^L and a permeability μ.

transverse electric field $\mathbf{E}^T$

$$[k^2-(\omega/c)^2]\mathbf{E}_{(f)}{}^T=[k^2-\epsilon^T(\omega/c)^2]\mathbf{E}^T. \qquad (22)$$

The longitudinal and transverse dielectric constants for the free electron gas can be computed using Eq. (16) and the previously stated expressions for the conductivities, Eq. (8). One obtains

$$\epsilon^L=1+3\frac{\omega_p{}^2}{v_F{}^2k^2}\left[1+\frac{1}{2}\frac{\omega}{v_Fk}\ln\left|\frac{1-\omega/v_Fk}{1+\omega/v_Fk}\right|\right]+i\tfrac{3}{2}\pi\frac{\omega_p{}^2}{v_F{}^2k^2}\frac{\omega}{v_Fk}\theta\left(1-\frac{\omega}{v_Fk}\right)$$

$$\epsilon^T=1-\frac{3}{2}\frac{\omega_p{}^2}{v_F{}^2k^2}\left[1+\frac{1}{2}\frac{(\omega/v_Fk)^2-1}{(\omega/v_Fk)}\ln\left|\frac{1-\omega/v_Fk}{1+\omega/v_Fk}\right|\right]+i\tfrac{3}{2}\pi\frac{\omega_p{}^2}{v_F{}^2k^2}\frac{v_Fk}{\omega}\left(1-\frac{\omega^2}{v_F{}^2k^2}\right)\theta\left(1-\frac{\omega}{v_Fk}\right). \qquad (23)$$

For low frequencies ($\omega/v_Fk\ll1$) these expressions reduce to

$$\epsilon^L=1+3(\omega_p{}^2/v_F{}^2k^2)[1+i\tfrac{1}{2}\pi(\omega/v_Fk)+\cdots]$$
$$\epsilon^T=i\tfrac{3}{4}\pi(\omega_p{}^2/v_F{}^2k^2)(v_Fk/\omega), \qquad (24)$$

whereas for high frequencies ($\omega/v_Fk\gg1$) they become

$$\epsilon^L=1-(\omega_p{}^2/\omega^2)[1+\tfrac{3}{5}(v_F{}^2k^2/\omega^2)+\cdots]$$
$$\epsilon^T=1-(\omega_p{}^2/\omega^2)[1+\tfrac{1}{5}(v_F{}^2k^2/\omega^2)+\cdots]. \qquad (25)$$

III. SCREENING

As a first example of the use of these ideas, let us calculate the electrostatic potential produced by a foreign point charge Ze placed in the electron gas. In free space the potential a distance r away from the charge is

$$\phi_{(f)}(r)=Ze/r \qquad (26)$$

with Fourier transform[13]

$$\phi_{(f)}(k)=4\pi Ze/k^2. \qquad (27)$$

If the charge is placed in the electron gas the electrons cluster about the charge, setting up an electric field which tends to cancel that due to the charge. This effect is known as "screening." As a result the potential $\phi_{(f)}(k)$ is reduced to $\phi=\phi_{(f)}/\epsilon^L$. Using the expression for the static dielectric constant $\epsilon^L(k, 0)$ given by Eq. (24) one finds

$$\phi(k)=4\pi Ze/(k^2+3\omega_p{}^2/v_F{}^2). \qquad (28)$$

This is the Fourier transform of the potential[14]

$$\phi(r)=Ze\exp(-k_{FT}r)/r \qquad (29)$$

where $k_{FT}=\sqrt{3}(\omega_p/v_F)$ is the Fermi–Thomas screening wave number. Thus, for the semi-classical free electron gas, the potential of the foreign charge is screened out by the electrons in a distance of the order of $k_{FT}{}^{-1}$. The description of the electron gas in terms of a continuum is valid if there are many electrons in a sphere of radius $k_{FT}{}^{-1}$, or in other words, if $k_{FT}r\ll1$ where r is the radius of a sphere with volume equal to the volume per electron. This is equivalent to the condition $r_s\ll1$ derived in the introduction using a somewhat different approach. As was stated there, this condition is not satisfied by the conduction electrons in metals. In Cu, for example, $r_s=2.67$, $r=1.41$ Å, and $k_{FT}{}^{-1}=0.553$ Å.

The induced electron charge density $\rho_{(e)}(r)$ which causes the potential to be changed from Eq. (26) to Eq. (29) is

$$\rho_{(e)}(r)=-Zek_{FT}{}^3\exp(-k_{FT}r)/k_{FT}r. \qquad (30)$$

It is infinite at the position of the foreign charge. This disagrees with the fact that the rate of annihilation of positrons in metals, which is proportional to the electron density at the position of the positron, is finite.[15]

The results are improved by a quantum-mechanical treatment. This more sophisticated approach shows that the static dielectric constant $\epsilon^L(k, 0)$ is given by Eq. (24) only in the limit of small wave numbers k. For wave numbers comparable to the Fermi wave number $k_F=(3\pi^2n_0)^{1/3}$ the following expression must be used[16]

$$\epsilon(k, 0)=1+(k_{FT}/k)^2g(k/2k_F), \qquad (31)$$

where the function $g(x)$ is given by

$$g(x)=\tfrac{1}{2}+[(x^2-1)/4x]\ln|(1-x)/(1+x)|. \qquad (32)$$

As a result of this modification to the dielectric constant the induced electron charge density at the position of the foreign charge becomes finite. Further, the induced charge density falls off more slowly with distance than is given by Eq. (30), and oscillates, behaving approximately as[17]

$$\rho_{(e)}(r)\simeq-\frac{4}{\pi}Ze\left(\frac{k_{FT}k_F}{k_{FT}{}^2+8k_F{}^2}\right)^2\frac{\cos 2k_Fr}{r^3} \qquad (33)$$

at large distances from the foreign charge.

IV. ENERGY LOSS

The computation of the rate of energy loss of a charged particle passing through various types of media

[13] $\phi_{(f)}(r)$ satisfies the equation $\nabla^2\phi_{(f)}=-4\pi\rho_{(f)}$. Taking the Fourier transform of this equation leads immediately to $\phi_{(f)}(k)$.

[14] To verify, observe that $\phi(r)$ satisfies $(\nabla-k_{FT}{}^2)\phi=-4\pi Ze\delta(\mathbf{x})$. Fourier transforming this equation gives $\phi(k)$.

[15] For theoretical work on positron annihilation in metals see: R. A. Ferrell, Rev. Mod. Phys. **28**, 308 (1956); S. Kahana, Phys. Rev. **117**, 123 (1960); **129**, 1622 (1963); J. P. Carbotte and S. Kahana Phys. Rev. **139**, A213 (1965).

[16] J. Lindhard, Ref. 1.

[17] J. S. Langer and S. H. Vosko, J. Phys. Chem. Solids **12**, 196 (1959).

has been the subject of numerous papers.[18–25] In this section the rate of energy loss is found using the dielectric constant formalism. For simplicity it is assumed that the mass of the particle is large in comparison to the mass of an electron, so that one can, to a first approximation, neglect recoil and assume that the particle moves with a constant velocity $\mathbf{v}$. If the charge of the particle is Ze, the foreign charge density $\rho_{(f)}(\mathbf{x}, t)$ at the space–time point $(\mathbf{x}, t)$ is then

$$\rho_{(f)}(\mathbf{x}, t) = Ze\delta(\mathbf{x}-\mathbf{v}t) \tag{34}$$

with Fourier transform

$$\rho_{(f)}(\mathbf{k}, \omega) = 2\pi Ze\delta(\omega-\mathbf{k}\cdot\mathbf{v}). \tag{35}$$

The Fourier transform of the electrostatic potential due to the foreign charge is

$$\phi_{(f)}(\mathbf{k}, \omega) = (4\pi/k^2)\rho_{(f)}(\mathbf{k}, \omega) = (8\pi^2 Ze/k^2)\delta(\omega-\mathbf{k}\cdot\mathbf{v}) \tag{36}$$

and the Fourier transform of its electric field is

$$\mathbf{E}_{(f)}(\mathbf{k}, \omega) = -i\mathbf{k}\phi_{(f)}(\mathbf{k}, \omega) = -(8\pi^2 iZe\mathbf{k}/k^2)\delta(\omega-\mathbf{k}\cdot\mathbf{v}). \tag{37}$$

In attempting to screen out this field, the electrons generate a field, the Fourier transform of which is

$$\mathbf{E}_{(e)}(\mathbf{k}, \omega) = [(1/\epsilon^L)-1]\mathbf{E}_{(f)}(\mathbf{k}, \omega)$$
$$= -(8\pi^2 iZe\mathbf{k}/k^2)[(1/\epsilon^L)-1]\delta(\omega-\mathbf{k}\cdot\mathbf{v}). \tag{38}$$

Taking the inverse Fourier transform of $\mathbf{E}_{(e)}(\mathbf{k}, \omega)$ one finds

$$\mathbf{E}_{(e)}(\mathbf{x}, t) = -\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{4\pi iZe\mathbf{k}}{k^2}[\epsilon^L(k, \mathbf{k}\cdot\mathbf{v})^{-1}-1] \times \exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{v}t)]. \tag{39}$$

The force which the electron gas exerts on the moving foreign charge is the product of the charge Ze of the particle and the electric field due to the electrons, evaluated at the instantaneous position $\mathbf{v}t$ of the moving charge. Using the facts that Re $(1/\epsilon^L)$ is even in $\mathbf{k}$ whereas Im $(1/\epsilon^L)$ is odd to simplify the resulting integral, one obtains

$$\mathbf{F} = Ze\mathbf{E}_{(e)}(\mathbf{v}t, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{4\pi Z^2e^2\mathbf{k}}{k^2}\,\mathrm{Im}\,[\epsilon^L(k, \mathbf{k}\cdot\mathbf{v})]^{-1}. \tag{40}$$

Thus the rate at which the foreign charge loses energy W is given by[26]

$$-\frac{dW}{dt} = -\mathbf{F}\cdot\mathbf{v} = -\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{4\pi Z^2e^2\mathbf{k}\cdot\mathbf{v}}{k^2} \times \mathrm{Im}\,[\epsilon^L(k, \mathbf{k}\cdot\mathbf{v})]^{-1}. \tag{41}$$

This expression can be written in the form

$$-\frac{dW}{dt} = -\frac{2}{\pi}\frac{Z^2e^2}{v}\int_0^{k_m}\frac{dk}{k}\int_0^{kv}\omega\, d\omega\,\mathrm{Im}\,[\epsilon^L(k, \omega)]^{-1}, \tag{42}$$

where k_m is a maximum cut-off wave number. An appropriate choice for this parameter is discussed presently.

The integrations over ω and k can be performed simply if the velocity of the foreign charge is either much greater than or much less than the Fermi velocity. Consider first the case $v \ll v_F$. In this limit one can use the low-frequency approximation to the dielectric constant, Eq. (24), to obtain

$$\mathrm{Im}\,\frac{1}{\epsilon^L} \simeq -\tfrac{1}{2}\pi\frac{\omega}{kv_F}\frac{k_{FT}^2/k^2}{(1+k_{FT}^2/k^2)^2}. \tag{43}$$

Substituting this expression into Eq. (42) and performing the integrations gives

$$-\frac{dW}{dt} = \frac{4}{3\pi}\frac{Z^2e^4m^2v^2}{\hbar^3}\left[\ln\left(1+\frac{k_m^2}{k_{FT}^2}\right)^{1/2} - \frac{1}{2}\frac{k_m^2/k_{FT}^2}{1+k_m^2/k_{FT}^2}\right]. \tag{44}$$

The wave number k_m is determined by requiring that $\hbar k_m$ be equal to the maximum momentum transferred from the foreign charge to an electron in a collision. The maximum momentum transfer ΔP occurs in a head-on collision, for which $\Delta P = 2m(v-v_{(e)})$, $v_{(e)}$ being the velocity of the electron. Since the maximum value of $v_{(e)}$ is v_F and since $v \ll v_F$, this condition gives $\hbar k_m = 2mv_F$, or $k_m = 2k_F$, where k_F is the Fermi wave number. Substituting this expression for k_m into Eq. (44), one obtains

$$-\frac{dW}{dt} = \frac{4}{3\pi}\frac{Z^2e^4m^2v^2}{\hbar^3}\left[\ln\left(1+\frac{\pi\hbar v_F}{e^2}\right)^{1/2} - \frac{1}{2}\frac{\pi\hbar v_F/e^2}{1+\pi\hbar v_F/e^2}\right] \tag{45}$$

in agreement with Fermi and Teller.[27] In this velocity region the rate of energy loss is proportional to the

[18] N. Bohr, Phil. Mag. **25**, 10 (1913); **30**, 581 (1915); Kgl. Danske Videnskab. Selskab Mat.-Fys. Medd. **18**, No. 8 (1948).
[19] H. A. Bethe, Ann. Physik **5**, 325 (1930); Z. Physik **76**, 293 (1932).
[20] E. Fermi, Phys. Rev. **57**, 485 (1940).
[21] H. A. Kramers, Physica **13**, 401 (1947).
[22] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **24**, No. 19 (1948).
[23] H. Frohlich and L. Pelzer, Proc. Phys. Soc. (London) **A68**, 525 (1955).
[24] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Addison-Wesley Publ. Co., Inc., Reading, Mass., 1960), Chap. 12.
[25] J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, Inc., New York, 1962), Chap. 13.

[26] This formula neglects the energy loss due to the transverse fields; see Refs. 1 and 2.
[27] E. Fermi and E. Teller, Phys. Rev. **72**, 399 (1947).

kinetic energy of the particle, the factor of proportionality being $35.1\times 10^{15}(m/M)Z^2[\ ]$/sec, where M is the mass of the particle and $[\ ]$ is the quantity in square brackets in Eq. (45). For electron densities present in metals, $[\ ]$ ranges between 0.10 and 0.35.

One can use Eq. (45) to derive an expression for the residual resistance of metals due to point charge impurities. Equation (45) gives the rate at which a beam of electrons moving with velocity v past a stationary charge loses energy. If there are N stationary charges per unit volume distributed randomly throughout the beam, the rate of energy loss is just N times that given by Eq. (45). Equating this to $(1/\sigma)J^2=(1/\sigma)n_0^2e^2v^2$, one finds for the dc residual resistivity $(1/\sigma)$

$$\frac{1}{\sigma}=\frac{4}{3\pi}\frac{NZ^2e^2m^2}{n_0^2\hbar^3}\left[\ln\left(1+\frac{\pi\hbar v_F}{e^2}\right)^{1/2}-\frac{1}{2}\frac{\pi\hbar v_F/e^2}{1+\pi\hbar v_F/e^2}\right]. \quad (46)$$

This expression, due originally to Mott,[28] gives the approximate dependence of $(1/\sigma)$ on the concentration of the impurity and valence difference Z of the impurity and solvent atoms. The numerical value is, however, too large. For Cu Eq. (46) gives (resistivity/atomic % impurity) $\sim 1.8\ Z^2\mu\Omega$-cm compared to an observed (resistivity/atomic % impurity) $\sim 0.4\ Z^2\mu\Omega$-cm. Theories which give better agreement with experiment are discussed by Friedel[29] and Blatt.[30]

Consider now the case $v\gg v_F$. In this limit one can use the high-frequency approximation to the dielectric constant, Eq. (25). At first it appears that Im $(1/\epsilon^L)$ vanishes, since ϵ^L is real. This, however, is not the case, since ϵ^L has a zero and $1/\epsilon^L$ a pole at $\omega=\pm\omega_p$. To obtain the proper expression for Im $(1/\epsilon^L)$ write $\epsilon^L=\epsilon_1^L+i\epsilon_2^L$ and consider the limits as ϵ_2^L tends to zero. This gives

$$\operatorname{Im}\frac{1}{\epsilon^L}=-\lim_{\epsilon_2^L\to 0}\frac{\epsilon_2^L}{(\epsilon_1^L)^2+(\epsilon_2^L)^2}=-\pi\delta(\epsilon_1^L) \quad (47)$$

and, using $\epsilon_1^L\simeq 1-\omega_p^2/\omega^2$, one finds

$$\operatorname{Im}(1/\epsilon^L)\simeq -\tfrac{1}{2}\pi\omega_p[\delta(\omega-\omega_p)-\delta(\omega+\omega_p)]. \quad (48)$$

Substituting this expression for Im $(1/\epsilon^L)$ into Eq. (42) and performing the integrations gives

$$-dW/dt=(4\pi n_0Z^2e^4/mv)\ln(vk_m/\omega_p). \quad (49)$$

The cut-off wave number k_m is again determined by setting $\hbar k_m$ equal to the maximum momentum transfer, which, since $v\gg v_F$, is given by $2mv$. One obtains

$$-dW/dt=(4\pi n_0Z^2e^4/mv)\ln(2mv^2/\hbar\omega_p) \quad (50)$$

in agreement with Kramers.[31]

[28] N. F. Mott, Proc. Cambridge Phil. Soc. **32**, 281 (1936); N. F. Mott and H. Jones, *Properties of Metals and Alloys* (Oxford University Press, Oxford, England, 1936), p. 294.
[29] J. Friedel, *Advances in Physics* (Taylor and Francis, Ltd., London, 1954), Vol. 3, p. 446.
[30] F. J. Blatt, Phys. Rev. **108**, 285 (1957).
[31] H. A. Kramers, Ref. 21.

V. PLASMA OSCILLATIONS

The condition for longitudinal oscillations in the density of the electron gas to exist in the absence of foreign fields can be obtained using the relation (15). If $\mathbf{E}_{(f)}^L$ is zero and $\mathbf{E}^L$ nonzero, the longitudinal dielectric constant must vanish. That is, the wave number k and frequency ω of the oscillation are related by[32]

$$\epsilon^L(k,\omega)=0. \quad (51)$$

For the free-electron gas at absolute zero this condition gives

$$1+\frac{k_{FT}^2}{k^2}\left[1+\frac{1}{2}\frac{\omega}{v_Fk}\ln\frac{\omega/v_Fk-1}{\omega/v_Fk+1}\right]=0. \quad (52)$$

At long wavelengths, $k\ll k_{FT}$, this equation reduces to

$$\omega^2\simeq\omega_p^2+(3/5)v_F^2k^2. \quad (53)$$

These oscillations in the density of the electron gas are called "plasma oscillations." According to quantum theory the energy is quantized in units of $\hbar\omega_p$, which in metals ranges between 3.4 and 20 eV. As a result plasma oscillations are not normally excited. They can, however, be excited by shooting an electron beam through a thin film of the metal. In many cases it is found that the energy loss of the electrons occurs in multiples of $\hbar\omega_p$, showing that the electrons lose energy through excitation of plasma oscillations.[33]

At nonzero temperatures the imaginary part of the dielectric constant is positive for all frequencies. Thus, there does not exist any real frequency for which ϵ^L vanishes. Regarded as a function of the complex variable ω, however, ϵ^L has a zero in the lower half-plane, $\omega=\omega_1-i\,|\,\omega_2\,|$. The amplitude of the oscillation, which contains a factor $\exp(-i\omega t)$, then behaves initially in time as $\exp(-|\,\omega_2\,|\,t)\exp(-i\omega_1t)$, and the oscillation is damped, the rate being determined by the imaginary part of the frequency. An approximate expression for the imaginary part of the frequency can be obtained as follows. For low temperatures $KT\ll E_F$ and long wavelengths $k\ll k_{FT}$, one expects the damping to be small, and thus a first approximation to the complex frequency at which ϵ^L vanishes is just ω_p. To obtain a better estimate, expand ϵ^L as a power series in ω about ω_p

$$\epsilon^L(k,\omega)=\epsilon^L(k,\omega_p)+[\partial\epsilon^L(k,\omega)/\partial\omega]_{\omega=\omega_p}(\omega-\omega_p)+\cdots$$
$$\simeq i\operatorname{Im}\epsilon^L(k,\omega_p)+(2/\omega_p)(\omega-\omega_p). \quad (54)$$

The frequency at which ϵ^L vanishes is thus shifted to

$$\omega\simeq\omega_p[1-\tfrac{1}{2}i\operatorname{Im}\epsilon^L(k,\omega_p)] \quad (55)$$

and the rate at which the plasma oscillation is initially damped is given by $\frac{1}{2}\omega_p\operatorname{Im}\epsilon^L(k,\omega_p)$. This problem was first studied by Landau,[34] and the phenomenon is

[32] J. Hubbard, Proc. Phys. Soc. (London) **A68**, 976 (1955).
[33] D. Pines, Rev. Mod. Phys. **28**, 184 (1956).
[34] L. D. Landau, J. Phys. U.S.S.R. **10**, 25 (1946).

known as "Landau damping".[35] For the highly degenerate electron gas one has

$$\frac{\mathrm{Im}\,\omega}{\omega_p} \simeq -\frac{\pi}{4\sqrt{3}}\left(\frac{k_{FT}}{k}\right)^3 \exp\left(-\frac{E_F k_{FT}^2}{3KTk^2}\right) \tag{56}$$

and, as expected, the damping is small for low temperatures and long wavelengths.

According to the semiclassical theory plasma oscillations can exist in the electron gas at absolute zero for arbitrarily large wave numbers. The quantum theory, which one must use for wave numbers comparable to or greater than k_F shows, however, that there exists a wave number k_c above which plasma oscillations can no longer exist.[36] The origin of this cut-off wave number can be seen without detailed calculations. Suppose that an electron with velocity $\mathbf{v}_{(i)}$, $|\mathbf{v}_i| < v_F$, absorbs a quantum of energy $\hbar\omega$ and momentum $\hbar\mathbf{k}$, as a result jumping to a state with velocity $\mathbf{v}_f$, $|\mathbf{v}_f| > v_F$. One says that the quantum creates an "electron–hole pair." Conservation of energy and momentum requires that

$$\omega = \mathbf{v}_i \cdot \mathbf{k} + \hbar k^2/2m. \tag{57}$$

The term $\mathbf{v}_i \cdot \mathbf{k}$ lies in the range $-v_F k < \mathbf{v}_i \cdot \mathbf{k} < v_F k$. Thus the quantum can be absorbed creating an electron–hole pair provided

$$(-v_F k + \hbar k^2/2m)\theta(-v_F k + \hbar k^2/2m) < \omega < v_F k + \hbar k^2/2m. \tag{58}$$

At small wave numbers the frequency $\omega(k)$ of the plasma oscillation is approximately ω_p and the oscillation is stable. As the wave number is increased, however, one reaches a wave number k_c at which

$$\omega(k_c) = v_F k_c + \hbar k_c^2/2m. \tag{59}$$

For wave numbers greater than k_c, the plasma oscillation is unstable and decays creating electron–hole pairs.

VI. SURFACE IMPEDANCE

The reflection of a plane electromagnetic wave by a conducting medium is discussed in most books on electromagnetic theory. In the usual treatment it is assumed that the relation between the current density and electric field is local. However, this assumption is not valid if the mean free path Λ of the electrons is long compared to the depth δ of penetration of the electromagnetic wave. The detailed theory of the reflection process, both in the normal limit $\delta \gg \Lambda$ and in the anomalous limit $\delta < \Lambda$ is given by Reuter and Sondheimer.[37] Some aspects of this problem are discussed in this section.

Consider a plane electromagnetic wave incident normally on a semi-infinite conducting medium which fills the half-space $z > 0$. It is convenient to characterize the medium by a surface impedance Z, defined by

$$Z = (4\pi/c)(E/B), \tag{60}$$

where E and B denote the (complex) amplitudes of the electric and magnetic fields at the surface of the medium. With the aid of this parameter one can completely describe the reflection process. For example, the absorption coefficient $\mathcal{A}$, or ratio of the energy absorbed by the medium per unit area per unit time to the incident energy flux, is given by

$$\mathcal{A} = \frac{4\,\mathrm{Re}\,(cZ/4\pi)}{|1+(cZ/4\pi)|^2} \simeq (c/\pi)\,\mathrm{Re}\,Z \tag{61}$$

and the reflection coefficient $\mathcal{R} = 1 - \mathcal{A}$, or ratio of the reflected energy flux to the incident energy flux, by

$$\mathcal{R} = \left|\frac{1-(cZ/4\pi)}{1+(cZ/4\pi)}\right|^2 \simeq 1 - (c/\pi)\,\mathrm{Re}\,Z. \tag{62}$$

The approximate expressions for $\mathcal{A}$ and $\mathcal{R}$ in Eqs. (61) and (62) hold for the case of a good reflector, for which $|cZ/4\pi| \ll 1$.

Another parameter which is often useful is the complex skin depth δ, defined by

$$\delta = -E/(dE/dz), \tag{63}$$

where E and dE/dz denote the amplitude of the electric field and its normal derivative evaluated at the surface of the medium. The parameter δ serves as a measure of the distance the electric field penetrates the medium. By using Faraday's law to express dE/dz in terms of B, one finds that δ and Z are related by

$$\delta = -(c^2/4\pi i\omega)Z. \tag{64}$$

The problem now reduces to relating Z to the properties of the medium. Because of the boundary at $z=0$ one cannot immediately use the techniques developed in Secs. I and II. The special case in which the electrons undergo specular reflection at the boundary can, however, be treated as follows. The current density in such a medium caused by an electric field $\mathbf{E}$ $(z>0)$ is the same as the current density in the region $z>0$ of an infinite medium caused by an electric field equal to $\mathbf{E}$ for $z>0$ and to the mirror image of $\mathbf{E}$ for $z<0$. Thus, if one solves a problem in an infinite medium and finds that E_x and E_y are even functions of z and E_z is an odd function, the solution for $z>0$ is also the solution to a problem in a semi-infinite medium, the surface $z=0$ of which is a specular reflector of electrons. In particular, the value of the electric field at $z=0$ in

[35] For recent work see T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill Book Co., Inc., New York, 1962), Chap. 7; S. Gartenhaus, *Elements of Plasma Physics* (Holt, Rinehart, and Winston, Inc., New York, 1964), Chap. 6.

[36] R. Ferrell, Bull. Am. Phys. Soc. **2**, 146 (1957); K. Sawada *et al.*, Phys. Rev. **108**, 507 (1957).

[37] G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) **A195**, 336 (1948); see also A. B. Pippard, Ref. 6.

the infinite case is the value of the electric field at the surface of the semi-infinite medium. With this in mind one calculates the electric field in an infinite medium produced by a sheet of current with density $J_{(f)}(z, t) = \mathrm{Re}\, J_{(f)}(z) \exp(-i\omega t)$ flowing in the x direction in the $z=0$ plane. The amplitude $J_{(f)}(z)$ of the current density is

$$J_{(f)}(z) = K_{(f)}\delta(z) \tag{65}$$

with Fourier transform with respect to z

$$J_{(f)}(k) = K_{(f)}. \tag{66}$$

By Eq. (21) the Fourier transform of the amplitude of the electric field produced by $J_{(f)}$ is

$$E(k) = \frac{4\pi i\omega}{c^2}\frac{K_{(f)}}{k^2-\epsilon^T(\omega/c)^2} \tag{67}$$

and thus the electric field is

$$E(z) = \frac{4i\omega K_{(f)}}{c^2}\int_0^\infty dk\,\frac{\cos kz}{k^2-\epsilon^T(\omega/c)^2}. \tag{68}$$

The field is even in z, and at $z=0+$ has the value

$$E(0+) = \frac{4i\omega K_{(f)}}{c^2}\int_0^\infty \frac{dk}{k^2-\epsilon^T(\omega/c)^2}. \tag{69}$$

This field, which is independent of x and y, is the same as is caused at the surface of a semi-infinite medium by a plane electromagnetic wave with its electric vector polarized in the x direction incident normally on the surface. The surface impedance is now obtained by dividing $(4\pi/c)E(0+)$, given by Eq. (69), by the magnetic field

$$B(0+) = -(2\pi/c)K_{(f)} \tag{70}$$

at the surface, as given by Ampere's law. One finds

$$Z = -\frac{8i\omega}{c^2}\int_0^\infty \frac{dk}{k^2-\epsilon^T(\omega/c)^2}. \tag{71}$$

At sufficiently low frequencies the field penetrates a considerable distance into the material and the change in the field over distances comparable to the electron mean free path Λ is small. Further, the period of the field is large compared to the mean time $\tau=\Lambda/v_F$ between collisions for the electrons. The conductivity reduces to the ordinary dc conductivity $\sigma_0=\omega_p^2\Lambda/4\pi v_F$, and the dielectric constant, which is related to the conductivity by Eq. (16), becomes

$$\epsilon^T \simeq 4\pi i\sigma_0/\omega. \tag{72}$$

In this limit the surface impedance, as given by Eq. (71), is

$$Z = (2\pi\omega/\sigma_0 c^2)^{1/2}(1-i) \tag{73}$$

and the complex skin depth is

$$\delta = \tfrac{1}{2}(c^2/2\pi\sigma_0\omega)^{1/2}(1+i). \tag{74}$$

These expressions hold so long as $|\delta| \gg \Lambda$, $\omega\tau \ll 1$, and $4\pi\sigma_0/\omega \gg 1$. If the mean free path of the electron is long, $\Lambda\omega_p/c > 1$, the condition $|\delta| \gg \Lambda$ dominates, and shows that the frequency must satisfy the condition $\omega \ll v_F c^2/\Lambda^3\omega_p^2$. In this region, which is the one commonly discussed, the absorption coefficient is proportional to $\omega^{1/2}$.

At higher frequencies, $\omega \gg v_F c^2/\Lambda^3\omega_p^2$, the ordinary skin depth is small compared to the electron mean free path. It is reasonable to treat the electrons in the material as a free electron gas, for which Eq. (24) gives for the dielectric constant

$$\epsilon^T \simeq (3\pi i/4)(\omega_p^2/v_F k\omega). \tag{75}$$

This approximation is valid if $\omega/v_F k \ll 1$. The wave numbers in Eq. (71) which contribute most strongly to Z are those for which $k^2 \sim \epsilon^T(\omega/c)^2$, or using Eq. (75), $k^3 \sim \omega_p^2\omega/c^2 v_F$. For the dominant wave numbers $\omega/v_F k \sim (\omega c/\omega_p v_F)^{2/3}$ and thus, as long as $\omega \ll (v_F/c)\omega_p$, one can use Eq. (75). It is possible to satisfy both this and the previous restriction on ω, namely $\omega \gg v_F c^2/\Lambda^3\omega_p^2$, if the mean free path of the electrons is long, $\Lambda\omega_p/c \gg 1$. In Cu at low temperatures mean free paths $\Lambda \sim 10^{-4}$ cm are easily obtained. For this value of the mean free path $\Lambda\omega_p/c \simeq 55$, and the restrictions on the frequency are 8.3×10^7 cps $\ll \omega/2\pi \ll 1.4\times 10^{13}$ cps. In this region the surface impedance is

$$Z = (8\pi/3\sqrt{3})(4v_F\omega^2/3\pi c^4\omega_p^2)^{1/3}(1-i\sqrt{3}) \tag{76}$$

and the absorption coefficient is proportional to $\omega^{2/3}$.

VII. ULTRASONIC ATTENUATION

Experiment[38] and theory[39–41] show that the dominant mechanism for the attenuation of ultrasonic waves propagating through pure metals at low temperatures is the interaction of the sound wave with the conduction electrons of the metal. In this section, the attenuation coefficient, or reciprocal of the distance over which the intensity of the wave decreases to $1/e$th its initial value, is calculated on the assumption that the conduction electrons can be treated as a free-electron gas. In particular collisions of the electrons with impurities in the lattice are neglected.

Consider first longitudinal waves. The ions oscillate back and forth along the direction of propagation with velocity $\mathbf{u}_{(i)}(\mathbf{x}, t) = \mathrm{Re}\, u_{(i)}(\mathbf{k}/k) \exp(i\mathbf{k}\cdot\mathbf{x}-i\omega t)$ giving an ion current $\mathbf{J}_{(i)} = -n_0 e\mathbf{u}_{(i)}$. The periodic compression and rarefaction of the ion density leads to a charge density, which generates a longitudinal

[38] H. E. Bömmel, Phys. Rev. **96**, 220 (1954); W. P. Mason and H. E. Bömmel, J. Acoust. Soc. Am. **28**, 930 (1956); R. W. Morse, in *Progress in Cryogenics*, K. Mendelssohn, Ed. (Heywood and Company, Ltd., London, 1959), Vol. 1.

[39] A. B. Pippard, Phil. Mag. **46**, 1104 (1955); Proc. Roy. Soc. (London) **A257**, 165 (1960); see also Ref. 6.

[40] M. H. Cohen, M. J. Harrison, and W. A. Harrison, Phys.Rev. **117**, 937 (1960).

[41] H. Stolz, Phys. Status Solidi **3**, 1153 (1963); **3**, 1493 (1963); **3**, 1957 (1963).

electric field $\mathbf{E}_{(i)}$. Using Eqs. (18) and (19), one finds

$$\mathbf{E}_{(i)}=(4\pi i/\omega)n_0e\mathbf{u}_{(i)}. \tag{77}$$

As far as the electrons are concerned, $\mathbf{E}_{(i)}$ is a foreign electric field, and by Eqs. (6), (16), and (19), it generates an electric current

$$\mathbf{J}_{(e)}=(\omega/4\pi i)(1-1/\epsilon_{\mathrm{k},\omega}{}^L)\mathbf{E}_{(i)}. \tag{78}$$

As a result of this interaction, energy is transferred from the ions to the electrons and the sound wave is attenuated. The attenuation coefficient α^L is obtained by dividing the time-averaged power transferred per unit volume, namely,

$$\tfrac{1}{2}\operatorname{Re}\mathbf{J}_{(e)}\cdot\mathbf{E}_{(i)}{}^*=-(\omega\mid\mathbf{E}_{(i)}\mid^2/8\pi)\operatorname{Im}(1/\epsilon_{\mathrm{k},\omega}{}^L) \tag{79}$$

by the time-averaged energy flux S^L of the sound wave

$$S^L=\tfrac{1}{2}\rho_m v_s{}^L\mid\mathbf{u}_{(i)}\mid^2$$
$$=\tfrac{1}{2}(\rho_m v_s{}^L\omega^2/16\pi^2n_0{}^2e^2)\mid\mathbf{E}_{(i)}\mid^2, \tag{80}$$

where ρ_m is the mass density of the metal and $v_s{}^L=\omega/k$ is the longitudinal sound velocity. This gives

$$\alpha^L=-(4\pi n_0{}^2e^2/\rho_m v_s{}^L\omega)\operatorname{Im}(1/\epsilon_{\mathrm{k},\omega}{}^L). \tag{81}$$

In typical metals and for frequencies of interest one has $v_s{}^L\ll v_F$ and $\omega\ll(v_s{}^L/v_F)\omega_p$. In Cu, for example, $v_s{}^L\simeq4.7\times10^5$ cm/sec, $v_F\simeq1.6\times10^8$ cm/sec, and $\omega_p\simeq 1.6\times10^{16}$/sec. Thus one can use the low-frequency, long-wavelength approximation to the dielectric constant, with

$$\operatorname{Im}(1/\epsilon_{\mathrm{k},\omega}{}^L)\simeq-(\pi/6)(v_F/v_s{}^L)(\omega^2/\omega_p{}^2). \tag{82}$$

Substituting this expression into Eq. (81) then gives the attenuation coefficient for longitudinal sound waves

$$\alpha^L\simeq(\pi/6)(n_0mv_F\omega/\rho_m v_s{}^{L2}). \tag{83}$$

This is the expression for the attenuation coefficient obtained by Pippard in the limit in which the mean free path of the electrons is much longer than the wavelength of the sound wave. In this limit the attenuation coefficient rises linearly with frequency.

Now consider transverse waves. The ions oscillate with velocity $\mathbf{u}_{(i)}(\mathbf{x},t)=\operatorname{Re}\mathbf{u}_{(i)}\exp(i\mathbf{k}\cdot\mathbf{x}-i\omega t)$ where $\mathbf{u}_{(i)}$ is perpendicular to $\mathbf{k}$, $\mathbf{k}\cdot\mathbf{u}_{(i)}=0$, giving an ion current $\mathbf{J}_{(i)}=-n_0e\mathbf{u}_{(i)}$. In this case there is no bunching of the ions and the metal remains charge free. However, the ion current $\mathbf{J}_{(i)}$ generates a time varying magnetic field and by induction an electric field $\mathbf{E}_{(i)}$. Using Eqs. (21) and (22) one finds

$$\mathbf{E}_{(i)}=[4\pi i\omega/(\omega^2-c^2k^2)]n_0e\mathbf{u}_{(i)}. \tag{84}$$

According to Eqs. (7), (16), and (22) this field generates an electron current

$$\mathbf{J}_{(e)}=\frac{\omega^2-c^2k^2}{4\pi i\omega}\left[1-\frac{\omega^2-c^2k^2}{\epsilon^T\omega^2-c^2k^2}\right]\mathbf{E}_{(i)}. \tag{85}$$

Dividing the time-averaged power transferred from the ions to the electrons per unit volume

$$\tfrac{1}{2}\operatorname{Re}\mathbf{J}_{(e)}\cdot\mathbf{E}_{(i)}=-\frac{(\omega^2-c^2k^2)^2\mid\mathbf{E}_{(i)}\mid^2}{8\pi\omega}\operatorname{Im}[\epsilon^T\omega^2-c^2k^2]^{-1} \tag{86}$$

by the time-averaged energy flux S^T of the sound wave

$$S^T=\tfrac{1}{2}\rho_m v_s{}^T\mid\mathbf{u}_{(i)}\mid^2$$
$$=\tfrac{1}{2}(\rho_m v_s{}^T/16\pi^2n_0{}^2e^2\omega^2)(\omega^2-c^2k^2)^2\mid\mathbf{E}_{(i)}\mid^2 \tag{87}$$

then gives the attenuation coefficient α^T for transverse sound waves

$$\alpha^T=-(4\pi n_0{}^2e^2\omega/\rho_m v_s{}^T)\operatorname{Im}[\epsilon^T\omega^2-c^2k^2]^{-1}. \tag{88}$$

As in the case of longitudinal waves one can use the low frequency approximation to the dielectric constant, setting

$$\epsilon^T\simeq(3\pi i/4)(v_s{}^T/v_F)(\omega_p{}^2/\omega^2). \tag{89}$$

One then finds[42]

$$\alpha^T\simeq\frac{4}{3\pi}\frac{n_0mv_F\omega}{\rho_m v_s{}^{T2}}\frac{\omega_0{}^4}{\omega_0{}^4+\omega^4} \tag{90}$$

where $\omega_0=(3\pi v_s{}^{T3}\omega_p{}^2/4v_Fc^2)^{1/2}$. In Cu, $\omega_0/2\pi\simeq1100$ Mc/sec. In most experiments the frequency is low in comparison to ω_0, and the attenuation coefficient is

$$\alpha^T\simeq(4/3\pi)(n_0mv_F\omega/\rho_m v_s{}^{T2}) \tag{91}$$

and rises linearly with frequency. At higher frequencies the attenuation coefficient increases less rapidly with frequency, reaches a maximum at $3^{-1/4}\omega_0$, and then drops, behaving approximately as

$$\alpha^T\simeq12\pi^3(n_0{}^3e^4/\rho_m mv_F)(v_s{}^T/c)^4\omega^{-3} \tag{92}$$

at very high frequencies. This ultrahigh-frequency behavior can be obtained more directly if one observes that at high frequencies the electrons are no longer able to keep up with the ions, and thus the field $\mathbf{E}_{(e)}$ generated by the electrons is small compared to the field $\mathbf{E}_{(i)}$ generated by the ions. Thus, in place of Eq. (85) one has $\mathbf{J}_{(e)}\simeq(\omega/4\pi i)(\epsilon^T-1)\mathbf{E}_{(i)}$. Using this relation one arrives after a short calculation at Eq. (92).

VIII. GENERALIZATIONS

The formalism developed in Secs. I and II can be applied to systems of particles interacting via forces other than Coulomb. Consider a system of particles, of mass M and equilibrium number density N_0, interacting via a potential $\varphi(\mathbf{x})$. That is, the force which a particle located at $\mathbf{x}'$ exerts on a particle located at $\mathbf{x}$ is $-\nabla\varphi(\mathbf{x}-\mathbf{x}')$. If these particles are acted on by a force, the net force acting on a particle located at $(\mathbf{x},t)$ being $\mathbf{F}(\mathbf{x},t)$, a flow of particles occurs, the number of particles crossing a unit area per unit time being $\mathbf{j}(\mathbf{x},t)$. Denoting the Fourier transforms of

[42] H. Stolz, Z. Naturforsch. **16a**, 446 (1961).

$\mathbf{F}(\mathbf{x}, t)$ and $\mathbf{j}(\mathbf{x}, t)$ by $\mathbf{F}(\mathbf{k}, \omega)$ and $\mathbf{j}(\mathbf{k}, \omega)$, one has

$$\mathbf{j}(\mathbf{k}, \omega) = N_0 \mu(\mathbf{k}, \omega)\mathbf{F}(\mathbf{k}, \omega), \tag{93}$$

where $\mu(\mathbf{k}, \omega)$ is the wave vector and frequency-dependent mobility. For a gas of bosons at absolute zero with distribution function $N_0\delta(\mathbf{v})$, the mobility can be calculated using the techniques described in the appendix, with the result

$$\mu(\mathbf{k}, \omega) = i/M\omega. \tag{94}$$

For a gas of fermions at absolute zero the result is somewhat more complicated. It can be obtained formally from the expression for the longitudinal conductivity, Eq. (8), by dividing σ^L by N_0e^2. For high frequencies ($\omega/v_Fk \gg 1$) one again obtains Eq. (94).

The force $\mathbf{F}(\mathbf{x}, t)$ is the sum of the force $\mathbf{F}_{(\mathrm{ex})}(\mathbf{x}, t)$ due to external agents and the force $\mathbf{F}_{(\mathrm{in})}(\mathbf{x}, t)$ which all the other particles exert on a particular particle at $(\mathbf{x}, t)$. This latter force is obtained from a potential $\Phi(\mathbf{x}, t)$

$$\mathbf{F}_{(\mathrm{in})}(\mathbf{x}, t) = -\nabla\Phi(\mathbf{x}, t), \tag{95}$$

where

$$\Phi(\mathbf{x}, t) = \int d^3\mathbf{x}'\varphi(\mathbf{x}-\mathbf{x}')N(\mathbf{x}', t). \tag{96}$$

Taking the Fourier transform of Eq. (96), one finds

$$\Phi(\mathbf{k}, \omega) = \varphi(\mathbf{k})N(\mathbf{k}, \omega), \tag{97}$$

where

$$\varphi(\mathbf{k}) = \int d^3\mathbf{x}\varphi(\mathbf{x}) \exp(-i\mathbf{k}\cdot\mathbf{x}) \tag{98}$$

is the Fourier transform of the interparticle potential. The Fourier transform of $\mathbf{F}_{(\mathrm{in})}$ is then

$$\mathbf{F}_{(\mathrm{in})}(\mathbf{k}, \omega) = -i\mathbf{k}\Phi(\mathbf{k}, \omega) = -i\mathbf{k}\varphi(\mathbf{k})N(\mathbf{k}, \omega). \tag{99}$$

Using Eq. (99) and the equation of continuity, one can eliminate $\mathbf{F}_{(\mathrm{in})}$ from Eq. (93) with the result

$$[1+(iN_0k^2\varphi\mu/\omega)]\mathbf{j} = N_0\mu\mathbf{F}_{(\mathrm{ex})}. \tag{100}$$

The factor in square brackets in this equation plays a role analogous to that played by the dielectric constant. In particular, the condition that longitudinal oscillations in the density of the system occur in the absence of external forces is

$$1+(iN_0k^2\varphi\mu/\omega) = 0. \tag{101}$$

Using Eq. (94) for the mobility one finds that the frequency and wave number of the oscillation are related by[43]

$$\omega/k = (N_0\varphi(\mathbf{k})/M)^{1/2}. \tag{102}$$

As an example of the use of this equation consider a system of particles each having a charge Ze interacting via a Coulomb potential $\varphi(\mathbf{x}) = Z^2e^2/|\mathbf{x}|$. The Fourier transform of $\varphi(\mathbf{x})$ is

$$\varphi(\mathbf{k}) = 4\pi Z^2e^2/k^2 \tag{103}$$

and the $\omega - k$ relation reads

$$\omega/k = (4\pi Z^2e^2N_0/Mk^2)^{1/2} \tag{104}$$

or

$$\omega = \Omega_p, \tag{105}$$

where $\Omega_p = (4\pi N_0Z^2e^2/M)^{1/2}$ is the plasma frequency appropriate to the system. This is just what one could expect on the basis of the previous calculations involving electrons.

As a second example consider the following rather naive model of a metal. Picture a metal as consisting of two gases, an electron gas and an ion gas. If it were not for the electrons the ions would simply interact via Coulomb forces and the ions would undergo plasma oscillations with frequency Ω_p. As was shown in Sec III, however, the electrons act to screen out the interionic potential and as a result the potential is changed to $\varphi(\mathbf{x}) = Z^2e^2 \exp(-k_{FT}|\mathbf{x}|)/|\mathbf{x}|$, where k_{FT} is the Fermi–Thomas screening wave number appropriate to the electrons. The Fourier transform of the interionic potential is now

$$\varphi(\mathbf{k}) = 4\pi Z^2e^2/(k^2+k_{FT}^2) \tag{106}$$

and the $\omega - k$ relation reads

$$\omega/k = \Omega_p/(k^2+k_{FT}^2)^{1/2}. \tag{107}$$

For long wavelengths, $k \ll k_{FT}$, this becomes

$$\omega/k = \Omega_p/k_{FT} = (Zm/3M)^{1/2}v_F. \tag{108}$$

In this limit the phase velocity of the wave is independent of the wave number. The right-hand side of Eq. (108) can be regarded as an approximate expression for the velocity of longitudinal sound waves in a metal.[44] Numerically, this expression is not too bad. For example, for Cu it gives 2.7×10^5 cm/sec, compared to an observed 4.7×10^5 cm/sec.

At short wavelengths Eq. (106) is no longer valid due to quantum effects. As is pointed out in Sec. III the short wavelength dielectric constant $\epsilon^L(k, 0)$ is $1+(k_{FT}/k)^2g(k/2k_F)$, and thus the interionic potential becomes

$$\varphi(\mathbf{k}) = 4\pi Z^2e^2/[k^2+k_{FT}^2g(k/2k_F)]. \tag{109}$$

The expression for the longitudinal sound velocity is then

$$\omega/k = \Omega_p/[k^2+k_{FT}^2g(k/2k_F)]. \tag{110}$$

The function $g(k/2k_F)$ has a logarithmic singularity at $k = 2k_F$. Although the phase velocity ω/k is continuous at $k = 2k_F$, the group velocity $d\omega/dk$ is infinite. As a

[43] A. Vlasov, J. Phys. U.S.S.R. **9**, 25 (1945); N. Bogolubov, *ibid.* **11**, 23 (1947).

[44] D. Bohm and T. Staver, Phys. Rev. **84**, 836 (1951); J. Bardeen and D. Pines, Phys. Rev. **99**, 1140 (1955).

result the $\omega - k$ dispersion curve has a kink at $k = 2k_F$. This is known as the Kohn effect.[45]

Both Eqs. (108) and (110) neglect the fact that ϵ^L and hence φ depends on the frequency. A somewhat better approximation to $\epsilon^L(k, \omega)$ at long wavelengths is $(k_{FT}/k^2)[1+(\pi i/2)(\omega/v_F k)]$, and thus

$$\varphi(\mathbf{k}, \omega) \simeq (4\pi Z^2 e^2/k_{FT}^2)[1-(\pi i/2)(\omega/v_F k)]. \quad (111)$$

The $\omega - k$ dispersion relation now reads

$$k \simeq (\omega/v_s^L) + (\pi i/4)(\omega/v_F), \quad (112)$$

where v_s^L is the longitudinal sound velocity as given by Eq. (108). Thus if one propagates a longitudinal sound wave with amplitude proportional to exp $(ikx - i\omega t)$, the amplitude of the wave decreases exponentially with distance along the direction of propagation. The attenuation coefficient is given by

$$\alpha^L = 2 \operatorname{Im} k = (\pi/2)(\omega/v_F). \quad (113)$$

One can check that this expression for the attenuation coefficient is consistent with the previously derived expression, Eq. (83), by substituting the expression (108) for the velocity of sound into Eq. (83). The result is Eq. (113).

ACKNOWLEDGMENT

The authors would like to thank Dr. M. H. Jericho for helpful discussions.

APPENDIX: CALCULATION OF THE CONDUCTIVITY TENSOR

The calculations in this paper which refer specifically to the degenerate electron gas are based on the expressions for the longitudinal and transverse conductivities which were stated without proof in Eq. (8). The purpose of this appendix is to provide a derivation of these expressions. There are a great many ways in which this can be done. Perhaps the most common involves solving the Boltzmann equation for the distribution function in the presence of an electric field. An account of this method can be found in books on plasma physics.[46] A somewhat different approach, which has the virtue of being more closely related to the work of Sec. I, is the following.[47]

First consider a gas of particles of charge e and mass m in equilibrium and in the absence of any electric field. Assume that one can neglect collisions. The number of particles at time t' in a volume $d^3\mathbf{x}'$ centered on the point $\mathbf{x}'$ with velocities in the range $d^3\mathbf{v} = v^2 dv d\Omega$, where $d\Omega$ is an element of solid angle is

$$n_0 f_0(\mathbf{v}) d^3\mathbf{x}' v^2 dv d\Omega. \quad (A1)$$

n_0 is the number of particles per unit volume, and $f_0(\mathbf{v})$ is the equilibrium velocity distribution function, normalized so that $\int d^3\mathbf{v} f_0(\mathbf{v}) = 1$. As time passes, those particles which at time t' were in $d^3\mathbf{x}'$ move out of the volume, and since by assumption there are no collisions, they move with their original velocities. At time t those particles with velocities $\mathbf{v} = (\mathbf{x}-\mathbf{x}')/(t-t') = \mathbf{R}/T$ pass the point $\mathbf{x}$. In the interval $t \to t+dt$ the number passing the point $\mathbf{x}$ and traveling in the solid angle $d\Omega$ is

$$\begin{aligned}\delta n &= n_0 f_0(\mathbf{R}/T) d^3\mathbf{x}'(R/T)^2(R dt/T^2) d\Omega \\ &= n_0(R^3/T^4) f_0(\mathbf{R}/T) d^3\mathbf{x}' dt d\Omega. \quad (A2)\end{aligned}$$

Thus the electric current density at the point $\mathbf{x}$ at the time t due to those particles which at time t' were in $d^3\mathbf{x}'$ is

$$\begin{aligned}\delta\mathbf{J}_{\text{in}}(\mathbf{x}, t) &= e\delta n(1/R^2 d\Omega)(1/dt)(\mathbf{R}/R) \\ &= e n_0(\mathbf{R}/T^4) f_0(\mathbf{R}/T) d^3\mathbf{x}'. \quad (A3)\end{aligned}$$

However, one knows that in equilibrium the current at any point must vanish. Thus, the current density at $\mathbf{x}$ at time t due to those particles which at t' were *not* in $d^3\mathbf{x}'$ is

$$\delta\mathbf{J}_{\text{out}}(\mathbf{x}, t) = -e n_0(\mathbf{R}/T^4) f_0(\mathbf{R}/T) d^3\mathbf{x}'. \quad (A4)$$

Now consider the effect of applying an electric field $\mathbf{E}(\mathbf{x}', t')$ for an interval $t' \to t'+dt'$ to those particles in $d^3\mathbf{x}'$. The effect is to change the velocity of all the particles in $d^3\mathbf{x}'$ by an amount $d\mathbf{v} = (e/m)\mathbf{E}(\mathbf{x}', t')dt'$, and thus to change the velocity distribution function for those particles to $f_0[\mathbf{v} - (e/m)\mathbf{E}(\mathbf{x}', t')dt']$. The current density at $\mathbf{x}$ at time t due to those particles which at t' were in $d^3\mathbf{x}'$ is thus changed to

$$\begin{aligned}&\delta\mathbf{J}_{\text{in}}(\mathbf{x}, t) \\ &= e n_0(\mathbf{R}/T^4) f_0[(\mathbf{R}/T) - (e/m)\mathbf{E}(\mathbf{x}', t')dt'] d^3\mathbf{x}'. \quad (A5)\end{aligned}$$

Those particles which were not in $d^3\mathbf{x}'$ at time t' are unaffected. Thus the total electric current density at $\mathbf{x}$ at time t caused by an electric field $\mathbf{E}(\mathbf{x}', t')$ acting on the particles in $d^3\mathbf{x}'$ and lasting a time interval $t' \to t'+dt'$ is

$$\begin{aligned}&\delta\mathbf{J}(\mathbf{x}, t) \\ &= e n_0(\mathbf{R}/T^4) \\ &\quad \times \{ f_0[(\mathbf{R}/T) - (e/m)\mathbf{E}(\mathbf{x}', t')dt'] - f_0(\mathbf{R}/T) \} d^3\mathbf{x}' \\ &= -(n_0 e^2/mT^3)\mathbf{v}\mathbf{E}(\mathbf{x}', t') \cdot \nabla_v f_0(\mathbf{v})|_{\mathbf{v}=\mathbf{R}/T} d^3\mathbf{x}' dt' \quad (A6)\end{aligned}$$

or, in component form

$$\begin{aligned}&\delta J^i(\mathbf{x}, t) \\ &= -(n_0 e^2/mT^3) v^i \partial f_0/\partial v^j |_{\mathbf{v}=\mathbf{R}/T} E^j(\mathbf{x}', t') d^3\mathbf{x}' dt'. \quad (A7)\end{aligned}$$

---

[45] W. Kohn, Phys. Rev. Letters **2**, 393 (1959); E. J. Woll, Jr., and W. Kohn, Phys. Rev. **126**, 1693 (1962).

[46] For example, S. Gartenhaus, *Elements of Plasma Physics* (Holt, Rinehart, and Winston, Inc., New York, 1964).

[47] See also R. G. Chambers, Proc. Phys. Soc. (London) **65**, 458 (1952).

The time $T=t-t'$ in Eq. (A7) must of course be greater than zero. That is, the current cannot precede the field which causes it. For $T<0$ the current $\delta\mathbf{J}$ vanishes. This condition of causality can be incorporated into the theory by multiplying the right-hand side of Eq. (A7) by the unit step function $\theta(T)$. One can then immediately read off the kernel in the integral relation between field and current, Eq. (1). It is

$$K_i^j(\mathbf{R},T)=-(n_0e^2/mT^3)v^i\partial f_0(\mathbf{v})/\partial v^j|_{\mathbf{v}=\mathbf{R}/T}\theta(T). \quad (A8)$$

For a Fermi gas at absolute zero the velocity distribution function is

$$f_0(\mathbf{v})=(3/4\pi v_F^3)\theta(v_F-v) \quad (A9)$$

and the kernel can be written

$$K_i^j(\mathbf{R},T)=(3/4\pi)(n_0e^2/mv_F)(R^iR_j/R^4)\delta(T-R/v_F). \quad (A10)$$

The relation between the electric field and current density is thus

$$\mathbf{J}(\mathbf{x},t)=\frac{3}{4\pi}\frac{n_0e^2}{mv_F}\int d^3\mathbf{x}'\frac{\mathbf{RR}\cdot[\mathbf{E}]}{R^4}, \quad (A11)$$

where $[\mathbf{E}]=\mathbf{E}(\mathbf{x}',t-R/v_F)$ is the electric field retarded in time an amount equal to the time required for a particle traveling at the Fermi velocity to move from the point $\mathbf{x}'$ of application of the field to the point $\mathbf{x}$ at which one observes the current.

Returning now to the general case, one can compute the conductivity tensor $\sigma_j^i(\mathbf{k},\omega)$, which is defined by Eq. (4) as the space–time Fourier transform of the kernel $K_j^i(\mathbf{R},T)$. Changing the integration over $\mathbf{R}$ to an integration over $\mathbf{v}=\mathbf{R}/T$ and performing the integration over T, one obtains

$$\sigma_j^i(\mathbf{k},\omega)=-\frac{in_0e^2}{m}\int d^3\mathbf{v}\frac{v^i\partial f_0/\partial v^j}{\omega-\mathbf{k}\cdot\mathbf{v}+io}, \quad (A12)$$

where the "io" in the denominator serves to move the pole in the integrand at $\omega\simeq\mathbf{k}\cdot\mathbf{v}$ an infinitesimal distance off the real axis into the lower half ω plane. After a certain amount of rearranging, Eq. (A12) can be written in the form given by Eq. (5) with longitudinal and transverse conductivities

$$\sigma^L=-\frac{in_0e^2\omega}{mk^2}\int d^3\mathbf{v}\frac{\mathbf{k}\cdot\nabla_v f_0(\mathbf{v})}{\omega-\mathbf{k}\cdot\mathbf{v}+io}$$

$$\sigma^T=\frac{in_0e^2}{m}\int d^3\mathbf{v}\frac{f_0(\mathbf{v})}{\omega-\mathbf{k}\cdot\mathbf{v}+io}. \quad (A13)$$

Performing the integrations over the two components of $\mathbf{v}$ perpendicular to $\mathbf{k}$, and using the formula

$$(x+io)^{-1}=P(x^{-1})-\pi i\delta(x)$$

where P denotes "principal value," one can reduce these expressions to

$$\sigma^L=\frac{in_0e^2\omega}{mk^2}P\int dv''\frac{f'_0(v'')}{v''-\omega/k}-\frac{\pi n_0e^2\omega}{mk^2}f'_0\left(\frac{\omega}{k}\right)$$

$$\sigma^T=-\frac{in_0e^2}{mk}P\int dv''\frac{f_0(v'')}{v''-\omega/k}+\frac{\pi n_0e^2}{mk}f_0\left(\frac{\omega}{k}\right), \quad (A14)$$

where $f_0(v'')=\int d^2\mathbf{v}_\perp f_0(\mathbf{v})$ is the velocity distribution function integrated over the two components of $\mathbf{v}$ perpendicular to $\mathbf{k}$, and $f'_0(v'')=df_0(v'')/dv''$ is its derivative with respect to the component of velocity v'' in the direction of $\mathbf{k}$.

For a Fermi gas at absolute zero one has

$$f_0(v'')=(3/4v_F^3)(v_F^2-v^2)\theta(v_F-v'').$$

Substituting this expression into Eq. (A14) and performing the integration over v'', one obtains Eq. (8).