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Angular Distributions of Gamma Rays in Terms of Phase-Defined Reduced Matrix Elements

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The theory of angular distributions of γ rays is developed systematically, aiming at a phase consistent derivation of angular distribution formulas for gamma rays emitted in the decay of an aligned initial state. The development starts from first principles, that is, the angular distribution formulas are derived directly from perturbation theory and all quantities introduced are carefully and explicitly defined. In particular the mixing ratios are phase consistently related to reduced matrix elements of interaction multipole operators which again are well defined in phase. Hence the mixing ratios become physical quantities which can be extracted from angular distribution measurements and then compared in both magnitude and sign with the predictions of nuclear models (especially the independent particle model). Critical stages in the theoretical development at which either a choice of phase convention has to be made or transformation properties enter are emphasized.

As a first step, the transition probability for emission of gamma radiation with wave vector $\mathbf{k}$ and polarization $\boldsymbol{\varepsilon}$ from an initial state $|\lambda\rangle$ to a final state $|\mu\rangle$ is calculated using time-dependent perturbation theory. This step makes no specification of the angular momentum of the initial and final states and no multipole expansion of the interaction. Particular attention is paid to the relation between emission and absorption. In the second step of the calculation the angular momentum of the initial and final states is specified, the interaction is expanded in a series of multipoles and the final angular distribution formula is derived. In order to describe *emission* of gamma radiation a definite and well-defined part of the interaction Hamiltonian must be expanded. This part of the Hamiltonian is determined by the order in which the initial and final states are written in transition matrix elements. The expansion yields a set of interaction multipole operators which are well defined in phase and transformation properties. There is no uncertainty in the relative phase of the electric and magnetic interaction multipole operators due to an arbitrariness in the phases of the vector potentials. These interaction multipole operators must be used to define the mixing ratios appearing in angular distribution formulas in this paper. It is shown that care must be taken when using Siegert's theorem and "effective" multipole operators.

In the last section, reduced matrix elements of the interaction multipole operators are given explicitly for one- and two-particle states and one- and two-holes states of the independent particle model.

I. INTRODUCTION

The technique of observing the mixing ratios δ, i.e., the ratios of reduced matrix elements of different multipoles contributing to a gamma transition between well-defined nuclear states has become a widely used source of information about the structure of nuclear levels. The mixing ratio is usually obtained from an angular distribution measurement of the gamma rays with respect to a fixed axis of quantization. This axis is defined in the experiment itself, as, for example, in the case of gamma rays following a nuclear reaction by the particle beam inducing the reaction. The information obtained is, in principle, twofold. In order to fit the measured angular distribution, a certain *magnitude* and a certain *sign* of the mixing ratio are needed.

Both these quantities can, in principle, be predicted if a nuclear model is used and can, therefore, both provide useful information about the validity of the model. One of the two pieces of information has so far been almost universally ignored: a comparison of the sign (i.e., the relative phase of two multipoles contributing to the transition) of the measured value of δ and its theoretical prediction (from a nuclear model) could not be made. This is because it is usually impossible to trace back to common ground the phase conventions made in angular distribution theory and in model calculations of electromagnetic transition matrix elements.

The methods used here are not new: We aim at giving a phase–consistent derivation of an angular distribution formula in terms of phase–defined electromagnetic reduced matrix elements and in terms of population parameters of the initial state in its magnetic substates. It covers the most common method used for determining mixing ratios from a distribution measurement of gamma rays with respect to an axis of cylinder symmetry. In order to make applications easy, the reduced matrix elements, which enter into the distribution formula, are explicitly given to cover the cases of single-particle, single-hole, two-particle, and two-hole transitions. We derive the formulas from first principles and try to proceed "step by step" to enable the reader to follow the conventions which are made, or the principles which are applied, to fix the phases in a "most consistent" manner.

II. THE COMMON STARTING-POINT

A. Transition Probabilities for Absorption and Emission of Gamma Rays

The fundamental starting point is the formula for a transition probability given by first-order perturbation theory for a transition from a discrete state to a continuum state (Di 47)

$$w_{a\to b}=(2\pi/\hbar)\mid\langle b\mid V\mid a\rangle\mid^2\rho_b, \qquad (2.1)$$

where $|\, a\rangle$ is the initial state and $|\, b\rangle$ is the final state. The interaction causing the transition is represented by V, and ρ_b is the density of final continuum states.

In our problem the initial state is a nuclear state $|\,\lambda\rangle$. The wave function representing this state may be a many-particle state of any complexity, and at this stage of the calculation it is not even necessary to specify its angular momentum. The final state $|\, b\rangle$ consists of a nuclear state $|\,\mu\rangle$ and a photon state $|\,\mathbf{k},\boldsymbol{\varepsilon}\rangle$, where $\mathbf{k}$ is the propagation vector of the photon, i.e., $|\,\mathbf{k}\,|=k$ the photon wave number, and $\boldsymbol{\varepsilon}$ is a unit vector giving the polarization. Special assumptions about the definition of $\boldsymbol{\varepsilon}$ are made later in this section when polarization conventions are specified. We may write the total final wave function as

$$|\, b\rangle=|\,\mu,(\mathbf{k},\boldsymbol{\varepsilon})\,\rangle$$

and

$$w(\lambda\to\mu+(\mathbf{k},\boldsymbol{\varepsilon}))=(2\pi/\hbar)\mid\langle\mu,(\mathbf{k},\boldsymbol{\varepsilon})\mid V\mid\lambda\rangle\mid^2\rho(E). \qquad (2.2)$$

In order to find the transition probability for emission of a photon with wave number $\mathbf{k}$ and polarization $\boldsymbol{\varepsilon}$ it is necessary to calculate the matrix element $\langle\mu,(\mathbf{k},\boldsymbol{\varepsilon})\mid V\mid\lambda\rangle$ and the density of states factor. If we normalize the photon wave functions in a cubical box with side L and periodic boundary conditions then the density of photon states with a definite polarization per unit energy per unit solid angle is

$$\rho(E)=L^3k^2/\hbar c(2\pi)^3, \qquad E=\hbar ck. \qquad (2.3)$$

Later in this section we need the analog of formula (2.2) for photon absorption. A nucleus in a state $|\,\mu\rangle$ is irradiated with gamma radiation with polarization $\boldsymbol{\varepsilon}$, wave vector $\mathbf{k}$ and intensity $I_n(\omega)\,d\omega$ photons cm^{-2} sec^{-1} in the frequency range $d\omega$. The probability per unit time for absorption of a gamma ray from the incident radiation with the associated nuclear transition $|\,\mu\rangle\to|\,\lambda\rangle$ is given by the "standard formula" of time-dependent perturbation theory (Di 47)

$$w(\mu+(\mathbf{k},\boldsymbol{\varepsilon})\to\lambda)=(2\pi/\hbar)(dn/dE)\mid\langle\lambda\mid V\mid\mu,(\mathbf{k},\boldsymbol{\varepsilon})\,\rangle\mid^2. \qquad (2.2')$$

The factor dn/dE is the number of photons per unit energy in the initial state at the absorption frequency $\omega_{\lambda\mu}$. If we again normalize the photon wave functions in a cubical box with side L then the number of photons in the box in the frequency range $d\omega$ is

$$dn=L^3I_n(\omega_{\lambda\mu})\,d\omega/c,$$

where c is the velocity of light. Hence

$$dn/dE=L^3I_n(\omega_{\lambda\mu})/\hbar c; \qquad (2.3')$$

$\omega_{\lambda\mu}=(E_\lambda-E_\mu)/\hbar$ is the frequency of the absorbed photon.

The photon intensity $I_n(\omega)$ is related to the energy intensity $I_e(\omega)$ by

$$I_e(\omega)=\hbar\omega I_n(\omega). \qquad (2.4)$$

The interaction V between the photons and the nucleons is self-adjoint if probability is to be conserved. This implies that

$$\langle\mu,(\mathbf{k},\boldsymbol{\varepsilon})\mid V\mid\lambda\rangle=\langle\lambda\mid V\mid\mu,(\mathbf{k},\boldsymbol{\varepsilon})\,\rangle^*. \qquad (2.5)$$

If the states in Eqs. (2.2) and (2.2′) are the *same* then formula (2.5) can be used to relate the emission and absorption process. The square modulus of the emission matrix element occurring in formula (2.2) can be replaced by the square modulus of the absorption matrix element occurring in (2.2′). This result is called the "principle of detailed balance." The emission and absorption matrix elements may also be related replacing the principle of detailed balance by the principle of time reversal invariance. This is necessary when, for example, the state $|\,\lambda\rangle$ is a continuum state (compared Sec. 2.2).

Up to this point we have not said anything about the *detailed* structure of the photon–nucleon interaction V. Two alternative approaches are possible for calculating the matrix element $\langle\mu,(\mathbf{k},\boldsymbol{\varepsilon})\mid V\mid\lambda\rangle$.

(a) The interaction V may be taken from quantum field theory and the matrix element $\langle\mu,(\mathbf{k},\boldsymbol{\varepsilon})\mid V\mid\lambda\rangle$ calculated explicitly.

(b) The transition probability for *absorption* of radiation can be calculated treating the radiation field classically. The result is compared with formula (2.2′) and a value for the square modulus of the matrix element $\mid\langle\lambda\mid V\mid\mu,(\mathbf{k},\boldsymbol{\varepsilon})\,\rangle\mid^2$ extracted. This quantity is then inserted in formula (2.2) for the emission probability making use of the *principle of detailed balance*.

The second approach has the advantage of a semiclassical treatment. The interaction matrix element is obtained from well-known time-dependent perturbation theory which makes use of familiar formulae only. It is unnecessary to erect the complex formalism of quantum field theory.[1]

In the following paragraphs we first develop the semiclassical approach to the absorption problem, treating the incident radiation as a classical electromagnetic field with an intensity distribution $I_e(\omega)$ (ergs per cm^2 and sec and unit angular frequency range). The Hamiltonian for a nucleus interacting with a classical electromagnetic field is[2]

$$H=H_0-(e\hbar/2mc)\sum_n\{2g_{ln}\mathbf{p}_n\cdot\mathbf{A}(\mathbf{r}_n)+g_{sn}\mathbf{s}_n\cdot\mathcal{H}(\mathbf{r}_n)\}$$
$$=H_0+H_{\text{int}}(t). \tag{2.6}$$

Here we have measured the momentum $\mathbf{p}_n$ of the nth particle in natural units of $\hbar$, i.e., $\mathbf{p}=-i\nabla$ and similarly the spin $\mathbf{s}$ is measured in units of $\hbar$, i.e., $\mathbf{s}=\frac{1}{2}\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ can be represented by the Pauli spin matrices. g_{ln} is the orbital g factor for the nth nucleon and g_{sn} is its spin g factor. $g_l=1$ or 0 for a proton or a neutron, $g_s=5.585$ or -3.826 for a proton or a neutron, respectively. For the sake of simplicity we now drop the subscript n and always understand H_{int} as a summation of single-particle operators, summed over all individual nucleons if many-nucleon systems are concerned.

If the classical field $\mathbf{A}$ is a plane wave, then

$$\mathbf{A}(\mathbf{r},t)=A_0\,\text{Re}\,[\boldsymbol{\varepsilon}\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)]$$
$$=\tfrac{1}{2}A_0[\boldsymbol{\varepsilon}\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)+\boldsymbol{\varepsilon}^*\exp(-i\mathbf{k}\cdot\mathbf{r}+i\omega t)], \tag{2.7}$$

with A_0 and ω being real, positive numbers. In general $\boldsymbol{\varepsilon}$ is a complex unit vector $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_1+i\boldsymbol{\varepsilon}_2$, $\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}^*=1$. Then Eq. (2.7) becomes simply

$$A(\mathbf{r},t)=A_0[\boldsymbol{\varepsilon}_1\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)-\boldsymbol{\varepsilon}_2\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)]. \tag{2.7$'$}$$

If the plane wave is plane polarized, then for example $\boldsymbol{\varepsilon}_1$ can be chosen as the direction of polarization, i.e., $\boldsymbol{\varepsilon}_2=0$. For a circular polarized wave we can introduce two mutually perpendicular unit vectors $\mathbf{e}_{x'}$ and $\mathbf{e}_{y'}$ perpendicular to $\mathbf{e}_{z'}=\mathbf{k}/|\mathbf{k}|$ in the right-hand screw sense, i.e., $\mathbf{e}_{x'}\times\mathbf{e}_{y'}=\mathbf{e}_{z'}$, ($\mathbf{e}_{z'}$ having the direction of $\mathbf{k}$), such that

$$\boldsymbol{\varepsilon}=\mathbf{e}_1=-\sqrt{2}^{-1}(\mathbf{e}_{x'}+i\mathbf{e}_{y'})$$

or

$$\boldsymbol{\varepsilon}=\mathbf{e}_{-1}=\sqrt{2}^{-1}(\mathbf{e}_{x'}-i\mathbf{e}_{y'})$$

or in a combined notation

$$\boldsymbol{\varepsilon}=\mathbf{e}_q=-(q/\sqrt{2})(\mathbf{e}_{x'}+iq\mathbf{e}_{y'}),\qquad q=1\ \text{or}\ -1, \tag{2.8}$$

where $q=1$ corresponds to right-handed, $q=-1$ to left-handed circular polarization with respect to $\mathbf{e}_{z'}$. In this case

$$\mathbf{A}(\mathbf{r},t)=A_0[-(q/\sqrt{2})\mathbf{e}_{x'}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)$$
$$+\sqrt{2}^{-1}\mathbf{e}_{y'}\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)]. \tag{2.9}$$

The over-all signs of $\mathbf{e}_1$ and $\mathbf{e}_{-1}$ are chosen so that these complex unit vectors have convenient transformation properties with respect to rotations.

The average energy flux $\bar{\mathbf{N}}$ in the wave may be calculated from the time average of the Poynting vector

$$\mathbf{N}=\frac{c}{4\pi}\langle\mathbf{E}\times\mathcal{H}\rangle_{\text{Av}},\qquad \mathbf{E}=-c^{-1}(\partial\mathbf{A}/\partial t),\qquad \mathcal{H}=\text{curl}\,\mathbf{A}. \tag{2.10}$$

Substituting for example formula (2.9) for $\mathbf{A}$ in these equations, we obtain (for a definite circular polarization q) the average energy flux carried by a circular polarized wave

$$\mathbf{N}=(\omega k/8\pi)A_0^2. \tag{2.11}$$

Substituting the plane wave of arbitrary polarization given by formula (2.7) into the expression for $H_{\text{int}}(t)$ defined in Eq. (2.6) we obtain the interaction Hamiltonian for a plane (circular or plane) polarized wave with the nucleons in a nuclear system. It is convenient to divide $H_{\text{int}}(t)$ into two parts with time dependence $\exp(-i\omega t)$ and $\exp(i\omega t)$.

$$H_{\text{int}}(t)=\tfrac{1}{2}A_0\{H_a(\mathbf{k},\boldsymbol{\varepsilon})\exp(-i\omega t)$$
$$+H_e(\mathbf{k},\boldsymbol{\varepsilon})\exp(i\omega t)\}, \tag{2.12}$$

where, for example,

$$H_a(\mathbf{k},\boldsymbol{\varepsilon})=-e\hbar/2mc\{2g_l\mathbf{p}\cdot\boldsymbol{\varepsilon}\exp(i\mathbf{k}\cdot\mathbf{r})$$
$$+g_s\mathbf{s}\cdot\nabla\times\boldsymbol{\varepsilon}\exp(i\mathbf{k}\cdot\mathbf{r})\}, \tag{2.13}$$

and

$$H_e(\mathbf{k},\boldsymbol{\varepsilon})=[H_a(\mathbf{k},\boldsymbol{\varepsilon})]^+, \tag{2.13$'$}$$

because $\mathbf{p}$ and $\mathbf{s}$ are self-adjoint and $\mathbf{A}$ is real [Eq.

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[1] The semiclassical treatment has the disadvantage that a *spontaneous emission* cannot be calculated in a completely consistent manner while the absorption can be treated satisfactorily. In a proper quantum-mechanical treatment this problem does not arise. Semiclassical derivations of the transition probability formula for spontaneous emission of electromagnetic radiation fall into two main categories. Some [Schiff (Sc 49), Mott and Sneddon (Mo 48)] introduce the Einstein probability coefficients A for spontaneous emission and B for *absorption* and *induced emission*. The absorption coefficient B is calculated by treating the radiation field classically and A is related to B by a thermodynamic argument. The present treatment is equivalent to this with the principle of detailed balance replacing the thermodynamic argument. Other derivations [Schiff (Sc 49), Blatt and Weisskopf (Bl 52)] begin by calculating the electromagnetic radiation from a classical oscillating charge–current distribution in the absence of external fields and then "somewhat arbitrarily rewrite the formulae in terms of quantum matrix elements to obtain a probability for *spontaneous emission*" [Schiff (Sc 49), p. 249].

Many derivations of formulas for transition probabilities begin by making a multipole expansion of the electromagnetic field [Blatt and Weisskopf (Bl 52), Preston (Pr 62)]. This approach introduces angular momentum algebra right from the start and makes it difficult to keep track of signs. We prefer to calculate the general formula for transition probabilities in a plane wave formalism and to introduce the angular momentum algebra at a later stage when calculating matrix elements.

[2] Strictly speaking, this form of the interaction Hamiltonian is true only when the nuclear forces are local. Compare however Sec. IIIB.

(2.7)]. Observing Eq. (2.13′) it is seen from Eq. (2.12) that $H_{\text{int}}(t)$ is (of course) self-adjoint.

If the radiation is switched on at $t=0$ then the probability of a transition at time t is given by time-dependent perturbation theory

$$W(\omega, t)=\frac{1}{\hbar^2}\left|\int_0^t dt \exp[i(\omega_\lambda-\omega_\mu)t]\langle\lambda\mid H_{\text{int}}(t)\mid\mu\rangle\right|^2, \tag{2.14}$$

with $\omega_\lambda-\omega_\mu=(E_\lambda-E_\mu)/\hbar$. Or, using Eq. (2.12),

$$W(\omega, t)=\frac{A_0^2}{4\hbar^2}\left|\int_0^t dt\,\{\exp[i(\omega_\lambda-\omega_\mu-\omega)t]\langle\lambda\mid H_a\mid\mu\rangle + \exp[i(\omega_\lambda-\omega_\mu+\omega)t]\langle\lambda\mid H_e\mid\mu\rangle\}\right|^2.$$

If we are considering an absorption process then $\omega_\lambda-\omega_\mu>0$. Since also $\omega>0$ only the first term in this equation contributes appreciably to the transition probability when $\omega\simeq\omega_\lambda-\omega_\mu$. Neglecting the second term and evaluating the time integral we get

$$W(\omega, t)=\frac{A_0^2}{\hbar^2}\mid\langle\lambda\mid H_a\mid\mu\rangle\mid^2\left(\frac{\sin(\omega_\lambda-\omega_\mu-\omega)t/2}{\omega_\lambda-\omega_\mu-\omega}\right)^2. \tag{2.15}$$

Formula (2.15) gives the transition probability for the transition $\mid\mu\rangle\rightarrow\mid\lambda\rangle$ if the incident radiation is monochromatic with frequency ω and intensity $\omega k A_0^2/8\pi$. If the radiation has a *continuous* energy distribution $I_e(\omega)$ then the transition probability at time t is

$$W(t)=\int d\omega W(\omega, t)I_e(\omega)\frac{8\pi}{\omega k A_0^2}.$$

The time-dependent function on the right-hand side of Eq. (2.15) is a sharply peaked function of ω centered at $\omega=\omega_\lambda-\omega_\mu$. In this situation all the slowly varying functions of ω may be taken outside the integral and evaluated at $\omega=\omega_\lambda-\omega_\mu=(E_\lambda-E_\mu)/\hbar$. We get

$$W(t)=\frac{8\pi}{\hbar^2 k\omega_{\lambda\mu}}I_e(\omega_{\lambda\mu})\mid\langle\lambda\mid H_a\mid\mu\rangle\mid^2\int_{-\infty}^{\infty}dx\left(\frac{\sin xt/2}{x}\right)^2.$$

The integral has the value $t\pi/2$. Hence the transition probability $W(t)$ is proportional to t and the transition probability per unit time is

$$w(\mu+(\mathbf{k}, \boldsymbol{\varepsilon})\rightarrow\lambda)=(2\pi/\hbar)I_n(\omega_{\lambda\mu})(2\pi/k)\mid\langle\lambda\mid H_a\mid\mu\rangle\mid^2. \tag{2.2″}$$

Equations (2.2′) and (2.2″) are two formulas for the same transition probability. Comparing them gives

$$\begin{aligned}\mid\langle\lambda\mid V\mid\mu, (\mathbf{k}, \boldsymbol{\varepsilon})\rangle\mid^2&=(2\pi\hbar c/L^3k)\mid\langle\lambda\mid H_a(\mathbf{k}, \boldsymbol{\varepsilon})\mid\mu\rangle\mid^2\\&=(2\pi\hbar c/L^3k)\mid\langle\mu\mid H_e(\mathbf{k}, \boldsymbol{\varepsilon})\mid\lambda\rangle\mid^2.\end{aligned} \tag{2.16}$$

using the relation $H_e=H_a^+$.

Equation (2.16) gives a formula for the square of the matrix element appearing in formula (2.2′) for the *absorption* of a photon. If $\mid\lambda\rangle$ and $\mid\mu\rangle$ are discrete nuclear states then the transition probability for *absorption* may be related to the transition probability for *emission* using the *principle of detailed balance* as discussed at the beginning of this section. Both the emission and absorption [Eqs. (2.2), (2.2′), and (2.2″)] were calculated using perturbation theory. The interaction V between nucleons and photons is self-adjoint. The matrix elements which occur in formula (2.2) for the *emission* probability of a photon $(\mathbf{k}, \boldsymbol{\varepsilon})$ in the transition $\mid\lambda\rangle\rightarrow\mid\mu\rangle$ and in formula (2.2′) for the *absorption* of a photon with the *same* quantum numbers $(\mathbf{k}, \boldsymbol{\varepsilon})$ in the inverse transition between the *same* nuclear states $\mid\mu\rangle\rightarrow\mid\lambda\rangle$ are related according to Eq. (2.5). Hence we may substitute for the square modulus of the matrix element of V from Eq. (2.16) into Eq. (2.2) and obtain a formula for the emission probability of a photon with wave number $\mathbf{k}$ and polarization $\boldsymbol{\varepsilon}$ in terms of the matrix element $\langle\lambda\mid H_a(\mathbf{k}, \boldsymbol{\varepsilon})\mid\mu\rangle$. The quantities in this matrix element involve only nucleon coordinates and the matrix element may be evaluated if the nuclear wave functions $\mid\lambda\rangle$ and $\mid\mu\rangle$ are known.

The alternative approach is to use an explicit expansion for V given by quantum field theory:

$$\begin{aligned}V&=-(e\hbar/2mc)\sum_n\{2g_{ln}\mathbf{p}_n\cdot\mathbf{A}(r_n)+g_{sn}\mathbf{s}_n\cdot\mathcal{H}(\mathbf{r}_n)\},\\\mathcal{H}(\mathbf{r}_n)&=\operatorname{curl}\mathbf{A}(\mathbf{r}_n),\end{aligned} \tag{2.17}$$

where the vector potential $\mathbf{A}$ is now a field operator[3]

$$\mathbf{A}(\mathbf{r})=\sum_{\mathbf{k}\eta}(2\pi\hbar c/L^3k)^{1/2}\{\boldsymbol{\varepsilon}_\eta\exp(i\mathbf{k}\cdot\mathbf{r})a_{\mathbf{k}\eta}+\boldsymbol{\varepsilon}_\eta^*\exp(-i\mathbf{k}\cdot\mathbf{r})a_{\mathbf{k}\eta}^+\}. \tag{2.18}$$

In (2.18) $a_{\mathbf{k}\eta}^+$ and $a_{\mathbf{k}\eta}$ are creation and annihilation operators for a photon with wave number $\mathbf{k}$ and polarization $\boldsymbol{\varepsilon}_\eta$ ($\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ are two orthogonal (i.e., $\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2=0$) polarization vectors). If (2.18) is substituted into (2.12) then V may be expressed in terms of the operators $H_a(\mathbf{k}, \boldsymbol{\varepsilon})$ and $H_e(\mathbf{k}, \boldsymbol{\varepsilon})$ given by Eq. (2.13),

$$V=\sum_{\mathbf{k}\eta}(2\pi\hbar c/L^3k)^{1/2}\{H_a(\mathbf{k}, \boldsymbol{\varepsilon}_\eta)a_{\mathbf{k}\eta}+H_e(\mathbf{k}, \boldsymbol{\varepsilon}_\eta)a_{\mathbf{k}\eta}^+\}. \tag{2.17′}$$

The emission matrix element $\langle\mu, (\mathbf{k}, \boldsymbol{\varepsilon})\mid V\mid\lambda\rangle$ may be evaluated by noting that the state $\mid\mu, (\mathbf{k}, \boldsymbol{\varepsilon})\rangle$ contains a photon with wave number and polarization $(\mathbf{k}, \boldsymbol{\varepsilon}_\eta)$ and the state $\mid\lambda\rangle$ contains no photon. Hence only the term in V containing the creation operator for the photon state $(\mathbf{k}, \boldsymbol{\varepsilon}_\eta)$ contributes to the matrix element. The matrix element of $a_{\mathbf{k}\eta}^+$ is unity, hence

$$\begin{aligned}\langle\mu, (\mathbf{k}, \boldsymbol{\varepsilon})\mid V\mid\lambda\rangle&=(2\pi\hbar c/L^3k)^{1/2}\langle\mu\mid H_e(\mathbf{k}, \boldsymbol{\varepsilon})\mid\lambda\rangle\\&=(2\pi\hbar c/L^3k)^{1/2}\langle\lambda\mid H_a(\mathbf{k}, \boldsymbol{\varepsilon})\mid\mu\rangle^*.\end{aligned} \tag{2.16′}$$

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[3] The form of $\mathbf{A}$ and the normalization constant $(2\pi\hbar c/L^3k)^{1/2}$ come from quantum field theory. [Cf. (Me 60), p. 898.]

Taking the square modulus of both sides we recover Eq. (2.16).

B. Time Reversal and the Relation between Emission and Absorption Amplitudes

The discussion of the previous section is adequate if perturbation theory is valid and if the nuclear states $|\lambda\rangle$ and $|\mu\rangle$ are discrete states. We assume throughout that perturbation theory may be used. In some problems, however, the initial or final nuclear state may be a continuum state, for example in the radiative capture of a nucleon by a nucleus and the inverse process of the photonuclear effect. The initial nuclear state $|\lambda\rangle$ in the capture process $|\lambda_c\rangle\rightarrow|\mu\rangle+$(photon) is not the same as the final state $|\lambda_p\rangle$ in the photonuclear process because the states $|\lambda_c\rangle$ and $|\lambda_p\rangle$ satisfy different boundary conditions. Then, Eqs. (2.2) and (2.2′) must be modified accordingly by including appropriate boundary conditions on the state $|\lambda\rangle$. In Eq. (2.2) $|\lambda\rangle=|\lambda_c\rangle$ has incoming plane and outgoing spherical waves and in Eq. (2.2′) $|\lambda\rangle=|\lambda_p\rangle$ has incoming spherical and outgoing plane waves. Consequently Eq. (2.5) can no longer be used to relate emission and absorption.

The approach based on quantum field theory is still correct, however. Formula (2.17′) is true and we may use Eq. (2.16′) for the matrix elements provided the continuum state $|\lambda\rangle$ obeys boundary conditions appropriate for the problem.

If the equations describing the nuclear structure and the interactions of the nuclear particles with the electromagnetic field are invariant with respect to time inversion we may always relate a photon emission process to an absorption process by using a time-reversal transformation.

We begin by giving a brief resumé of the time reversal invariance principle.[4] If $\psi(\mathbf{r},t)$ is the wave function of a single particle without spin; i.e., $\psi(\mathbf{r},t)$ is a solution of the Schrödinger equation

$$H(\mathbf{r},t)\psi(\mathbf{r},t)=i\hbar[\partial\psi(\mathbf{r},t)/\partial t], \qquad (2.19)$$

then the time-reversed wave function is

$$\psi_R(\mathbf{r},t)=\psi^*(\mathbf{r},-t)\equiv\boldsymbol{\theta}\psi(\mathbf{r},-t). \qquad (2.20)$$

This wave function can be interpreted as describing a state of reversed motion. $\boldsymbol{\theta}$ is called the time-reversal operator and in this case is simply complex conjugation. The time-reversed wave function satisfies the original Schrödinger equation, provided

$$\boldsymbol{\theta}H(\mathbf{r},-t)\boldsymbol{\theta}^{-1}\equiv H^*(\mathbf{r},-t)=H(\mathbf{r},t). \qquad (2.21)$$

If in addition $\psi_R(\mathbf{r},t=0)=\psi(\mathbf{r},t=0)$ then the time-reversed solution is identical with $\psi(\mathbf{r},t)$ for all times. If condition (2.21) is satisfied the Hamiltonian is said to be invariant under time reversal.

In a more complicated system whose dynamics are invariant with respect to time reversal it is always possible to define a time-reversal operator $\boldsymbol{\theta}$ with the following properties:

(i) $\boldsymbol{\theta}H(-t)\boldsymbol{\theta}^{-1}=H(t)$ [Eq. (2.21)]; i.e., the Hamiltonian is time-reversal-invariant.

(ii) $\boldsymbol{\theta}$ is equivalent to complex conjugation followed by a unitary transformation; i.e., $\boldsymbol{\theta}$ is an anti-unitary operator.

(iii) The time-reversed wave function $\psi_R(\mathbf{r},t)=\boldsymbol{\theta}\psi(\mathbf{r},-t)$ is a solution of the Schrödinger equation of the system if $\psi(\mathbf{r},t)$ is a solution.

For example, in the case of a system of (nonrelativistic) nucleons with spin, the Hamiltonian may depend explicitly on the spin $\mathbf{s}_n=\frac{1}{2}\boldsymbol{\sigma}_n$ of the single nucleons ($\boldsymbol{\sigma}$ for example being represented by the Pauli spin matrices σ_x, σ_y, σ_z) and $\mathbf{l}_n$, the orbital angular momentum of the single nucleons; then the time-reversal operator for a nucleon can simply be represented by

$$\boldsymbol{\theta}=\sigma_y\mathbf{K}_0, \qquad (2.22)$$

where $\mathbf{K}_0$ means complex conjugation. [For many nucleons, a product of all $(\sigma_y)_n$ enters.] $\boldsymbol{\theta}$ of Eq. (2.22) has the effect that $\boldsymbol{\theta}\mathbf{p}\boldsymbol{\theta}^{-1}=-\mathbf{p}$, $\boldsymbol{\theta}\mathbf{l}\boldsymbol{\theta}^{-1}=-\mathbf{l}$, $\boldsymbol{\theta}\mathbf{s}\boldsymbol{\theta}^{-1}=-\mathbf{s}$ but $\boldsymbol{\theta}\mathbf{r}\boldsymbol{\theta}^{-1}=\mathbf{r}$ and $\boldsymbol{\theta}$ does *not* act on t.

Suppose $|\alpha JM\rangle$ is a discrete eigenstate of a time-reversal invariant Hamiltonian H, which has only angular momentum degeneracy. In general

$$\boldsymbol{\theta}\,|\alpha JM\rangle=(-)^{R-M}\,|\alpha J-M\rangle,$$

where the phase R is independent of M but depends on the choice of phase of the wave function $|\alpha JM\rangle$. It is always possible to choose this phase so that

$$\boldsymbol{\theta}\,|\alpha JM\rangle=(-)^{J-M}\,|\alpha J-M\rangle. \qquad (2.23)$$

Throughout this work we use this phase convention for all eigenfunctions. The choice is invariant with respect to angular momentum coupling. If the wave functions $|j_1m_1\rangle$ and $|j_2m_2\rangle$ transform under time reversal according to Eq. (2.23) then

$$|JM\rangle=\sum_{m_1m_2}|j_1m_1\rangle\,|j_2m_2\rangle(j_1j_2m_1m_2\,|\,JM)$$

does also, because

$$(j_1j_2-m_1-m_2\,|\,J-M)=(-)^{j_1+j_2-J}(j_1j_2m_1m_2\,|\,JM),$$

if the phases of the vector coupling coefficients conform to the Condon and Shortley convention [cf. Refs. (Co 35), (Br 62)].

We may get some information about the time reversal of the photon state $|\mathbf{k},\boldsymbol{\varepsilon}\rangle$ by looking at the transformation properties of the classical electromagnetic field. If we make the transformation $t\rightarrow-t$; $\mathbf{r}\rightarrow\mathbf{r}$, then Maxwell's equations are invariant provided we transform the charge and current densities ρ and $\mathbf{j}$ and the

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[4] For a more detailed discussion cf. (Wi 59).

fields $\mathbf{E}$, $\mathcal{H}$, and $\mathbf{A}$ as follows:

$$\rho(\mathbf{r},t)\to\rho(\mathbf{r},-t),\qquad \mathbf{E}(\mathbf{r},t)\to\mathbf{E}(\mathbf{r},-t);$$
$$\mathbf{j}(r,t)\to-\mathbf{j}(\mathbf{r},-t),\qquad \mathcal{H}(\mathbf{r},t)\to-\mathcal{H}(\mathbf{r},-t);$$
$$\mathbf{A}(\mathbf{r},t)\to-\mathbf{A}(\mathbf{r},-t).$$

If

$$\begin{aligned}\mathbf{A}(\mathbf{r},t)&=A_0\,\mathrm{Re}\,\{\boldsymbol{\varepsilon}\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]\}\\&=\tfrac{1}{2}A_0\{\boldsymbol{\varepsilon}\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]\\&\qquad+\boldsymbol{\varepsilon}^*\exp[-i(\mathbf{k}\cdot\mathbf{r}-\omega t)]\},\end{aligned}\tag{2.7}$$

then

$$\begin{aligned}A_R(\mathbf{r},t)&=-\mathbf{A}(\mathbf{r},-t)\\&=\tfrac{1}{2}A_0\{-\boldsymbol{\varepsilon}^*\exp[i(-\mathbf{k}\cdot\mathbf{r}-\omega t)]\\&\qquad-\boldsymbol{\varepsilon}\exp[-i(-\mathbf{k}\cdot\mathbf{r}-\omega t)]\}.\end{aligned}\tag{2.24}$$

Comparing the last two equations we see that if the field $\mathbf{A}$ is a plane wave with wave vector $\mathbf{k}$ and polarization $\boldsymbol{\varepsilon}$, then the time reversed field $\mathbf{A}_R$ is a plane wave with wave vector $-\mathbf{k}$ and polarization $-\boldsymbol{\varepsilon}^*$. We require that the quantum numbers specifying the photon state transform in the same way

$$\mathbf{k}\to-\mathbf{k}\quad\text{and}\quad\boldsymbol{\varepsilon}\to-\boldsymbol{\varepsilon}^*.$$

Formally we can include this classical transformation property into a quantum-mechanical theory by a time-reversal operator $\boldsymbol{\theta}_T$ for the total system of nucleons and photons. Acting on a photon wave function $\boldsymbol{\theta}_T$ then has the property $\boldsymbol{\theta}_T\,|\,\mathbf{k},\boldsymbol{\varepsilon}\rangle=|\,-\mathbf{k},-\boldsymbol{\varepsilon}^*\rangle$ while on a nuclear wave function $\boldsymbol{\theta}_T$ has the same action as the operator $\boldsymbol{\theta}$ defined in Eq. (2.22).

In the previous section we have related absorption to emission using the principle of detailed balance. At the beginning of this section we have said that it may be sometimes necessary to use time reversal invariance to establish this relation. In order to do this [and also to derive the transformation properties of $H_e(\mathbf{k},\boldsymbol{\varepsilon})$ and $H_a(\mathbf{k},\boldsymbol{\varepsilon})$ under time reversal] it is convenient to introduce the emission and absorption amplitudes

$$T(\lambda\to\mu+(\mathbf{k},\boldsymbol{\varepsilon}))=\langle\mu,(\mathbf{k},\boldsymbol{\varepsilon})\,|\,V\,|\,\lambda\rangle\quad\text{(emission)},\tag{2.25}$$

$$T(\mu+(\mathbf{k},\boldsymbol{\varepsilon})\to\lambda)=\langle\lambda\,|\,V\,|\,\mu,(\mathbf{k},\boldsymbol{\varepsilon})\rangle\quad\text{(absorption)}.$$

The transition amplitude $T(\mu+(\mathbf{k},\boldsymbol{\varepsilon})\to\lambda)$ for absorption of a photon with wave number $\mathbf{k}$ and polarization $\boldsymbol{\varepsilon}$ by a nuclear state $|\,\mu\rangle$ to form a final state $|\,\lambda\rangle$ may be related to the transition amplitude for the (inverse) emission process $T(\lambda\to\mu+(\mathbf{k},\boldsymbol{\varepsilon}))$ by time-reversal invariance

$$\begin{aligned}T(\lambda\to\mu+(\mathbf{k},\boldsymbol{\varepsilon}))&=T(\mu_R+(\mathbf{k}_R,\boldsymbol{\varepsilon}_R)\to\lambda_R)\\&=T(\boldsymbol{\theta}_T\{\mu+(\mathbf{k},\boldsymbol{\varepsilon})\}\to\boldsymbol{\theta}_T\lambda).\end{aligned}\tag{2.26}$$

Here $|\,\lambda_R\rangle$ and $|\,\mu_R\rangle$ are the time reverses of the nuclear states $|\,\lambda\rangle$ and $|\,\mu\rangle$ and $|\,\mathbf{k}_R,\boldsymbol{\varepsilon}_R\rangle$ is the time reverse of the photon state $|\,\mathbf{k},\boldsymbol{\varepsilon}\rangle$.

Equation (2.26) can be proved by using the antiunitary property of any time-reversal operator, $\boldsymbol{\theta}=U\mathbf{K}_0$ and the invariance of a scalar product under a unitary transformation U

$$\langle\boldsymbol{\theta}\phi,\boldsymbol{\theta}\psi\rangle=\langle U\phi^*,U\psi^*\rangle=\langle\phi^*,\psi^*\rangle=\langle\phi,\psi\rangle^*=\langle\psi,\phi\rangle.\tag{2.27}$$

It follows from this that the matrix element of an operator B has the following property:

$$\langle\boldsymbol{\theta}\phi\,|\,\boldsymbol{\theta}B\boldsymbol{\theta}^{-1}\,|\,\boldsymbol{\theta}\psi\rangle=\langle\phi\,|\,B\,|\,\psi\rangle^*.\tag{2.28}$$

The absorption amplitude for time reversed states is

$$T(\mu_R+(\mathbf{k}_R,\boldsymbol{\varepsilon}_R)\to\lambda_R)=\langle\boldsymbol{\theta}_T\lambda\,|\,V\,|\,\boldsymbol{\theta}_T\{\mu+(\mathbf{k},\boldsymbol{\varepsilon})\}\rangle$$

by definition. Because V is invariant[5] under time reversal, i.e., $\boldsymbol{\theta}_TV\boldsymbol{\theta}_T^{-1}=V$, it follows that

$$T(\mu_R+(\mathbf{k}_R,\boldsymbol{\varepsilon}_R)\to\lambda_R)=\langle\boldsymbol{\theta}_T\lambda\,|\,\boldsymbol{\theta}_TV\boldsymbol{\theta}_T^{-1}\,|\,\boldsymbol{\theta}_T\{\mu+(\mathbf{k},\boldsymbol{\varepsilon})\}\rangle.$$

Using relation (2.28) we obtain

$$T(\mu_R+(\mathbf{k}_R,\boldsymbol{\varepsilon}_R)\to\lambda_R)=\langle\lambda\,|\,V\,|\,\mu,(\mathbf{k},\boldsymbol{\varepsilon})\rangle^*,$$

and because V is Hermitian; i.e., $V^+=V$,

$$T(\mu_R+(\mathbf{k}_R,\boldsymbol{\varepsilon}_R)\to\lambda_R)=\langle\mu,(\mathbf{k},\boldsymbol{\varepsilon})\,|\,V\,|\,\lambda\rangle,$$

which by definition is the transition amplitude $T(\lambda\to\mu+(\mathbf{k},\boldsymbol{\varepsilon}))$ for emission. This proves Eq. (2.26). The relation also holds if the transition amplitudes are not calculated from perturbation theory. The preceding equations still contain the photon eigenfunctions. However, for the absorption amplitude we have already calculated that

$$|\,T(\mu+(\mathbf{k},\boldsymbol{\varepsilon})\to\lambda)\,|^2=(2\pi\hbar c/L^3k)\,|\,\langle\lambda\,|\,H_a(\mathbf{k},\boldsymbol{\varepsilon})\,|\,\mu\rangle\,|^2,\tag{2.16}$$

where wave functions and operators on the right-hand side depend on nuclear coordinates only.

To use relation (2.26) we remember that $\boldsymbol{\theta}_T$ transforms $\mathbf{k}\to-\mathbf{k}$ and $\boldsymbol{\varepsilon}\to-\boldsymbol{\varepsilon}^*$, or in other words,

$$\boldsymbol{\theta}_T\,|\,\mathbf{k},\boldsymbol{\varepsilon}\rangle=|\,-\mathbf{k},-\boldsymbol{\varepsilon}^*\rangle,$$

therefore

$$T(\lambda\to\mu+(\mathbf{k},\boldsymbol{\varepsilon}))=T(\boldsymbol{\theta}_T\mu+(-\mathbf{k},-\boldsymbol{\varepsilon}^*)\to\boldsymbol{\theta}_T\lambda),$$

and consequently (2.16) can be written

$$\begin{aligned}&|\,T(\lambda\to\mu+(\mathbf{k},\boldsymbol{\varepsilon}))\,|^2\\&\quad=(2\pi\hbar c/L^3k)\,|\,\langle\boldsymbol{\theta}\lambda\,|\,H_a(-\mathbf{k},-\boldsymbol{\varepsilon}^*)\,|\,\boldsymbol{\theta}\mu\rangle\,|^2.\end{aligned}\tag{2.29}$$

In this last matrix element $\boldsymbol{\theta}$ acts only on the nuclear coordinates and may therefore be represented just by $\boldsymbol{\theta}=\sigma_y\mathbf{K}_0$.

We can simplify the expression on the right-hand side of Eq. (2.29) by calculating $H_a(\mathbf{p},\mathbf{s},-\mathbf{k},-\boldsymbol{\varepsilon}^*)$

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[5] For a possible, C,T noninvariance in the electromagnetic interaction cf. (Be 65, He 66).

explicitly and comparing it with the result obtained by calculating $\boldsymbol{\theta}H_a(\mathbf{p}, \mathbf{s}, \mathbf{k}, \boldsymbol{\varepsilon})\boldsymbol{\theta}^{-1}$ explicitly (where now $\boldsymbol{\theta}$ just acts on *nuclear* coordinates and operators). $H_a(\mathbf{p}, \mathbf{s}, \mathbf{k}, \boldsymbol{\varepsilon})$ was given by

$$H_a(\mathbf{p}, \mathbf{s}, \mathbf{k}, \boldsymbol{\varepsilon}) \equiv H_a(\mathbf{k}, \boldsymbol{\varepsilon}) = (-e\hbar/2mc)\{2g_l\mathbf{p}\cdot\boldsymbol{\varepsilon} \exp(i\mathbf{k}\cdot\mathbf{r}) + g_s\mathbf{s}\cdot\operatorname{curl}[\boldsymbol{\varepsilon}\exp(i\mathbf{k}\cdot\mathbf{r})]\}. \tag{2.13}$$

Therefore

$$\begin{aligned} H_a(\mathbf{p}, \mathbf{s}, -\mathbf{k}, -\boldsymbol{\varepsilon}^*) &= (-e\hbar/2mc)\{2g_l\mathbf{p}\cdot(-\boldsymbol{\varepsilon}^*) \exp(-i\mathbf{k}\cdot\mathbf{r}) + g_s\mathbf{s}\cdot\operatorname{curl}[-\boldsymbol{\varepsilon}^*\exp(-i\mathbf{k}\cdot\mathbf{r})]\} \\ &= -\{H_a(\mathbf{p}, \mathbf{s}, \mathbf{k}, \boldsymbol{\varepsilon})\}^+. \end{aligned}$$

On the other hand, with $\boldsymbol{\theta} = \sigma_y\mathbf{K}_0$

$$\begin{aligned} \boldsymbol{\theta}H_a(\mathbf{p}, \mathbf{s}, \mathbf{k}, \boldsymbol{\varepsilon})\boldsymbol{\theta}^{-1} &= -e\hbar/2mc\{2g_l(-\mathbf{p})\cdot\boldsymbol{\varepsilon}^* \exp(-i\mathbf{k}\cdot\mathbf{r}) + g_s(-\mathbf{s})\cdot\operatorname{curl}[\boldsymbol{\varepsilon}^*\exp(-i\mathbf{k}\cdot\mathbf{r})] \\ &= -\{H_a(\mathbf{p}, \mathbf{s}, \mathbf{k}, \boldsymbol{\varepsilon})\}^+. \end{aligned}$$

Comparing the two results we see that

$$\boldsymbol{\theta}H_a(\mathbf{p}, \mathbf{s}, \mathbf{k}, \boldsymbol{\varepsilon})\boldsymbol{\theta}^{-1} = H_a(\mathbf{p}, \mathbf{s}, -\mathbf{k}, -\boldsymbol{\varepsilon}^*) = -\{H_a(\mathbf{p}, \mathbf{s}, \mathbf{k}, \boldsymbol{\varepsilon})\}^+ \tag{2.30}$$

or in the shorter notation used previously

$$\boldsymbol{\theta}H_a(\mathbf{k}, \boldsymbol{\varepsilon})\boldsymbol{\theta}^{-1} = H_a(-\mathbf{k}, -\boldsymbol{\varepsilon}^*) = -\{H_a(\mathbf{k}, \boldsymbol{\varepsilon})\}^+ \tag{2.30}$$

and similarly for $H_e(\mathbf{k}, \boldsymbol{\varepsilon})$. In simple words: the operation with the time-reversal operator on $H_a(\mathbf{k}, \boldsymbol{\varepsilon})$ or $H_e(\mathbf{k}, \boldsymbol{\varepsilon})$ is equivalent to reversing the signs of all quantum-mechanical operators related to momenta ($\mathbf{p}$, $\mathbf{l}$, $\mathbf{s}$) and taking the complex conjugate of everything else ["time-reversal rule (2.30)"].

Using relation (2.30) we can now rewrite Eq. (2.29)

$$|T(\lambda\rightarrow\mu+(\mathbf{k}, \boldsymbol{\varepsilon}))|^2 = (2\pi\hbar c/L^3k)\,|\langle\boldsymbol{\theta}\lambda\,|\,\boldsymbol{\theta}H_a(\mathbf{k}, \boldsymbol{\varepsilon})\boldsymbol{\theta}^{-1}\,|\,\boldsymbol{\theta}\mu\rangle|^2$$

which by relation (2.28) is

$$|T(\lambda\rightarrow\mu+(\mathbf{k}, \boldsymbol{\varepsilon}))|^2 = (2\pi\hbar c/L^3k)\,|\langle\lambda\,|\,H_a(\mathbf{k}, \boldsymbol{\varepsilon})\,|\,\mu\rangle^*|^2, \tag{2.29'}$$

where the complex conjugation indicated by the asterisk of course makes no difference.[6]

In Eq. (2.29′) we have arrived at a relation for the square modulus rather than for the emission amplitude itself, because a semiclassical calculation of the absorption probability was made. Since $H_a(\mathbf{k}, \boldsymbol{\varepsilon}) = H_e(\mathbf{k}, \boldsymbol{\varepsilon})^+$ it follows that

$$T(\lambda\rightarrow\mu+(\mathbf{k}, \boldsymbol{\varepsilon})) = (2\pi\hbar c/L^3k)^{1/2}\langle\mu\,|\,H_e(\mathbf{k}, \boldsymbol{\varepsilon})\,|\,\lambda\rangle.$$

It is seen that it is purely a matter of convenience whether the initial state $|\lambda\rangle$ for the emission process is written on the left or the right side in the matrix element provided one uses the correct operator H_a or H_e, respectively. Because H_a depends in the plane wave representation on $\exp(i\mathbf{k}\cdot\mathbf{r})$ while H_e depends on $\exp(-i\mathbf{k}\cdot\mathbf{r})$ we prefer to use the operator H_a and therefore have to use the convention of writing the initial state of the emission process always on the left. We feel—in view of a later expansion of the plane wave into multipole fields—that using H_a it is easier to keep track of signs.

The results of Sec. IIB may be summarized: Whichever way we approach the problem, we always obtain the result that the transition probability per unit time for the emission of a photon with wave number $\mathbf{k}$ and polarization $\boldsymbol{\varepsilon}$ associated with a transition from an initial state $|\lambda\rangle$ to a final state $|\mu\rangle$ is given by

$$w(\lambda\rightarrow\mu+(\mathbf{k}, \boldsymbol{\varepsilon})) = (k/2\pi\hbar)\,|\langle\lambda\,|\,H_a(\mathbf{k}, \boldsymbol{\varepsilon})\,|\,\mu\rangle|^2. \tag{2.31}$$

The problem of calculating the transition probability reduces to calculating the nuclear matrix element appearing in Eq. (2.31).

C. Time Reversal and Phase Conventions of Spherical Tensor Operators

In the previous section we have stated the phase convention for eigenfunctions which is adopted throughout this work, namely, that under time reversal

$$\boldsymbol{\theta}\,|\,\alpha JM\rangle = (-)^{J-M}\,|\,\alpha J-M\rangle. \tag{2.23}$$

This choice had the advantage of being invariant with respect to angular momentum coupling.

In a similar manner we can fix the phase of spherical tensor operators. Let T_{LM} be a spherical tensor, i.e., a quantity of rank L represented by $2L+1$ components which transform according to the irreducible representation $\mathfrak{D}_{MN}{}^L$ of the rotation group[7]

$$T_{LN}' = \sum_M T_{LM}\mathfrak{D}_{MN}{}^L(\alpha\beta\gamma) \tag{2.32}$$

$(\alpha\beta\gamma)$ are the Euler angles of the rotation taking the old, unprimed axes into the new, primed axes. Thus Eq. (2.32) expresses a component T_{LN}' of the tensor

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[6] In a consistent field theoretical treatment we would have arrived at a relation for the emission amplitude

$$T(\lambda\rightarrow\mu+(\mathbf{k}, \boldsymbol{\varepsilon})) = (2\pi\hbar c/L^3k)^{1/2}\langle\lambda\,|\,H_a(\mathbf{k}, \boldsymbol{\varepsilon})\,|\,\mu\rangle^*.$$

[7] This definition is chosen to agree with the transformation law for spherical harmonics on rotation of axes. Throughout this work the $\mathfrak{D}_{MN}{}^L(\alpha\beta\gamma)$ as defined by Brink and Satchler (Br 62) and their definition of Euler angles

$$\mathfrak{D}_{MN}{}^L(\alpha\beta\gamma) = \langle LM\,|\,\exp(-i\alpha J_z)\exp(-i\beta J_y)\exp(-i\gamma J_z)\,|\,LN\rangle$$

is used. The same definition is used by Messiah (Me 60).

operator with respect to the new axes in terms of the components T_{LM} defined with respect to the old axes. For the tensors under consideration

$$\boldsymbol{\theta}T_{LM}\boldsymbol{\theta}^{-1}=(-)^{R-M}T_{L-M}, \tag{2.33}$$

where the phase R is independent of M but depends on the choice of phase of T_{LM}. It is always possible to choose this phase so that

$$\boldsymbol{\theta}T_{LM}\boldsymbol{\theta}^{-1}=(-)^{L-M}T_{L-M}. \tag{2.33$'$}$$

The choice of phase defined by formula (2.33′) ensures that the matrix elements of T_{LM} are real if the wave functions transform according to Eq. (2.23). (In general if $R-L$ is even then the matrix elements of T_{LM} will be real; if it is odd they will be pure imaginary.)

Later we expand the transverse plane wave $\mathbf{A}$ in a series of spherical vector fields $\mathbf{A}_{LM}{}^{m}$ and $\mathbf{A}_{LM}{}^{e}$ and, by this expansion, express the interaction Hamiltonian in a series of tensor operators $T_{LM}{}^{m}$ and $T_{LM}{}^{e}$. These operators are essentially products of momenta $\mathbf{p}$ and spins $\mathbf{s}$ with the fields $\mathbf{A}_{LM}{}^{m}$ and $\mathbf{A}_{LM}{}^{e}$. Since both $\mathbf{p}$ and $\mathbf{s}$ change sign under time reversal, it is most convenient to adopt a phase convention for the spherical vector fields which under time reversal yields a phase factor $(-)^{L-M+1}$. The vector fields $\mathbf{A}_{LM}{}^{\langle\pi\rangle}$ act on nuclear coordinates only and "time reversal" reduces to just taking the complex conjugate. Then the expansion terms $T_{LM}{}^{e}$ and $T_{LM}{}^{m}$ of the interaction Hamiltonian $H_a(\mathbf{k},\boldsymbol{\varepsilon})$ will transform according to Eq. (2.33′) and therefore will yield real reduced matrix elements. This matter of convenience also was the reason why we have adopted the formula (2.8) to describe circular polarization. It leads in a convenient way to an expansion in terms of "natural" sets $\mathbf{A}_{LM}{}^{e}$ and $\mathbf{A}_{LM}{}^{m}$ which have the required property.

III. THE TRANSITION PROBABILITY IN TERMS OF MULTIPOLE MATRIX ELEMENTS

A. Mathematical Definitions

The functions

$$\phi_{LM}=i^{L}(2L+1)j_{L}(kr)C_{LM}(\theta,\phi) \tag{3.1}$$

satisfy the scalar wave equation $\nabla^2\phi+k^2\phi=0$ and form a complete set.

The scalar plane wave in the z direction has the well-known expansion in terms of ϕ_{LM}

$$\exp(ikz)=\sum_{L}\phi_{L0}. \tag{3.2}$$

The spherical Bessel functions $j_L(kr)$ can be expressed in terms of Bessel functions of half-odd integer order

$$j_L(kr)=(\pi/2kr)^{1/2}J_{L+1/2}(kr) \tag{3.3}$$

and if $kr\ll 1$ (long wavelength approximation) then

$$j_L(kr)\simeq(kr)^{L}/(2L+1)!!.$$

The modified spherical harmonics $C_{LM}(\theta,\phi)$ are used throughout this article because of their convenient normalization. They are related to the usual spherical harmonics

$$C_{LM}=(4\pi/2L+1)^{1/2}Y_{LM}, \tag{3.4}$$

where

$$Y_{LM}(\theta,\phi)=\Theta_{LM}(\theta)\Phi_M(\phi),$$

with

$$\Theta_{LM}(\theta)=(-)^{M}\left[\frac{2L+1}{2}\frac{(L-M)!}{(L+M)!}\right]^{1/2}P_L{}^{M}(\theta),\quad \text{if } M\geq 0;$$

$$\Theta_{LM}(\theta)=(-)^{M}\Theta_{L|M|}(\theta),\qquad \text{if } M<0;$$

$$\Phi_M(\phi)=(2\pi)^{-1/2}\exp(iM\phi).$$

$P_L{}^{M}(\theta)\ (M\geq 0)$ is the associated Legendre polynomial of Jahnke and Emde (Ja 45).

The definition of the $Y_{LM}(\theta,\phi)$ involves an arbitrary choice of phase and we follow Condon and Shortley (Co 35). With this choice

$$C_{LM}{}^{*}=(-)^{M}C_{L-M}\quad\text{and}\quad C_{L0}(\theta,\phi)=P_L(\cos\theta).$$

The vector plane wave $\mathbf{A}$ which solves the vector wave equation

$$\nabla^2\mathbf{A}+k^2\mathbf{A}=0$$

can be similarly expanded, remembering that the three sets of vector fields

$$\boldsymbol{\nabla}\phi_{LM},\ \mathbf{L}\phi_{LM},\ \text{and}\ \boldsymbol{\nabla}\times\mathbf{L}\phi_{LM} \tag{3.5}$$

form a complete set of solutions of the vector wave equation. Normalizing these three sets in the same way as the ϕ_{LM} are normalized, we have

$$\begin{aligned}\mathbf{A}_{LM}&=(ik)^{-1}\boldsymbol{\nabla}\phi_{LM},\\ \mathbf{A}_{LM}{}^{e}&=\{k[L(L+1)]^{1/2}\}^{-1}\boldsymbol{\nabla}\times\mathbf{L}\phi_{LM},\\ \mathbf{A}_{LM}{}^{m}&=\{[L(L+1]^{1/2}\}^{-1}\mathbf{L}\phi_{LM},\end{aligned} \tag{3.6}$$

where

$$\boldsymbol{\nabla}=\mathbf{e}_x(\partial/\partial x)+\mathbf{e}_y(\partial/\partial y)+\mathbf{e}_z(\partial/\partial z), \tag{3.7}$$

and $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are unit vectors along the x, y, and z axes. Also

$$\mathbf{L}=-i\mathbf{r}\times\boldsymbol{\nabla}. \tag{3.8}$$

The choice of phase of the vector fields defined in Eq. (3.6) is such that they transform under time reversal according to the requirements discussed in Sec. 2.3. Furthermore, the fields have the following properties:

$$\begin{aligned}&\boldsymbol{\nabla}\cdot\mathbf{A}_{LM}{}^{e}=\boldsymbol{\nabla}\cdot\mathbf{A}_{LM}{}^{m}=0,\\ &\boldsymbol{\nabla}\times\mathbf{A}_{LM}{}^{e}=k\mathbf{A}_{LM}{}^{m},\qquad \boldsymbol{\nabla}\times\mathbf{A}_{LM}{}^{m}=k\mathbf{A}_{LM}{}^{e},\\ &\mathbf{L}\cdot\mathbf{A}_{LM}{}^{e}=0,\qquad \mathbf{L}\cdot\mathbf{A}_{LM}{}^{m}=[L(L+1)]^{1/2}\phi_{LM}.\end{aligned} \tag{3.9}$$

The fields $\mathbf{A}_{LM}{}^{e}$ and $\mathbf{A}_{LM}{}^{m}$ are known as the electric and magnetic multipole components of the transverse field; they are solenoidal and have parities $(-)^{L+1}$ and $(-)^{L}$, respectively. The fields $\mathbf{A}_{LM}$ are irrotational, have parity $(-)^{L+1}$ and are longitudinal fields.

The plane, transverse, circular polarized wave $\mathbf{e}_q \exp(i\mathbf{k}\cdot\mathbf{r})$ may be expanded in terms of the $\mathbf{A}_{LM}{}^m$ and $\mathbf{A}_{LM}{}^e$. In particular

$$\mathbf{e}_q \exp(ikz) = -\sqrt{2}^{-1} \sum_L (q\mathbf{A}_{Lq}{}^m + \mathbf{A}_{Lq}{}^e), \qquad q = \pm 1. \tag{3.10}$$

We may pass from Eq. (3.10) to the expansion of a general transverse plane wave $\mathbf{e}_q' \exp(i\mathbf{k}\cdot\mathbf{r})$ by making a rotation of axes

$$\mathbf{e}_q' \exp(i\mathbf{k}\cdot\mathbf{r}) = -\sqrt{2}^{-1} \sum_{LM} (q\mathbf{A}_{LM}{}^m + \mathbf{A}_{LM}{}^e)\mathfrak{D}_{Mq}{}^L(R). \tag{3.11}$$

In Eq. (3.11) the unit vectors $\mathbf{e}_q'$ are referred to a set of axes with z' axis parallel to $\mathbf{k}$ and the x' and y' axes forming a right-hand set with respect to z'. $R = (\alpha, \beta, \gamma)$ is the rotation with Euler angles α, β, γ taking the z axis to the direction of $\mathbf{k}$ (i.e., the z' axis).

B. Multipole Expansion of the Interaction

Inserting (3.11) into Eq. (2.13) gives $H_a(\mathbf{k}, \boldsymbol{\varepsilon})$ in terms of $\mathbf{A}_{LM}{}^m$ and $\mathbf{A}_{LM}{}^e$. In order to reduce this multipole expansion to standard form we have to make the long-wavelength approximation

$$j_L(kr) \simeq (kr)^L/(2L+1)!!$$

or

$$\phi_{LM} \simeq X_L r^L C_{LM}(\theta, \phi), \qquad X_L = (ik)^L/(2L-1)!!. \tag{3.12}$$

In this approximation

$$\begin{aligned}\mathbf{A}_{LM}{}^e &= \{X_L/k[L(L+1)]^{1/2}\}\,\nabla\times\mathbf{L}[r^L C_{LM}(\theta, \phi)]\\ &= (iX_L/k)(L+1/L)^{1/2}\nabla(r^L C_{LM}),\end{aligned} \tag{3.13}$$

where, in order to obtain the last line, we have used the relation

$$\nabla\times\mathbf{L}(r^L C_{LM}) = i(L+1)\nabla(r^L C_{LM}).$$

Also

$$\begin{aligned}\mathbf{A}_{LM}{}^m &= \{X_L/[L(L+1)]^{1/2}\}\mathbf{L}(r^L C_{LM})\\ &= \{iX_L/[L(L+1)]^{1/2}\}\nabla(r^L C_{LM})\times\mathbf{r},\end{aligned} \tag{3.14}$$

where the last line was obtained using Eq. (3.8). Substituting in Eq. (3.11) the long-wavelength expressions for $\mathbf{A}_{LM}{}^e$ and $\mathbf{A}_{LM}{}^m$ given by formulas (3.13) and (3.14) and inserting this into (2.13) we obtain, after remembering that

$$(\nabla(r^L C_{LM})\times\mathbf{r})\cdot\mathbf{p} = \nabla(r^L C_{LM})\cdot(\mathbf{r}\times\mathbf{p}) = \nabla(r^L C_{LM})\cdot\mathbf{l},$$

$$\begin{aligned}H_a(\mathbf{k}, \mathbf{e}_q) = -\sum_{LM} \{&\alpha_L{}^e(Q_{LM}+Q_{LM}')\\ &+q\alpha_L{}^m(M_{LM}+M_{LM}')\}\mathfrak{D}_{Mq}{}^L(R),\end{aligned} \tag{3.15}$$

where

$$\alpha_L{}^e = [(ik)^L/(2L-1)!!](L+1/2L)^{1/2}, \qquad \alpha_L{}^m = -i\alpha_L{}^e \tag{3.16}$$

are the multipole expansion coefficients, and

$$\begin{aligned}Q_{LM} &= -2g_l\beta(i/k)\nabla(r^L C_{LM})\cdot\mathbf{p},\\ Q_{LM}' &= -g_s\beta[k/(L+1)]\mathbf{L}(r^L C_{LM})\cdot\mathbf{s},\\ M_{LM} &= 2g_l\beta(L+1)^{-1}\nabla(r^L C_{LM})\cdot\mathbf{l},\\ M_{LM}' &= g_s\beta\nabla(r^L C_{LM})\cdot\mathbf{s}\end{aligned} \tag{3.17}$$

are the electric and magnetic multipole operators[8], $\beta = e\hbar/2mc$ is the nuclear magneton and $\mathbf{l}$, $\mathbf{s}$, and $\mathbf{p}$ are measured in their natural units of $\hbar$, which has been incorporated into β.

If O_{LM} stands for any of the operators (3.17), then

$$O_{LM}{}^+ = (-)^{C-M} O_{L-M} \tag{3.18}$$

with $C=1$ for the electric and $C=0$ for the magnetic terms. The time-reversal properties of the operators (3.17) are

$$\theta O_{LM}\theta^{-1} = (-)^{R-M}O_{L-M} = (-)^{C-M+1}O_{L-M}, \tag{3.19}$$

again $C=1$ for the electric and $C=0$ for the magnetic terms, i.e. $R=0$ and $R=1$, respectively.

For actual calculations of reduced matrix elements of the operators (3.17) it is more convenient to write the electric multipole operator Q_{LM} as

$$Q_{LM} = 2g_l(m\beta/\hbar^2 k)[H, r^L C_{LM}]. \tag{3.17'}$$

This expression is completely equivalent to the original form of Q_{LM} in Eq. (3.17), provided the nuclear potentials are local and commute[9] with $r^L C_{LM}$. The equiv-

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[8] We have used the usual notation Q_{LM}, M_{LM}, etc. for these operators, although they are not identical with the ones defined in Ref. (Bl 53) (cf. Sec. VC).

[9] The basic interaction between the nucleus and the electromagnetic field must be gauge invariant, i.e., must be consistent with the equation of continuity for charge and current. The interaction (2.6) postulated in this article does not have this property if nuclear forces have an exchange character or if velocity-dependent forces (for example spin–orbit forces) are present. If velocity-dependent or exchange forces are present, the following derivation of Eq. (3.17′) in the text is not valid and also the interaction (2.6) between the nucleus and the electromagnetic field must be modified. Siegert's theorem [cf. (Sa 53)] shows that for the electric multipole terms these two changes compensate one another. The electric multipole interaction Q_{LM} is no longer represented by Eq. (3.17 a) because the interaction is changed, but formula (3.17′) is still correct.

alence can be easily shown using the result[10]

$$[T, f(r)] = -i(\hbar^2/m)(\nabla f)\cdot\mathbf{p},$$

if T is the kinetic energy operator, $\nabla^2 f=0$ and $\mathbf{p}$ is measured in units of $\hbar$. Then

$$Q_{LM} = 2g_l(m\beta/\hbar^2 k)[T, r^L C_{LM}].$$

We have proved Eq. (3.17′) for one-particle systems. The result holds for many-particle systems provided H is the total Hamiltonian of the system and the transition operators $r^L C_{LM}$ are summed over all particles.

The usual form for the electric multipole operators is just $eg_l r^L C_{LM}$. Expression (3.17′) reduces to this standard form, because for a matrix element between eigenstates $|\lambda\rangle$ and $|\mu\rangle$ of H we get

$$\langle\lambda | Q_{LM} | \mu\rangle = 2g_l(m\beta/\hbar^2 k)(E_\lambda - E_\mu)\langle\lambda | r^L C_{LM} | \mu\rangle.$$

If the transition is from $|\lambda\rangle$ to $|\mu\rangle$ *emitting* a gamma ray then

$$E_\lambda - E_\mu = \hbar\omega = \hbar c k > 0$$

and we obtain

$$\langle\lambda | Q_{LM} | \mu\rangle = \langle\lambda | (Q_{LM})_{\text{eff}} | \mu\rangle, \qquad (3.17'')$$

with

$$(Q_{LM})_{\text{eff}} = eg_l r^L C_{LM}. \qquad (3.17''')$$

However, it is wrong to conclude from (3.17″) that $Q_{LM} = (Q_{LM})_{\text{eff}}$. The transformation property of the effective electric multipole operator $(Q_{LM})_{\text{eff}}$ under Hermitian conjugation is different from that of Q_{LM}. If the Hermitian conjugate $\langle\lambda | Q_{LM}^+ | \mu\rangle$ is taken, then it is *not* true that $\langle\lambda | Q_{LM}^+ | \mu\rangle = \langle\lambda | (Q_{LM})_{\text{eff}}^+ | \mu\rangle$, but

$$\begin{aligned}\langle\lambda | Q_{LM}^+ | \mu\rangle &= \langle\mu | Q_{LM} | \lambda\rangle^* \\ &= 2g_l(m\beta/\hbar^2 k)(E_\mu - E_\lambda)\langle\mu | r^L C_{LM} | \lambda\rangle^* \\ &= 2g_l(m\beta/\hbar^2 k)(E_\mu - E_\lambda)\langle\lambda | (r^L C_{LM})^+ | \mu\rangle \\ &= -\langle\lambda | (Q_{LM})_{\text{eff}}^+ | \mu\rangle,\end{aligned}$$

i.e. the factor $(E_\lambda - E_\mu)$ changes sign.

Formula (3.17″) is valid only when $E_\lambda > E_\mu$. It is possible to make (3.17″) true in general by enforcing transformation laws under Hermitian conjugation on $(Q_{LM})_{\text{eff}}[(Q_{LM})_{\text{eff}}^+ = (-)^{M+1}(Q_{L-M})_{\text{eff}}]$ which are different from those of the spherical harmonics. This is an artificial manipulation. One is likely to produce errors in phase unless one uses Q_{LM} as defined by Eq. (3.17′). This applies especially when one has to "switch" the states in the matrix elements or when one wants to describe nuclear states composed of many particles as "hole states" in closed shells (compare Sec. 4.4). Using the operator Q_{LM} of Eq. (3.17′) throughout always gives the energy factor $(E_\lambda - E_\mu)$ with the correct sign.

Table I. Multipole expansion coefficients for $L=1, 2, 3$.

L	α_L^e	α_L^m
1	ik	k
2	$-k^2/2\sqrt{3}$	$ik^2/2\sqrt{3}$
3	$-ik^3\sqrt{2}/3.5\sqrt{3}$	$-k^3\sqrt{2}/3.5\sqrt{3}$

Because of the different transformation properties of the electric and magnetic terms of Eq. (3.17) under Hermitian conjugation and time reversal it is more convenient to include the multipole expansion coefficients $\alpha_L^{\langle\pi\rangle}$ of Eq. (3.16) into the definition of the multipole operators and define in this way a set of operators all having the *same* transformation properties. We call this set the *interaction multipole operators* $T_{LM}^{\langle\pi\rangle}$ and define

$$\begin{aligned}T_{LM}^e &= \alpha_L^e(Q_{LM} + Q_{LM}'), \\ T_{LM}^m &= \alpha_L^m(M_{LM} + M_{LM}'). \qquad (3.20)\end{aligned}$$

The multipole expansion coefficients $\alpha_L^{\langle\pi\rangle}$ are listed up to $L=3$ in Table I, and by the use of this table the multipole operators of Eqs. (3.17) and (3.17′) can be easily converted into interaction multipole operators. The Hamiltonian $H_a(\mathbf{k}, \mathbf{e}_q)$ can now be written [see Eqs. (3.11) and (3.15)]

$$H_a(\mathbf{k}, \boldsymbol{\varepsilon}_q) = -\sum_{LM\pi} q^\pi T_{LM}^{\langle\pi\rangle}\mathfrak{D}_{Mq}^L(R). \qquad (3.21)$$

The summation runs over all possible L, M, π which can contribute to the transition in question. The $T_{LM}^{\langle\pi\rangle}$ are one-particle operators, and if the system described by the eigenfunctions $|\lambda\rangle$ and $|\mu\rangle$ is composed of many particles an appropriate particle index n has to be introduced and the summation extended over all particles.

$\pi=0$ for electric interaction multipole operators. The superscript $\langle\pi=0\rangle$ or $\langle 0\rangle$ then simply means "electric."

$\pi=1$ for magnetic interaction multipole operators. The superscript $\langle\pi=1\rangle$ or $\langle 1\rangle$ then simply means "magnetic."

Writing the factor q^π in front of each operator leads to a compact notation of the interaction Hamiltonian. Because $\pi=0$ for all electric interaction multipole operators the contributions of these to the Hamiltonian are independent of whether $q=1$ or -1. The magnetic contributions are not independent of circular polarization.[11]

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[10] The relation follows simply by acting with the left side on a wave function ϕ:

$$\begin{aligned}[T, f]\phi &= -(\hbar^2/2m)[\nabla^2(f\phi) - f\nabla^2\phi] \\ &= -(\hbar^2/2m)[(\nabla^2 f)\phi + 2(\nabla f)\cdot(\nabla\phi) + f\nabla^2\phi - f\nabla^2\phi] \\ &= -(i\hbar^2/m)(\nabla f)\cdot\mathbf{p}\phi,\end{aligned}$$

where $\mathbf{p} = -i\nabla$ and $\nabla^2 f = 0$.

[11] We could also have included the factor q, which occurs in Eq. (3.15), into the multipole expansion coefficients α_L^m themselves. This would have led to an even more compact notation of the interaction Hamiltonian (3.21). However, then the $T_{LM}^{\langle\pi\rangle}$ would have depended implicitly upon q and some of the subsequent discussions would have become rather bulky because of constant reminders to the hidden dependence of the magnetic interaction multipole operators on q.

The interaction multipole operators $T_{LM}^{<\pi>}$ have identical transformation properties: under rotations they transform like spherical tensors of rank L, while their Hermitian adjoints are given by

$$T_{LM}^{<\pi>+}=(-)^{L-M+1}T_{L-M}^{<\pi>} \tag{3.22}$$

and their time reversals

$$\boldsymbol{\theta}T_{LM}^{<\pi>}\boldsymbol{\theta}^{-1}=(-)^{L-M}T_{L-M}^{<\pi>} \tag{3.23}$$

in accord with Eq. (2.33′). Therefore the matrix elements of $T_{LM}^{<\pi>}$ are real, provided that the choice of phase for the eigenfunctions entering the matrix elements is made according to (2.23).

For convenience we collect at the end of this section in their final form all the expressions which enter $H_a(\mathbf{k}, \boldsymbol{\varepsilon}_q)$:

Multipole operators

$$G_{LM}^e=Q_{LM}+Q_{LM}', \qquad G_{LM}^m=M_{LM}+M_{LM}';$$

$$Q_{LM}=2g_l(m\beta/\hbar^2k)[H, r^LC_{LM}],$$

$$Q_{LM}'=-g_s\beta[k/(L+1)]\mathbf{L}(r^LC_{LM})\cdot\mathbf{s},$$

$$M_{LM}=2g_l\beta(L+1)^{-1}\boldsymbol{\nabla}(r^LC_{LM})\cdot\mathbf{l},$$

$$M_{LM}'=g_s\beta\boldsymbol{\nabla}(r^LC_{LM})\cdot\mathbf{s}. \tag{3.17 final}$$

Interaction multipole operators:

$$T_{LM}^e=\alpha_L^eG_{LM}^e,$$

$$\alpha_L^e=[(ik)^L/(2L-1)!!](L+1/2L)^{1/2};$$

$$T_{LM}^m=\alpha_L^mG_{LM}^m,$$

$$\alpha_L^m=-i\alpha_L^e. \tag{3.20 final}$$

C. The Angular Distribution Formula for Emission of Gamma Rays from a Nuclear State Aligned with Respect to an Axis of Cylindrical Symmetry

The states $|\lambda\rangle$ and $|\mu\rangle$ may now be specified by their total angular momentum quantum numbers. We write the initial state, which decays by emission of a photon, as $|\lambda\rangle=|J_1M_1\rangle$ and the final state as $|\mu\rangle=|J_2M_2\rangle$. The magnetic quantum numbers M_i refer to the axis of cylindrical symmetry. The probability for a transition from a state $|J_1M_1\rangle$ to a state $|J_2M_2\rangle$ with emission of a circular polarized photon along the direction of $\mathbf{k}$ is given by Eq. (2.31).

In order to simplify the discussion in the following paragraphs we introduce the probability amplitude $A_{M_1M_2}^q(\mathbf{k})$ for emission of a photon along $\mathbf{k}$ with circular polarization q associated with the nuclear transition $|J_1M_1\rangle\rightarrow|J_2M_2\rangle$. The probability amplitude is related to the transition amplitude for emission introduced previously by

$$A_{M_1M_2}^{q*}(\mathbf{k})=(L^3k^2/4\pi^2\hbar^2c)^{1/2}T(J_1M_1\rightarrow J_2M_2+(\mathbf{k}, \mathbf{e}_q)).$$

Formula (2.31) can then be rewritten

$$w(J_1M_1\rightarrow J_2M_2+(\mathbf{k}, \mathbf{e}_q))=|A_{M_1M_2}^q(\mathbf{k})|^2.$$

After expansion of $H_a(\mathbf{k}, \mathbf{e}_q)$ into interaction multipole operators [cf. Eq. (3.21)]

$$A_{M_1M_2}^q(\mathbf{k})=-(k/2\pi\hbar)^{1/2} \times\sum_{L_1M_1\pi} q^\pi\langle J_1M_1|T_{LM}^{<\pi>}|J_2M_2\rangle\mathfrak{D}_{Mq}^L(R). \tag{3.24}$$

The rotation $R=(\alpha\beta\gamma)$ takes the z axis to the direction of $\mathbf{k}$ as described in Sec. IIIA. Since we limit ourselves to the case that the z axis is an axis of cylindrical symmetry there is no need to discuss specific representations of the $\mathfrak{D}_{Mq}^L$ provided only they follow the same reduction laws as applied in the subsequent paragraphs.

If we do not observe the orientation of the spin J_2 of the final system we must sum over M_2 and the probability for emission $w(\lambda\rightarrow\mu+(\mathbf{k}, \boldsymbol{\varepsilon}))=P_{M_1}^q(\mathbf{k})$ from the initial state $|J_1M_1\rangle$ becomes proportional to

$$\sum_{M_2}|A_{M_1M_2}^q(\mathbf{k})|^2.$$

If linear and not circular polarization is observed we must take a coherent superposition of $q=\pm1$ terms A^q. Thus $\sqrt{2}^{-1}(A^{q=-1}-A^{q=+1})$ describes the emission of linear polarized photons with the plane of polarization parallel to the x' direction while $\sqrt{2}^{-1}(A^{q=-1}+A^{q=+1})$ gives the respective description of the plane polarization being parallel to y'. A polarization insensitive measurement requires an incoherent sum over q.

If the radiating system is in a cylindrically symmetric environment so that M_1 is a constant of the motion if the z axis is chosen along the symmetry axis, the total radiative probability for photons along $\mathbf{k}$ is obtained by weighting each $P_{M_1}^q(\mathbf{k})$ with the population parameters (i.e., relative probabilities of population) $w(M_1)$ of the substate M_1

$$P^q(\mathbf{k})=\sum_{M_1}w(M_1)P_{M_1}^q(\mathbf{k})=\sum_{M_1}w(M_1)\sum_{M_2}|A_{M_1M_2}^q(\mathbf{k})|^2. \tag{3.25}$$

The normalization is

$$\sum_{M_1}w(M_1) = 1.$$

Using the Wigner–Eckart theorem

$$\langle J_1M_1|T_{LM}|J_2M_2\rangle=(-)^{2L}(J_2LM_2M|J_1M_1)\langle J_1||\mathbf{T}_L||J_2\rangle,$$

and the reduction formula for a product of rotation matrices

$$\mathfrak{D}_{m_1n_1}{}^{j_1}\mathfrak{D}_{m_2n_2}{}^{j_2}=\sum_{KHN}(j_1j_2m_1m_2\mid KH)(j_1j_2n_1n_2\mid KN)\mathfrak{D}_{HN}{}^{K}$$

and remembering that

$$\mathfrak{D}_{mn}{}^{j*}=(-)^{m-n}\mathfrak{D}_{-m-n}{}^{j};\qquad \mathfrak{D}_{00}{}^{K}=P_K(\cos\theta)$$

we obtain

$$\begin{aligned}P_{M_1}{}^{q}(\mathbf{k})=\sum_{M_2}\mid A_{M_1M_2}{}^{q}\mid^2=(k/2\pi\hbar)\sum_{LL'\pi\pi'K}&(LL'q-q\mid KO)\\ \times\sum_{M_2M}(-)^{-M+q}(LL'M-M\mid KO)&(J_2LM_2M\mid J_1M_1)(J_2L'M_2M\mid J_1M_1)\\ &\times P_K(\cos\theta)q^{\pi+\pi'}\langle J_1\parallel \mathbf{T}_L{}^{\langle\pi\rangle}\parallel J_2\rangle\langle J_1\parallel \mathbf{T}_{L'}{}^{\langle\pi'\rangle}\parallel J_2\rangle^*.\end{aligned}\tag{3.26}$$

The product of the three CG coefficients which appears in the sum over M_2 and M can be reduced in the usual way to a product of one CG and one Racah coefficient

$$\begin{aligned}[(2J_1+1)^{1/2}]^2(-)^{J_1-M_1}(J_1J_1M_1-M_1\mid KO)W(J_1J_1LL';KJ_2)(-)^{L'-L+J_1-J_2-K}&\\ =\sum_{M_2M}(-)^{M}(LL'M-M\mid KO)(J_2LM_2M\mid J_1M_1)(J_2L'M_2M\mid J_1M_1)&,\end{aligned}$$

and we obtain the angular distribution formula for the emission of photons from a state J_1 leading to a state J_2

$$\begin{aligned}P^q(\mathbf{k})=(k/2\pi\hbar)\sum_{KLL'\pi\pi'}B_K(J_1)P_K(\cos\theta)(-)^{q+J_1-J_2+L'-L-K}(2J_1+1)^{1/2}(LL'q-q\mid KO)W(J_1J_1LL';KJ_2)&\\ \times q^{\pi+\pi'}\langle J_1\parallel \mathbf{T}_L{}^{\langle\pi\rangle}\parallel J_2\rangle\langle J_1\parallel \mathbf{T}_{L'}{}^{\langle\pi'\rangle}\parallel J_2\rangle^*,&\end{aligned}\tag{3.27}$$

with

$$B_K(J_1)=\sum_{M_1}w(M_1)(-)^{J_1-M_1}(2J_1+1)^{1/2}(J_1J_1M_1-M_1\mid KO),\tag{3.28}$$

$$B_0(J_1)=1,\qquad \sum_{M_1}w(M_1)=1.$$

The reduced matrix elements in Eq. (3.27) are real if the eigenstates $\mid J_iM_i\rangle$ and the operators $T_{LM}{}^{\langle\pi\rangle}$ transform under time reversal according to Eqs. (2.23) and (2.33′). $B_K(J_1)$ is a statistical tensor for the cylinder–symmetrical case.

So far we have not made any special assumptions about the states $\mid J_1M_1\rangle$ and $\mid J_2M_2\rangle$ except that they are composed of individual nucleons which interact with the electromagnetic field according to Eq. (2.6). We deal in Sec. IV with the calculation of reduced matrix elements from a model, and give explicit formulas for the cases when the states $\mid J_1\rangle$ and $\mid J_2\rangle$ can be described as single-particle or two-particle states outside closed shells or as "single-hole" or "two-hole" states in a closed shell.

To conclude this section we introduce the mixing ratios in terms of the above reduced matrix elements, which we shall also relate to the radiative width Γ_γ of the transition. We finally rewrite the fundamental formula (3.27) in terms of suitable coefficients and compilations for the $B_K(J_1)$ as given in the Appendix. We also discuss various methods for producing an aligned initial state $\mid J_1M_1\rangle$ and investigate the restrictions placed on the population parameter $w(M_1)$ or $B_K(J_1)$ by the production process. The restriction that the observed gamma ray is the one following directly from the decay of the initial aligned state will be removed.

This leaves us with a set of formulas covering most of the practical cases whenever a cylinder–symmetrical environment is used to determine mixing ratios. Apart from the definition of the mixing ratios in terms of explicitly defined matrix elements of interaction multipole operators between states $\mid J_1\rangle$ and $\mid J_2\rangle$ of any complexity, the mixing ratios may be looked upon as formal quantities or parameters. Fitting the respective formula to a set of experimental data one arrives at (positive or negative) values for these parameters. How and if the mixing ratios are interpreted in terms of model descriptions of the states $\mid J_1\rangle$ and $\mid J_2\rangle$ affects neither the original definition nor the phase with which these parameters stand in the distribution formula.

D. Gamma Widths

Integrating Eq. (3.27) over all directions of $\mathbf{k}$ and summing over polarizations q gives the total transition probability per unit time; i.e., the reciprocal lifetime of a state $\mid J_1\rangle$ for gamma decay. Remembering that $\int P_K(\cos\theta)\,d\Omega=0$ unless $K=0$ (in which case it gives 4π) and that the CG and Racah coefficients for $K=0$ are

$$(LLq-q\mid 00)=(-)^{L-q}(2L+1)^{-1/2}$$

and

$$W(J_1J_1LL;OJ_2)=(-)^{J_1-J_2+L}[(2J_1+1)(2L+1)]^{-1/2}$$

we obtain

$$\frac{1}{\tau}=\sum_q \int P^q(\mathbf{k})\, d\Omega$$

$$=2\times\frac{2}{\hbar}\, k \sum_{L\pi} |\, \langle J_1 \,||\, \mathbf{T}_L{}^{<\pi>} \,||\, J_2\rangle\, |^2/(2L+1) \quad (3.29)$$

because the integrated transition probability $\int P^q(\mathbf{k})\, d\Omega$ is independent of q. For applications it is sometimes more convenient to eliminate the energy dependence of the matrix elements $\langle J_1 \,||\, \mathbf{T}_L{}^{<\pi>} \,||\, J_2\rangle$ by using the effective electric and magnetic multipole operators (3.17″) instead of the $\mathbf{T}_L{}^{<\pi>}$. We then obtain the well-known Weisskopf formula[12]

$$\tau^{-1}=\frac{2}{\hbar}\sum_{L\pi}\frac{(L+1)}{L[(2L-1)\,!!]^2}\, k^{2L+1}\, \frac{|\, \langle J_1 \,||\, \mathbf{G}_L{}^{<\pi>} \,||\, J_2\rangle\, |^2}{(2L+1)} \quad (3.30)$$

and the gamma width of the state $|\, J_1\rangle$ is given by $\Gamma_\gamma=\hbar/\tau$.

E. Mixing Ratios

For practical computations of angular distributions it is most convenient to rewrite the basic formula (3.27) in terms of suitable coefficients. This compact notation is still quite general; i.e., the only specialization is [as for formula (3.27)] that the initial state has definite J_1 and that the problem has cylindrical symmetry. We then restrict ourselves to the case that circular polarization is not observed, introduce the mixing ratios and rewrite the formula in terms of these ratios. The section is concluded with a discussion of two questions: (i) The question of alignment or polarization of the initial state and (ii) The question of restrictions following from the assumption that all the nuclear states involved have definite parity.[13]

To rewrite formula (3.27) in a compact notation we introduce the coefficient

$$R_K{}^q(LL'J_1J_2)=(-)^{q+J_1-J_2+L'-L-K}(2J_1+1)^{1/2}(2L+1)^{1/2}(2L'+1)^{1/2} \times (LL'q-q\,|\,KO)\,W(J_1J_1LL';KJ_2). \quad (3.31)$$

Formula (3.27) then becomes

$$P^q(\mathbf{k})=\frac{k}{2\pi\hbar}\sum_{LL'\pi\pi'K} B_K(J_1)\,R_K{}^q(LL'J_1J_2)\,P_K(\cos\theta) \times q^{\pi+\pi'}\, \frac{\langle J_1 \,||\, \mathbf{T}_L{}^{<\pi>} \,||\, J_2\rangle}{(2L+1)^{1/2}}\, \frac{\langle J_1 \,||\, \mathbf{T}_{L'}{}^{<\pi'>} \,||\, J_2\rangle^*}{(2L'+1)^{1/2}}. \quad (3.32)$$

We note that:

(i) Formula (3.32) is restricted to the case that the initial state has definite J_1. It does not necessarily have to have definite parity.

(ii) The sum runs independently over all possible L, L', π, π', and K. There is no fixed relation between π and L or π' and L'.

(iii) K can be odd as well as even. The highest possible value of K is fixed by the well-known triangular conditions governing CG and Racah coefficients contained in $B_K(J_1)$ and $R_K{}^q(LL'J_1J_2)$.

(iv) The allowed (integer) values of L, L' are $|\, J_1-J_2\,| \leq L,\, L' \leq J_1J_2$. Of course $L,\, L' \neq 0$.

(v) $\boldsymbol{\theta} T_{LM}{}^{<\pi>}\boldsymbol{\theta}^{-1}=(-)^{L-M}T_{L-M}{}^{<\pi>}$ and if $\boldsymbol{\theta}\,|\, J_1M_1\rangle=(-)^{J-M}\,|\, J_1-M_1\rangle$, then the reduced matrix elements are real. The asterisk may be omitted.

The coefficient $R_K{}^q(LL'J_1J_2)$ is symmetric in L and L'. Also

$$R_K{}^{-q}(LL'J_1J_2)=(-)^{L+L'-K}R_K{}^q(LL'J_1J_2). \quad (3.33)$$

With this definition the $R_K{}^q(LL'J_1J_2)$ coefficient has the orthonormal property

$$R_0{}^q(LL'J_1J_2)=\delta(L,L'). \quad (3.34)$$

Due to the choice of normalization according to Eq. (3.34) and due to the definition of multipole interaction operators and reduced matrix elements as used in this article[14] a slight aesthetic distortion is caused in Eq. (3.32) by the appearance of square roots $(2L+1)^{1/2}$ and $(2L'+1)^{1/2}$ (written in the denominator of the reduced matrix elements). However, from a practical point of view it is important *not* to avoid the appearance of the square roots in Eq. (3.32). This is because the modulus of the quantity $\langle J_1 \,||\, \mathbf{T}_L{}^{<\pi>} \,||\, J_2\rangle/(2L+1)^{1/2}$ may be thought of [except for a factor $4k$, compare

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[12] Strictly speaking, the contribution from $\mathbf{Q}_L{}'$ which is contained in $\mathbf{G}_L{}^e$ is still energy-dependent. But usually its contribution to the reduced matrix element of $\mathbf{G}_L{}^e$ is negligible when compared to the one from $\mathbf{Q}_L$.

[13] The initial state need not have definite parity if it is a continuum state with an energy between overlapping resonances of opposite parity.

[14] The $T_{LM}{}^{<\pi>}$ are the "most natural" choice for the set of multipole operators to be used in a phase-consistent treatment, because they lead to the simple expansion (3.21) of the interaction Hamiltonian and all have the same transformation properties under time reversal and Hermitian conjugation. With the definition of reduced matrix elements we are in accord with Refs. (Br 62, Ro 57). Racah (Ra 42, 43), Edmonds (Ed 57), and Messiah (Me 60) define reduced matrix elements, which have the same phase but are $(2J_1+1)^{1/2}$ times ours.

Eq. (3.29)] as the square root of the partial gamma width due to the multipole transition (L, π). Thus, the product of the two reduced matrix elements divided by the corresponding $(2L+1)^{1/2}$ can, in modulus, (apart from $4k$) be interpreted as the product of the square roots of two partial gamma widths. Consequently the mixing ratio as defined in Eq. (3.39) will be, *in modulus*, interpretable as the square root of the ratio of two partial gamma widths. Some authors [cf. (De 57), (Li 61), (Po 65)] define their mixing ratio as the ratio of two reduced matrix elements. The square roots then are absorbed into the reduced matrix elements which are written $\langle J_b \| L \| J_a \rangle$. We feel that it would have been somewhat artificial to break with the "natural" operators $T_{LM}^{<\pi>}$ in the present treatment or modify a quite commonly adopted form of the Wigner–Eckart theorem only to make the reduced matrix elements equivalent (in modulus) to square roots of gamma widths. In fact, since the phases of matrix elements $\langle J_b \| L \| J_a \rangle$ anyway are not well defined, and since it is usually difficult (if not impossible) to discover the definition of this basic quantity (and that of the "operator L", cf. Sec. V), it would have been more appropriate to write just $\{\Gamma_\gamma(L, \pi)\}^{1/2}$ and avoid the purely symbolic quantity $\langle J_b \| L \| J_a \rangle$.

Formula (3.32) gives the probability per unit time and unit solid angle that the nucleus emits a gamma ray in the direction $\mathbf{k}$ with polarization q. If the polarization is not measured then we sum over the circular polarization quantum q

$$P(\mathbf{k}) = P^1(\mathbf{k}) + P^{-1}(\mathbf{k})$$

to get the total probability per unit time per unit solid angle for the emission of a gamma ray in the direction $\mathbf{k}$. Using the properties of the coefficients $R_K{}^q(LL'J_1J_2)$ in Eq. (3.33) we get

$$P(\mathbf{k}) = \frac{k}{2\pi\hbar} \sum_{LL'K\pi\pi'} B_K(J_1) R_K(LL'J_1J_2) P_K(\cos\theta) \times \{1+(-1)^{L+L'+\pi+\pi'-K}\} \frac{\langle J_1 \| \mathbf{T}_L{}^{<\pi>} \| J_2 \rangle}{(2L+1)^{1/2}} \times \frac{\langle J_1 \| \mathbf{T}_{L'}{}^{<\pi'>} \| J_2 \rangle^*}{(2L'+1)^{1/2}}. \tag{3.35}$$

The coefficient $R_K(LL'J_1J_2)$ is the $R_K{}^q(LL'J_1J_2)$ coefficient for $q=1$ [cf. Eq. (3.31)]. Explicitly

$$R_K(LL'J_1J_2) \equiv R_K{}^{q=1}(LL'J_1J_2) = (-)^{1+J_1-J_2+L'-L-K} \times \{(2J_1+1)(2L+1)(2L'+1)\}^{1/2} \times (LL'1-1 \mid KO) W(J_1J_1LL'; KJ_2). \tag{3.36}$$

It is related to the $F_K(LL'J_2J_1)$ coefficient of Ferentz and Rosenzweig (Fe 55) by

$$R_K(LL'J_1J_2) = (-)^{L-L'+K} F_K(LL'J_2J_1). \tag{3.37}$$

Tables for $R_K(LL'J_1J_2)$ are given in the Appendix.

We now introduce the "mixing ratios." The reduced matrix elements occurring in formula (3.32) contain the "nuclear information." They can be calculated from a model, which gives eigenfunctions for the states J_1 and J_2. Experimentally the sum of their square moduli can be obtained from a lifetime measurement[15] [Eq. (3.29)]. Consequently one is only interested in determining the *ratios* of the reduced matrix elements from measurements of angular distributions. In other words, when measuring angular distributions one wants to interpret them in terms of ratios of reduced matrix elements. The experiment gives from a sequence of measurements taken at different angles θ with respect to the symmetry axis an angular distribution of the form[16]

$$W_{\exp}(\theta) = \sum_K a_K P_K(\cos\theta). \tag{3.38}$$

We therefore rewrite Eq. (3.35) in terms of *ratios* of reduced matrix elements and define the mixing ratios[17]

$$\delta_L{}^{<\pi>} = \frac{\langle J_1 \| \mathbf{T}_L{}^{<\pi>} \| J_2 \rangle / (2L+1)^{1/2}}{\langle J_1 \| \mathbf{T}_{\bar{L}}{}^{<\bar{\pi}>} \| J_2 \rangle / (2\bar{L}+1)^{1/2}}, \tag{3.39}$$

$\bar{L}$, $\bar{\pi}$ stand for the lowest-order multipolarity occurring in the transition $J_1 \rightarrow J_2$. Using this notation we get a formula for the angular distribution of gamma rays

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[15] In the case that the initial state is formed by particle absorption, an absolute yield measurement of the reaction (p, γ), (α, γ), etc. is equivalent to this.

[16] The coefficients a_K may either be thought of as "attenuated values", i.e., it is assumed that corrections for efficiencies etc. of the gamma detectors have been made, or—more convenient for computer analysis and more logically related to a general treatment of less special cases—the attenuation coefficients Q_K as used for example in (Li 61), (Po 65) may be introduced as multiplicative factors into the theoretical distribution formula itself. We leave this point with the reader.

[17] The theorem which states that the ratio of the reduced matrix elements of electromagnetic multipole operators must be real [usually referred to as Lloyd's theorem (Lo 51)] makes use of the invariance of the interaction Hamiltonian under time reversal and assumes definite parity of J_1 and J_2. It is also satisfied by the operators $G_{LM}{}^{<\pi>}$ [Eq. (3.17″)], which do *not* transform under time reversal as the $T_{LM}{}^{<\pi>}$ do. In other words, to make the *ratio* of two reduced matrix elements of multipole operators with *equal* parity real, it is neither necessary that the multipole operators *must* transform as $\boldsymbol{\theta} T_{LM}{}^{<\pi>} \boldsymbol{\theta}^{-1} = (-)^{L-M} T_{L-M}{}^{<\pi>}$ nor that the eigenfunctions obey $\boldsymbol{\theta} \mid JM \rangle = (-)^{J-M} \mid J-M \rangle$. These choices make the mixing ratio real *a fortiori*. Assumptions about definite parities of J_1 and J_2 then do not have to be made, i.e., also the ratio of two reduced matrix elements of multipole operators of *different* parities will be real. This would not be the case for the $G_{LM}{}^{<\pi>}$ although they satisfy Lloyd's theorem in the above sense. That the reduced matrix elements of electromagnetic multipole operators themselves can be made real was shown in a most general form by Biedenharn and Rose (cf. Bi 53). We like to emphasize that the choice of phase of the operators $T_{LM}{}^{<\pi>}$ is essential for a comparison of the mixing ratios with model calculations. However, if the eigenfunctions do not transform according to $\boldsymbol{\theta} \mid JM \rangle = (-)^{J-M} \mid J-M \rangle$ (but for example according to $\boldsymbol{\theta} \mid JM \rangle = (-)^{p-M} \mid J-M \rangle$ with p arbitrary) then the mixing ratio is still real and its phase is not influenced by this different choice. We can therefore omit all asterisks on mixing ratios which are composed of reduced matrix elements of the operators $T_{LM}{}^{<\pi>}$. It is still useful and desirable to adopt a choice of phase for the eigenfunctions according to Eq. (2.23), even for the case when formulas are quoted in terms of mixing ratios.

in terms of mixing ratios when circular polarization is not observed

$$W(\theta)=\sum_{LL'\pi\pi'K}\{B_K(J_1)R_K(LL'J_1J_2)P_K(\cos\theta)\times[1+(-1)^{L+L'+\pi+\pi'-K}]\delta_L{}^{<\pi>}\delta_{L'}{}^{<\pi'>}\}/2\sum_{L\pi}|\delta_L{}^{<\pi>}|^2. \tag{3.40}$$

The normalization of $W(\theta)$ is chosen so that the coefficient of $P_0(\cos\theta)$ in the expansion of $W(\theta)$ is unity.

Formula (3.40) is restricted to the case that the initial state has a definite J_1 and has an axis of symmetry. It may be polarized or aligned and it is not even necessary that the initial state has definite parity.

We speak of an alignment of the initial state J_1 if

$$w(-M_1)=w(M_1)$$

and call this the *alignment condition.* If this condition is satisfied then it follows directly from definition (3.28) that all $B_K(J_1)$ vanish when $K=$odd. Consequently only terms with even K can occur in the angular distribution. This is true regardless of whether circular polarization is observed or not [i.e., for Eqs. (3.32) (3.40)].

If the alignment condition is not satisfied, i.e. when for example the $+M_1$ substates are stronger populated than the $-M_1$ ones, we speak of a *polarization* of the initial state J_1. Then there is no restriction on K from $B_K(J_1)$; K may be odd and even. If circular polarization is observed, then Eq. (3.32) must be used. Odd order K terms can occur regardless of whether the initial state J_1 (and the final, J_2) has definite parity or not, and in general the distribution of right-hand circular polarized quanta will be different from that of left-hand circular polarized quanta. However, the odd order K terms can propagate into the angular distribution of the gamma rays when circular polarization is *not* observed [i.e., into Eq. (3.40)] *only* when the states do not have definite parity.

If the initial (and also the final) state has definite parity, then formula (3.40) may be simplified. Because the electromagnetic interaction conserves parity the sum in Eq. (3.40) [as well as in Eq. (3.32)] no longer runs independently over all possible $LL'\pi\pi'$ because there is a fixed relation between the L values and π values which is:

(a) If J_1 and J_2 have the same parity, then in combination with $\pi=0$ only even values of L can occur, while in combination with $\pi=1$ only odd values of L can occur.

(b) If J_1 and J_2 have opposite parities, then in combination with $\pi=0$ only odd values of L can occur, while in combination with $\pi=1$ only even values of L can occur. In other words, for case (a) only $M1$, $E2$, $M3$, etc. are allowed and in case (b) only $E1$, $M2$, $E3$, etc. are allowed.

Hence all terms appearing in Eq. (3.40) have $L+L'+\pi+\pi'=$even. The factor $[1+(-1)^{L+L'+\pi+\pi'-K}]$ is zero if K is odd and 2 if K is even. Hence Eq. (3.40) reduces to

$$W(\theta)=\sum_{\substack{(L\pi)(L'\pi')\\ K\text{ even}}}\frac{\{B_K(J_1)R_K(LL'J_1J_2)\delta_L{}^{<\pi>}\delta_{L'}{}^{<\pi'>}P_K(\cos\theta)\}}{\sum_L|\delta_L{}^{<\pi>}|^2}. \tag{3.41}$$

Thus if polarization is not observed, and if all nuclear states have definite parity, then the angular distribution contains only terms with even K, and the odd K values drop out. The "unpolarized distribution" of the gamma ray cannot reflect the polarization of the initial state. Note that:

(i) Formula (3.41) is restricted to the case that J_1 and J_2 have definite parity.

(ii) The sum over $(L\pi)$ is over all multipoles consistent with conservation of angular momentum and parity. The allowed values of L, L' are

$$|J_1-J_2|\leq L,\ L'\leq(J_1+J_2),\qquad L,\ L'\neq0.$$

(iii) The sum is taken over even values of K only. Even if the initial state is polarized and not just aligned [i.e., $B_K(J_1)\neq0$ for $K=$odd] this is not reflected in the "unpolarized" gamma-ray angular distribution.

With the above restrictions, we can also rewrite formula (3.35) and obtain the transition probability per unit time and unit solid angle in terms of "mixing products"

$$P(\mathbf{k})=\frac{k}{\pi\hbar}\sum_{\substack{(L\pi)(L'\pi')\\ K\text{ even}}}B_K(J_1)R_K(LL'J_1J_2)P_K(\cos\theta)\times\frac{\langle J_1\,\|\,\mathbf{T}_L{}^{<\pi>}\,\|\,J_2\rangle}{(2L+1)^{1/2}}\frac{\langle J_1\,\|\,\mathbf{T}_{L'}{}^{<\pi'>}\,\|\,J_2\rangle}{(2L'+1)^{1/2}}. \tag{3.42}$$

It is possible to remove from this formula the restriction that J_1 has definite parity. One then has to equip it with an "*ad hoc* recipe" of how to use it. This "*ad hoc* recipe" is sometimes worked into Z coefficients [which replace $R_K(LL'J_1J_2)$]. This approach can lead to confusion of phases (compare Sec. V).

Clearly formula (3.41) consists of two multiplicative parts. One, $B_K(J_1)$, depends on the nuclear alignment only. The other,[18]

$$R_K(J_1J_2)=\sum_{(L\pi)(L'\pi')}\frac{\{R_K(LL'J_1J_2)\delta_L{}^{<\pi>}\delta_{L'}{}^{<\pi'>}\}}{\sum_L|\delta_L{}^{<\pi>}|^2} \tag{3.43}$$

depends only on quantities which characterize the nuclear transition.

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[18] A similar notation was introduced in (Po 65).

Formula (3.41) gives the angular distribution in terms of mixing ratios for a gamma ray originating from the decay of the state J_1, whose alignment was described by $B_K(J_1)$. Quite frequently the decay of the state J_1 proceeds through a gamma cascade to the ground state and it is desirable to obtain a distribution formula for any one of the subsequent gamma rays in terms of the initial populations $w(M_1)$ of the magnetic substates of J_1. We specify our notation in Fig. 1. The observed gamma ray is the one without subscripts, the others are assumed to be unobserved. For the case displayed in Fig. 1 the state J_2 is the initial state and J_3 the final for the gamma ray to be observed. The populations $w(M_2)$ of J_2 can easily be worked out in terms of the populations $w(M_1)$ of the magnetic substates of J_1. Following closely the notation of (Po 65) one can introduce the coefficient

$$U_K(J_1J_2)=\sum_{L_{12}}(\delta_{L_{12}})^2U_K(L_{12}J_1J_2)\Big/\sum_{L_{12}}(\delta_{L_{12}})^2, \qquad (3.44)$$

where

$$\begin{aligned}U_K(L_{12}J_1J_2)&=(-)^K\frac{W(J_1J_2J_1J_2;\,L_{12}K)}{W(J_1J_2J_1J_2;\,L_{12}0)} \qquad (3.45)\\ &=(-)^K\frac{W(J_1J_1J_2J_2;\,KL_{12})}{W(J_1J_1J_2J_2;\,0L_{12})}.\end{aligned}$$

Tables of these coefficients are given in the Appendix. Clearly $U_0(L_{12}J_1J_2)=1$, and $U_K(L_{12}J_1J_2)=U_K(L_{12}J_2J_1)$. With the restrictions under which formula (3.41) is valid, $K=$even and the phase factor $(-)^K$ may be omitted. As a consequence of the nonobservation of L_{12} there are no interference terms between different multipoles in Eq. (3.44). The distribution formula for the second gamma ray, using the above notations, is then

$$W(\theta)=\sum_K B_K(J_1)\,U_K(J_1J_2)R_K(J_2J_3)P_K(\cos\theta) \qquad (3.46)$$

and a generalization of this formula in order to obtain an expression for the nth gamma ray in a cascade can simply be made by putting further coefficients $U_K(J_iJ_{i+1})$ into Eq. (3.46) for each preceding nonobserved gamma ray.

To conclude this section we make a final specialization to the most common case in practice when only two multipoles contribute to the transition. Then formula (3.41) becomes simply ($\bar{L}$ and L being the lowest order multipoles in ascending order, respectively)

$$W(\theta)=\sum_K B_K(J_1)P_K(\cos\theta)\times\frac{\{R_K(\bar{L}\bar{L}J_1J_2)+2\delta R_K(\bar{L}LJ_1J_2)+\delta^2R_K(LLJ_1J_2)\}}{(1+\delta^2)} \qquad (3.47)$$

with

$$\delta=\frac{\langle J_1\,||\,\mathbf{T}_L{}^{<\pi>}\,||\,J_2\rangle/(2L+1)^{1/2}}{\langle J_1\,||\,\mathbf{T}_{\bar{L}}{}^{<\bar{\pi}>}\,||\,J_2\rangle/(2\bar{L}+1)^{1/2}}. \qquad (3.48)$$

Similarly Eq. (3.44) reduces to

$$U_K(J_1J_2)=\frac{\{U_K(\bar{L}_{12}J_1J_2)+(\delta_{L_{12}})^2U_K(L_{12}J_1J_2)\}}{[1+(\delta_{L_{12}})^2]}, \qquad (3.49)$$

with

$$(\delta_{L_{12}})^2=\left|\frac{\langle J_1\,||\,\mathbf{T}_{L_{12}}{}^{<\pi_{12}>}\,||\,J_2\rangle/(2L_{12}+1)^{1/2}}{\langle J_1\,||\,\mathbf{T}_{\bar{L}_{12}}{}^{<\bar{\pi}_{12}>}\,||\,J_2\rangle/(2\bar{L}_{12}+1)^{1/2}}\right|^2. \qquad (3.50)$$

F. Methods of Alignment and Population Parameters in Special Cases

The coefficients $B_K(J_1)$ describe the nuclear alignment and depend on the population parameters $w(M_1)$ of the magnetic substates of the initial state J_1 according to Eq. (3.28):

$$B_K(J_1)=\sum_{M_1}w(M_1)(-)^{J_1-M_1}(2J_1+1)^{1/2}\times(J_1J_1M_1-M_1\,|\,K0), \qquad (3.28)$$

$$B_0(J_1)=1.$$

The sum over M_1 runs from $-J_1$ to $+J_1$ and the normalization of the population parameters $w(M_1)$ is such that

$$\sum_{M_1}w(M_1)=1.$$

The values which the population parameters may assume in a specific case depend on the specific process by which the state $|J_1M_1\rangle$ is formed. In general we can only say that if the state $|J_1M_1\rangle$ is formed in a nuclear reaction of any kind and if neither the beam of the incident particles nor the target nuclei are polarized then there usually will be alignment with respect to the beam axis, if $|J_1M_1\rangle$ has definite parity and if the beam axis provides an axis of cylinder symmetry.

The interpretation of gamma-ray angular distributions and correlations following particle–particle reactions in terms of undetermined population parameters of the magnetic substates of the initial state J_1 and the determination of these parameters from a combination of measurements in order to determine the spin

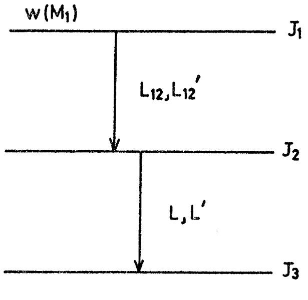

FIG. 1. Notations for the case that the initially populated state J_1 decays via a cascade. L, L' is observed, L_{12}, L_{12}' is unobserved.

of a nuclear state was first made by Warburton and Rose (Wa 58). The method was then generalized and extended by Litherland and Ferguson (Li 61) and has since been applied to numerous determinations of nuclear spins and mixing ratios of nuclear gamma-ray transitions. The present treatment of the "cylinder-symmetrical case" in terms of phase-defined operators and reduced matrix elements, which has yielded formula (3.27) (and its subsequent specializations), in practice mainly applies to what is called "method II" by Litherland and Ferguson. In this method the number of open parameters is quite restricted by the detection of particles in the direction of the axis of cylindrical symmetry. In the following paragraphs we discuss some of the "typical" practical applications of the cylinder-symmetrical case, where the population parameters are rather restricted or even completely determined. These are the cases: (i) The alignment is achieved by particle capture. (ii) The alignment is achieved by a particle–particle reaction. (iii) The alignment is achieved by observation of a gamma ray originating from a randomly populated state and followed by further gamma rays ("source case"). We do not describe methods of analysis in this paper. This is done for the above cases in detail in a laboratory report, in which also the technique of computer analysis used at Oxford are described in detail (Ha 66). The report is in all phases in accord with the present article.

1. The Alignment is Achieved by Resonant Capture of a Particle from the Initial Beam

The resonant capture of a particle h_1 from the incident beam by the target X leads to an excited state Y^* of the compound nucleus Y so that Y^* is a state with definite angular momentum J_1 and parity. If the beam is unpolarized and the target nuclei are randomly orientated then the beam direction is an axis of symmetry and may be conveniently taken as quantization axis. If only one partial wave l and one channel spin s contribute to the formation of the compound state Y^* then the population parameters are uniquely determined. The channel spin s and the orbital angular momentum l ($m_l=0$) couple to give the compound nucleus angular momentum (J_1M_1). Consequently

$$w(M_1)\sim(slM_10 \mid J_1M_1)^2.$$

Remembering the normalization requirement

$$\sum_M w(M)=1$$

and the symmetry properties of the CG coefficients we can write this as

$$w(M_1)=(sJ_1M_1-M_1 \mid l0)^2. \qquad (3.51)$$

If more than one channel spin can contribute to the formation of J_1 the contributions of different channel spins add incoherently and we can introduce the channel spin intensities $T(s)$ to describe the relative contributions to the population parameters $w(M_1)$ by multiplying the respective contribution from each s by $T(s)$, i.e., (cf. Li 61)

$$w(M_1)=\sum_s (sJ_1M_1-M_1 \mid l0)^2T(s). \qquad (3.52)$$

We then obtain for the coefficients $B_K(J_1)$ which describe the alignment with respect to the axis of cylinder symmetry

$$B_K(J_1)=\sum_s (-)^{s-J_1}[(2l+1)^{1/2}]^2(2J_1+1)^{1/2} \times(ll00 \mid K0)W(J_1J_1ll;Ks)T(s). \qquad (3.53)$$

The normalization of channel spin intensities is

$$\sum_s T(s)=1. \qquad (3.54)$$

We can remove the condition that the state $\mid J_1M_1\rangle$ is formed by one partial wave l of the incoming particle beam only. Since in practice usually not more than two partial waves need to be taken into consideration[19] we avoid writing the formula in its most general form (the extension to more than two partial waves is trivial).

If two partial waves l and l' contribute to the formation with channel spin s then clearly

$$w(M_1)=\mid (sJ_1M_1-M_1 \mid l0)A(l) +(sJ_1M_1-M_1 \mid l'0)A(l') \mid^2T(s). \qquad (3.55)$$

Here $A(l)$ and $A(l')$ are the amplitudes with which the partial waves l and l' contribute to the formation of $\mid J_1M_1\rangle$ through channel spin s. These particle absorption amplitudes are proportional to the reduced matrix elements for absorbing a particle specified by $\mathbf{l}$, $\mathbf{s}$ from the incident beam by a target nucleus having total angular momentum $\mathbf{J}_X$ to form $\mathbf{J}_1$. These matrix elements are in general not real (nor pure imaginary) but complex numbers and therefore $A(l)$ and $A(l')$ are complex, i.e.,

$$A(l)=\mid A(l)\mid \exp(i\phi_l);$$
$$A(l')=\mid A(l')\mid \exp(i\phi_{l'}). \qquad (3.56)$$

The normalization is

$$\mid A(l)\mid^2+\mid A(l')\mid^2=1. \qquad (3.57)$$

Because the interaction which causes the particle capture is not known (unless specific models are assumed) no specific conclusions about the (relative) magnitude and phases of $A(l)$ and $A(l')$ can be drawn (unlike the case of absorption or emission of electromagnetic quanta, where the interaction is known). For model calculations Eq. (3.55) is to be taken as a *definition* of the amplitudes $A(l)$ and $A(l')$.

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[19] The assumption that $\mid J_1M_1\rangle$ corresponds to an isolated resonance with defined spin and parity requires that $\mid l-l'\mid = 2, 4\cdots$.

From Eq. (3.28) we obtain for the coefficients $B_K(J_1)$, which describe the alignment with respect to the axis of cylinder symmetry,

$$B_K(J_1)=\sum_s \{(-)^{s-J_1}[(2l+1)^{1/2}]^2(2J_1+1)^{1/2}(ll00\mid KO)W(J_1J_1ll;Ks)\mid A(l)\mid^2$$

$$+(-)^{s-J_1}\,2(2l+1)^{1/2}(2l'+1)^{1/2}(2J_1+1)^{1/2}(ll'00\mid KO)W(J_1J_1ll';Ks)\mid A(l)\mid\mid A(l')\mid\cos(\phi_l-\phi_{l'})$$

$$+(-)^{s-J_1}[(2l'+1)^{1/2}]^2(2J_1+1)^{1/2}(l'l'00\mid KO)W(J_1J_1l'l';Ks)\mid A(l')\mid^2\}T(s). \quad (3.58)$$

Equation (3.58) shows that for each channel spin which contributes, two undetermined parameters enter $B_K(J_1)$. These are the relative magnitude of the amplitudes $A(l)$, $A(l')$ and the cosine of the relative phases $\phi_l-\phi_{l'}$.

With the notation

$$S_K(ll'J_1s)=(-)^{s-J_1}(2l+1)^{1/2}(2l'+1)^{1/2}(2J_1+1)^{1/2}(ll'00\mid KO)W(J_1J_1ll';Ks), \quad (3.59)$$

we can rewrite Eq. (3.58)

$$B_K(J_1)=\sum_s \{S_K(llJ_1s)\mid A(l)\mid^2+2S_K(ll'J_1s)\mid A(l)\mid\mid A(l')\mid\cos(\phi_l-\phi_{l'})+S_K(l'l'J_1s)\mid A(l')\mid^2\}T(s). \quad (3.58')$$

The coefficient $S_K(ll'J_1s)$ is tabulated in the Appendix. It is related to the various Z coefficients in use (cf. Table II). $S_0(llJ_1s)=1$ and $S_0(ll'J_1s)=0$ for $l\neq l'$. Therefore, because of $\mid A(l)\mid^2+\mid A(l')\mid^2=1$ and $\sum T(s)=1$, it follows that $B_0(J_1)=1$. $S_K(ll'J_1s)=S_K(l'lJ_1s)$ for $K=$even.

An estimate of the relative phase $\phi_l-\phi_{l'}$ can sometimes be useful. If we consider the target nucleus to be well described by the model of a hard-core scatterer with a Coulomb field the relative phase is given by

$$\phi_l-\phi_{l'}=-\arctan[F_l(R)/G_l(R)]+\sigma_l+\arctan[F_{l'}(R)/G_{l'}(R)]-\sigma_{l'}+n\pi \quad (3.60)$$

n is an integer, but one requires a phase–consistent theory of nuclear reactions to decide its numerical value. $F_l(R)$ and $G_l(R)$ are the regular and irregular solutions of the Coulomb wave equation and $\sigma_l=$argument $\Gamma(l+1+i\eta)$, $\eta=(Z_1Z_2e^2/\hbar v)$. The physics is now distilled into the choice of the channel radius R. However, the arbitrariness of this choice is limited, and useful limits on $\phi_l-\phi_{l'}$ may be obtained in this way.

We may now go one step further in this case of particle absorption and remove the assumption again that $\mid J_1M_1\rangle$ has definite parity. Equation (3.55) shows that the alignment condition $w(M_1)=w(-M_1)$ breaks down if $\mid J_1M_1\rangle$ does not have definite parity. Calculating

$$w(-M_1)=\mid (sJ_1-M_1M_1\mid l0)A(l)+(sJ_1-M_1M_1\mid l'0)A(l')\mid^2$$

$$=\mid (-)^{s+J_1-l}\{(sJ_1M_1-M_1\mid l0)A(l)+(-)^{l-l'}(sJ_1M_1-M_1\mid l'0)A(l')\}\mid^2 \quad (3.61)$$

and comparing it to Eq. (3.55) it is seen that *only* when $l-l'=$even (i.e., definite parity of J_1) $w(-M_1)=w(M_1)$. Consequently, if J_1 has mixed parity (i.e., $l-l'$ can be odd) the distribution of the right-hand circular polarized quanta $P^1(\mathbf{k})$ is no longer identical with that of the left-hand circular polarized quanta; i.e., $P^1(\mathbf{k})\neq P^{-1}(\mathbf{k})$ (compare Sec. IIIE) and the nonpolarized distribution can no longer be represented by formulas (3.41) or (3.42). One must go back to Eq. (3.40) or (3.35), respectively.

Table II. R_K and S_K coefficients in terms of previously defined coefficients.

(1) $R_K(LL'J_1J_2)$
$=(-)^{J_2-J_1-K/2}i^{L-\pi-L'+\pi'}Z_1(LJ_1L'J_1;\,J_2K)/(2J_1+1)^{1/2}$
(2) $R_K(LL'J_1J_2)=(-)^{J_2-J_1}\bar{Z}_1(LJ_1L'J_1;\,J_2K)/(2J_1+1)^{1/2}$
(3) $R_K(LL'J_1J_2)=(-)^{L-L'+K}F_K(LL'J_2J_1)$
(4) $S_K(ll'J_1s)=(-)^{s-J_1}i^{l-l'-K}Z(lJ_1l'J_1;\,sK)/(2J_1+1)^{1/2}$
(5) $S_K(ll'J_1s)=(-)^{s-J_1}\bar{Z}(lJ_1l'J_1;\,sK)/(2J_1+1)^{1/2}$

However, remembering the discussion following Eq. (3.40) we can determine the composition of the angular distribution in this case of nondefined parity *when circular polarization is not observed.* There will be interference terms of multipoles with equal parity as well as with opposite parity. The former occur in combination with terms of even K only, the latter only in combination with odd K. Looking now at Eq. (3.58) for $B_K(J_1)$ and assuming that $l-l'=$odd (i.e., no definite parity of J_1), we notice that the mixed term containing $\mid A(l)\mid\mid A(l')\mid\cos(\phi_l-\phi_{l'})$ will give rise to odd values of K only, while the ones containing $\mid A(l)\mid^2$ and $\mid A(l')\mid^2$ give rise to even K only. Consequently in combination with the "particle interference term" only multipole mixture terms of opposite parity occur, and in combination with the "clean" partial-wave intensities $\mid A(l)\mid^2$ and $\mid A(l')\mid^2$ only multipole mixture terms of equal parity occur.

Formulas (3.41) and (3.42) describe the angular distribution when polarization is not observed under the restriction that the initial state J_1 is aligned (but not polarized) and that J_1 has definite parity.

The above discussion shows that formula (3.41) and likewise (3.42), although they are restricted formulas, can still be applied *provided one equips them with an "ad hoc recipe" of how to use them.* This recipe is: Remove again the relation between L and π as fixed by parity conservation; then, for all *even K omit* the interference terms of all multipoles with *different parity* but *keep* the ones with *equal parity*; for all *odd K omit* the interference terms of all multipoles with *equal parity* but *keep* the ones with *different parity.* This recipe follows directly from Eq. (3.35) or (3.40). The extension to cascades for the usual case, that all nuclear states have definite parity and that circular polarization is not observed, is again made using the $U_K(J_iJ_{i+1})$ coefficients as described in Sec. IIIE.

2. *The Alignment is Achieved by a Particle–Particle Reaction*

By a particle–particle reaction we mean a reaction of the type $X(h_1h_2)Y^*$, where X is the target nucleus, h_1 the incident particle and h_2 the outgoing particle perhaps after the formation of some undefined compound state) leaving the final nucleus in an excited state Y^* which is the initial state $|J_1M_1\rangle$ for subsequent gamma transitions.

If the particle h_2 is not detected, the cylindrical symmetry (in this case the beam axis) of the problem is not destroyed. In order to predict the population parameters $w(M_1)$ uniquely in this quite general case, a complete knowledge of the process itself would be required, which usually is not available. If beam and target nuclei are unpolarized and if J_1 has definite parity, then always $w(M_1)=w(-M_1)$; i.e., the state J_1 is aligned but not polarized. The alignment condition $w(M_1)=w(-M_1)$ and the normalization $\sum w(M_1)=1$ together reduce the number of undetermined parameters entering the coefficients $B_K(J_1)$. If the spin J_1 is integer then only J_1 open parameters remain; if J_1 is odd, only $J_1-\frac{1}{2}$ ones remain.

The expression (3.28) for the coefficients $B_K(J_1)$ can be somewhat simplified for the case of alignment. With the normalization $\sum w(M_1)=1$ and $w(M_1)=w(-M_1)$ we can replace the summation from $M_1=-J_1$ to $M_1=J_1$ by a summation from $M_1=0$ to $M_1=J_1$ and Eq. (3.28) becomes

$$B_K(J_1)=\sum_{M_1=0\text{ or }1/2}^{M_1=J_1} w(M_1)(2-\delta_{M_1,0})(-)^{J_1-M_1}\times(2J_1+1)^{1/2}(J_1J_1M_1-M_1\mid KO)$$
$$=\sum_{M_1=0\text{ or }1/2}^{M_1=J_1} w(M_1)\rho_K(J_1M_1). \qquad (3.62)$$

Obviously $B_0(J_1)=1$ and $B_K(J_1)=0$ for K odd. The coefficients

$$\rho_K(J_1M_1)=(2-\delta_{M_1,0})(-)^{J_1-M_1}(2J_1+1)^{1/2}\times(J_1J_1M_1-M_1\mid KO) \qquad (3.63)$$

are identical with the ones defined in Eq. (3) of Poletti and Warburton (Po 65) [so is $B_K(J_1)$ with $\rho_K(a)$ of this reference]. The $\rho_K(J_1M_1)$ were tabulated by these authors. For convenience we include in the Appendix a tabulation for spin values in the range $0\leq J_1\leq 10$ and $0\leq J_1\leq 19/2$.

In what is called "method II" (Li 61) the particles h_2 are detected at 0 or 180° with respect to the incident beam of particles h_1 (polarization of h_2 is of course not detected). Then neither the cylindrical symmetry of the problem nor the alignment condition is destroyed. The remaining undetermined population parameters (cf. the preceding paragraph) have a further restriction. Because of the detection of h_2 coaxial with h_1, no contributions to the possible M_1 values can come from orbital angular momenta; obviously then only those magnetic substates M_1 can be populated which fulfill

$$|M_1|\leq J_X+s_{h_1}+s_{h_2}. \qquad (3.64)$$

For example, if the target nucleus is a $J_X=0$ nucleus, h_1 a proton (or ^{3}He) and the outgoing particle h_2 an α particle, then only $M_1=\pm\frac{1}{2}$ can be populated. Since $w(\frac{1}{2})=w(-\frac{1}{2})$, the population parameters are even uniquely determined in this case. In less favorable cases, when the outgoing particle h_2 does not have $s_{h_2}=0$ or when $J_X\neq 0$, all M_1 values permitted by Eq. (3.64) can be assumed. However, even in these less favorable cases the experiment can lead to conclusions about spins and mixing ratios involved in the transition $J_1\rightarrow J_2$ as has been widely demonstrated in recent years. Sometimes the state J_1 shows further decay modes and a determination of the remaining population parameters is possible from the observation of the angular distributions of these additional gamma rays.

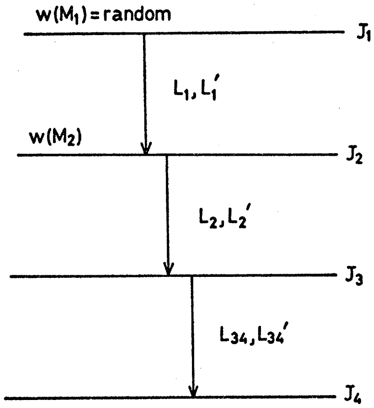

Fig. 2. Notations for the case of a gamma cascade originating from a randomly populated state J_1. The alignment of J_2 is achieved by observation of the transition J_1-J_2.

3. The Alignment is Achieved by Observation of a Gamma Ray

By this we mean that an initial state J_1 is randomly populated and then decays by emission of a sequence of two or more gamma rays (gamma cascade).

First we treat the simple case that the two gamma rays which are observed (in coincidence) are the first and second one in the cascade originating from the decay of J_1. This case with the respective notations is shown in Fig. 2. Again double subscripts mean that the respective gamma rays are not observed. If J_1 is randomly populated then no direction in space is preferred. We may choose the direction of propagation of the first gamma ray emitted as the axis of quantization (z axis). The magnetic substates M_1 of J_1 may be specified with respect to this axis and certainly the eigenfunctions $|J_1M_1\rangle$ can be chosen so that they are eigenstates of $\mathbf{J}_z$ (more elegantly—although we have not made explicit use of the density matrix formalism in this article, because it is not necessary for the treatment of cylinder-symmetrical problems—the density matrix $\rho_{J_1M_1,J_1M_1'}$ is diagonal). The population parameter $w(M_1)$ is then just the statistical weight $(2J_1+1)^{-1}$.

The circular polarization of a gamma ray is specified with respect to its direction of propagation. The first gamma ray serves as quantization axis, therefore its circular polarization quantum number q refers to the axis of quantization. Consequently the first gamma ray can change the magnetic quantum number M_1 only by ± 1, regardless of what its angular momentum is. Since in principle we could measure its circular polarization we can in principle strictly relate to each value M_1 one (and only one) value of M_2, namely, $M_2+q=M_1$. Since M_1 was sharp with respect to z, and q is sharp with respect to z, so will then be M_2. Or in other words, a set $|J_2M_2\rangle$ exists such that M_2 is an eigenvalue of $\mathbf{J}_{2z}$ (again more elegantly, if $\rho_{J_1M_1,J_1M_1'}$ is diagonal, so is $\rho_{J_2M_2,J_2M_2'}$, provided there is no other preselected axis in space but the one set by the direction of propagation of the gamma quantum originating from the transition $J_1\rightarrow J_2$). If only one of the further subsequent gamma rays is detected (at an angle θ relative to the first one) then the above quantization axis is an axis of cylindrical symmetry. Consequently the angular distribution of the observed second gamma ray ($J_2\rightarrow J_3$) can be described by formula (3.32), with the respective replacements of J_1 by J_2, and J_2 by J_3. Again, most importantly, the distribution formula is split up into two multiplicative parts, one which depends on the populations $w(M_2)$ of the state J_2 only, and another one which depends on the observed transition $J_2\rightarrow J_3$ only. All that we are left with is to calculate the populations $w(M_2)$.

After what we have said above this is easily done. Take the probability amplitude $A^{q*}{}_{M_1M_2}(0)$ for a transition from M_1 to M_2 with emission of a circularly polarized gamma ray of polarization q in the z direction (compare Sec. IIIC). Since for a definite q there is only one M_2 corresponding to each M_1, the probability for populating M_2 from M_1 is obtained by taking its square modulus, then weight each of these probabilities with the statistical weight of M_1, i.e., multiply by $w(M_1)=(2J_1+1)^{-1}$ and sum over all M_1. This gives

$$w^{q_1}(M_2)\sim\sum_{M_1}(2J_1+1)^{-1}\,|\,A^{q_1}{}_{M_1M_2}(0)\,|^2. \qquad (3.65)$$

The index q_1 indicates that this is the population achieved if the circular polarization of the populating gamma rays is measured. Using Eq. (3.24) we obtain

$$\begin{aligned} w^{q_1}(M_2)&\sim(k_1/2\pi\hbar)\,(2J_1+1)^{-1}\sum_{M_1}\sum_{L_1L_1',\pi_1\pi_1'}q_1{}^{\pi+\pi'}\langle J_1M_1(|\,T_{L_1q_1}{}^{<\pi_1>}\,|\,J_2M_2\rangle\,\langle J_1M_1\,|\,T_{L_1'q_1}{}^{<\pi_1'>}\,|\,J_2M_2\rangle^* \\ &\sim(k_1/2\pi\hbar)\,(2J_1+1)^{-1}\sum_{M_1}\sum_{L_1L_1',\pi_1\pi_1'}(J_2L_1M_2q_1\,|\,J_1M_1)\,(J_2L_1'M_2q_1\,|\,J_1M_1) \\ &\qquad\times q_1{}^{\pi_1+\pi_1'}\langle J_1\,||\,\mathbf{T}_{L_1}{}^{<\pi_1>}\,||\,J_2\rangle\,\langle J_1\,||\,\mathbf{T}_{L_1'}{}^{<\pi_1'>}\,||\,J_2\rangle^*. \qquad (3.66)\end{aligned}$$

Again with the phase conventions for the eigenfunctions under time reversal according to Eq. (2.23) the reduced matrix elements are real and the asterisk may be omitted.

The calculation of $B_K(J_2)$ is straightforward, observing that in the definition (3.28) the quantities J_1, M_1 have now to be replaced by J_2, M_2. Inserting $w(M_2)$ from Eq. (3.66) into the respective expression for $B_K(J_2)$ obtained from (3.28) and shuffling around some CG coefficients we obtain

$$\begin{aligned} B_K{}^{q_1}(J_2)&\sim\frac{k_1}{2\pi\hbar}\,(-)^{J_2-J_1-q_1}(2J_2+1)^{1/2}\sum_{L_1'L_1',\pi_1'\pi_1'}(L_1L_1'q_1-q_1\,|\,KO) \\ &\qquad\times W(J_2J_2L_1L_1';KJ_1)\,q_1{}^{\pi_1+\pi_1'}\langle J_1\,||\,\mathbf{T}_{L_1}{}^{<\pi_1>}\,||\,J_2\rangle\,\langle J_1\,||\,\mathbf{T}_{L_1'}{}^{<\pi_1'>}\,||\,J_2\rangle^*. \qquad (3.67)\end{aligned}$$

This is the "statistical tensor" for the cylinder–symmetric case as produced when the polarization q_1 of the populating gamma ray is observed. We can write formula (3.67) in a more compact form using the coefficient $R_K{}^q(LL'JJ)$ as defined by Eq. (3.31). We get

$$B_K{}^{q_1}(J_2)\sim\frac{k_1}{2\pi\hbar}\sum_{L_1L_1',\pi_1\pi_1'}(-)^{L_1-L_1'+K}R_K{}^{q_1}(L_1L_1'J_2J_1)\,q_1{}^{\pi_1+\pi_1'}\frac{\langle J_1\,||\,\mathbf{T}_{L_1}{}^{<\pi_1>}\,||\,J_2\rangle}{(2L_1+1)^{1/2}}\,\frac{\langle J_1\,||\,\mathbf{T}_{L_1'}{}^{<\pi_1'>}\,||\,J_2\rangle^*}{(2L_1'+1)^{1/2}}. \qquad (3.68)$$

If the proportional sign in Eq. (3.68) were replaced by an equality, $B_K^{q_1}(J_2)$ would not be normalized in the way required by Eq. (3.28), i.e., $B_0^q(J_2)=1$. We can make this normalization following along the lines which led to Eq. (3.41), i.e., we introduce the mixing ratios according to definition (3.39). We restrict ourselves to the quotation of formulas which are valid when all states involved have definite parity. We then obtain from formula (3.68)

$$B_K^{q_1}(J_2)=\sum_{(L_1\pi_1)(L_1'\pi_1')}\{(-)^{L_1-L_1'+K}\,R_K^{q_1}(L_1L_1'J_2J_1)q_1^{\pi_1+\pi_1'}\,\delta_{L_1}^{<\pi_1>}\,\delta_{L_1'}^{\langle\pi_1'\rangle}\}/\sum_{N_1}(\delta_{N_1}^{<\pi_1>})^2. \qquad (3.69)$$

In the case that the population of the initial state was determined by the absorption of a particle from a *non-polarized* beam the expression for $B_K(J_1)$ was governed by a coefficient $S_K(ll'J_1s)$. This coefficient was proportional to a CG coefficient $(ll'00\,|\,KO)$ and the appearance of odd K values was bound to J_1 not having definite parity because $(ll'00\,|\,KO)=0$ for $l+l'+K=\text{odd}$. The coefficient $R_K^q(LL'J_2J_1)$ which regulates the population of the state J_2 [Eq. (3.69)] now reflects that the corresponding statistical tensor $B_K^{q_1}(J_2)$ depends on the (circular) polarization of the populating gamma ray. It is proportional to a CG coefficient $(LL'q-q\,|\,KO)$. This coefficient has finite values for $L+L'+K=\text{even}$ integer as well as for $L+L'+K=\text{odd}$. Consequently, even under the restriction of definite parity, odd values of K are allowed. This reflects that the observation of circular polarization of the populating gamma ray leads effectively (i.e., for the observation of subsequent gamma rays *in coincidence* with the populating one) to a polarization of J_2. When the circular polarization of the populating gamma ray is *not* observed, this effect must vanish.

If circular polarization is not observed we sum over q_1 in formula (3.69). This gives

$$B_K(J_2)=\sum_{(L_1\pi_1)(L_1'\pi_1')}\{[(-1)^{L_1-L_1'+K}+(-1)^{\pi_1+\pi_1'}]R_K(L_1L_1'J_2J_1)\,\delta_{L_1}^{<\pi_1>}\,\delta_{L_1'}^{<\pi_1'>}\}/2\sum_{N_1}(\delta_{N_1}^{<\pi_1>})^2. \qquad (3.70)$$

Parity was assumed to be definite for all states. Therefore, if $L_1-L_1'=\text{odd}$, also $\pi_1+\pi_1'=\text{odd}$ $(=1)$. Consequently, unless $K=\text{even}$, the respective terms vanish. Similarly, if $L_1-L_1'=\text{even}$, $\pi_1+\pi_1'=0$, and again K must be even. We therefore obtain

$$B_K(J_2)=\sum_{(L_1\pi_1)(L_1'\pi_1')}\{(-)^{L_1-L_1'}\,R_K(L_1L_1'J_2J_1)\,\delta_{L_1}^{<\pi_1>}\,\delta_{L_1'}^{<\pi_1'>}\}/\sum_{N_1}(\delta_{N_1}^{<\pi_1>})^2$$
$$\text{if } K=\text{even}, \quad \text{and} \quad =0 \quad \text{if } K=\text{odd}. \qquad (3.71)$$

Formula (3.71) is restricted to cases where all nuclear states have definite parity.

The usual case is again that only two multipolarities need to be considered in formula (3.71). Then we get, with $\bar{L}_1$ and L_1 being the lowest possible multipolarities in ascending order,

$$B_K(J_2)=\{R_K(\bar{L}_1\bar{L}_1J_2J_1)+(-)^{\bar{L}_1-L_1}\,2\delta_1R_K(\bar{L}_1L_1J_2J_1)+\delta_1^2\,R_K(L_1L_1J_2J_1)\}/(1+\delta^2),$$

with

$$\delta_1=\frac{\langle J_1\,||\,\mathbf{T}_{L_1}^{<\pi_1>}\,||\,J_2\rangle/(2L_1+1)^{1/2}}{\langle J_1\,||\,\mathbf{T}_{\bar{L}_1}^{<\bar{\pi}_1>}\,||\,J_2\rangle/(2\bar{L}_1+1)^{1/2}}. \qquad (3.72)$$

Inserting $B_K(J_2)$ as given by Eq. (3.71) or (3.72) into Eqs. (3.41) or (3.47) gives the formulas for the angular distribution of the gamma ray associated with the transition $J_2\rightarrow J_3$ with respect to the direction of observation of the gamma ray associated with the transition $J_1\rightarrow J_2$, when J_1 was randomly populated. We quote it, using formula (3.47), for the usual case that the populating and the subsequent gamma ray are contributed to by two multipolarities

$$W(\theta)=\{R_K(\bar{L}_1\bar{L}_1J_2J_1)+(-)^{\bar{L}_1-L_1}\,2\delta_1R_K(\bar{L}_1L_1J_2J_1)+\delta_1^2\,R_K(L_1L_1J_2J_1)\}(1+\delta_1^2)^{-1}$$
$$\times\{R_K(\bar{L}_2\bar{L}_2J_2J_3)+2\delta_2R_K(\bar{L}_2L_2J_2J_3)+\delta_2^2R_K(L_2L_2J_2J_3)\}(1+\delta_2^2)^{-1}P_K(\cos\theta). \qquad (3.73)$$

The extension to the observation of the nth gamma ray in coincidence with the one originating from the transition $J_1\rightarrow J_2$ is made easily using the $U_K(J_iJ_{i+1})$ coefficients as described in Sec. IIIE.

Equation (3.73) shows that the term arising from the mixture of two multipoles in the *populating* transition is multiplied by a phase factor $(-)^{\bar{L}_1-L_1}$. Both mixing ratios δ_1 and δ_2 are of course defined in terms of matrix elements written in the *same* order, i.e.,

$$\delta_1=\frac{\langle J_1\,||\,\mathbf{T}_{L_1}^{<\pi_1>}\,||\,J_2\rangle/(2L_1+1)^{1/2}}{\langle J_1\,||\,\mathbf{T}_{\bar{L}_1}^{<\bar{\pi}_1>}\,||\,J_2\rangle/(2\bar{L}_1+1)^{1/2}},$$

and

$$\delta_2=\frac{\langle J_2\,||\,\mathbf{T}_{L_2}^{<\pi_2>}\,||\,J_3\rangle/(2L+1)^{1/2}}{\langle J_2\,||\,\mathbf{T}_{\bar{L}_2}^{<\bar{\pi}_2>}\,||\,J_3\rangle/(2\bar{L}+1)^{1/2}}.$$

The initial state for each transition stands always on the left.

Biedenharn (Bi 60) has followed the convention of writing the intermediate state (in this case J_2) always on the right. In view of Eq. (3.73) this was a logical convention, because it makes the distribution formula more symmetric. The order of the spins entering the R_K coefficients is for the populating (or the "first") gamma ray (in the cascade) anyway J_2, J_1. When the operators transform under *Hermitian conjugation* as the $T_{LM}^{<\pi>}$ do [cf. Eq. (3.22)] or in a way

$$T_{LM}^{<\pi>+}=(-)^{L-M+\mathrm{const}}\,T_{L-M}^{<\pi>},$$

where *const* is *independent* of L, M, π, hent "switching around" the reduced matrix elements in δ_2 produces a phase factor $(-)^{\bar{L}_2-L_2}$. Rewriting Eq. (3.73) in terms of F_K coefficients (which Biedenharn (Bi 60) uses) all extra phase factors cancel. However, the price one pays for this is that the mixing ratios entering (3.73) have now got *different* definitions. This would not be so serious if there were only gamma–gamma correlations to be investigated. One could adopt it as a convention. However, when triple cascades have to be investigated logic begins to fail already. But more seriously: even in a cylinder–symmetric environment the mixing ratio of the populating gamma transition in the gamma–gamma correlation may be determined from two different experiments:

(a) J_1 is for example populated in a particle–particle reaction [case (ii) of this section]. Then the transition $J_1\rightarrow J_2$ is described by formula (3.41). In this formula the order is J_1, J_2 in the R_K coefficient and "logically" also J_1, J_2 in the reduced matrix elements composing the mixing ratio.

(b) The mixing ratio of the multipoles composing the transition $J_1\rightarrow J_2$ may sometimes also be determined from a gamma-gamma "correlation" experiment to be described by formula (3.73). It is therefore *important* to have a unique definition of mixing ratios throughout, regardless of whether Eq. (3.41) or (3.73) describes the experiment in question (compare also Sec. VB).

IV. REDUCED MATRIX ELEMENTS

A. General Remarks

In favorable cases it is possible to extract the multipole mixing ratios $\delta_L^{<\pi>}$ from experimental gamma-ray angular distributions. The mixing ratios are ratios of reduced matrix elements of the multipole interaction operators $T_{LM}^{<\pi>}$ [cf. definition (3.39)] and these in turn are defined by the Wigner–Eckart theorem as stated in Sec. IIIC from the matrix elements $\langle J_1M_1\mid T_{LM}^{<\pi>}\mid J_2M_2\rangle$ between the initial nuclear state $\mid J_1M_1\rangle$ and the final state $\mid J_2M_2\rangle$. Generally these nuclear wave functions are not known exactly but are represented by a model and the method of calculating the matrix elements depends on the model. It is not possible to summarize results for all models in this review. We consider some general questions and then give in Sec. IVB the reduced matrix elements for single–particle transitions explicitly. In Sec. IVC we then give the reduction formula for two-particle cases in $j-j$ coupling to single-particle reduced matrix elements and a conversion formula from L–S to j–j coupling. Sometimes the complex states J_1 and J_2 can be described as hole states in closed shells. In Sec. IVD we show how reduced matrix elements between such states can be evaluated in a phase–consistent manner.

The choice of the over-all phase of the wave functions $\mid J_1M_1\rangle$ and $\mid J_2M_2\rangle$ is not important. A change in phase of the wave functions only produces a phase change in the transition amplitude $A^q{}_{M_1M_2}(\mathbf{k})$ [Eq. (3.24)]. This phase cancels when the amplitude is squared to obtain the transition probability $P^q(\mathbf{k})$ [Eq. (3.25)]. Alternatively the gamma-ray angular distribution depends only on the mixing ratios $\delta_L^{<\pi>}$ which are ratios of reduced matrix elements [Eq. (3.39)]. Any alteration in the phase of the wave functions of the initial or final state cancels in the mixing ratios. On the other hand the reduced matrix elements are real if the wave functions are required to transform under time reversal according to Eq. (2.23) and, other things being equal, it is good to satisfy this condition consistently. We do this in the special cases considered later in this section.

The choice of phase of the multipole operators is of *vital importance*. Equation (3.27) for the angular distribution is likely to give the wrong results unless the choice of phase of the operators $T_{LM}^{<\pi>}$ defined in Eqs. (3.16), (3.17), and (3.20) is adhered to. The definition of *reduced matrix elements* is also important. Ours are defined by the Wigner–Eckart theorem of Sec. IIIC. The reduced matrix elements of Racah (Ra 42, 43), Edmonds (Ed 57), and Messiah (Me 60) are a factor $(2J_1+1)^{1/2}$ times ours, but have the *same phase*. The numerical factor $(2J_1+1)^{1/2}$ cancels in the mixing ratios. The *order of the states* in the matrix elements is also important. In our convention the initial state always stands to the left and the final state to the right (cf. Sec. II). Because our operators $T_{LM}^{<\pi>}$ have the Hermitian property given in Eq. (3.22) the order of the initial and final states in the reduced matrix element may be changed in a well-defined way. In general for any Hermitian tensor operator O_{LM} which satisfies

$$O_{LM}^{+}=(-)^{C-M}O_{L-M}$$

we have the relation

$$(2J_1+1)^{1/2}\langle J_1\,||\,\mathbf{O}_L\,||\,J_2\rangle = (-)^{J_1-J_2-C}(2J_2+1)^{1/2}\langle J_2\,||\,\mathbf{O}_L\,||\,J_1\rangle^{*}. \quad (4.1)$$

The phase C is arbitrary unless the phase O_{LM} is specified. For the operators $T_{LM}^{<\pi>}$ we have [cf. Eq. (3.22)]

$C=L+1$. Hence

$$(2J_1+1)^{1/2}\langle J_1\,\|\,\mathbf{T}_L^{<\pi>}\,\|\,J_2\rangle$$
$$=(-)^{J_1-J_2-(L+1)}(2J_2+1)^{1/2}\langle J_2\,\|\,\mathbf{T}_L^{<\pi>}\,\|\,J_1\rangle^* \quad (4.2)$$

and when the order of states is changed in the definition of the mixing ratios the relation is

$$\delta_L=\frac{\langle J_1\,\|\,\mathbf{T}_L^{<\pi>}\,\|\,J_2\rangle/(2L+1)^{1/2}}{\langle J_1\,\|\,\mathbf{T}_{\bar{L}}^{<\bar{\pi}>}\,\|\,J_2\rangle/(2\bar{L}+1)^{1/2}}$$
$$=(-)^{\bar{L}-L}\frac{\langle J_2\,\|\,\mathbf{T}_L^{<\pi>}\,\|\,J_1\rangle^*/(2L+1)^{1/2}}{\langle J_2\,\|\,\mathbf{T}_{\bar{L}}^{<\bar{\pi}>}\,\|\,J_1\rangle^*/(2\bar{L}+1)^{1/2}}. \quad (4.3)$$

The ratio of the reduced matrix elements (cf. Ref. 17) is real because of the time-reversal properties of the operators $T_{LM}^{<\pi>}$ [Eq. (3.23)]; but the factor $(-)^{\bar{L}-L}$ must be included if the initial state is written on the right of the reduced matrix elements.

B. Single-Particle Reduced Matrix Elements

Writing the wave function for a spin-$\frac{1}{2}$ particle with spin–orbit coupling in a central field

$$|\tfrac{1}{2}jm\rangle=\sum_\sigma|\tfrac{1}{2}\sigma\rangle u_l(r)i^lY_{l\,m-\sigma}(\theta,\phi)(l\tfrac{1}{2}m-\sigma\sigma\,|\,jm) \quad (4.4)$$

where i^l is included to give the required time-reversal properties [Eq. (2.23)], we can calculate the reduced matrix elements of the operators $T_{LM}^{<\pi>}$. We obtain (cf. Br 62) in terms of the operators (3.17″)

$$\langle l_1\tfrac{1}{2}j_1\,\|\,(\mathbf{Q}_L)_{\text{eff}}\,\|\,l_2\tfrac{1}{2}j_2\rangle=eg_lI(L)b_{12},$$
$$\langle l_1\tfrac{1}{2}j_1\,\|\,\mathbf{Q}_L'\,\|\,l_2\tfrac{1}{2}j_2\rangle=\tfrac{1}{2}g_s\beta kI(L)(a_1-a_2)b_{12}/(L+1),$$
$$\langle l_1\tfrac{1}{2}j_1\,\|\,\mathbf{M}_L\,\|\,l_2\tfrac{1}{2}j_2\rangle=2\beta g_li^{l_1-l_2}(-)^{j_1+1/2}I(L-1)$$
$$\times(l_1l_201\,|\,L1)W(j_1l_1j_2l_2;\tfrac{1}{2}L)[(2j_2+1)(2l_1+1)(2l_2+1)l_2(l_2+1)L]^{1/2}[(2L+1)(L+1)]^{-1/2},$$
$$\langle l_1\tfrac{1}{2}j_1\,\|\,\mathbf{M}_L'\,\|\,l_2\tfrac{1}{2}j_2\rangle=\tfrac{1}{2}g_s\beta I(L-1)(L-a_1-a_2)b_{12}, \quad (4.5)$$

where $I(L)$ is the radial integral

$$I(L)=\int u_1(r)r^{L+2}u_2(r)\,dr,$$

and

$$b_{12}=i^{l_1-l_2}(-)^{j_1-1/2}(2j_2+1)^{1/2}(2L+1)^{-1/2}(j_1j_2\tfrac{1}{2}-\tfrac{1}{2}\,|\,LO),$$
$$a_1=(l_1-j_1)(2j_1+1). \quad (4.6)$$

The sequence of coupling of angular momenta and spins is $\mathbf{l}+\mathbf{s}$ as can be seen from the CG coefficient in Eq. (4.4). The single-particle reduced matrix elements of the operators $G_{LM}^{<\pi>}$ are according to Eqs. (3.20) simply the sums of those of $M_{LM}+M_{LM}'$ or $Q_{LM}+Q_{LM}'$. The matrix elements of the operators $T_{LM}^{<\pi>}$ are readily obtained from Eqs. (4.5) by remembering the definition (3.20) and using formula (3.16) or Table I for the multipole expansion coefficients $\alpha_L^{<\pi>}$. To conserve parity $L+l_1+l_2$ must be even for electric operators and odd for magnetic ones, and it is easily seen then that indeed all reduced matrix elements of the operators $T_{LM}^{<\pi>}$ turn out to be real.

We do not enlist reduced single–particle matrix elements for the special operators $M1$, $E1$, and $E2$ as done in (Br 62). In this reference these especially simple cases are evaluated from Eqs. (4.5) for transitions $j\rightarrow j$, $j\rightarrow j+1$ or $j\rightarrow j+2$. Sometimes the reverse order, for example $j+1\rightarrow j$, is wanted, and "switching around" the order of states in the specialized formula implies that the transformation properties of the operators have to be carefully observed. In a phase–consistent treatment the switching operations might be a source of errors while inserting the states in the correct order ("initial one left") into Eqs. (4.5) is a "foolproof" operation. The first of the matrix elements (4.5) is that of the operator $(Q_{LM})_{\text{eff}}$. It is equal to the one of the "correct" operator Q_{LM} only when the state to the left is higher in energy than that of the state to the right (compare also Secs. IIIB and IVD).

C. Reduction of Two-Particle Reduced Matrix Elements

In this section we give reduced matrix elements for transitions between two-particle states. These may be derived from the general reduction formula for a system composed of two parts [cf. (Br 62)]:

$$\langle j_a(1)j_b(2)J\,\|\,\mathbf{T}_L(1)+\mathbf{T}_L(2)\,\|\,j_c(1)j_d(2)J'\rangle=(2J'+1)^{1/2}(-)^{j_c-j_b-L}$$
$$\times\{\delta(j_bj_d)(-)^J(2j_a+1)^{1/2}W(j_aj_cJJ';Lj_b)\langle j_a\,\|\,\mathbf{T}_L(1)\,\|\,j_c\rangle$$
$$+\delta(j_aj_c)(-)^{J'}(2j_b+1)^{1/2}W(j_bj_dJJ';Lj_a)\langle j_b\,\|\,T_L(2)\,\|\,j_d\rangle\}.$$

The two components of the system have angular momenta j_a, j_b coupled to J in the initial state and j_c, j_d coupled to J' in the final state. The electromagnetic transition operator is $\mathbf{T}_L^{<\pi>}(1)$ for the first component and $\mathbf{T}_L^{\langle\pi\rangle}(2)$ for the second.

We avoid explicit use of isobaric spin quantum numbers in labeling single-particle states and isobaric spin dependence of operators. (The total isobaric spin quantum number is used only in order to distinguish between symmetric and antisymmetric two-nucleon wave functions.) This is to avoid unnecessary complications especially when transformation properties of operators are concerned; also it seems unnecessary to erect a further formalism in a section of which the aim is mainly to provide formulas for practical applications.

The wave function of a system of two nucleons must be symmetric or antisymmetric with respect to exchange of the two particles in order to give eigenstates of isobaric spin. A symmetric wave function has isobaric spin $T=0$ while an antisymmetric wave function has isobaric spin $T=1$ (only the antisymmetric $T=1$ state is allowed for identical nucleons). The symmetrized wave functions are

$$|j_aj_bJMT\rangle=N\{\,|j_a(1)j_b(2)JM\rangle+(-)^T\,|j_a(2)j_b(1)JM\rangle\}. \tag{4.8}$$

The normalization coefficient N of the symmetrized states depends on whether the single-particle states j_a and j_b (or j_c and j_d) are identical.

As

$$|j_a(1)j_b(2)JM\rangle=\sum_{m_am_b}(j_aj_bm_am_b\,|\,JM)\,|j_a(1)m_a(1)\rangle\,|j_b(2)m_b(2)\rangle, \tag{4.9}$$

and similarly for $|j_a(2)j_b(1)JM\rangle$, the normalization constant N is easily determined from

$$\begin{aligned}1&=\langle j_aj_bJMT\,|\,j_aj_bJMT\rangle\\ &=2N^2\{1+(-1)^T\sum_{m_am_bm_a'm_b'}(j_aj_bm_am_b\,|\,JM)(j_aj_bm_a'm_b'\,|\,JM)\langle j_a(1)m_a(1)\,|\,j_b(1)m_b'(1)\rangle\\ &\qquad\times\langle j_b(2)m_b(2)\,|\,j_a(2)m_a'(2)\rangle\}.\end{aligned} \tag{4.10}$$

For $j_a\neq j_b$ we simply obtain $N=1/\sqrt{2}$ and for $j_a=j_b$, $N=\frac{1}{2}$. In this case it also follows from evaluating the right-hand side of Eq. (4.10) that for $T=0$ only odd J, and for $T=1$ only even J values are allowed. Combining the results obtained for the normalization coefficient N we can write

$$N=\sqrt{2}^{-1}[1+\delta(j_aj_b)]^{-1/2}. \tag{4.11}$$

After symmetrizing and normalizing correctly, we get

$$\begin{aligned}\langle j_aj_bJT\,\|\,\mathbf{T}_L(1)+\mathbf{T}_L(2)\,\|\,j_cj_dJ'T'\rangle&=(-)^{L+j_c-j_b}(2J'+1)^{1/2}\,\tfrac{1}{2}\{[1+\delta(j_aj_b)][1+\delta(j_cj_d)]\}^{-1/2}\\ &\times\{\delta(j_bj_d)(-)^J(2i_a+1)^{1/2}W(j_aj_cJJ';Lj_b)M(j_aj_c)+\delta(j_aj_c)(-)^{J'+T+T'}(2j_b+1)^{1/2}W(j_bj_dJJ';Lj_a)M(j_bj_d)\\ &+\delta(j_aj_d)(-)^{T+1}(2j_b+1)^{1/2}W(j_bj_cJJ';Lj_a)M(j_bj_c)+\delta(j_bj_c)(-)^{J+J'+T'+1}(2j_a+1)^{1/2}W(j_aj_dJJ';Lj_b)M(j_aj_d)\},\end{aligned} \tag{4.12}$$

where

$$M(j_aj_c)=[\langle j_a\,\|\,\mathbf{T}_L(1)\,\|\,j_c\rangle+(-)^{T+T'}\langle j_a\,\|\,\mathbf{T}_L(2)\,\|\,j_c\rangle]. \tag{4.13}$$

If nucleon "1" is a proton and "2" is a neutron, then

$$M(j_aj_c)=[\langle j_a\,\|\,\mathbf{T}_L(p)\,\|\,j_c\rangle+(-)^{T+T'}\langle j_a\,\|\,\mathbf{T}_L(n)\,\|\,j_c\rangle] \tag{4.14}$$

where $\mathbf{T}_L(p)$ and $\mathbf{T}_L(n)$ are the multipole operators for protons and neutrons, respectively. It is arbitrary in this case of a proton–neutron two-body problem whether we call nucleon "1" the proton and "2" the neutron or vice versa. A relabeling in the opposite manner introduces an over-all phase change $(-)^{T+T'}$ of *all* matrix elements (4.13) and therefore is irrelevant. Only, once nucleon "1" is called the proton and nucleon "2" the neutron then this labeling must be kept in the evaluation of all matrix elements (4.13).

On the other hand, if the two nucleons are identical we must have $T=T'=1$ and

$$M(j_aj_c)=2\langle j_a\,\|\,\mathbf{T}_L(p)\,\|\,j_c\rangle\qquad\text{for two protons} \tag{4.15}$$

or

$$2\langle j_a\,\|\,\mathbf{T}_L(n)\,\|\,j_c\rangle\qquad\text{for two neutrons.} \tag{4.16}$$

Sometimes wave functions for "two-body" nuclei are available in L–S coupling expansion. To apply the formula (4.12) it is then necessary to convert them into a j–j coupling expansion.

If $\mathbf{l}_1$ and $\mathbf{l}_2$ are the orbital angular momenta of the two nucleons, $\mathbf{s}_1$ and $\mathbf{s}_2$ are their spins, coupled such that $\mathbf{l}_1+\mathbf{l}_2=\mathbf{L}$ and $\mathbf{s}_1+\mathbf{s}_2=\mathbf{S}$, then

$$|\, l_1l_2L,\, s_1s_2S,\, JM\rangle=\sum_{j_1j_2} A\begin{pmatrix} l_1s_1j_1 \\ l_2s_2j_2 \\ LSJ \end{pmatrix} |\, l_1s_1j_1,\, l_2s_2j_2,\, JM\rangle \tag{4.17}$$

or

$$|\, l_1s_1j_1,\, l_2s_2j_2,\, JM\rangle=\sum_{LS} A\begin{pmatrix} l_1s_1j_1 \\ l_2s_2j_2 \\ LSJ \end{pmatrix} |\, l_1l_2L,\, s_1s_2S,\, JM\rangle, \tag{4.18}$$

where j_1 and j_2 are again the total angular momenta of the individual nucleons and J is the total angular momentum of the system. The A coefficient is the one of Kennedy and Cliff (Ke 55) and tabulated by them. It is related to the X coefficient of Fano (Fa 51) or Wigner's 9-j symbol by

$$A\begin{pmatrix} l_1s_1j_1 \\ l_2s_2j_2 \\ LSJ \end{pmatrix}=[(2L+1)(2S+1)(2j_1+1)(2j_2+1)]^{1/2}\, X\begin{pmatrix} l_1s_1j_1 \\ l_2s_2j_2 \\ LSJ \end{pmatrix}. \tag{4.19}$$

For the calculation of reduced matrix elements from eigenfunctions given in the literature it is necessary to know the coupling scheme used by the author. This is important for two- (or many-) body systems when several configurations can contribute to the eigenfunction or when the coupling situation is an intermediate one. Also, extreme care must be exercised when for example matrix elements for a transition from a particle–hole state in a (doubly) closed shell $|\, c\rangle$ to another particle–hole state in the same (doubly) closed shell $|\, c\rangle$ are to be calculated. A similar procedure which led to formula (4.12) has then to be applied, starting from particle–hole wave functions

$$|\, j_a j_b JMT\rangle=\sqrt{2}^{-1}\sum_{m_am_b}(j_a j_b m_a m_b \,|\, JM)(-)^{j_a-m_a}[a_{j_a-m_a}{}^{p}a_{j_bm_b}{}^{p+}+(-)^{T}a_{j_a-m_a}{}^{n}a_{j_bm_b}{}^{n+}]\,|\, c\rangle$$

which now replaces formula (4.8). One arrives at a result which replaces formula (4.12) and it is seen that in fact formally the result is identical with (4.12) when only the terms with $\delta(j_aj_c)$ and $\delta(j_bj_d)$ are retained. This result is physically obvious, because in the first term the "hole" and in the second the particle makes the transition. The matrix elements which occur now in the formula corresponding to Eq. (4.12) are those of Eq. (4.14). *However*, only when the hole acts as a bystander then the single-particle elements (4.5) can be substituted as they stand, while when the particle acts as a bystander the electric multipole matrix element $(Q_L)_{\text{eff}}$ of formula (4.5) must be substituted with the opposite sign (cf. Sec. IVD).

D. Hole Matrix Elements

The angular distribution formula (3.27) holds for states J_1 and J_2 of any complexity. Only the total angular momentum properties of the states J_1 and J_2 were used. The reduced matrix elements occurring in formula (3.27) or the mixing ratios in formula (3.41) are of corresponding complexity, i.e., the operators $\mathbf{T}_L^{\langle\pi\rangle}$ may be sums over many individual particle operators of multipolarity L. All that matters for the symmetry of the angular distribution formula are the total spins J_1 and J_2, the multipolarities L, L' involved and the ratios of the formal quantities called mixing ratios $\delta_L^{\langle\pi\rangle}$, regardless of whether these latter quantities are interpreted in terms of a model at all or, if so, in terms of which model. In certain cases the many-particle states J_1 and J_2 can be well described by a simple model of one- (or two-) hole states in closed shells. If this simple model is applicable then the many-particle reduced matrix elements may be expressed in terms of single- (or two-) particle matrix elements. This reduction can also be made using the formalism of fractional parentage (not speaking of "holes" at all), but in the case of one or two-hole states in a closed shell it is simpler to apply the particle–hole conjugation symmetry [cf. (Br 56)].

Let $|\, 0\rangle$ denote a closed-shell state in which certain levels are filled but the subshells $j_l\cdots j_r$ are empty, and let $|\, c\rangle$ denote the state in which these levels are also filled. Both the states $|\, 0\rangle$ and $|\, c\rangle$ have zero total angular momentum. We denote the creation operator for the single-particle state $|jm\rangle$ by $a_{jm}{}^{+}$ and the corresponding annihilation operator by a_{jm}. The state $a_{jm}{}^{+}|0\rangle$ describing a single particle outside the closed shell $|0\rangle$ has the same angular momentum and transformation properties as the single-particle state $|jm\rangle$.

If the closed-shell state $|c\rangle$ contains n particles then the state $a_{jm}|c\rangle$ represents a state of $n-1$ particles. As it is formed by removing a particle with angular momentum (j, m) from the state $|c\rangle$ with zero total angular momentum, it has angular momentum $(j, -m)$. However, its phase is such that it does not transform like a state $|j, -m\rangle$ under rotations. On the other hand, the $n-1$ particle state

$$|(n-1)jm\rangle=(-)^{j-m}a_{j-m}|c\rangle \qquad (4.20)$$

has the correct transformation laws for a state with angular momentum (j, m). Often such $(n-1)$-particle states with one particle removed from a closed shell are called "one-hole" states. It is sometimes convenient to define "hole" creation and annihilation operators

$$b_{jm}{}^{+}=(-)^{j-m}a_{j-m}, \qquad b_{jm}=(-)^{j-m}a_{j-m}{}^{+}. \qquad (4.21)$$

The operators $b_{jm}{}^{+}$ and b_{jm} have the same anticommutation relations as $a_{jm}{}^{+}$ and a_{jm} and the $(n-1)$-particle (or one-hole) state

$$|(n-1)jm\rangle=b_{jm}{}^{+}|c\rangle=(-)^{j-m}a_{j-m}|c\rangle \qquad (4.22)$$

has the correct transformation properties for a state with angular momentum (jm).

Before going on to more complicated systems we first consider the case that the states J_1 and J_2 may be well described as "one-hole" states in a closed shell, i.e., we consider a tensor operator

$$O_{LM}=\sum_{i=1}^{i=(n-1)} O_{LM}(i)$$

which is a sum of single-particle operators, and express its matrix elements between closed shell minus one-particle states [$(n-1)$-particle states or "one-hole" states] in terms of single-particle matrix elements. More exactly we show that

$$\begin{aligned}\langle(n-1)j_1m_1|O_{LM}|(n-1)j_2m_2\rangle\\ &=(-)^{L-C+1}\langle j_1m_1|O_{LM}|j_2m_2\rangle^{*}\\ &=(-)^{R-C+1}\langle j_1m_1|O_{LM}|j_2m_2\rangle, \qquad (4.23)\end{aligned}$$

if the transformation law of the operator O_{LM} under Hermitian conjugation is

$$O_{LM}{}^{+}=(-)^{C-M}O_{L-M}, \qquad (4.24)$$

and if the operator O_{LM} transforms under time reversal as

$$\boldsymbol{\theta}O_{LM}\boldsymbol{\theta}^{-1}=(-)^{R-M}O_{L-M}. \qquad (4.25)$$

(The matrix element is real if $L-R$ is even, and pure imaginary if $L-R$ is odd.)

In Eq. (4.23) the operator on the left-hand side is (as stated above) a sum over $(n-1)$ individual particle operators $O_{LM}(i)$, while on the right-hand side the operator is simply a single-particle operator and the respective matrix element simply a single-particle matrix element. Consequently Eq. (4.23) simplifies the calculation of the reduced matrix element $\langle(n-1)j_1\|\mathbf{O}_L\|(n-1)j_2\rangle$ considerably, provided that the states $|(n-1)j\rangle$ are "good single-hole states."

Equation (4.23) may be proved as follows. Because O_{LM} is a sum of individual particle operators we can expand it in terms of single-particle matrix elements and creation and annihilation operators,

$$O_{LM}=\sum_{jmj'm'} a_{jm}{}^{+}a_{j'm'}\langle jm|O_{LM}|j'm'\rangle, \qquad (4.26)$$

where $|jm\rangle$ and $|j'm'\rangle$ are single-particle states. Using the definition of the one-hole states (4.22) we get

$$\begin{aligned}\langle(n-1)j_1m_1|O_{LM}|(n-1)j_2m_2\rangle\\ &=\sum_{jmj'm'}(-)^{j_1-m_1+j_2-m_2}\langle jm|O_{LM}|j'm'\rangle\\ &\qquad\times\langle c|a^{+}{}_{j_1-m_1}a^{+}{}_{jm}a_{j'm'}a_{j_2-m_2}|c\rangle\\ &=(-)^{j_1-m_1+j_2-m_2+1}\langle j_2-m_2|O_{LM}|j_1-m_1\rangle.\end{aligned} \qquad (4.27)$$

In this equation the additional factor of -1 comes from the anticommutation law for the creation and annihilation operators. We now remember that $m_1-m_2=M$ and use the Wigner–Eckart theorem

$$\begin{aligned}\langle j_2-m_2|O_{LM}|j_1-m_1\rangle\\ &=(j_1L-m_1M|j_2-m_2)\langle j_2\|\mathbf{O}_L\|j_1\rangle,\end{aligned}$$

and the symmetry relation for vector addition coefficients

$$(j_1L-m_1M|j_2-m_2)=(-)^{j_1+L-j_2}(j_1Lm_1-M|j_2m_2),$$

and obtain

$$\begin{aligned}\langle(n-1)j_1m_1|O_{LM}|(n-1)j_2m_2\rangle\\ &=(-)^{L-M+1}\langle j_2m_2|O_{L-M}|j_1m_1\rangle. \qquad (4.28)\end{aligned}$$

If the transformation property of O_{LM} is that of Eq. (4.24) then

$$\begin{aligned}\langle(n-1)j_1m_1|O_{LM}|(n-1)j_2m_2\rangle\\ &=(-)^{L-C+1}\langle j_1m_1|O_{LM}|j_2m_2\rangle^{*}, \qquad (4.29)\end{aligned}$$

and correspondingly for the reduced matrix elements

$$\begin{aligned}\langle(n-1)j_1\|\mathbf{O}_L\|(n-1)j_2\rangle\\ &=(-)^{L-C+1}\langle j_1\|\mathbf{O}_L\|j_2\rangle^{*}. \qquad (4.30)\end{aligned}$$

These equations (4.29) and (4.30) reduce to a very simple form for the electromagnetic transition operators $T_{LM}{}^{(\pi)}$. These operators satisfy the Hermitian conjugation property (4.24) with $C=L+1$ [cf. Eq. (3.22)], hence

$$\langle (n-1)j_1m_1 \mid T_{LM}^{\langle\pi\rangle} \mid (n-1)j_2m_2\rangle = \langle j_1m_1 \mid T_{LM}^{\langle\pi\rangle} \mid j_2m_2\rangle^*. \quad (4.31)$$

Also the operator $T_{LM}^{\langle\pi\rangle}$ transforms according to Eq. (4.25) with $R=L$ under time reversal. Thus if the single-particle states $\mid j_1m_1\rangle$ have standard transformation properties (2.23) under time reversal then the single-particle matrix elements are real and we get

$$\langle (n-1)j_1m_1 \mid T_{LM}^{\langle\pi\rangle} \mid (n-1)j_2m_2\rangle = \langle j_1m_1 \mid T_{LM}^{\langle\pi\rangle} \mid j_2m_2\rangle. \quad (4.32)$$

In other words, the matrix elements of the electromagnetic interaction multipole operators (3.20 final) between one-hole states are identical with those between one-particle states.

Great care must be exercised when using relation (4.31) or (4.32) because the relations were derived assuming *certain specific transformation properties* for the operators and states, namely those of Eqs. (3.22), (3.23) and (2.23).

For evaluating the reduced matrix element

$$\langle (n-1)j_1 \,\|\, \mathbf{T}_L^{\langle\pi\rangle} \,\|\, (n-1)j_2\rangle = \langle j_1 \,\|\, \mathbf{T}_L^{\langle\pi\rangle} \,\|\, j_2\rangle, \quad (4.33)$$

from the single-particle model a difficulty arises for the *electric* multipole matrix elements. This is because Siegert's theorem is always used for this evaluation. More precisely, the electric multipole operator $T_{LM}^e=\alpha_L^eQ_{LM}$ with Q_{LM} defined by Eq. (3.17′) is used for the evaluation. The application of the operator (3.17′) introduces new operators which do not have the required transformation property (3.22). Also, the energy factors which occur through the application of the Hamiltonian H from formula (3.17′) require careful consideration.

We illustrate these points using the example of a "single-hole" transition before going on to establish relations between matrix elements of many-particle and many-hole states. Let us assume that n particles form a closed-shell state c and let us consider the relation between the energy states of the $(n-1)$-particle system and the related [in the sense of relation (4.33)] one-particle system. To have a concrete example we assume that the $(n-1)$-particle states under consideration are "one-hole" states in the doubly closed p shell, i.e. a $j_1=\frac{3}{2}$ and a $j_2=\frac{1}{2}$ hole in O^{16}. To outline the principle it is sufficient to assume that the total energies of the 15-particle system are composed according to the independent particle model. Then,

$$E_{3/2}=4\epsilon(s_{1/2})+7\epsilon(p_{3/2})+4\epsilon(p_{1/2})=E_{O^{16}}-\epsilon(p_{3/2})$$

$$E_{1/2}=4\epsilon(s_{1/2})+8\epsilon(p_{3/2})+3\epsilon(p_{1/2})=E_{O^{16}}-\epsilon(p_{1/2}).$$

All independent particle energies $\epsilon(i)$ are negative. The energy difference

$$E_{3/2}-E_{1/2}=-\epsilon(p_{3/2})+\epsilon(p_{1/2}) \quad (4.34)$$

is positive.

There are two ways to evaluate the matrix element

$$\langle (n-1)j_1m_1 \mid T_{LM}^e \mid (n-1)j_2m_2\rangle:$$

(i) Insert for T_{LM}^e the expression $\alpha_L^eQ_{LM}$ with Q_{LM} defined by Eq. (3.17′). Observe that the operator T_{LM}^e or Q_{LM}^e is a sum over 15 single-particle operators and correspondingly H is the Hamiltonian of the 15-particle system. Then act with the Hamiltonian on the $(n-1)$-particle states j_1 and j_2, get the corresponding energies in front and use relation (4.29) [*and not* relation (4.32)!] to convert the remaining matrix element of $(Q_{LM})_{\text{eff}}=eg_lr^LC_{LM}$ into a single-particle matrix element.

(ii) Use relation (4.32). Then insert for $T_{LM}^e=\alpha_L^eQ_{LM}$ with Q_{LM} defined by Eq. (3.17′). But Q_{LM} is now a single-particle operator and H is a single-particle Hamiltonian. Act with H on the single-particle states j_1 and j_2 and the corresponding single-particle energies occur in front of a single-particle matrix element of $(Q_{LM})_{\text{eff}}=eg_lr^LC_{LM}$. The energy difference of the single-particle energies (which for themselves are somewhat fictive) must be equal in magnitude but opposite in sign to the energy difference $(E_{3/2}-E_{1/2})$ as obtained using (i).

Probably (i) is the "more physical" approach, because the Hamiltonian H from (3.17′) is that of the total 15-particle system and the energies produced by it are the true energies of the states under observation.

Proceeding as stated under (i) we get

$$\begin{aligned}\langle (n-1)j_1m_1 \mid T_{LM}^e \mid (n-1)j_2m_2\rangle &= 2g_l(m\beta/\hbar^2k)\,(E_1-E_2)\,\langle (n-1)j_1m_1 \mid \alpha_L^er^LC_{LM} \mid (n-1)j_2m_2\rangle\\ &= 2g_l(m\beta/\hbar^2k)\,(E_1-E_2)\,(-)^1\langle j_1m_1 \mid \alpha_L^er^LC_{LM} \mid j_2m_2\rangle\\ &= -\alpha_L^eeg_l\langle j_1m_1 \mid r^LC_{LM} \mid j_2m_2\rangle. \end{aligned} \quad (4.35)$$

The second line is true because $\alpha_L^er^LC_{LM}$ transforms with $C=L$ [cf. Eq. (4.24)]. The asterisk originating from the application of formula (4.29) has been omitted in this calculation, because the operator $\alpha_L^er^LC_{LM}$ transforms under time reversal with $R=L$ [cf. Eq. (4.25)] and the eigenfunctions are assumed to obey $\boldsymbol{\theta} \mid j_1m_1\rangle=(-)^{j_1-m_1} \mid j_1-m_1\rangle$. This latter assumption is observed in the calculation of the single-particle matrix elements given in Sec. IVB.

Proceeding as stated under (ii) we get

$$\begin{aligned}&\langle (n-1)j_1m_1 \mid T_{LM}^e \mid (n-1)j_2m_2\rangle\\ &\quad=\langle j_1m_1 \mid T_{LM}^e \mid j_2m_2\rangle\\ &\quad=2g_l(m\beta/\hbar^2k)\,(\epsilon_1-\epsilon_2)\,\alpha_L^e\langle j_1m_1 \mid r^LC_{LM} \mid j_2m_2\rangle \end{aligned} \quad (4.36)$$

which is because of (4.34) the same result as arrived at by using (i).

The "hole" operators b_{jm}^+ may be used to establish a correspondence between more complicated particle states and hole states. For example the $(n-2)$-particle state (two-hole state)

$$| (n-2)JM\rangle = \sum_{m_1m_2} (j_1j_2m_1m_2 \mid JM) b^+_{j_1m_1} b^+_{j_2m_2} \mid c\rangle$$

is conjugate to the two-particle state

$$| JM\rangle = \sum_{m_1m_2} (j_1j_2m_1m_2 \mid JM) a^+_{j_1m_1} a^+_{j_2m_2} \mid 0\rangle. \qquad (4.37)$$

This concept of particle–hole conjugation may be extended to more complicated states in an obvious way. In general a state $| rA\rangle$ of r particles outside a closed shell $| 0\rangle$ may be written as a polynomial in the particle creation operators a_{jm}^+ acting on $| 0\rangle$. The conjugate state $| (n-r)A\rangle$ may then be defined by the polynomial in the b_{jm}^+ with the same coefficients acting on the state $| c\rangle$.

Let O_{LM} be a multipole operator which is a sum of single-particle operators and whose Hermitian conjugation and time-reversal properties are related by

$$\theta O_{LM} \theta^{-1} = (-)^{C-R} (O_{LM})^+. \qquad (4.38)$$

It is possible to show that

$$\langle (n-r)A \mid O_{LM} \mid (n-r)B\rangle = (-)^{C-R+1} \langle rA \mid O_{LM} \mid rB\rangle \qquad (4.39)$$

if $(n-r)A$ and $(n-r)B$ are the states conjugate to rA and rB and provided the single-particle states have standard time reversal transformation properties [Eq. (2.23)].

To prove relation (4.39) we replace the operators $a^+_{j_1m_1}$ and $a_{j_2m_2}$ in (4.26) by the corresponding operators for holes from Eq. (4.21) and use the relations (4.24) and (4.25) between the Hermitian conjugation and time-reversal transformations of O_{LM} and obtain

$$O_{LM} = (-)^{C-R+1} \sum_{j_1m_1,\ j_2m_2} b^+_{j_1m_1} b_{j_2m_2} \langle j_1m_1 \mid O_{LM} \mid j_2m_2\rangle. \qquad (4.40)$$

except for the phase $(-)^{C-R+1}$, O_{LM} has identical expansions in the particle creation and annihilation operators and in the hole creation and annihilation operators. Matrix elements of O_{LM} between r-particle states $| rA\rangle$ and $| rB\rangle$ may be evaluated using Eq. (4.26) and the anticommutation relations of the particle creation and annihilation operators. Matrix elements between the conjugate hole states may be evaluated in an analogous way. Because the expansions of O_{LM} in Eqs. (4.26) and (4.40) have the same form and the anticommutation relations for hole operators are the same as those for particles, the matrix elements between conjugate states are identical except for the phase $(-)^{C-R+1}$. The relation which we calculated explicitly for one-hole states is a special case of this general result. The discussion in this section did not specify the type of particle and therefore may be applied separately to protons or neutrons. To treat states composed of protons *and* neutrons one must introduce either creation and annihilation operators a^{p+}, a^p and a^{n+}, a^n etc., separately for protons and neutrons, or—in more complicated cases—the isotopic spin formalism. Then the operators of Eqs. (3.17) must be reformulated accordingly using isotopic spin operators. The simple two-particle case involving proton–neutron states discussed in the previous section can of course still be treated without making use of isotopic spin formalism. One arrives at Eq. (4.12) by using the two-particle eigenfunctions written in second quantization formalism and can show by making a particle–hole conjugation that for the calculation of mixing ratios formula (4.12) also holds for two-hole states. Of course, care has to be taken again when matrix elements of the electric interaction multiple operators are converted into matrix elements of effective electric multiple operators as described above.

V. COMMENTS

A. General

We begin this section by quoting from an article by Huby (Hu 58).

"There have been many extensive papers and collections of formulae on the theory of angular distributions, using highly developed mathematical techniques, but in making practical applications of the formulas, the writer has encountered the following difficulties:

(1) The conventions of various authors differ greatly. Often different authors use essentially the same symbol for different quantities, e.g., the "radiation parameter" $c_{k\tau}(LL')$ as defined by Racah (1951) and Biedenharn and Rose (1953) differs from the $c_{\nu\tau}(LL')$ of Frauenfelder (1955) and Devons and Goldfarb (1957); the relation (if the wave functions used are the same in each case) is:

$$c_{k\tau}(LL')\ \text{(Racah)} = (-)^{2L+k-\tau} c^*_{k-\tau}(LL')\ \text{(Frauenfelder)}.$$

(2) Often it is very difficult to discover the basic definitions of quantities used. This applies particularly to reduced matrix elements, such as are written $\langle I_b \| L \| I_a\rangle$, or multipole mixture ratios δ, which one may wish to calculate from a model.

(3) A number of compilations contain errors (not surprisingly). These difficulties become serious if, for example, one wishes to compare the sign of a measured γ-mixture ratio δ with that predicted by a model. This and other questions of phase are significant mainly in processes which involve the mixture of several

channels. It would be desirable to carry out the following program:

(1) Point out the various crucial stages in the theoretical development at which a choice of convention has to be made, and compare the choices of different authors.
(2) Give a fresh systematic development of angular distribution formulas, explaining all the conventions used carefully.
(3) Take the symbols (e.g., reduced matrix elements) appearing in the final angular distribution formulas of the various authors, and present expressions for them in the form needed for calculating them from a model."

Huby (Hu 58) then presents some comparisons of phase conventions of different authors. We have quoted from his paper because it points out why a "fresh approach" had to be made and why this had to be done in a rather explicit form. The method of dealing with each formula from the viewpoint of population parameters $w(M_1)$ (*which can be positive numbers only*!) should enable the reader to see clearly what factors can influence his phases. The approach starts from first principles. All conventions which have to be made are explicitly stated. The electromagnetic vector potentials $\mathbf{A}_{LM}^{e}$ and $\mathbf{A}_{LM}^{m}$ are terms of a consistent expansion of a plane (circular polarized) wave $\mathbf{e}_q \exp(i\mathbf{k}\cdot\mathbf{r})$ which interacts with the nucleus. The choice of the relative phases of these two quantities is of no importance as long as the expansion is correct. A consistent perturbation treatment of the plane wave interaction starting either from field theory and calculating an emission process, or from time-dependent perturbation theory calculating an absorption process and relating this one via "detailed balance" or "time reversal" consistently to the emission process, leads to a set of interaction multipole operators, which are terms of an expansion of a definite and well-defined interaction Hamiltonian. The set of interaction multipole operators is explicitly defined and has well-defined Hermitian and time reversal properties. When the initial state of a transition is always written on the left of a matrix element and when this set of interaction multipole operators is consistently used then an emission process is described. One arrives at an angular distribution formula in terms of reduced matrix elements of the interaction multipole operators which have well-defined relative phases and which are calculable from nuclear models. There are two more important features to be included into this retrospective summary. Firstly, to define mixing ratios the interaction multipole operators $T_{LM}^{<\pi>}$ must be used for reasons of internal consistency. Secondly, the electric interaction multipole operator $T_{LM}^{e}=\alpha_L^{e}Q_{LM}$ should not be reduced to "standard form" of the operator Q_{LM}, namely $(Q_{LM})_{\text{eff}}=eg_l r^L C_{LM}$ but should always be written as the commutator

$$Q_{LM}=2g_l(m\beta/\hbar^2k)[H, r^LC_{LM}].$$

Otherwise the correct transformation property under Hermitian conjugation is lost and one produces errors in phase, especially when "switching" of the order of the states is needed or when a calculation of reduced matrix elements from nuclear models is made using the concept of "hole states".

When it comes to the point of comparing different authors, three types of problems arise.

(a) The same types of experiments were carried out by different authors on different transitions. Formulas from different papers on angular distribution theory were used for analyzing the data. How can one compare the phase of the mixing ratios?

This is "the trivial case" of comparison. The mixing ratios x or δ may be treated as formal quantities, i.e., as parameters of distribution formulas, which were used to fit the data. Provided each author has stated which angular distribution paper he has used to fit his data, the comparison can simply be made by comparing the final formulas.

(b) Different types of experiments were done to determine mixing ratios. For example, one mixing ratio may have been determined by using "method II" and applying Litherland and Ferguson's formula (23) (Li 61), and another one may have been determined from a gamma–gamma correlation measurement using formulas of Biedenharn (Bi 60) or Ferentz and Rosenzweig (Fe 55). How do the phases compare?

(c) A comparison is wanted of a mixing ratio (which was extracted from the fit of an angular distribution formula to experimental data) with the prediction calculated by another author from a certain nuclear model. How can one compare the phases?

In order to solve (b), first one ought to write both formulas using the same coefficients. Then it is necessary to know the relation between mixing ratios and reduced matrix elements. The states in the reduced matrix elements appearing in the two formulas describing the two different experiments then have to be arranged in the same order and a unique relation has to be established throughout between mixing ratios and reduced matrix elements. For rearranging the states (if necessary) one must know the transformation properties of the operators under Hermitian conjugation. Finally apply faith: If the operators in the reduced matrix elements [such as are all written now for example $\langle b \mid L \mid a\rangle$] are not defined in *both* theories (and they are not), then hope that they originate from the expansion of the *same* Hamiltonian [in our terminology either $H_a(\mathbf{k}, \boldsymbol{\varepsilon})$ or $H_e(\mathbf{k}, \boldsymbol{\varepsilon})$] in both cases, i.e., that they differ only by an over-all phase, independent of multipolarity.

To solve (c) it is necessary to know two things in addition. First, on behalf of the angular distribution theory, that the operators are originating from the *correct* interaction Hamiltonian to describe an emission process, i.e., for example from $H_a(\mathbf{k}, \boldsymbol{\varepsilon})$ when the initial state is always written on the left. Second, that the operators and reduced matrix elements occurring in the angular distribution theory as well as the ones used in the theoretical calculation are explicitly defined.

There is only one "safe way" of making comparisons of types (b) and (c), and that is: Take a consistent theory in which things are well and explicitly defined, and which covers for comparisons of type (b) the practical cases in question, either explicitly or to a degree from which the case in question can be worked out without losing track of phases. In other words, take the final formulas of the present approach which describe the types of experiments to be compared. Then rewrite these formulas in terms of those coefficients in which the other formulas are given. Now identify by direct comparison the meaning of mixing ratios in terms of the well-defined reduced matrix elements of the present approach.

In the following section we make some comparisons of this kind, rather to illuminate the method than to achieve completeness. Nevertheless these comparisons should cover frequently occurring practical cases where mixing ratios are extracted from measurements taken in cylinder-symmetrical environments. In Sec. VB we relate the signs of mixing ratios when extracted from an experiment of the type "method II" by application of Poletti and Warburton's (Po 65) [or Litherland and Ferguson's (Li 61)] formulas to those of the present theory. Furthermore, when gamma-gamma directional correlations were used by application of the formulas of Biedenharn (Bi 60) or Ferentz and Rosenzweig (Fe 55), we also relate this to our theory. This comparison will also give the mixing ratios of these authors in terms of our reduced matrix elements, i.e., it will redefine them in terms of explicitly defined and therefore calculable quantities. In Sec. VC we give relations between our operators and reduced matrix elements and those used by Lane (La 60) and Alder *et al.* (Al 56).

We realize that we have restricted the discussion to the simplest case, i.e., a cylinder-symmetrical environment. However, the basic difficulties of a phase-consistent approach have been overcome, once Eq. (3.24) of the present treatment has been established. The extension of the present treatment to noncylinder-symmetrical cases (i.e., "method I") requires only straightforward application of density matrix formalism, starting with Eq. (3.24).

B. Comparison of Mixing Ratios

In Sec. IV we have given formulas which are adapted to three major experimental methods of determinations of mixing ratios, when a cylinder-symmetric environment is used. Of these three methods the one which is called "method II" by Litherland and Ferguson (Li 61) is probably the most widely used technique in nuclear structure work. The basic formula to fit experimental results obtained by "method II" is formula (23) of Litherland and Ferguson (Li 61). This formula has been rewritten in terms of F_K coefficients by Poletti and Warburton (Po 65) and thus been brought into a most suitable form for the analysis of experimental data. Formula (1) of (Po 65) is identical[20] with formula (23) of (Li 61), and therefore the phase convention of Poletti and Warburton is that of Litherland and Ferguson. Phase conventions, referring to earlier work, means always and only that the formal quantities called mixing ratios occur with a certain sign in the distribution formula. The formal quantities x of these authors are defined as ratios of reduced matrix elements $\langle b \| L \| a \rangle$. It is difficult to trace down the exact definition of these matrix elements and of the operators "L".

When mixing ratios δ are extracted from experimental data using our formula (3.47) [substituting for $B_K(J_1)$ expression (3.62)] or Eq. (11) of Poletti and Warburton (Po 65), the mixing ratios δ will agree in sign and magnitude for the case of electric radiation of multipole order $(L+1)$ mixed with magnetic radiation of order (L).

Because of the use of Z_1 coefficients in the treatment of Litherland and Ferguson (Li 61) there is in formula (23) of (Li 61) and formula (11) of (Po 65) a dependence of the phase of the mixing ratios δ on whether the mixed transitions are "naturally occurring ones" or not. Thus in formula (11) of (Po 65) $\sigma=0$ for $E(L+1)/M(L)$ mixture, while

$$\sigma=1 \quad \text{for} \quad M(L+1)/E(L).$$

Consequently for $\sigma=1$ the mixing ratios extracted from the experiment using formula (11) of (Po 65) have the opposite sign to those one would get using our formula (3.47). No attention ought to be paid to the interpretation of the mixing ratios in terms of reduced matrix elements such as are written $\langle b \| L \| a \rangle$ in Eq. (5) or (6) of (Po 65) or (Li 61), respectively. We have shown how the mixing ratios x or δ used in these papers are related to ours. Hence they may be expressed in terms of our reduced matrix elements.

Some confusion about "phases" has arisen from the

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[20] In fact this statement of (Po 65) is only true apart from an overall phase. When formula (1) of (Po 65) is derived from formula (23) of (Li 61) the phase factor $(-)^{J_1-M_1}$ in the statistical tensor $\rho_K(a)$ of (Po 65) [$\rho_K(a)$ is identical with our $B_K(J_1)$] is not arrived at, but one gets $(-)^{J_1+M_1}$. This in turn means that there is an overall phase difference $(-)^{2J_1}$ between formula (23) of (Li 61) and formula (1) of (Po 65), which has of course no consequences for the signs of mixing ratios. Nevertheless, formula (1) of (Po 65) is more correct, not only because it agrees in overall phase with our formula (3.63) but because formula (23) of (Li 61) can lead to negative counting rates.

use of Z_1 coefficients into which imaginary powers had been incorporated. The reason for working these powers $i^{L'-\pi'-L+\pi+K+2}$ into them seems to be that restricted formulas like our formulas (3.41) and (3.42) can be "made to work" in more general cases than they have been derived for, if one equips them with what we have called in Sec. IV an "*ad hoc* recipe." The complex factor just makes those contributions disappear, which one does not want when the formulas are applied to a case where the initial state does not have definite parity. The price which one has to pay for this "automatization" is that the "*ad hoc* factor" $i^{L'-\pi'-L+\pi+K+2}$ yields opposite signs for "naturally occurring" and "unnaturally (?) occurring" mixtures. In later work [cf. (Fe 65)] the imaginary powers were cut out again and $\bar{Z}_1$ coefficients were introduced. Similar things happened to the Z coefficient which appears in the case where particle absorption followed by gamma rays is to be described. It contained originally a factor $i^{l_2-l_1+K}$ which worked in combination with the imaginary powers of the Z_1 coefficients. The factor $i^{l_2-l_1+K}$ then disappeared from the Z coefficient for the reason "of a correction to the wave functions" (Fe 65).

We now compare our formula (3.73), which describes gamma–gamma directional correlations, with those presented previously by Biedenharn (Bi 60) and Ferentz and Rosenzweig (Fe 55). These authors write their formulas for a transition from a state J_1 to a state J_2 with multipolarities L_1 and L_1' and from J_2 to a state J_3 with multipolarities L_2 and L_2'

$$W(\theta)=\sum_K A_K(1)A_K(2)P_K(\cos\theta)$$

with

$$A_K(1)=F_K(L_1L_1J_1J_2)+2\delta_1F_K(L_1L_1'J_1J_2)+\delta_1^2F_K(L_1'L_1'J_1J_2),$$

$$A_K(2)=F_K(L_2L_2J_3J_2)+2\delta_2F_K(L_2L_2'J_3J_2)+\delta_2^2F_K(L_2'L_2'J_3J_2).$$

Rewriting our formula (3.73) in terms of F_K coefficients [compare relation (3.37)] we find that

$$\delta_1(\text{Bi})=\delta_1(\text{ours}) \quad\text{and}\quad \delta_2(\text{Bi})=(-)^{L_2-L_2'}\,\delta_2(\text{ours}). \tag{5.1}$$

We can go even one step further and establish from these relations a correspondence between Biedenharn's (Bi 60) operators $T(LM)$ and our $T_{LM}^{\langle\pi\rangle}$.

From relation (5.1) it follows that

$$\delta_1(\text{Bi})=\frac{\langle J_1\|\,T_{L_1'}^{<\pi_1'>}\,\|J_2\rangle/(2L_1'+1)^{1/2}}{\langle J_1\|\,T_{L_1}^{<\pi_1>}\,\|J_2\rangle/(2L_1+1)^{1/2}}\,;$$

$$\delta_2(\text{Bi})=(-)^{L_2-L_2'}\,\frac{\langle J_2\|\,T_{L_2'}^{<\pi_2'>}\,\|J_3\rangle/(2L_2'+1)^{1/2}}{\langle J_2\|\,T_{L_2}^{<\pi_2>}\,\|J_3\rangle/(2L_2+1)^{1/2}}\,.$$

Rearranging the states in the second expression we can write

$$\delta_2(\text{Bi})=\frac{\langle J_3\|\,T_{L_2'}^{<\pi_2'>}\,\|J_2\rangle/(2L_2'+1)^{1/2}}{\langle J_3\|\,T_{L_2}^{<\pi_2>}\,\|J_2\rangle/(2L_2+1)^{1/2}}\,.$$

Biedenharn states that his mixing ratios δ_1 and δ_2 are defined differently in terms of his reduced matrix elements. He always writes the intermediate state J_2 on the right; i.e.,

$$\delta_1(\text{Bi})=\frac{\langle J_1\,|\,L_1'\,|\,J_2\rangle}{\langle J_1\,|\,L_1\,|\,J_2\rangle}\qquad \delta_2(\text{Bi})=\frac{\langle J_3\,|\,L_2'\,|\,J_2\rangle}{\langle J_3\,|\,L_2\,|\,J_2\rangle}\,.$$

From comparison we can therefore conclude that

$$\langle J_1\,|\,L\,|\,J_2\rangle_{\text{Biedenharn}}=N\langle J_1\,\|\,T_L^{<\pi>}\,\|\,J_2\rangle/(2L+1)^{1/2} \tag{5.2}$$

where N is an over-all normalization factor which is independent of L and π.

The mixing ratios of Ferentz and Rosenzweig (Fe 55) are the same as those of Biedenharn (Bi 60), although they are not defined in terms of reduced matrix elements. Knowing that the Wigner–Eckart theorem used by Biedenharn (Bi 60) is the same as ours, we can conclude from Eq. (5.2) that his operators $T(LM)$ must have the following relation to ours[21]

$$T(LM)=NT_{LM}^{<\pi>}/(2L+1)^{1/2}.$$

We conclude this section with Table II in which the relations of the coefficients R_K and S_K used in the present paper to the various coefficients in use are stated. We have introduced the R_K coefficient (and tabulated) to avoid any unnecessary phase factors in the formulas of a phase-consistent treatment.

C. Multipole Operators

In Sec. VB we have shown how to relate mixing ratios of other authors to reduced matrix elements of our operators $T_{LM}^{<\pi>}$. These matrix elements may be calculated from model wave functions and the theoretical predictions compared with measured mixing ratios. A calculation of electromagnetic transition matrix elements may already exist in the literature, but the operators used would not in general be the same as the $T_{LM}^{<\pi>}$ used here. Such a calculation may be used to give matrix elements of the $T_{LM}^{<\pi>}$ if the operators can be related. In order to facilitate comparison with existing theoretical calculations we relate our operators $T_{LM}^{<\pi>}$ with other operators in common use.

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[21] This relation holds for the operators of Biedenharn (Bi 60), but not for those of Biedenharn and Rose (Bi 53). There the relation is

$$T(LM\pi)=N'(-)^L(T_{LM}^{<\pi>})^+/(2L+1)^{1/2}.$$

The transition operators used by Bohr and Mottelson (Bo 53) and by Alder *et al.* (Al 56) are written $\mathfrak{M}(E\lambda, \mu)$ for electric radiation and $\mathfrak{M}(M\lambda, \mu)$ for magnetic radiation. They are related to our operators by

$$\mathfrak{M}(E\lambda, \mu) = (2\lambda+1/4\pi)^{1/2}(Q_{\lambda\mu})_{\text{eff}}$$

$$= (2\lambda+1/4\pi)^{1/2}(G_{\lambda\mu}{}^{e})_{\text{eff}}$$

neglecting the small spin contribution to $G_{\lambda\mu}{}^{e}$, and

$$\mathfrak{M}(M\lambda, \mu) = (2\lambda+1/4\pi)^{1/2}G_{\lambda\mu}{}^{m}.$$

The definition of the reduced matrix element used by these authors has the same phase as the one used here but differs by a factor $(2I_i+1)^{1/2}$;

$$\langle I_i \| \mathfrak{M}(E\lambda) \| I_f \rangle_{BM} = \left(\frac{2\lambda+1}{4\pi(2I_i+1)}\right)^{1/2} \langle I_i \| (\mathbf{Q}_\lambda)_{\text{eff}} \| I_f \rangle$$

$$\langle I_i \| \mathfrak{M}(M\lambda) \| I_f \rangle_{BM} = \left(\frac{2\lambda+1}{4\pi(2I_i+1)}\right)^{1/2} \langle I_i \| \mathbf{G}_\lambda{}^{m} \| I_f \rangle.$$

The reduced transition probability defined by (Bo 53) and (Al 56) is

$$B(\lambda, I_i \rightarrow I_f) = [(2\lambda+1)/4\pi] \,|\, \langle I_i \| \mathbf{G}_\lambda \| I_f \rangle \,|^2.$$

The operators used by Lane [Eqs. (31), (32), (33) of (La 60)] are identical with those of Bohr and Mottelson. Those introduced by Blatt and Weisskopf (Bl 52) are the Hermitian conjugates of those of Bohr and Mottelson.

One should remember that the matrix element of Q_{eff} in a γ emission process is equal to a matrix element of Q_{LM} only if the initial state is written on the left of the matrix element (cf. Sec. IIIC). In Table II relations (1) and (4) hold only when K is even, L and L' are integers and correspond to multipole transitions of equal parity. Relations (2), (3), and (5) are restricted to K, L, L' being integers.

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APPENDIX

Angular Distribution Coefficients $R_k(LL'J_1J_2)$

J_1	J_2	$k=2$			4			6			8		
		LL^a	LL'	$L'L'$	LL	LL'	$L'L'$	LL	LL'	$L'L'$	LL	LL'	$L'L'$
1	0	0.7071	0.0000	0.0000									
	1	−0.3536	1.0607	−0.3536									
	2	0.0707	−0.4743	0.3536									
	3	−0.1010	−0.3780	0.5303									
	4	−0.1768	−0.3062	0.6010									
2	0	−0.5976	0.0000	0.0000	−1.0690	0.0000	0.0000						
	1	0.4183	0.9354	−0.2988	0.0000	0.0000	0.7127						
	2	−0.4183	0.6124	0.1281	0.0000	0.0000	−0.3054						
	3	0.1195	−0.6547	0.3415	0.0000	0.0000	0.0764						
	4	−0.1707	−0.5051	0.4482	−0.0085	0.0627	−0.0297						
	5	−0.2988	−0.4009	0.4618	0.0040	0.0815	−0.1093						
3	0	−0.8660	0.0000	0.0000	0.2132	0.0000	0.0000	1.3056	0.0000	0.0000			
	1	−0.4949	0.4629	−0.6495	−0.4467	−1.0446	0.0355	0.0000	0.0000	−0.9792			
	2	0.3464	0.9487	−0.1237	0.0000	0.0000	0.6701	0.0000	0.0000	0.0000			
	3	−0.4330	0.4330	0.2268	0.0000	0.0000	−0.4467	0.0000	0.0000	0.0000			
	4	0.1443	−0.7217	0.3093	0.0000	0.0000	0.1489	0.0000	0.0000	0.0000			
	5	−0.2062	−0.5455	0.3608	−0.0203	0.1343	−0.0549	0.0000	0.0000	−0.0099			
	6	−0.3608	−0.4270	0.3346	0.0097	0.1720	−0.1946	0.0008	−0.0663	−0.0014			
4	1	−0.7835	0.2714	−0.8234	0.1453	−0.7549	0.3017	0.4214	1.0218	−0.0034	0.0000	0.0000	1.1847
	2	−0.4477	0.5297	−0.4701	−0.3044	−0.9004	−0.0484	0.0000	0.0000	−0.7585	0.0000	0.0000	0.0000
	3	0.3134	0.9402	−0.0448	0.0000	0.0000	0.6088	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	4	−0.4387	0.3354	0.2646	0.0000	0.0000	−0.4981	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.1595	−0.7568	0.2849	0.0000	0.0000	0.1937	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	6	−0.2279	−0.5641	0.2991	−0.0298	0.1844	−0.0687	0.0000	0.0000	−0.0265	0.0000	0.0000	0.0000
	7	−0.3989	−0.4369	0.2466	0.0142	0.2337	−0.2363	0.0024	−0.0181	−0.0036	0.0000	0.0000	0.0010
5	2	−0.7360	0.3291	−0.6825	0.1159	−0.7774	0.0802	0.2420	0.7574	−0.0290	0.0000	0.0000	0.7785
	3	−0.4206	0.5563	−0.3680	−0.2428	−0.8030	−0.0773	0.0000	0.0000	−0.6049	0.0000	0.0000	0.0000
	4	0.2944	0.9309	0.0000	0.0000	0.0000	0.5666	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	5	−0.4416	0.2739	0.2831	0.0000	0.0000	−0.5230	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	6	0.1698	−0.7783	0.2669	0.0000	0.0000	0.2241	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	7	−0.2426	−0.5742	0.2548	−0.0374	0.2210	−0.0773	0.0000	0.0000	−0.0419	0.0000	0.0000	0.0000
	8	−0.4246	−0.4413	0.1837	0.0178	0.2779	−0.2592	0.0041	−0.0299	−0.0056	0.0000	0.0000	0.0035
													0.0000
6	3	−0.7051	0.3575	−0.5915	0.0997	−0.7581	−0.0296	0.1708	0.6062	−0.0342	0.0000	0.0000	0.5502
	4	−0.4029	0.5698	−0.3022	−0.2088	−0.7383	−0.0902	0.0000	0.0000	−0.5124	0.0000	0.0000	0.0000
	5	0.2820	0.9232	0.0288	0.0000	0.0000	0.5370	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	6	−0.4432	0.2315	0.2935	0.0000	0.0000	−0.5370	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	7	0.1773	−0.7928	0.2533	0.0000	0.0000	0.2461	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	8	−0.2533	−0.5803	0.2216	−0.0434	0.2488	−0.0829	0.0000	0.0000	−0.0553	0.0000	0.0000	0.0000
	9	−0.4432	−0.4432	0.1370	0.0207	0.3109	−0.2727	0.0058	−0.0407	−0.0072	0.0000	0.0000	0.0066
7	4	−0.6834	0.3743	−0.5281	0.0894	−0.7349	−0.0929	0.1340	0.5137	−0.0348	0.0000	0.0000	0.4248
	5	−0.3905	0.5776	−0.2563	−0.1874	−0.6929	−0.0969	0.0000	0.0000	−0.4522	0.0000	0.0000	0.0000
	6	0.2734	0.9169	0.0488	0.0000	0.0000	0.5154	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Angular Distribution Coefficients $R_k(LL'J_1J_2)$ (*Continued*)

J_1	J_2	$k=2$			4			6			8		
		LL^a	LL'	$L'L'$	LL	LL'	$L'L'$	LL	LL'	$L'L'$	LL	LL'	$L'L'$
7	7	−0.4442	0.2004	0.3001	0.0000	0.0000	−0.5457	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	8	0.1829	−0.8033	0.2426	0.0000	0.0000	0.2627	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	9	−0.2613	−0.5843	0.1960	−0.0484	0.2706	−0.0869	0.0000	0.0000	−0.0667	0.0000	0.0000	0.0000
	10	−0.4573	−0.4440	0.1010	0.0231	0.3364	−0.2810	0.0074	−0.0504	−0.0086	0.0000	0.0000	0.0098
8	5	−0.6673	0.3853	−0.4813	0.0824	−0.7138	−0.1332	0.1119	0.4522	−0.0343	0.0000	0.0000	0.3483
	6	−0.3813	0.5825	−0.2224	−0.1727	−0.6595	−0.1007	0.0000	0.0000	−0.4103	0.0000	0.0000	0.0000
	7	0.2669	0.9117	0.0636	0.0000	0.0000	0.4989	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	8	−0.4449	0.1768	0.3044	0.0000	0.0000	−0.5514	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	9	0.1873	−0.8111	0.2341	0.0000	0.0000	0.2757	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	10	−0.2676	−0.5870	0.1756	−0.0525	0.2880	−0.0898	0.0000	0.0000	−0.0766	0.0000	0.0000	0.0000
9	6	−0.6549	0.3930	−0.4453	0.0773	−0.6959	−0.1606	0.0973	0.4089	−0.0335	0.0000	0.0000	0.2977
	7	−0.3742	0.5858	−0.1965	−0.1620	−0.6340	−0.1031	0.0000	0.0000	−0.3797	0.0000	0.0000	0.0000
	8	0.2620	0.9075	0.0748	0.0000	0.0000	0.4860	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	9	−0.4453	0.1581	0.3074	0.0000	0.0000	−0.5555	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	10	0.1909	−0.8172	0.2272	0.0000	0.0000	0.2861	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	7	−0.6451	0.3986	−0.4169	0.0734	−0.6808	−0.1803	0.0871	0.3767	−0.0326	0.0000	0.0000	0.2621
	8	−0.3686	0.5881	−0.1759	−0.1539	−0.6138	−0.1046	0.0000	0.0000	−0.3563	0.0000	0.0000	0.0000
	9	0.2580	0.9039	0.0838	0.0000	0.0000	0.4757	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	10	−0.4457	0.1430	0.3096	0.0000	0.0000	−0.5584	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3/2	1/2	0.5000	0.8660	−0.5000									
	3/2	−0.4000	0.7746	0.0000									
	5/2	0.1000	−0.5916	0.3571									
	7/2	−0.1429	−0.4629	0.5000									
	9/2	−0.2500	−0.3708	0.5409									
5/2	1/2	−0.5345	0.3780	−0.8018	−0.6172	−1.0911	0.1543						
	3/2	0.3742	0.9487	−0.1909	0.0000	0.0000	0.7054						
	5/2	−0.4276	0.5071	0.1909	0.0000	0.0000	−0.3968						
	7/2	0.1336	−0.6944	0.3245	0.0000	0.0000	0.1176						
	9/2	−0.1909	−0.5297	0.4009	−0.0147	0.1019	−0.0444						
	11/2	−0.3341	−0.4173	0.3924	0.0070	0.1314	−0.1602						
7/2	1/2	−0.8183	0.2113	−0.9274	0.1709	−0.6621	0.5128	0.6528	1.1798	0.0435			
	3/2	−0.4676	0.5051	−0.5455	−0.3582	−0.9671	−0.0190	0.0000	0.0000	−0.8704			
	5/2	0.3273	0.9449	−0.0779	0.0000	0.0000	0.6367	0.0000	0.0000	0.0000			
	7/2	−0.4364	0.3780	0.2494	0.0000	0.0000	−0.4775	0.0000	0.0000	0.0000			
	9/2	0.1528	−0.7416	0.2962	0.0000	0.0000	0.1737	0.0000	0.0000	0.0000			
	11/2	−0.2182	−0.5563	0.3273	−0.0253	0.1614	−0.0627	0.0000	0.0000	−0.0183			
	13/2	−0.3819	−0.4330	0.2867	0.0121	0.2056	−0.2188	0.0015	−0.0121	−0.0025			
9/2	3/2	−0.7569	0.3062	−0.7444	0.1281	−0.7774	0.1693	0.3077	0.8714	−0.0213	0.0000	0.0000	0.9633
	5/2	−0.4325	0.5455	−0.4129	−0.2684	−0.8465	−0.0660	0.0000	0.0000	−0.6714	0.0000	0.0000	0.0000
	7/2	0.3028	0.9354	−0.0197	0.0000	0.0000	0.5857	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	9/2	−0.4404	0.3015	0.2752	0.0000	0.0000	−0.5125	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	11/2	0.1651	−0.7687	0.2752	0.0000	0.0000	0.2102	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	13/2	−0.2359	−0.5698	0.2752	−0.0338	0.2040	−0.0735	0.0000	0.0000	−0.0344	0.0000	0.0000	0.0000
	15/2	−0.4129	−0.4395	0.2127	0.0161	0.2575	−0.2493	0.0032	−0.0241	−0.0046	0.0000	0.0000	0.0022

Angular Distribution Coefficients $R_k(LL'J_1J_2)$ (*Continued*)

J_1	J_2	$k=2$			4			6			8		
		LL[a]	LL'	$L'L'$	LL	LL'	$L'L'$	LL	LL'	$L'L'$	LL	LL'	$L'L'$
11/2	5/2	−0.7191	0.3454	−0.6326	0.1067	−0.7692	0.0171	0.1998	0.6718	−0.0326	0.0000	0.0000	0.6460
	7/2	−0.4109	0.5641	−0.3319	−0.2237	−0.7676	−0.0849	0.0000	0.0000	−0.5532	0.0000	0.0000	0.0000
	9/2	0.2876	0.9269	0.0158	0.0000	0.0000	0.5505	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	11/2	−0.4425	0.2509	0.2890	0.0000	0.0000	−0.5309	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	13/2	0.1738	−0.7862	0.2596	0.0000	0.0000	0.2359	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	15/2	−0.2483	−0.5776	0.2371	−0.0406	0.2358	−0.0804	0.0000	0.0000	−0.0488	0.0000	0.0000	0.0000
	17/2	−0.4346	−0.4424	0.1588	0.0194	0.2956	−0.2668	0.0050	−0.0354	−0.0064	0.0000	0.0000	0.0050
13/2	7/2	−0.6934	0.3669	−0.5572	0.0940	−0.7464	−0.0651	0.1498	0.5548	−0.0348	0.0000	0.0000	0.4791
	9/2	−0.3962	0.5742	−0.2774	−0.1970	−0.7138	−0.0940	0.0000	0.0000	−0.4793	0.0000	0.0000	0.0000
	11/2	0.2774	0.9199	0.0396	0.0000	0.0000	0.5254	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	13/2	−0.4438	0.2148	0.2972	0.0000	0.0000	−0.5418	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	15/2	0.1803	−0.7984	0.2476	0.0000	0.0000	0.2550	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	17/2	−0.2575	−0.5825	0.2080	−0.0460	0.2603	−0.0851	0.0000	0.0000	−0.0612	0.0000	0.0000	0.0000
	19/2	−0.4507	−0.4437	0.1179	0.0220	0.3245	−0.2773	0.0066	−0.0457	−0.0079	0.0000	0.0000	0.0082
15/2	9/2	−0.6748	0.3803	−0.5030	0.0856	−0.7240	−0.1151	0.1217	0.4801	−0.0346	0.0000	0.0000	0.3823
	11/2	−0.3856	0.5803	−0.2382	−0.1794	−0.6751	−0.0991	0.0000	0.0000	−0.4295	0.0000	0.0000	0.0000
	13/2	0.2699	0.9142	0.0567	0.0000	0.0000	0.5066	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	15/2	−0.4446	0.1879	0.3024	0.0000	0.0000	−0.5488	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	17/2	0.1852	−0.8074	0.2382	0.0000	0.0000	0.2696	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	19/2	−0.2646	−0.5858	0.1852	−0.0506	0.2797	−0.0885	0.0000	0.0000	−0.0718	0.0000	0.0000	0.0000
17/2	11/2	0.6607	0.3894	−0.4622	0.0797	−0.7045	−0.1481	0.1039	0.4288	−0.0339	0.0000	0.0000	0.3206
	13/2	0.3776	0.5843	−0.2087	−0.1670	−0.6459	−0.1021	0.0000	0.0000	−0.3939	0.0000	0.0000	0.0000
	15/2	0.2643	0.9095	0.0696	0.0000	0.0000	0.4921	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	17/2	−0.4451	0.1669	0.3060	0.0000	0.0000	−0.5536	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	19/2	0.1892	−0.8143	0.2305	0.0000	0.0000	0.2812	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
19/2	13/2	−0.6497	0.3960	−0.4303	0.0753	−0.6880	−0.1712	0.0918	0.3917	−0.0331	0.0000	0.0000	0.2784
	15/2	−0.3713	0.5870	−0.1856	−0.1577	−0.6233	−0.1039	0.0000	0.0000	−0.3672	0.0000	0.0000	0.0000
	17/2	0.2599	0.9056	0.0796	0.0000	0.0000	0.4806	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	19/2	−0.4455	0.1502	0.3086	0.0000	0.0000	−0.5570	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

[a] L is the lowest multipolarity allowed by J_1 and J_2, $L'=L+1$.

Angular Distribution Coefficients $U_k(L_{12}J_1J_2)$ and $U_k(L_{12}+1J_1J_2)$

J_1	J_2	$k=2$		4		6		8	
1	1	−0.5000	0.1000						
	2	0.5916	−0.5916						
	3	0.4899	−0.6124						
	4	0.4432	−0.6205						
	5	0.4163	−0.6245						
3/2	3/2	0.2000	−0.6000						
	5/2	0.7483	−0.1069						
	7/2	0.6547	−0.2182						
	9/2	0.6055	−0.2752						
	11/2	0.5752	−0.3097						
2	2	0.5000	−0.2143	−0.6667	0.2857				
	3	0.8281	0.2070	0.4179	−0.6268				
	4	0.7491	0.0749	0.2847	−0.5694				
	5	0.7037	0.0000	0.2271	−0.5300				
	6	0.6742	−0.0482	0.1953	−0.5023				
5/2	5/2	0.6571	0.1000	−0.1429	−0.5000				
	7/2	0.8748	0.4082	0.5803	−0.4513				
	9/2	0.8092	0.2795	0.4349	−0.5140				
	11/2	0.7687	0.2010	0.3624	−0.5296				
	13/2	0.7412	0.1482	0.3192	−0.5320				
3	3	0.7500	0.3167	0.1667	−0.5000	−0.7500	0.4167		
	4	0.9047	0.5428	0.6814	−0.2271	0.3227	−0.5809		
	5	0.8498	0.4249	0.5436	−0.3624	0.1853	−0.4633		
	6	0.8142	0.3489	0.4675	−0.4230	0.1308	−0.3924		
	7	0.7891	0.2959	0.4195	−0.4545	0.1026	−0.3463		
7/2	7/2	0.8095	0.4667	0.3651	−0.3333	−0.3333	−0.3333		
	9/2	0.9250	0.6367	0.7495	−0.0292	0.4714	−0.5571		
	11/2	0.8787	0.5311	0.6245	−0.1990	0.3060	−0.5415		
	13/2	0.8473	0.4600	0.5501	−0.2881	0.2295	−0.5048		
	15/2	0.8246	0.4088	0.5010	−0.3410	0.1864	−0.4715		
4	4	0.8500	0.5734	0.5000	−0.1494	−0.0500	−0.5136	−0.8000	0.5091
	5	0.9394	0.7045	0.7977	0.1330	0.5742	−0.4306	0.2629	−0.5257
	6	0.9000	0.6107	0.6861	−0.0490	0.4053	−0.5066	0.1300	−0.3715
	7	0.8723	0.5452	0.6157	−0.1539	0.3179	−0.5167	0.0820	−0.2868
	8	0.8517	0.4968	0.5674	−0.2207	0.2656	−0.5090	0.0588	−0.2352
9/2	9/2	0.8788	0.6515	0.5960	0.0152	0.1515	−0.4848	−0.4545	−0.1818
	11/2	0.9500	0.7551	0.8332	0.2635	0.6492	−0.2830	0.3963	−0.5792
	13/2	0.9161	0.6718	0.7340	0.0816	0.4868	−0.4219	0.2261	−0.4973
	15/2	0.8915	0.6118	0.6684	−0.0306	0.3954	−0.4730	0.1533	−0.4239
	17/2	0.8729	0.5666	0.6220	−0.1055	0.3378	−0.4918	0.1150	−0.3691
5	5	0.9000	0.7103	0.6667	0.1538	0.3000	−0.3821	−0.2000	−0.4462
	6	0.9580	0.7938	0.8601	0.3686	0.7059	−0.1412	0.4947	−0.5230
	7	0.9286	0.7196	0.7718	0.1930	0.5537	−0.3184	0.3118	−0.5300
	8	0.9067	0.6649	0.7113	0.0790	0.4625	−0.4008	0.2237	−0.4921
	9	0.8899	0.6229	0.6672	0.0000	0.4023	−0.4426	0.1738	−0.4519

Angular Distribution Coefficients $U_k(L_{12}J_1J_2)$ and $U_k(L_{12}+1J_1J_2)$ (*Continued*)

J_1	J_2	$k=2$		4		6		8	
11/2	11/2	0.9161	0.7554	0.7203	0.2687	0.4126	−0.2587	−0.0070	−0.5105
	13/2	0.9643	0.8240	0.8809	0.4538	0.7498	−0.0136	0.5705	−0.4253
	15/2	0.9385	0.7578	0.8022	0.2874	0.6091	−0.2117	0.3869	−0.5069
	17/2	0.9189	0.7079	0.7465	0.1750	0.5202	−0.3161	0.2901	−0.5095
	19/2	0.9036	0.6689	0.7050	0.0946	0.4595	−0.3760	0.2321	−0.4913
6	6	0.9286	0.7909	0.7619	0.3636	0.5000	−0.1364	0.1429	−0.4805
	7	0.9692	0.8481	0.8974	0.5235	0.7845	0.0981	0.6302	−0.3151
	8	0.9464	0.7887	0.8270	0.3676	0.6552	−0.1092	0.4521	−0.4521
	9	0.9288	0.7431	0.7758	0.2586	0.5700	−0.2280	0.3513	−0.4918
	10	0.9149	0.7069	0.7369	0.1786	0.5100	−0.3014	0.2879	−0.4973
13/2	13/2	0.9385	0.8192	0.7949	0.4423	0.5692	−0.0231	0.2615	−0.4077
	15/2	0.9732	0.8675	0.9107	0.5810	0.8124	0.1948	0.6782	−0.2056
	17/2	0.9529	0.8140	0.8474	0.4357	0.6939	−0.0140	0.5086	−0.3809
	19/2	0.9371	0.7723	0.8004	0.3313	0.6129	−0.1414	0.4068	−0.4515
7	7	0.9464	0.8422	0.8214	0.5079	0.6250	0.0786	0.3571	−0.3177
	8	0.9765	0.8835	0.9216	0.6290	0.8352	0.2784	0.7174	−0.1025
	9	0.9583	0.8351	0.8645	0.4940	0.7266	0.0727	0.5574	−0.3026
	10	0.9439	0.7968	0.8212	0.3946	0.6501	−0.0591	0.4568	−0.3975
15/2	15/2	0.9529	0.8611	0.8431	0.5630	0.6706	0.1686	0.4353	−0.2235
	17/2	0.9792	0.8967	0.9305	0.6693	0.8541	0.3506	0.7499	−0.0079
	19/2	0.9628	0.8528	0.8788	0.5440	0.7545	0.1509	0.5998	−0.2228
8	8	0.9583	0.8768	0.8611	0.6096	0.7083	0.2478	0.5000	−0.1316
	9	0.9814	0.9078	0.9381	0.7036	0.8699	0.4132	0.7770	0.0777
	10	0.9667	0.8678	0.8910	0.5873	0.7784	0.2211	0.6367	−0.1447
17/2	17/2	0.9628	0.8899	0.8762	0.6494	0.7399	0.3173	0.5542	−0.0449
	19/2	0.9833	0.9172	0.9444	0.7328	0.8833	0.4676	0.7999	0.1546
9	9	0.9667	0.9011	0.8889	0.6835	0.7667	0.3784	0.6000	0.0353
	10	0.9850	0.9253	0.9499	0.7580	0.8947	0.5151	0.8195	0.2235
19/2	19/2	0.9699	0.9107	0.8997	0.7129	0.7895	0.4322	0.6391	0.1087
10	10	0.9727	0.9189	0.9091	0.7385	0.8091	0.4797	0.6727	0.1755

Population Tensor $S_k(ll'J_1s)$

l	l'	J_1	s	$k=0$	2	4	6	8
0	0	0	0	1.00000	···	···	···	···
1	1	1	0	1.00000	−1.41421	···	···	···
2	2	2	0	1.00000	−1.19523	1.60357	···	···
3	3	3	0	1.00000	−1.15470	1.27920	−1.74078	···
4	4	4	0	1.00000	−1.13961	1.20687	−1.34840	1.85110
0	0	1	1	1.00000	···	···	···	···
1	1	0	1	1.00000	···	···	···	···
1	1	1	1	1.00000	0.70711	···	···	···
1	1	2	1	1.00000	−0.83666	···	···	···
2	2	1	1	1.00000	−0.70711	···	···	···
2	2	2	1	1.00000	−0.59761	−1.06904	···	···
2	2	3	1	1.00000	−0.98974	0.67006	···	···
3	3	2	1	1.00000	−0.95618	0.53452	···	···
3	3	3	1	1.00000	−0.86603	0.21320	1.30558	···
3	3	4	1	1.00000	−1.04464	0.87163	−0.56183	···
4	4	3	1	1.00000	−1.03098	0.82235	−0.43519	···
4	4	4	1	1.00000	−0.96866	0.60344	0.06742	−1.48088
4	4	5	1	1.00000	−1.07052	0.96275	−0.77424	0.48656
0	2	1	1	···	1.00000	···	···	···
1	3	2	1	···	0.29277	−1.30931	···	···
2	4	3	1	···	0.14286	−0.52753	1.50756	···
0	0	2	2	1.00000	···	···	···	···
1	1	1	2	1.00000	−0.14142	···	···	···
1	1	2	2	1.00000	0.83666	···	···	···
1	1	3	2	1.00000	−0.69282	···	···	···
2	2	0	2	1.00000	···	···	···	···
2	2	1	2	1.00000	0.70711	···	···	···
2	2	2	2	1.00000	0.25612	0.45816	···	···
2	2	3	2	1.00000	−0.24744	−1.00509	···	···
2	2	4	2	1.00000	−0.89540	0.45657	···	···
3	3	1	2	1.00000	−0.56569	···	···	···
3	3	2	2	1.00000	−0.23905	−0.80178	···	···
3	3	3	2	1.00000	−0.36566	−0.63960	−0.72532	···
3	3	4	2	1.00000	−0.62678	−0.29054	1.01130	···
3	3	5	2	1.00000	−0.98131	0.69532	−0.32260	···
4	4	2	2	1.00000	−0.85373	0.34362	···	···
4	4	3	2	1.00000	−0.61859	−0.27412	0.78335	···
4	4	4	2	1.00000	−0.65342	−0.18025	0.69259	0.94238
4	4	5	2	1.00000	−0.80289	0.16046	0.58068	−0.97313
0	2	2	2	···	1.00000	···	···	···
0	4	2	2	···	···	1.00000	···	···
1	3	1	2	···	1.03923	···	···	···
1	3	2	2	···	0.71714	0.53452	···	···
1	3	3	2	···	0.45356	−1.02353	···	···
2	4	2	2	···	0.45816	−0.85373	···	···
2	4	3	2	···	0.39123	−0.77051	−0.82572	···
2	4	4	2	···	0.25612	−0.71234	0.96424	···
0	0	3	3	1.00000	···	···	···	···
1	1	2	3	1.00000	−0.23905	···	···	···
1	1	3	3	1.00000	0.86603	···	···	···
1	1	4	3	1.00000	−0.62678	···	···	···
2	2	1	3	1.00000	−0.20203	···	···	···
2	2	2	3	1.00000	0.68299	−0.11454	···	···
2	2	3	3	1.00000	0.45363	0.67006	···	···
2	2	4	3	1.00000	−0.08954	−0.91314	···	···
2	2	5	3	1.00000	−0.84112	0.36422	···	···
3	3	0	3	1.00000	···	···	···	···
3	3	1	3	1.00000	0.70711	···	···	···
3	3	2	3	1.00000	0.43825	0.53452	···	···
3	3	3	3	1.00000	0.19245	−0.21320	0.29013	···
3	3	4	3	1.00000	−0.11396	−0.60750	−0.91936	···
3	3	5	3	1.00000	−0.49065	−0.46355	0.80650	···
4	4	1	3	1.00000	−0.50508	···	···	···
4	4	2	3	1.00000	−0.08537	−0.68724	···	···
4	4	3	3	1.00000	−0.11247	−0.57315	−0.71214	···
4	4	4	3	1.00000	−0.24716	−0.57209	−0.20839	−0.47119
4	4	5	3	1.00000	−0.44556	−0.44787	0.37968	1.04799
0	2	3	3	···	1.00000	···	···	···
0	4	3	3	···	···	1.00000	···	···

Population Tensor $S_k(ll'J_1s)$ (*Continued*)

l	l'	J_1	s	$k=0$	2	4	6	8
1	3	2	3	···	0.93895	−0.11664	···	···
1	3	3	3	···	0.80178	0.72375	···	···
1	3	4	3	···	0.52489	−0.89214	···	···
2	4	1	3	···	1.04978	···	···	···
2	4	2	3	···	0.80992	0.60368	···	···
2	4	3	3	···	0.67006	−0.11247	0.32141	···
2	4	4	3	···	0.51224	−0.59362	−0.96424	···
2	4	5	3	···	0.31944	−0.75448	0.73510	···
0	0	4	4	1.00000	···	···	···	···
1	1	3	4	1.00000	−0.28868	···	···	···
1	1	4	4	1.00000	0.87750	···	···	···
1	1	5	4	1.00000	−0.58878	···	···	···
2	2	2	4	1.00000	−0.34149	0.01273	···	···
2	2	3	4	1.00000	0.61859	−0.22335	···	···
2	2	4	4	1.00000	0.52910	0.74711	···	···
2	2	5	4	1.00000	···	−0.84984	···	···
3	3	1	4	1.00000	−0.23570	···	···	···
3	3	2	4	1.00000	0.59761	−0.17817	···	···
3	3	3	4	1.00000	0.57735	0.62668	−0.07913	···
3	3	4	4	1.00000	0.35708	0.07924	0.51076	···
3	3	5	4	1.00000	0.02265	−0.51109	−0.93058	···
4	4	0	4	1.00000	···	···	···	···
4	4	1	4	1.00000	0.70711	···	···	···
4	4	2	4	1.00000	0.50448	0.56229	···	···
4	4	3	4	1.00000	0.35241	0.07476	0.39563	···
4	4	4	4	1.00000	0.17020	−0.28152	−0.48561	0.18123
4	4	5	4	1.00000	−0.05147	−0.51840	−0.48390	−0.74856
0	2	4	4	···	1.00000	···	···	···
0	4	4	4	···	···	1.00000	···	···
1	3	3	4	···	0.88641	−0.21822	···	···
1	3	4	4	···	0.83324	0.78680	···	···
1	3	5	4	···	0.56569	−0.81650	···	···
2	4	2	4	···	0.89540	−0.18202	···	···
2	4	3	4	···	0.86504	0.66792	−0.08299	···
2	4	4	4	···	0.74711	0.17020	0.54091	···
2	4	5	4	···	0.57735	−0.45455	−0.94901	···
0	0	1/2	1/2	1.00000	···	···	···	···
1	1	1/2	1/2	1.00000	···	···	···	···
1	1	3/2	1/2	1.00000	−1.00000	···	···	···
2	2	3/2	1/2	1.00000	−1.00000	···	···	···
2	2	5/2	1/2	1.00000	−1.06904	0.92582	···	···
3	3	5/2	1/2	1.00000	−1.06904	0.92582	···	···
3	3	7/2	1/2	1.00000	−1.09109	1.02565	−0.87039	···
4	4	7/2	1/2	1.00000	−1.09109	1.02565	−0.87039	···
4	4	9/2	1/2	1.00000	−1.10096	1.06436	−0.98473	0.82784
0	0	3/2	3/2	1.00000	···	···	···	···
1	1	1/2	3/2	1.00000	···	···	···	···
1	1	3/2	3/2	1.00000	0.80000	···	···	···
1	1	5/2	3/2	1.00000	−0.74833	···	···	···
2	2	1/2	3/2	1.00000	···	···	···	···
2	2	3/2	3/2	1.00000	···	···	···	···
2	2	5/2	3/2	1.00000	−0.38180	−1.05808	···	···
2	2	7/2	3/2	1.00000	−0.93522	0.53724	···	···
3	3	3/2	3/2	1.00000	−0.80000	···	···	···
3	3	5/2	3/2	1.00000	−0.58797	−0.46291	···	···
3	3	7/2	3/2	1.00000	−0.72739	−0.11396	1.16052	···
3	3	9/2	3/2	1.00000	−1.00922	0.76871	−0.41030	···
4	4	5/2	3/2	1.00000	−0.95450	0.59517	···	···
4	4	7/2	3/2	1.00000	−0.81052	0.14652	0.69631	···
4	4	9/2	3/2	1.00000	−0.87577	0.33866	0.42523	−1.20413
0	2	3/2	3/2	···	1.00000	···	···	···
1	3	3/2	3/2	···	0.60000	···	···	···
1	3	5/2	3/2	···	0.39279	−1.13389	···	···
2	4	5/2	3/2	···	0.28057	−0.80992	···	···
2	4	7/2	3/2	···	0.20912	−0.65526	1.16775	···

Population Tensor $S_k(ll'J_1s)$ (*Continued*)

l	l'	J_1	s	$k=0$	2	4	6	8
0	0	5/2	5/2	1.00000	···	···	···	···
1	1	3/2	5/2	1.00000	−0.20000	···	···	···
1	1	5/2	5/2	1.00000	0.85524	···	···	···
1	1	7/2	5/2	1.00000	−0.65465	···	···	···
2	2	1/2	5/2	1.00000	···	···	···	···
2	2	3/2	5/2	1.00000	0.71429	···	···	···
2	2	5/2	5/2	1.00000	0.38180	0.59517	···	···
2	2	7/2	5/2	1.00000	−0.15587	−0.95510	···	···
2	2	9/2	5/2	1.00000	−0.86504	0.40266	···	···
3	3	1/2	5/2	1.00000	···	···	···	···
3	3	3/2	5/2	1.00000	0.20000	···	···	···
3	3	5/2	5/2	1.00000	0.03563	−0.46291	···	···
3	3	7/2	5/2	1.00000	−0.21822	−0.64578	−0.87039	···
3	3	9/2	5/2	1.00000	−0.55048	−0.39600	0.89521	···
4	4	3/2	5/2	1.00000	−0.71429	···	···	···
4	4	5/2	5/2	1.00000	−0.38180	−0.59517	···	···
4	4	7/2	5/2	1.00000	−0.39959	−0.51948	0.14243	···
4	4	9/2	5/2	1.00000	−0.53618	−0.35248	0.53713	1.05361
0	2	5/2	5/2	···	1.00000	···	···	···
0	4	5/2	5/2	···	···	1.00000	···	···
1	3	3/2	5/2	···	0.97980	···	···	···
1	3	5/2	5/2	···	0.77143	0.65983	···	···
1	3	7/2	5/2	···	0.49487	−0.94761	···	···
2	4	3/2	5/2	···	0.69985	···	···	···
2	4	5/2	5/2	···	0.59517	−0.38180	···	···
2	4	7/2	5/2	···	0.46238	−0.68051	−0.93891	···
2	4	9/2	5/2	···	0.29161	−0.74037	0.82988	···
0	0	7/2	7/2	1.00000	···	···	···	···
1	1	5/2	7/2	1.00000	−0.26726	···	···	···
1	1	7/2	7/2	1.00000	0.87287	···	···	···
1	1	9/2	7/2	1.00000	−0.60553	···	···	···
2	2	3/2	7/2	1.00000	−0.28571	···	···	···
2	2	5/2	7/2	1.00000	0.64906	−0.17635	···	···
2	2	7/2	7/2	1.00000	0.49878	0.71632	···	···
2	2	9/2	7/2	1.00000	−0.03932	−0.87852	···	···
3	3	1/2	5/2	1.00000	···	···	···	···
3	3	3/2	7/2	1.00000	0.66667	···	···	···
3	3	5/2	7/2	1.00000	0.53452	0.61721	···	···
3	3	7/2	7/2	1.00000	0.29096	−0.04144	0.42201	···
3	3	9/2	7/2	1.00000	−0.03670	−0.55906	−0.93251	···
4	4	1/2	7/2	1.00000	···	···	···	···
4	4	3/2	7/2	1.00000	0.28571	···	···	···
4	4	5/2	7/2	1.00000	0.20131	−0.21643	···	···
4	4	7/2	7/2	1.00000	0.05668	−0.42624	−0.63301	···
4	4	9/2	7/2	1.00000	−0.13583	−0.55717	−0.39079	−0.64838
0	2	7/2	7/2	···	1.00000	···	···	···
0	4	7/2	7/2	···	···	1.00000	···	···
1	3	5/2	7/2	···	0.90914	−0.17496	···	···
1	3	7/2	7/2	···	0.82065	0.76190	···	···
1	3	9/2	7/2	···	0.54772	−0.84984	···	···
2	4	3/2	7/2	···	0.95831	···	···	···
2	4	5/2	7/2	···	0.84946	0.66877	···	···
2	4	7/2	7/2	···	0.71632	0.05668	0.45455	···
2	4	9/2	7/2	···	0.54908	−0.51831	−0.96161	···

Statistical Tensor Coefficients $\rho_k(J_1M_1)$

J_1	k	$M_1=0$	1	2	3	4	5	6
1	2	−1.4142	1.4142					
2	2	−1.1952	−1.1952	2.3905				
	4	1.6036	−2.1381	0.5345				
3	2	−1.1547	−1.7321	0.0000	2.8868			
	4	1.2792	0.4264	−2.9848	1.2792			
	6	−1.7408	2.6112	−1.0445	0.1741			
4	2	−1.1366	−1.9373	−0.9117	0.7977	3.1909		
	4	1.2069	1.2069	−1.4751	−2.8160	1.8774		
	6	−1.3484	0.1348	2.9665	−2.2923	0.5394		
	8	1.8511	−2.9618	1.4809	−0.4231	0.0529		
5	2	−1.1323	−2.0381	−1.3587	−0.2265	1.3587	3.3968	
	4	1.1767	1.5689	−0.3922	−2.3534	−2.3534	2.3534	
	6	−1.2524	−0.7515	2.2544	1.8161	−3.0059	0.9393	
	8	1.4085	−0.5634	−2.7365	2.9377	−1.2475	0.2012	
6	2	−1.1282	−2.0951	−1.6116	−0.8058	0.3223	1.7728	3.5456
	4	1.1609	1.7689	0.3040	−1.4925	−2.6534	−1.8242	2.7363
	6	−1.2098	−1.2098	1.3308	2.6010	0.4839	−3.3269	1.3308
	8	1.2941	0.3697	−2.6251	−0.7025	3.2907	−2.0336	0.4067
7	2	−1.1256	−2.1306	−1.7688	−1.1658	−0.3216	0.7638	2.0904
	4	1.1514	1.8916	0.7646	−0.7585	−2.1444	−2.6470	−1.3068
	6	−1.1863	−1.4829	0.5931	2.3370	2.0879	−0.6525	−3.3928
	8	1.2411	0.8865	−1.9502	−2.2055	1.1134	2.9431	−2.6736
8	2	−1.1239	−2.1541	−1.8732	−1.4049	−0.7493	0.0937	1.1239
	4	1.1453	1.9725	1.0817	−0.1909	−1.5271	−2.4815	−2.4815
	6	−1.1717	−1.6600	0.0391	1.8162	2.4997	1.2694	−1.5233
	8	1.2109	1.2109	−1.2801	−2.5256	−0.8649	2.2488	2.2488
9	2	−1.1227	−2.1706	−1.9461	−1.5718	−1.0479	−0.3742	0.4491
	4	1.1411	2.0287	1.3083	0.2421	−0.9682	−2.0402	−2.6107
	6	−1.1621	−1.7818	−0.3768	1.2835	2.3805	2.1745	0.4120
	8	1.1918	1.4301	−0.7151	−2.3495	−1.9409	0.5108	2.7581
10	2	−1.1219	−2.1825	−1.9990	−1.6930	−1.2646	−0.7139	−0.0408
	4	1.1381	2.0693	1.4753	0.5748	−0.4982	−1.5558	−2.3567
	6	−1.1552	−1.8694	−0.6931	0.8083	2.0552	2.4539	1.6141
	8	1.1787	1.5859	−0.2572	−1.9931	−2.3710	−0.8857	1.5824
11	2	−1.1212	−2.1915	−2.0386	−1.7838	−1.4270	−0.9683	−0.4077
	4	1.1359	2.0997	1.6019	0.8340	−0.1112	−1.1041	−1.9779
	6	−1.1503	−1.9345	−0.9379	0.4030	1.6699	2.4011	2.2152
	8	1.1694	1.7009	0.1112	−1.5925	−2.4259	−1.7593	0.2145
12	2	−1.1207	−2.1984	−2.0690	−1.8535	−1.5518	−1.1638	−0.6897
	4	1.1342	2.1229	1.6999	1.0390	0.2062	−0.7059	−1.5783
	6	−1.1465	−1.9843	−1.1304	0.0627	1.2873	2.1863	2.4280
	8	1.1624	1.7883	0.4088	−1.2038	−2.2816	−2.2150	−0.8773

Statistical Tensor Coefficients $\rho_k(J_1M_1)$ (*Continued*)

J_1	k	$M_1=1/2$	3/2	5/2	7/2	9/2	11/2	13/2
3/2	2	−2.0000	2.0000					
5/2	2	−2.1381	−0.5345	2.6726				
	4	1.8516	−2.7775	0.9258				
7/2	2	−2.1822	−1.3093	0.4364	3.0551			
	4	2.0513	−0.6838	−2.9630	1.5954			
	6	−1.7408	3.1334	−1.7408	0.3482			
9/2	2	−2.2019	−1.6514	−0.5505	1.1010	3.3029		
	4	2.1287	0.3548	−2.0105	−2.6018	2.1287		
	6	−1.9695	1.4771	2.4618	−2.7080	0.7385		
	8	1.6557	−3.3113	2.3652	−0.8278	0.1183		
11/2	2	−2.2125	−1.8332	−1.0746	0.0632	1.5803	3.4768	
	4	2.1678	0.9291	−1.0065	−2.5549	−2.0904	2.5549	
	6	−2.0684	0.4137	2.5854	1.1376	−3.2059	1.1376	
	8	1.8992	−2.0077	−1.7635	3.2286	−1.6550	0.2984	
13/2	2	−2.2188	−1.9415	−1.3868	−0.5547	0.5547	1.9415	3.6056
	4	2.1904	1.2778	−0.2637	−1.8659	−2.6772	−1.5617	2.9003
	6	−2.1221	−0.2653	1.9629	2.4086	−0.1167	−3.3847	1.5173
	8	2.0121	−1.0061	−2.6157	0.2817	3.1791	−2.3743	0.5231
15/2	2	−2.2229	−2.0112	−1.5878	−0.9527	−0.1059	0.9527	2.2229
	4	2.2048	1.5049	0.2683	−1.1782	−2.3448	−2.5781	−1.0616
	6	−2.1550	−0.7183	1.2930	2.4998	1.6953	−1.1206	−3.3618
	8	2.0767	−0.2967	−2.4327	−1.5783	1.7681	2.6226	−2.9311
17/2	2	−2.2257	−2.0587	−1.7249	−1.2241	−0.5564	0.2782	1.2798
	4	2.2145	1.6609	0.6543	−0.6040	−1.8119	−2.5668	−2.3655
	6	−2.1767	−1.0340	0.7173	2.1718	2.3796	0.8361	−1.8651
	8	2.1178	0.2118	−1.9665	−2.3296	−0.1513	2.5716	1.8455
19/2	2	−2.2276	−2.0926	−1.8226	−1.4176	−0.8776	−0.2025	0.6075
	4	2.2214	1.7726	0.9405	−0.1440	−1.2846	−2.2195	−2.6215
	6	−2.1919	−1.2620	0.2491	1.7282	2.4666	1.9122	0.0077
	8	2.1459	0.5852	−1.4631	−2.4663	−1.4392	1.0928	2.8306
21/2	2	−2.2291	−2.1177	−1.8947	−1.5604	−1.1146	−0.5573	0.1115
	4	2.2264	1.8553	1.1576	0.2220	−0.8183	−1.7856	−2.4579
	6	−2.2030	−1.4319	−0.1271	1.2837	2.2750	2.3640	1.2964
	8	2.1659	0.8664	−0.9997	−2.3064	−2.1187	−0.3244	1.9769
23/2	2	−2.2302	−2.1367	−1.9495	−1.6688	−1.2945	−0.8266	−0.2651
	4	2.2302	1.9183	1.3257	0.5147	−0.4211	−1.3569	−2.1367
	6	−2.2113	−1.5618	−0.4299	0.8777	1.9688	2.4475	2.0232
	8	2.1809	1.0828	−0.5957	−2.0186	−2.3839	−1.3345	0.7115