A General Graphical Method for Angular Momentum

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A graphical method is presented allowing the representations of "*j-m*" and "$3nj$" coefficients, spherical harmonics, irreducible tensor operators, and rotation matrices. Rules are established which permit calculations on expressions with the above elements. As the main difficulty of using "$3j$" Wigner coefficients is the construction of the phase, an algorithm is proposed which simplifies this problem. Concrete examples are given for this general method.

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I. INTRODUCTION

The book of Yutsis, Levinson, and Vanagas[1] (YLV) first attracted our attention to a graphical method for describing "$3j$" Wigner coefficients. However it seems to us that their work was not complete on two points. First it does not permit the calculation of expressions where both vector-coupling coefficients and spherical harmonics are present. We have extended their graphical method to spherical harmonics, irreducible tensor operators, and rotation matrices. The second point was that (YLV)'s method could be applied only after writing a formal expression of vector-coupling coefficients and then transforming it in a graphical manner. So we have given an algorithm which allows immediately a graphical representation of a scheme coupling with its phase.

A similar method to (YLV)'s has been proposed by Kotansky *et al.*[2] On the other hand, Yutsis *et al.*[3–5] have published a graphical method based on the Clebsch–Gordan coefficient representation for simplifying phase considerations in a calculus. Our generalization of (YLV)'s work has been done with the first sort of conventions, because of their simplicity.

After a summary of (YLV)'s method we set out our extension of this work and give a concrete example.

II. SUMMATION OF "njm" AND "$3nj$"

A. The "$3jm$" Symbol

When coupling two angular momenta $\mathbf{j}_1$ and $\mathbf{j}_2$, one obtains a vector $\mathbf{J}=\mathbf{j}_1+\mathbf{j}_2$, the coefficients which transform one basis to another one are the Clebsch–Gordan coefficients;

$$|JM\rangle=\sum_{m_1m_2}\langle j_1m_1j_2m_2\,|\,JM\rangle\,|j_1m_1\rangle\,|j_2m_2\rangle.$$

We define a "$3jm$" coefficient which is the usual "$3j$" Wigner coefficient by

$$\langle j_1m_1j_2m_2\,|\,JM\rangle=(-)^{j_1-j_2+j}\hat{J}(-)^{J-M}\begin{pmatrix} j_1 & j_2 & J\\ m_1 & m_2 & -M\end{pmatrix},$$

where $\hat{J}=(2j+1)^{1/2}$. We call it the "$3jm$" coefficient because it depends on the magnetic quantum numbers. We keep the notation "$3nj$" ($n=1, 2\cdots$) for coefficients like "$6j$" which do not depend on m.

The following formula recalls some symmetry properties of "$3jm$" coefficients.

$$\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}=\begin{pmatrix} j_3 & j_1 & j_2\\ m_3 & m_1 & m_2\end{pmatrix}$$

$$=(-)^{j_1+j_2+j_3}\begin{pmatrix} j_2 & j_1 & j_3\\ m_2 & m_1 & m_3\end{pmatrix}$$

$$=(-)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3\\ -m_1 & -m_2 & -m_3\end{pmatrix}.$$

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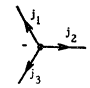
[1] A. P. Yutsis, I. B. Levinson, and V. V. Vanagas, *Mathematical Apparatus of the Theory of Angular Momentum* (Israël Program for Scientific Translation, Jerusalem, 1962).
[2] A. Kotansky, Acta Phys. Polon. **26**, 109 (1964).
[3] A. Yutsis, A. Banzaitis, and J. Vizbaraite, Lietuvos Fiz. Rinkinys **1-2**, 74 (1962).
[4] A. Yutsis, A. Banzaitis, and J. Vizbaraite, Lietuvos Fiz. Rinkinys **1**, 91 (1962).
[5] A. Yutsis, Z. B. Rudzikas, and A. Bandzaitis, Lietuvos Fiz. Rinkinys **5**, 4 (1965).

Yutsis, Levinson, and Vanagas[1] represent the "$3jm$" coefficient

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

by the diagram



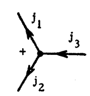
The three lines are called kinetic lines. Each of these lines denotes a kinetic moment j and its free end the projection m of j on the quantization axis. If $m>0$, the free line is directed outwards from the node, and if $m<0$, the free line is directed inward.

The sign of the node denotes the cyclic order in which the lines are read; that is, + sign if the order is anticlockwise.

We denote the "$3jm$" representation

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}$$

by the following:



The symmetry properties of "$3jm$" coefficients give several topological rules:

A change in the orientation of three lines together gives a phase $(-)^{j_1+j_2+j_3}$.

A change in the sign of a node gives the same phase $(-)^{j_1+j_2+j_3}$.

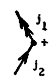
It is forbidden to change the direction of a free line alone.

A rotation or a geometrical deformation of a diagram does not affect the "$3jm$" represented by the diagram.

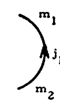
Particular cases. If one moment is zero

$$\begin{pmatrix} j_1 & j_2 & 0 \\ m_1 & -m_2 & 0 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 \\ m_1 & -m_2 \end{pmatrix} = \hat{j}_1^{-1}(-)^{j_1-m_1}\delta_{j_1j_2}\delta_{m_1m}$$

is represented by

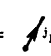


In the same way as in (YLV) we can extract the node and its value $\hat{j}_1^{-1}\delta_{j_1j_2}$, we obtain the equivalence



$= (-)^{j_1-m_1}\delta_{m_1m_2}.$

If two moments are now equal to zero

$$\begin{pmatrix} j_1 & 0 & 0 \\ m_1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} j_1 \\ m_1 \end{pmatrix} = \delta_{j_10}\delta_{m_10} =$$


B. Summation on "$3jm$"

By summation over a magnetic quantum number of the expression

$$\sum_{m_1} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = F\left(\begin{matrix} j_2 & j_3 \\ m_2 & m_3 \end{matrix} \middle| j_1\right)$$

we can transform it into

$$(-)^{j_2-m_2+j_3-m_3}\sum_{m_1}(-)^{j_1-m_1}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} = F\left(\begin{matrix} j_2 & j_3 \\ m_2 & m_3 \end{matrix} \middle| j_1\right). \quad (1)$$

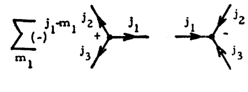
On writing

$$(-)^{j_2-m_2+j_3-m_3}F\left(\begin{matrix} j_2 & j_3 \\ m_2 & m_3 \end{matrix} \middle| j_1\right) = G\left(\begin{matrix} j_2 & j_3 \\ m_2 & m_3 \end{matrix} \middle| j_1\right)$$

we get

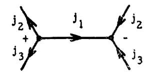
$$\sum_{m_1}(-)^{j_1-m_1}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} = G\left(\begin{matrix} j_2 & j_3 \\ m_2 & m_3 \end{matrix} \middle| j_1\right)$$

given by



Graphically we sum this expression by joining the ends of free corresponding lines; but the phase $(-)^{j_1-m_1}$ must be always present before a summation over m_1, and the directions of lines to sum must be opposite. So we obtain for

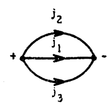
$$G\begin{pmatrix} j_2 & j_3 & \Big| & j_1 \\ m_2 & m_3 & \Big| & \end{pmatrix}$$



Continuing the summation over m_2 and m_3 of expression (1)

$$\sum_{m_2m_3}(-)^{j_2-m_2+j_3-m_3}G\begin{pmatrix} j_2 & j_3 & \Big| & j_1 \\ m_2 & m_3 & \Big| & \end{pmatrix}=F(j_1j_2j_3)$$

the diagram representing $F(j_1j_2j_3)$ is



$F(j_1j_2j_3)$ equals one if $|j_1-j_2|\leq j_3\leq j_1+j_2$ and zero otherwise. We call $F(j_1j_2j_3)$ a triangular delta $\{j_1j_2j_3\}$ and note that it is the first "$3nj$" coefficient with $n=1$.

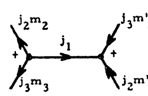
Whenever two nodes are joined by a line j we can change the direction of j but it is necessary to multiply the result by $(-)^{2j}$.

We can find the second rule of orthogonality of "$3jm$" coefficients by noting that

$$(-)^{j_2-m'_2+j_3-m'_3}\sum_{j_1m_1}\hat{j}_1^2(-)^{j_1-m_1}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m'_2 & -m'_3 \end{pmatrix}=\delta_{m_2m'_2}\delta_{m_3m'_3}$$

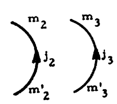
which is given by

$$\sum_{j_1}\hat{j}_1^2$$



$$=(-)^{j_2-m'_2}\delta_{m'_2m_2}(-)^{j_3-m'_3}\delta_{m'_3m_3}.$$

But we know that $(-)^{j_2-m'_2}\delta_{m'_2m_2}$ and $(-)^{j_3-m'_3}\delta_{m'_3m_3}$ are represented by



So we have a rule of summation over j:

The expression to sum must show the term $\hat{j}_1^2$ without any phase over j_1.

All the lines must be convergent at a node and divergent at the other one.

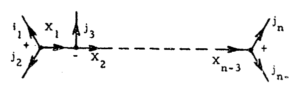
When these conditions are fulfilled we drop the line j_1 with its nodes and join the corresponding lines (with the same kinetic moment).

C. The "*njm*" Coefficients

We call an "njm" coefficient an expression obtained by the superposition of "$3jm$" symbols so as to obtain a tree. This condition is fulfilled when for the coupling of n moments j_i there are $(n-3)$ intermediate moments X_k:

$$\begin{pmatrix} j_1 & j_2\cdots j_n & \Big| & X_1\cdots X_{n-3} \\ m_1 & m_2\cdots m_n & \Big| & \end{pmatrix}_A=\sum_{x_i}(-)^{\Sigma_i X_i-x_i}\begin{pmatrix} j_1 & j_2 & X_1 \\ m_1 & m_2 & x_1 \end{pmatrix}\begin{pmatrix} X_1 & j_3 & X_2 \\ -x_1 & m_3 & x_2 \end{pmatrix}\cdots\begin{pmatrix} X_{n-3} & j_{n-1} & j_n \\ -x_{n-3} & m_{n-1} & m_n \end{pmatrix},$$

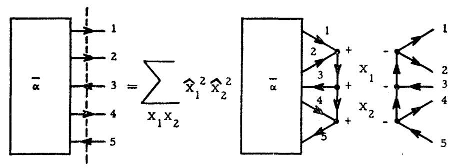
where A defines the mode of coupling chosen. Graphically it is represented by



The "njm" has orthogonality relations like "$3jm$" coefficients which are given in (YLV).[1]

If the structure of a diagram is not interesting for a certain problem, we can close it in a block which we call a "closed block" (and we denote such a block by $\bar{\alpha}$ if inside it there are no free lines); and we call a block open if inside it some free lines remain (denoted then by α).

Let us consider a diagram in which free lines are outside of a closed block $\bar{\alpha}$. (YLV) show that we can extract an "njm" coefficient by closing all free lines with the aid of $(n-3)$ intermediate moments X_k. For example with $n=5$

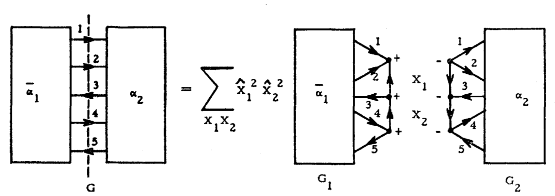


$$F\begin{pmatrix} j_1 & j_2 & j_3 & j_4 & j_5 & \bigg| \\ m_1 & m_2 & -m_3 & m_4 & -m_5 & \bigg| \end{pmatrix}\!\!\begin{matrix}\bar{\alpha}\end{matrix}\Bigg) \qquad G\begin{pmatrix} j_1 & j_2 & j_3 & j_4 & j_5 \\ & & X_1 & X_2 & \end{pmatrix}\cdot\text{"}5jm\text{"}$$

$$F\left(\begin{matrix} j_1 & j_2 & j_3 & j_4 & j_5 \\ m_1 & m_2 & -m_3 & m_4 & -m_5 \end{matrix}\middle| \bar{\alpha}\right) = \sum_{X_1X_2} \hat{X}_1^2\hat{X}_2^2 G\begin{pmatrix} j_1 & j_2 & j_3 & j_4 & j_5 \\ & & X_1 & X_2 & \end{pmatrix}\left(\begin{matrix} j_1 & j_2 & j_3 & j_4 & j_5 \\ m_1 & m_2 & -m_3 & m_4 & -m_5 \end{matrix}\middle| X_1 \quad X_2\right)_A .$$

The signs of the nodes and the directions of the lines are opposite in the G block and in the "njm" diagram.

If a closed block $\bar{\alpha}_1$ is connected to a second block α_2 (α_2 can be closed or open) by n lines we can use the same rule. That is, we close the first and the second block on the same intermediate moments:



$$G = \sum_{m_1\cdots m_5} (-)^{j_1-m_1+\cdots+j_5-m_5} F_1\left(\begin{matrix} j_1 & j_2 & j_3 & j_4 & j_5 \\ + & + & - & + & - \end{matrix}\middle| \bar{\alpha}_1\right) F_2\left(\begin{matrix} j_1 & j_2 & j_3 & j_4 & j_5 \\ - & - & + & - & + \end{matrix}\middle| \alpha_2\right),$$

$$G = \sum_{X_1X_2} \hat{X}_1^2\hat{X}_2^2 G_1\left(\begin{matrix} j_1\cdots j_5 \\ X_1 \quad X_2 \end{matrix}\middle| \bar{\alpha}_1\right) G_2\left(\begin{matrix} j_1\cdots j_5 \\ X_1 \quad X_2 \end{matrix}\middle| \alpha_2\right),$$

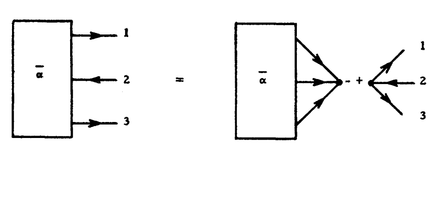
with

$$G_1\left(\begin{matrix} j_1\cdots j_5 \\ X_1 \quad X_2 \end{matrix}\middle| \bar{\alpha}_1\right) = {}_\delta\sum_{m_1\cdots m_5} (-)^{j_1-m_1+\cdots+j_5-m_5} F_1\left(\begin{matrix} j_1 & j_2 & j_3 & j_4 & j_5 \\ + & + & - & + & - \end{matrix}\middle| \bar{\alpha}_1\right)\left(\begin{matrix} j_1 & j_2 & j_3 & j_4 & j_5 \\ - & - & + & - & + \end{matrix}\middle| X_1 \quad X_2\right)_A$$

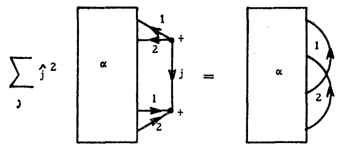
$$G_2\left(\begin{matrix} j_1\cdots j_5 \\ X_1 \quad X_2 \end{matrix}\middle| \alpha_2\right) = \sum_{m_1\cdots m_5} (-)^{j_1-m_1+\cdots+j_5-m_5} F_2\left(\begin{matrix} j_1 & j_2 & j_3 & j_4 & j_5 \\ - & - & + & - & + \end{matrix}\middle| \alpha_2\right)\left(\begin{matrix} j_1 & j_2 & j_3 & j_4 & j_5 \\ + & + & - & + & - \end{matrix}\middle| X_1 \quad X_2\right)_A .$$

These relations are reversible. In the case where $n=3$ there are no more intermediate summations but it is necessary for conservation of the phase that the three lines have the same directions if tied.

When three lines are free, we pinch them with their respective signs but in the closed diagram we have to put the three tied lines in the same direction.



The rule of summation on a kinetic moment is not affected by the existence of blocks.



D. The "$3nj$" Coefficients

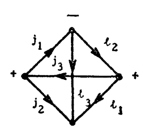
Just as (YLV) have called "jm" coefficient a diagrammatic expression with free lines they call "j-coefficient" a diagram with all lines tied. Such a coefficient has $3n$ lines and $2n$ nodes. If it is separable on no less than four lines it is called a "$3nj$" coefficient.

For $n=1$ we find again the triangular delta $\{j_1 j_2 j_3\}$.

For $n=2$ we get the so-called "$6j$" coefficient related to the "$3jm$" coefficients by

$$\begin{Bmatrix} j & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \sum_{m_i n_i} (-)^{\Sigma_i j_i - m_i + l_i - n_i} \times \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \begin{pmatrix} j_1 & l_2 & l_3 \\ -m_1 & n_2 & n_3 \end{pmatrix} \times \begin{pmatrix} l_1 & l_2 & j_3 \\ n_1 & -n_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & j_2 & l_3 \\ -n_1 & -m_2 & -n_3 \end{pmatrix}$$

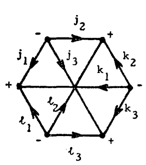
and represented by the diagram



In the same way the "$9j$"

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 \end{Bmatrix}$$

is represented by



A complete study of these diagrams for $n>3$ is given by (YLV) with their diagrammatic representations.

III. SPHERICAL AND MULTIPOLAR HARMONICS

A. The Spherical Harmonics Functions

If the spherical harmonic $Y_{lm}(\Omega)$ is written with Dirac's formalism, we obtain $Y_{lm}(\Omega) = \langle \Omega \mid lm \rangle$. So $Y_{lm}(\Omega)$ is the juxtaposition of the ket $\mid lm\rangle$ and the bra $\langle \Omega \mid$. It becomes therefore natural to give for $Y_{lm}(\Omega)$ the representation .

The full line is the kinetic line and the dotted line will be called the "angular line." The properties of the kinetic lines are those of the preceding section. For the angular line we adopt the convention that an outward direction denotes a positive solid angle $\Omega = (\theta, \phi)$ and $-\Omega = (\pi - \theta, \phi + \pi)$ is represented by an inward direction.[6]

As obviously $Y_{lm}(-\Omega) = (-)^l Y_{lm}(\Omega)$, we say that a change in the direction of the angular line gives a phase $(-)^l$ in the result. On the other hand the relation $Y_{lm}^*(\Omega) = (-)^m Y_{l-m}(\Omega)$ permits a diagrammatic representation of a conjugate spherical harmonic. We can note that $Y_{00}(\Omega) = Y_{00}(-\Omega) = (4\pi)^{-1/2}$, so that suppressing a kinetic line of an Y_{lm} gives the constant value $(4\pi)^{-1/2}$.

With these conventions and those of the preceding section we now see how we can make an algebraic summation or an integration.

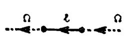
As

$$\sum_m Y_{lm}^*(\Omega) Y_{lm}(\Omega') = (\hat{l}^2/4\pi) P_l(\cos(\Omega, \Omega'))$$

can be written

$$\sum_m (-)^{l-m} Y_{l-m}(\Omega) Y_{lm}(-\Omega') = (\hat{l}^2/4\pi) P_l \cos(\Omega, \Omega')$$

(YLV)'s convention for summation over m gives the diagram



which represents $(l^2/4\pi) P_l(\cos(\Omega, \Omega'))$.

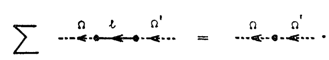
Note that when $\Omega = \Omega' P_l(\cos(\Omega, \Omega')) = 1$ and

 $= \hat{l}^2/4\pi$.

The summation over l gives

$$\sum_l (\hat{l}^2/4\pi) P_l(\cos(\Omega, \Omega')) = \delta(\Omega - \Omega')$$

and graphically



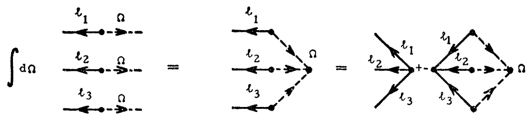
For these calculations we have simply adopted Yutsis' conventions and rules.

Let us consider now the integration of three spherical harmonics

$$\int Y_{l_1m_1}(\Omega) Y_{l_2m_2}(\Omega) Y_{l_3m_3}(\Omega) d\Omega = \frac{\hat{l}_1\hat{l}_2\hat{l}_3}{(4\pi)^{1/2}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.$$

[6] J.-N. Massot, E. El Baz, and J. Lafoucrière, Nucl. Phys. **83**, 449 (1966).

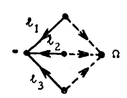
As the result does not depend on solid angle Ω we join the ends of free angular lines with the same direction to represent an integration. Thus



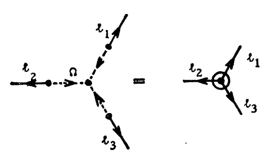
The last part of the above diagram has been obtained by applying the rules of the preceding section. On comparing the analytic result and the diagrams obtained, it is easily seen that the value

$$\frac{l_1 l_2 l_3}{(4\pi)^{1/2}}\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}$$

must be attached to the diagram



For simplicity's sake we replace it by a circle put on the corresponding node, and call it a "marking circle." Therefore the integration of three Y_{lm} becomes very simple graphically



When such a "marking circle" exists it is not necessary to give a sign to a node since $l_1+l_2+l_3$ is even because of the "$3jm$"

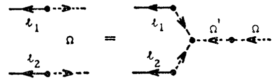
$$\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}.$$

The integration is independent of the fact that kinetic lines belong to a block or not. For the integration of less than three spherical harmonics if is sufficient to take the preceding case and to drop the kinetic lines equal to zero.

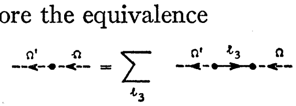
We shall now do graphically the composition of two spherical harmonics of the same direction. As

$$Y_{l_1m_1}(\Omega)\,Y_{l_2m_2}(\Omega)=\int Y_{l_1m_1}(\Omega')\,Y_{l_2m_2}(\Omega')\,\delta(\Omega-\Omega')\;d\Omega'.$$

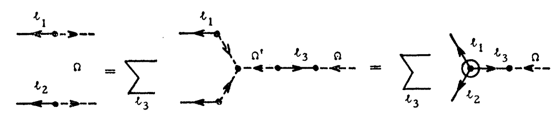
It follows that

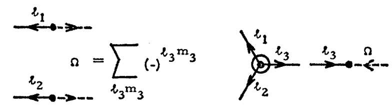


We saw before the equivalence



We therefore obtain



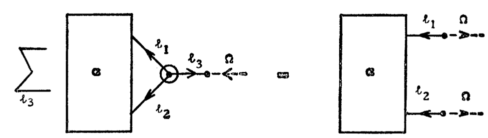


that is analytically

$$Y_{l_1m_1}(\Omega)\,Y_{l_2m_2}(\Omega)$$

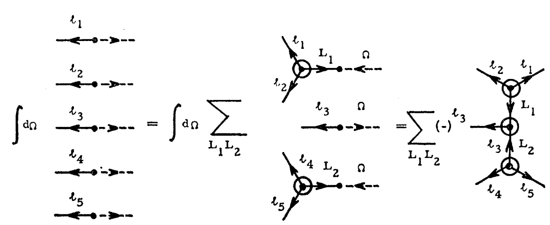
$$=\sum_{l_3m_3}\frac{l_1 l_2 l_3}{(4\pi)^{1/2}}\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} Y^*_{l_3m_3}(\Omega)$$

more generally



To integrate more than three spherical harmonics, we can then group them two by two as before and integrate after.

The result of the integration of five spherical harmonics is, for example,



which is easily translated analytically.

B. Multipolar Harmonics

Since we have defined spherical harmonics by $\langle\Omega\mid lm\rangle$ with l being an integer, we can say that $\langle u\mid \alpha jm\rangle$ represents a monopolar harmonic where j can be an integer or a half-integer, and $|jm\rangle$ is an eigenfunction of J^2J_z, the quantum number α completing the chosen basis.

We note that $\langle u\mid \alpha jm\rangle = M_{\alpha jm}(u)$ and represent it graphically as

We continue by calling the dotted line an "angular line" and the full line a kinetic line.

To keep the same phase convention as for spherical harmonics and time reversal properties, we write

$$M^*_{\alpha jm}(u)=(-)^{j-m}M_{\alpha j-m}(-u).$$

We put that

$$M_{\alpha jm}(u)=(-)^{j}M_{\alpha jm}(-u)$$

so that

$$M^*_{\alpha jm}(u)=(-)^{-m}M_{\alpha j-m}(u).$$

We can then extract the following rules:

We can change the direction of the u line by introducing a phase $(-)^{\pm j}$ depending on whether the u line is initially positive or negative.

To obtain the conjugate of a monopolar harmonic we have to change the direction of the kinetic line and add $(-)^{\pm m}$ depending on whether the final kinetic line is positive or negative.

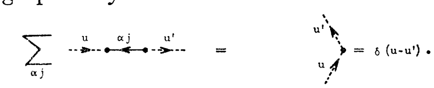
The closure relation

$$\sum_{\alpha jm}\langle u' \mid \alpha jm\rangle\langle\alpha jm \mid u\rangle=\delta(u-u')$$

can be rewritten

$$\sum_{\alpha jm}(-)^{j-m}M_{\alpha j-m}(-u)M_{\alpha jm}(u')=\delta(u-u').$$

It gives graphically



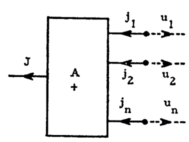
Both lines must be convergent or divergent at a node. We can now define multipolar harmonics by

$$\langle u_1u_2\cdots u_n \mid j_1\cdots j_nJM\rangle_A=X_1\cdots\hat{X}_{n-2}\hat{J}(-)^{f_A(j_1\cdots j_nJ)}\sum_{m_1\cdots m_n}(-)^{j_1-m_1+\cdots+j_n-m_n}\times\left(\begin{matrix}j_1\cdots j_n & J\\ -m_1\cdots -m_n & M\end{matrix}\middle| X_1\cdots X_{n-2}\right)_A\langle u_1 \mid j_1m_1\rangle\cdots\langle u_n \mid j_nm_n\rangle,$$

where $\langle u_1\cdots u_n \mid j_1\cdots j_nJM\rangle_A$ is the multipolar harmonic of n rank and $\langle u_1 \mid j_1m_1\rangle\cdots$ monopolar harmonics. A is a chosen mode of coupling; $f_A(j_1\cdots j_nJ)$ is a linear combination of its parameters depending on the scheme of addition A.

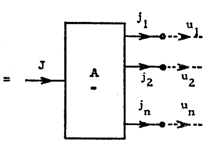
It can be easily seen that multipolar harmonics have closure and orthogonality relations as monopolar harmonics.

The graphical representation of a multipolar harmonic will be, following its analytical definition,

$$\hat{X}_1\cdots\hat{X}_{n-2}\hat{J}(-)^{f_A(j_1\cdots j_nJ)}$$  .

The complex conjugate of a multipolar harmonic is obtained by conjugation of all monopolar harmonics it contains. This is expressed graphically by changing the direction of all kinetic lines and the sign of the nodes (symbolically represented by + on the above diagram) and adding the phase $(-)^{-m_1-m_2\cdots-m_n}=(-)^{-M}$

$$[\langle u_1\cdots u_n \mid j_1\cdots j_nJM\rangle_A]^*=(-)^{f_A(j_1\cdots j_nJ)}\hat{X}_1\hat{X}_2\cdots\hat{X}_{n-2}\hat{J}(-)^{-M}G_3$$

$G_3=$ 

We see in the next section the advantage of such a representation.

IV. IRREDUCIBLE TENSOR OPERATORS

The spherical harmonics $Y_{lm}(\Omega)$ which we have previously introduced are irreducible tensor operators of l rank. Thus the graphical representation of $Y_{lm}(\Omega)$ must be that of an irreducible tensor operator.[7] Let u be the coordinates of a certain space ϵ where a tensor T_{lm} (l integer) is acting without going outside of ϵ so that

$$\langle u' \mid T_{lm} \mid u\rangle=T_{lm}(u)\delta(u-u').$$

Then $T_{lm}(u)$ is represented by . The symmetry and conjugate properties of $T_{lm}(u)$ will be found the same as for spherical harmonics; that is

$$T_{lm}^+(u)=(-)^{-m}T_{l-m}(u)$$

shown by

$(-)^{-m}$ 

We write $T_{lm}(u)=(-)^{l}T_{lm}(-u)$ which defines the inversion of the direction of a u line.

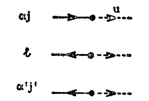
A. The Wigner–Eckart Theorem

Let us evaluate a matrix element of a T_{lm} tensor operator $\langle\alpha j\mu \mid T_{lm} \mid \alpha'j'\mu'\rangle$. By writing it in the ϵ space, we get

$$\langle\alpha j\mu \mid T_{lm} \mid \alpha'j'\mu'\rangle=\iint du\,du'\langle\alpha j\mu \mid u\rangle\langle u \mid T_{lm} \mid u'\rangle\langle u' \mid \alpha'j'\mu'\rangle=\int du\langle\alpha j\mu \mid u\rangle T_{lm}(u)\langle u \mid \alpha'j'\mu'\rangle.$$

[7] E. El Baz, J.-N. Massot, and J. Lafoucrière, Nucl. Phys. **82**, 189 (1966).

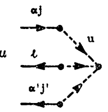
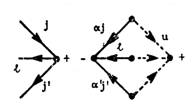
We recognize here the monopolar harmonics which we have defined in the preceding section. We get in that way

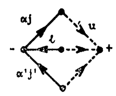
$$\langle \alpha j\mu \mid T_{lm} \mid \alpha' j' \mu' \rangle = (-)^{-\mu} \int du$$ 

The sign $(-)^{-\mu}$ comes from the conjugation of the monopolar harmonic $\langle \alpha j\mu \mid j\rangle$.

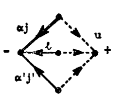
We know that an integration consists in joining u lines of the same direction. Thus

$\langle \alpha k\mu \mid T_{lm} \mid \alpha' j' \mu' \rangle$

$$= (-)^{-\mu}$$  $= (-)^{-\mu}$ 

$$= (-)^{-\mu} \begin{pmatrix} j & l & j' \\ -\mu & m & \mu' \end{pmatrix}$$ 

Note that the diagram remaining does not depend on magnetic quantum numbers (as it is closed). Therefore we can identify it as being the reduced matrix element of the tensor operator and to obtain the usual phase of Wigner–Eckart we put

 $= (-)^{j} \langle \alpha j \parallel T_l \parallel \alpha' j' \rangle.$

The monopolar harmonics are defined as to give usual spherical harmonics when j becomes integer. It does not affect at all the result given in this part. It is even possible to define $M_{jm}^*(u) = (-)^{j-m} M_{j-m}(u)$ so that the marking circle takes the simplest value of the reduced matrix element. The results obtained are evidently identical to the preceding one.

We can simplify the graphical representation by substituting in this diagram a "marking circle" as in the preceding section. Let us note that if

$$T_{l_2m_2} \equiv Y_{l_2m_2}$$

$$\langle l_1 \parallel Y_{l_2} \parallel l_3 \rangle = (-)^{l_1} \frac{\hat{l}_1\hat{l}_2\hat{l}_3}{(4\pi)^{1/2}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}$$

so that the value

$$\frac{l_1 l_2 \hat{l}_3}{(4\pi)^{1/2}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}$$

of the marking circle previously defined is coherent.

The Wigner–Eckart is now written

$$\langle \alpha j\mu \mid T_{lm} \mid \alpha' j' \mu' \rangle = (-)^{-\mu}$$ 

To read this matrix element we have always to start from the entrance kinetic line and to follow the cyclic order given by the sign of the node.

B. Tensorial Product of Two Tensor Operators

The tensorial product $\prod_{lm}$ of two tensor operators $T_{l_1m_1}^{(1)}$ and $T_{l_2m_2}^{(2)}$ will be

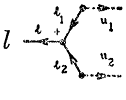
$$\prod_{lm} = [T^{(1)}{}_{l_1m_1} \otimes T^{(2)}{}_{l_2m_2}]_{lm} = \sum_{m_1m_2} \langle l_1m_1l_2m_2 \mid lm \rangle T^{(1)}{}_{l_1m_1} T^{(2)}{}_{l_2m_2}$$

with "$3jm$" symbols it becomes

$$\prod_{lm} = l \sum_{m_1m_2} (-)^{l_1-m_1+l_2-m_2} \begin{pmatrix} l_1 & l & l_2 \\ -m_1 & m & -m_2 \end{pmatrix} T^{(1)}{}_{l_1m_1} T^{(2)}{}_{l_2m_2}.$$

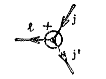
If $T^{(1)}{}_{l_1m_1}$ acts in the ϵ_1 space and $T^{(2)}{}_{l_2m_2}$ in the ϵ_2 space, which can be identical of not, $\prod_{lm}$ will act in the ϵ space, tensorial product of ϵ_1 and ϵ_2.

We obtain a clear representation of $\prod_{lm}(u_1, u_2)$

$$\prod_{lm}(u_1, u_2) = l$$  .

This tensor operator has the same structure as a multipolar harmonic with $n=2$.

$\prod_{lm}$ being a tensor operator, we can apply to it the Wigner–Eckart theorem

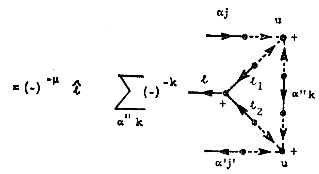
$$\langle \alpha j\mu \mid \prod_{lm} \mid \alpha' j' \mu' \rangle = (-)^{-\mu} \quad (2)$$ 

Let us find the reduced matrix element of $\prod_{lm}$ with the aid of the matrix elements of $T_{l_1m_1}$ and $T_{l_2m_2}$ tensors.

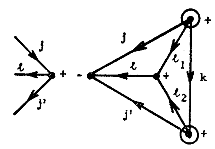
If $T_{l_1m_1}$ and $T_{l_2m_2}$ acts in the same space $u_1=u_2=u$

$$\langle \alpha j\mu \mid \prod_{lm} \mid \alpha' j' \mu' \rangle = (-)^{-\mu} \int M_{\alpha j-\mu}(u) \prod_{lm}(u) M_{\alpha' j' \mu'}(u)\, du$$

represented by

$$\langle \alpha j\mu \mid \prod_{lm} \mid \alpha' j' \mu' \rangle$$ 

The intermediate kinetic line must have a direction as to permit the construction of reduced matrix elements (at each angular node there must exist one entrance and two exit directions).

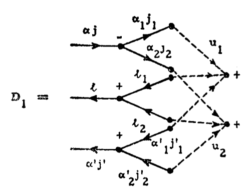
$$\langle \alpha j\mu \mid \prod_{lm} \mid \alpha' j' \mu' \rangle = (-)^{-\mu} \hat{l} \sum_{\alpha'' k} (-)^{-k}$$  (3)

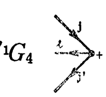
By comparing (2) and (3) we get

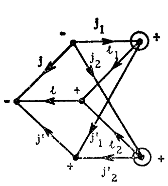
$$\langle \alpha j \parallel \prod_{l} \parallel \alpha' j' \rangle = \hat{l} (-)^{j+j'+l} \sum_{\alpha'' k} \langle \alpha j \parallel T_{l_1} \parallel \alpha'' k \rangle \langle \alpha'' k \parallel T_{l_2} \parallel \alpha' j' \rangle \begin{Bmatrix} l_1 & l_2 & l \\ j' & j & k \end{Bmatrix}.$$

If now $u_1 \neq u_2$, that is, $T^{(1)}{}_{l_1m_1}$ and $T^{(2)}{}_{l_2m_2}$ act on different spaces, in place of monopolar harmonic we have to introduce bipolar harmonics before integration over u_1 and u_2 coordinates. This easily gives the result

$$\langle \alpha j\mu \mid \prod_{lm} \mid \alpha' j' \mu' \rangle = (-)^{-\mu+2j_1+2j'_1} \hat{j} \hat{l} \hat{j}' \cdot D_1$$



$$\langle \alpha j\mu \mid \prod_{lm} \mid \alpha' j' \mu' \rangle = \hat{j} \hat{l} \hat{j}' (-)^{-\mu+2j_1+2j'_1} G_4$$ 

$G_4=$ 

We get the reduced matrix element of $\prod_{lm}$ as previously done:

$$\langle \alpha_1 j_1 \alpha_2 j_2 \alpha j \parallel \prod_{l} \parallel \alpha'_1 j'_1 \alpha'_2 j'_2 \alpha' j' \rangle = \hat{j} \hat{l} \hat{j}' \langle \alpha_1 j_1 \parallel T_{l_1} \parallel \alpha'_1 j'_1 \rangle \langle \alpha_2 j_2 \parallel T_{l_2} \parallel \alpha'_2 j'_2 \rangle \begin{Bmatrix} j_1 & j'_1 & l_1 \\ j_2 & j'_2 & l_2 \\ j & j' & l \end{Bmatrix}.$$

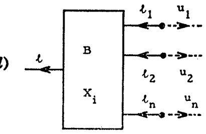
C. Tensorial Product of n Tensor Operators

As we have constructed a tensorial product of two tensor operators, we can generalize the method now to obtain the tensorial product of n tensor operators after the choice of a coupling scheme B has been made

$$\prod_{lm} = [T^{(1)}{}_{l_1m_1} \otimes T^{(2)}{}_{l_2m_2} \otimes \cdots \otimes T^{(n)}{}_{l_nm_n}]_{lm} = \hat{X}_1 \cdots \hat{X}_{n-2} \hat{l} (-)^{f_B(l_1 \cdots l_n l)} \sum_{m_1 \cdots m_n} (-)^{l_1-m_1+\cdots+l_n-m_n}$$
$$\times \begin{pmatrix} l_1 \cdots l_n & \\ -m_1 \cdots -m_n & \end{pmatrix} \left| X_1 \cdots X_{n-2} \right)_B T^{(1)}{}_{l_1m_1} \cdots T^{(n)}{}_{l_nm_n}.$$

If we characterize an ϵ space as one where each of these tensors acts, we obtain the graphical representation

$$\prod_{lm} = \hat{X}_1 \cdots \hat{X}_{n-2} l (-)^{f_B(l_1 \cdots l_n l)}$$

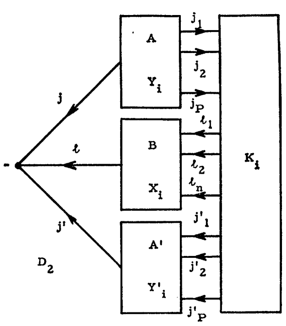


To evaluate the reduced matrix element of such an operator, we have to introduce multipolar harmonics (Sec. III) the order of which equals the number of independent spaces where tensor operators $T_{l_i m_i}$ act, and to join corresponding lines in threes with $(n-p)$ intermediate moments K_i

$$\langle \alpha j \mu \mid \Pi_{lm} \mid \alpha' j' \mu' \rangle = (-)^{-\mu} \hat{X}_1 \cdots \hat{X}_{n-2} \hat{Y}_1 \cdots \hat{Y}_{n-2} \hat{Y}'_1 \cdots \hat{Y}'_{n-2} \hat{l} \hat{j} \hat{j}' (-)^{f_A(j_1 \cdots j_n J)} (-)^{f_{A'}(j'_1 \cdots j'_n J')} (-)^{f_B(l_1 \cdots l_n l)} \times \begin{pmatrix} j & l & j' \\ -\mu & m & \mu' \end{pmatrix} G_5,$$

where G_5 has the representation

$$G_5 = \sum_{K_1 \cdots K_{n-p}} (-)^{K_1 + \cdots K_{n-p}} \cdot D_2,$$

$D_2 =$ 

In the last block of this graph we find all "marking circles"; that is, all reduced matrix elements.

V. ROTATION MATRICES

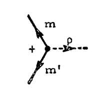
It now remains to give a graphical representation of rotation matrices in order to have a complete description of the elements of rotation group R_3.

Let $D^j_{mm'}(\rho)$ be an element of a matrix[8]

$$D(\rho) \; \langle jm \mid D(\rho) \mid jm' \rangle,$$

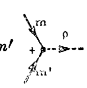
where ρ is related to a reference system. [For instance $\rho = (\alpha, \beta, \gamma)$ if the system taken is that of Euler's angles.]

We say that the graphical representation of $D^j_{mm'}(\rho)$ is



where the dotted line is still an "angular" line and the full lines are the kinetic lines, attached now to a same kinetic moment j; this is the reason why we note it (mm') and no longer j. The indices are read from the angular line by following the cyclic order given by the sign of the node.

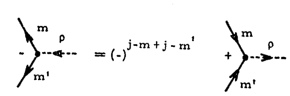
The complex conjugate $D^{j*}_{mm'}(\rho)$ being equal to $(-)^{j-m+j-m'} D^j_{-m-m'}(\rho)$ it will be easy to have a representation of it.

$$D^{j*}_{mm'}(\rho) = (-)^{j-m+j-m'}$$ 

We represent by $-\rho$ the angular coordinates obtained for the inverse rotation R^{-1}. Thus

$$D^j_{m'm}(-\rho) = D^{j*}_{mm'}(\rho)$$

and graphically



The above equality gives the rule for changing the direction of an angular line:

We must change the sign of the node and the direction of the kinetic line.

Also we add the phase $(-)^{j-m+j-m'}$ to the result.

For integration we keep the same method as for spherical harmonics. The relation

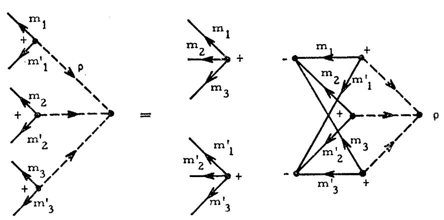
$$\int D^{j_1}_{m_1 m'_1}(\rho) D^{j_2}_{m_2 m'_2}(\rho) D^{j_3}_{m_3 m'_3}(\rho) \, d\rho = 8\pi^2 \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix}$$

is represented by joining angular lines of the same direction.

We can close this diagram on its kinetic lines in the usual manner but it must be carefully noted that two

[8] E. El Baz, J.-N. Massot, and J. Lafoucrière, Nucl. Phys. **86**, 625 (1966).

kinetic lines issuing from the same node are a projection of the same j. They cannot be tied together. Thus we have to tie only corresponding lines (lines issuing from different nodes with the same order starting from the angular lines). Thus we get



The identification with the analytic formula attributes the value $8\pi^2$ to the closed diagram.

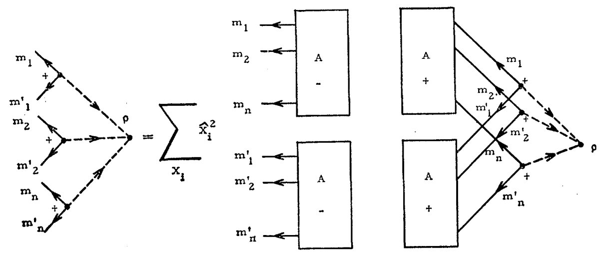
A generalization can be given for the integration of n rotation matrix elements.

$$\int D^{j_1}{}_{m_1m'_1}(\rho)\cdots D^{j_n}{}_{m_nm'_n}(\rho)\, d\rho$$

$$=8\pi^2 \sum_{X_1\cdots X_{n-2}} \hat{X}_1{}^2\cdots\hat{X}_{n-3}{}^2 \begin{pmatrix} j_1\cdots j_n \\ m_1\cdots m_n \end{pmatrix}\!\!\left|\, X_1\cdots X_{n-3}\right)_A$$

$$\times\begin{pmatrix} j_1\cdots j_n \\ m'_1\cdots m'_n \end{pmatrix}\!\!\left|\, X_1\cdots X_{n-3}\right)_A .$$

Graphically, by joining the angular lines and shutting the corresponding lines with "njm" coefficients, we get

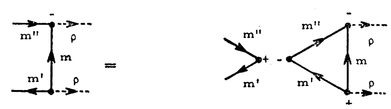


The closed diagram here also gets the volume of rotation group $8\pi^2$.

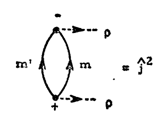
We can write the unitarity of rotation matrices as

$$\sum_m (-)^{j-m} D^j{}_{-m-m''}(\rho)\, D^j{}_{mm'}(\rho) = (-)^{j-m'}\delta_{m'm''}$$

represented by



The extraction of the node from the first graph gives the value j to the second one and

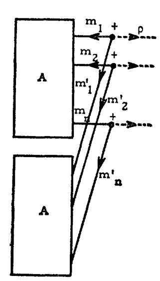


It is easy now to obtain expressions where both rotation matrices and "$3jm$" coefficients are present.

$$G_6=\sum_{\substack{m_im'_i\\X_k}} \hat{X}_k{}^2(-)^{\Sigma_i j_i-m_i+j_i-m'_i} D^{j_1}{}_{m_1m'_1}(\rho)\cdots D^{j_n}{}_{m_nm'_n}(\rho)$$

$$\times\begin{pmatrix} j_1\cdots j_n \\ -m_1\cdots -m_n \end{pmatrix}\!\!\left|\, X_1\cdots X_{n-3}\right)_A \begin{pmatrix} j_1\cdots j_n \\ -m'_1\cdots -m'_n \end{pmatrix}\!\!\left|\, X_1\cdots X_{n-3}\right)_A$$

shown by the following graph



has 1 for value as[1]

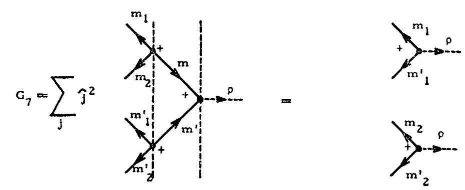
$$D^{j_1}{}_{m_1m'_1}(\rho)\cdots D^{j_n}{}_{m_nm'_n}(\rho) = \sum_{\substack{JMM'\\X'_k}} \hat{\jmath}^2\hat{X}_k{}^2(-)^{j-M+J-M'} \begin{pmatrix} j_1\cdots j_n & J \\ m_1\cdots m_n & M \end{pmatrix}\!\!\left|\, X'_k\right)_A \begin{pmatrix} j_1\cdots j_n & J \\ m'_1\cdots m'_n & -M \end{pmatrix}\!\!\left|\, X'_k\right)_A D^J{}_{MM'}(\rho).$$

A graphical integration over angular variables of the preceding diagram gives to the graph the previously obtained value of $8\pi^2$.

Let us show how the summation over kinetic moment must be done. For this purpose let us consider the expression:

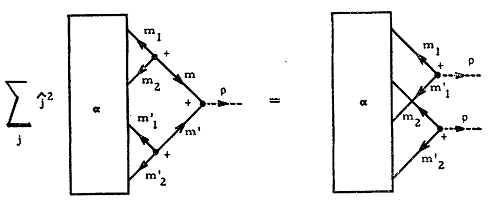
$$G_7=\sum_{jmm'}\hat{j}^2(-)^{j-m+j-m'}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m\end{pmatrix}\begin{pmatrix} j_1 & j_2 & j \\ m'_1 & m'_2 & m'\end{pmatrix}D^j{}_{-m-m'}(\rho).$$

If we remember that magnetic lines represent the same j we shall have to apply (YLV)'s rule simultaneously on both lines and join the corresponding lines that remain



$$G_7=D^{j_1}{}_{m_1m'_1}(\rho)D^{j_2}{}_{m_2m'_2}(\rho).$$

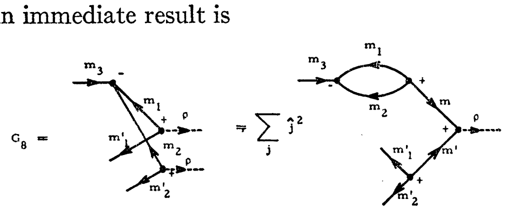
This method is not affected by the presence of a block tied to the kinetic lines

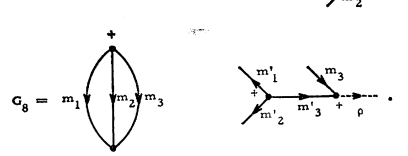


By using this rule we can find some other summation like

$$G_8=\sum_{m_1m_2}(-)^{j_1-m_1+j_2-m_2}D^{j_1}{}_{m_1m'_1}(\rho)\;D^{j_2}{}_{m_2m'_2}(\rho)\times\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3\end{pmatrix}.$$

An immediate result is





Thus we get the value of G_8

$$G_8=\sum_{m_3}(-)^{j_3-m_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m'_1 & m'_2 & m'_3\end{pmatrix}D^{j_3}{}_{-m_3-m'_3}(\rho).$$

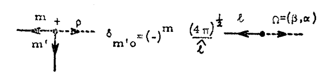
Before the end of this section we note that when j is an integer (say l) and when m' equals zero, the rotation matrix becomes a spherical harmonic. Following the phase convention adopted by Rose,[9] Messiah,[10] Brink[11] or Rose,[12] Wigner,[13] Fano,[14] Racah,[15] Edmonds[16] we obtain the equivalence

$$D^l{}_{m0}(\alpha\beta\gamma)=[(4\pi)^{1/2}/\hat{l}]Y^*{}_{lm}(\beta\alpha)$$
$$D^l{}_{0m'}(\alpha\beta\gamma)=[(4\pi)^{1/2}/\hat{l}](-)^{m'}Y^*{}_{lm'}(\beta\gamma)$$

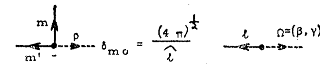
with the first conventions.

With the second sort of conventions we get easily

$$D^l{}_{m0}(\alpha\beta\gamma)=(-)^m\frac{(4\pi)^{1/2}}{\hat{l}}Y_{lm}(\beta\alpha)$$



$$D^l{}_{0m}(\alpha\beta\gamma)=\frac{(4\pi)^{1/2}}{\hat{l}}Y_{lm}(\beta\gamma)$$



We go from the second convention to the first by a simple change in the direction of kinetic lines.

This graphical method applied to rotation matrices of the same angle ρ does not present any difficulty. However when problems use different angles $\rho_1\rho_2$ the manipulation of the diagrammatic representation of its rotation matrices must be done carefully, expecially when one is finding the product of two rotations.

VI. VECTORIAL DIAGRAMS AND THEIR USE FOR THE COUPLING OF ANGULAR MOMENTA

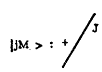
So far we have talked about "njm" coefficients without specifying the coupling scheme which was enclosed in a block diagram. When the coupling scheme has been chosen we can find analytically the phase $(-)^{f_A(j_1\cdots j_n)}$ and the form of the tree enclosed in the block. As a graphical method is faster to use than

[9] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).
[10] A. Messiah, *Mécanique Quantique* (Dunod Cie, Paris, 1960).
[11] D. M. Brink and G. R. Satchler, *Angular Momentum* (Oxford University Press, London, 1962).
[12] M. E. Rose, *Multipole Fields* (John Wiley & Sons, Inc., New York, 1955).
[13] E. P. Wagner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic Press Inc., New York, 1959).
[14] V. Fano and G. Racah, *Irreducible Tensorial Sets* (Academic Press Inc., New York, 1959).
[15] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N.J., 1957).
[16] D. Robson, Nucl. Phys. **22**, 34 (1961).

an analytical method, we have constructed a method resolving the above problem.

A. Clebsch–Gordan Coefficients and Triangular Diagrams

The basic idea is to associate to a ket $|jm\rangle$, or a vector $\mathbf{J}$ an oriented line. The following diagram will correspond to

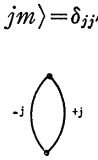


the $+$ sign shows that it represents a ket and not a bra (defined by $-$ sign.)

The orthogonality

$$\langle j'm' \mid jm \rangle = \delta_{jj'}\delta_{mm'}$$

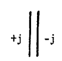
will be represented by



Also, an elementary projector on a state with j fixed

$$\sum_m |jm\rangle\langle jm| = 1$$

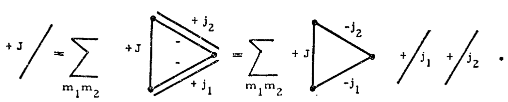
is shown as

$\sum_m$ 

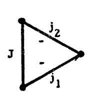
A ket $|JM\rangle$ eigenfunction of J^2Jz is related to $|j_1m_1\rangle|j_2m_2\rangle$ eigenfunctions of $J_1^2J_{1z}J_2^2J_{2z}$ when $\mathbf{J}=\mathbf{j}_1+\mathbf{j}_2$ by the relation:

$$|JM\rangle = \sum_{m_1m_1} |j_1m_1\, j_2m_2\rangle\langle j_1m_1\, j_2m_2 \mid JM\rangle$$

and the associated diagram will be



It is clear that the C–G $\langle j_1m_1\, j_2m_2 \mid JM\rangle$ is attached to the triangular diagram



which is read in the direct order; the last moment being read must be that of the "unique" sign (for instance here J).

$$\langle j_1m_1\, j_2m_2 \mid JM\rangle = \hat{J}(-)^{j_1-j_2+J}(-)^{J-M} \times \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} \quad (4)$$

$$\langle j_1m_1\, j_2m_2 \mid JM\rangle = \hat{J}(-)^{j_1-j_2-J}(-)^{j_1-m_1+j_2-m_2} \times \begin{pmatrix} j_1 & j_2 & J \\ -m_1 & -m_2 & M \end{pmatrix}. \quad (5)$$

From these equations we can get the rule giving a graphical "$3jm$" coefficient starting from a triangular diagram:

To each triangular diagram we tie a node inside it.

From this node go out three lines cutting the side of the triangle. These lines are the so-called kinetic lines.

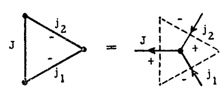
The sign of the node follows the cyclic order of the triangular diagram.

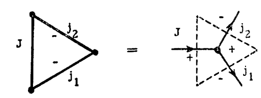
The direction of a kinetic line is given by the sign attached to the side of the triangle cutted.

To each entrance arrow is attached the phase $(-)^{j-m}$.

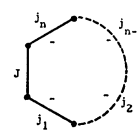
The remaining phase is given by (4) or (5) relations.

Thus we obtain

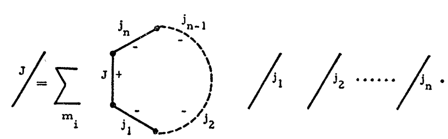
 $\hat{J}(-)^{j_1-j_2-J}(-)^{j_1-m_1+j_2-m_2}$

 $\hat{J}(-)^{j_1-j_2+J}(-)^{J-M}$.

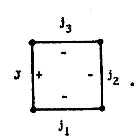
The preceding remarks may be generalized to a coupling of n angular momenta: $\mathbf{J}-\mathbf{j}_1-\mathbf{j}_2\cdots\mathbf{j}_n=0$. We symbolize this by



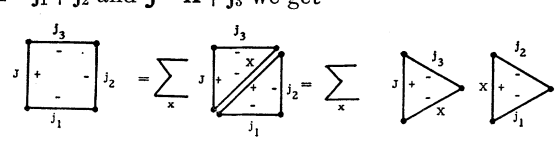
and this triangular diagram obviously represents a "generalized Clebsch–Gordan" as we can see



Let us take a concrete example with $n=3$. We obtain then the "generalized Clebsch–Gordan"



We can now give it with elementary triangular diagrams by using an adequate projector. If we choose $\mathbf{X}=\mathbf{j}_1+\mathbf{j}_2$ and $\mathbf{J}=\mathbf{X}+\mathbf{j}_3$ we get

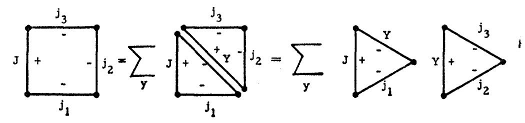


that is

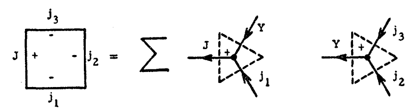
$$\langle j_1m_1\, j_2m_2\, j_3m_3 \mid (j_1j_2j_3)XJM\rangle = \sum_x \langle j_1m_1\, j_2m_2 \mid Xx\rangle\langle Xx j_3m_3 \mid JM\rangle$$

and we can give the above diagram with "$3jm$" coefficients by using relations (4) or (5).

With the other coupling scheme $\mathbf{j}_2+\mathbf{j}_3=\mathbf{Y}$ $\mathbf{J}=\mathbf{j}_1+\mathbf{Y}$ we would have obtained



Let us translate this symbol analytically. It comes with (5)



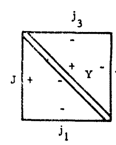
$$=\sum_y \hat{J}(-)^{j_1-Y-J}(-)^{j_1-m_1+Y-y}\begin{pmatrix} j^1 & Y & J \\ -m_1 & -y & M \end{pmatrix}\hat{Y}(-)^{j_2-j_3-Y}\begin{pmatrix} j_2 & j_3 & Y \\ -m_2 & -m_3 & y \end{pmatrix}(-)^{j_2-m_2+j_3-m_3}$$

$$=\hat{J}\hat{Y}(-)^{\varphi}(-)^{j_1-m_1+j_2-m_2+j_3-m_3}\sum_y(-)^{Y-y}\begin{pmatrix} j_1 & Y & J \\ -m_1 & -y & M \end{pmatrix}\begin{pmatrix} j_2 & j_3 & Y \\ -m_2 & -m_3 & y \end{pmatrix}$$

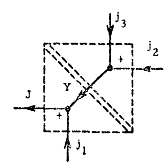
with $\varphi=j_1+j_2-j_3-J-2Y$.

Graphically it is not necessary to decompose all these steps and even to give the analytical expression represented.

We go immediately from the triangular representation to the "*njm*" representation with the preceding rules. Thus



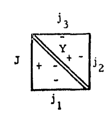
$$=\hat{J}\hat{Y}(-)^{j_1-m_1+j_2-m_2+j_3-m_3}(-)^{\varphi}$$



We see later how to obtain easily the phase $(-)^{\varphi}$.

We can then write

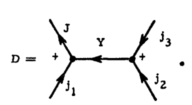
$$|JM\rangle=\sum_y$$



$$|j_1m_1\rangle\,|j_2m_2\rangle\,|j_3m_3\rangle$$

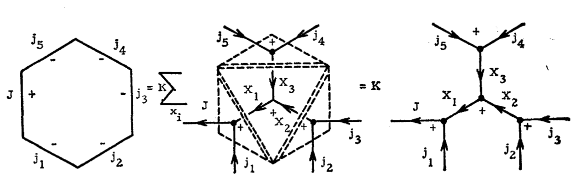
or

$$|(j_1j_2j_3)YJM\rangle=\sum_y \hat{J}\hat{Y}(-)^{j_1-m_1+j_2-m_2+j_3-m_3}(-)^{\varphi}\,|j_1m_1j_2m_2j_3m_3\rangle\cdot D$$



For more complicated problems the result can be easily found when a coupling scheme has been chosen.

For instance if $\mathbf{J}=\mathbf{j}_1+\mathbf{j}_2+\mathbf{j}_3+\mathbf{j}_4+\mathbf{j}_5$, one gets



The K coefficient will be defined with the help of next paragraph.

B. An Algorithm for finding the K Coefficient

The name K coefficient designates one which permits the transition from a triangular diagram to a "*njm*" coefficient.

We can find K step by step, following the (5) and (4) relations.

$$K=\hat{J}(-)^{j_1-X_1-J+j_1-m_1}\hat{X}_1(-)^{X_2-X_3-X_1}$$
$$\times\hat{X}_2(-)^{j_2-j_3-X_2+j_2-m_2+j_3-m_3}$$
$$\times\hat{X}_3(-)^{j_4-j_5-X_3+j_4-m_4+j_5-m_5}$$

$$K=\hat{J}\hat{X}_1\hat{X}_2\hat{X}_3(-)^{j_1+j_2-j_3+j_4-j_5-J-2X_1-2X_3}$$
$$\times(-)^{j_1-m_1+j_2-m_2+j_3-m_3+j_4-m_4+j_5-m_5}.$$

It would be convenient to have a general rule for finding it directly.

Let us make a few remarks here:

(a) All free entrance kinetic line j bring a factor $(-)^{j-m}$

(b) All kinetic lines J or X whose direction is "unique" when we consider the three lines all together, bring a factor $\hat{J}$ or X.

(c) All kinetic line j or X whose direction is "unique" and *positive* bring a phase $(-)^{-j}$ or $(-)^{-X}$.

Let us call 1 the "unique" line (if positive it will be $1+$ and negative $1-$). When starting from this line and following the cyclic order given by the sign of the node we meet a second line (called $2+$ or $2-$ according to its direction) and a third one ($3+$ or $3-$).

(d) All kinetic lines j or X which are of the $3\pm$ type bring the phase $(-)^{-j}$ or $(-)^{-X}$.

With these remarks it becomes easy to find the resulting K factor.

This noted it is sufficient to number the $1\pm$ $1+$ $3\pm$ and free negative lines. This work may be done directly on the triangular diagram because it is simpler.

Thus we construct the following table (K table):

$\hat{A}$	$(-)^{j-m}$	$(-)^{\phi(J)}(-)^{\phi(x)}$	
$1\pm$	Free negative	$1+$	$3\pm$
···	···	···	···
···	···	···	···

The first column gives $\hat{j}$ or $\hat{X}$ coefficients.

The second column gives $(-)^{j_i-m_i}$ phase.

The phase on J will be a sum of all j_i and J with the $(-)$ sign if we meet them in columns $1+$ or $3\pm$.

The phase on X will be a sum of all $2X_i$, dropping the X_i which appears only once in columns $1+$ or $3\pm$.

C. Example of Application

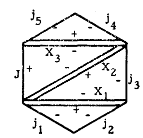
Let us give a concrete example: we want the ket of a momentum **J** sum of five moments.

$$\mathbf{J}=\mathbf{j}_1+\mathbf{j}_2+\mathbf{j}_3+\mathbf{j}_4+\mathbf{j}_5$$

with the coupling scheme imposed

$$A=\{[(1+2)+3]+(4+5)\}.$$

The corresponding triangular diagram will be



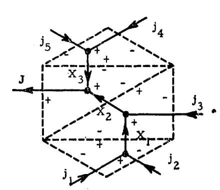
To get the K factor we immediately construct the K table

$\hat{A}$	$(-)^{j_i-m_i}$	$(-)^{\phi(J)}$	$(-)^{\phi(x)}$
$1\pm$	free ⟨0	$1+$	$3\pm$
J	j_1	J	X_3
X_1	j_2	X_1	j_2
X_2	j_3	X_2	j_3
X_3	j_4	X_3	j_5
	j_5		

The K factor will be.

$$K=\hat{J}\hat{X}_1\hat{X}_2\hat{X}_3(-)^{j_1-m_1+j_2-m_2+j_3-m_3+j_4-m_4+j_5-m_5}(-)^{+j_1-j_2-j_3+j_4-j_5-J}(-)^{2X_1+2X_2+2X_3}.$$

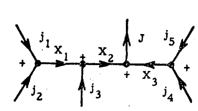
Now we write the dual of the triangular diagram



and obtain finally

$$|(j_1j_2j_3j_4j_5)_AX_1X_2X_3JM\rangle=\sum_{\substack{x_1x_2x_3\\m_1\ldots m_5}}\langle j_1m_1j_2m_2j_3m_3j_4m_4j_5m_5|(j_1\cdots j_5)_AX_1\cdots X_3JM\rangle|j_1m_1\rangle\cdots|j_5m_5\rangle$$

$$\langle j_1m_1\cdots j_5m_5|(j_1\cdots j_5)_AX_1X_2X_3JM\rangle=K$$



$$K=\hat{J}\hat{X}_1\hat{X}_2\hat{X}_3(-)^{j_1-m_1+\cdots+j_5-m_5}(-)^{j_1-j_2-j_3+j_4-j_5-J}(-)^{2X_1+2X_2+2X_3}$$

If the analytical result is desired it is easy to write it.

Instead of a conclusion, let us now give as a general concrete example: the calculus of angular distribution in a reaction $A(dp)B$ when spin–orbit potentials are included in the entrance and exit channels.

The amplitude for the reaction $A(dp)B$ using distorted waves is given by[17]

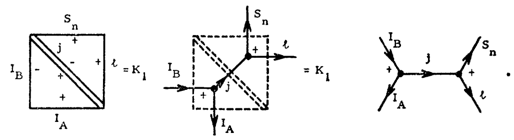
$$I_{M_BM_A}=\int\psi_p(k_p\mathbf{r}'_pS_p)^*\chi_{I_BM_B}(\xi\mathbf{r}_nS_n)^*V_{np}(|\mathbf{r}_p-\mathbf{r}_n|)\phi_d(\mathbf{r}_n-\mathbf{r}_p)\psi_d(k_d\mathbf{r}_p\mathbf{r}_nS_pS_n)\chi_{I_AM_A}(\xi)\,d\tau.$$

If we write

$$\chi^*_{I_BM_B}(\xi\mathbf{r}_nS_n)=\langle I_BM_B\mid\xi\mathbf{r}_n\rangle$$

we have the coupling $\mathbf{I}_B=\mathbf{I}_A+\mathbf{l}+\mathbf{S}_n$ and the imposed intermediate coupling $\mathbf{j}=\mathbf{l}+\mathbf{S}_n$.

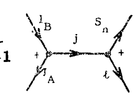
It is easy with the aid of the rules given in this section to write the state $\langle I_BM_B\mid$.

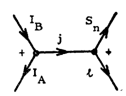


with

$$K_1=\hat{I}_B\hat{j}(-)^{I_B-M_B}(-)^{I_B+I_A+l-S_n}.$$

So that

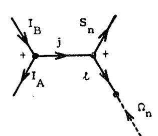
$$\langle I_BM_B\mid=\sum_{M_Am\mu_n}K_1$$  $$\langle lm\mid\langle I_AM_A\mid\langle S_n\mu_n\mid$$

$$\langle I_BM_B\mid\xi\mathbf{r}_n\rangle=\sum_{M_Am\mu_n}K_1$$  $$\langle lm\mid\mathbf{r}_n\rangle\langle I_AM_A\mid\xi\rangle\langle S_n\mu_n\mid.$$

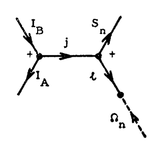
Writing

$$\langle lm\mid\mathbf{r}_n\rangle=\langle lm\mid\Omega_n\rangle\langle l\mid r_n\rangle=Y^*_{lm}(\Omega_n)u^*_l(r_n)$$
$$=(-)^{l-m}Y_{l-m}(-\Omega_n)u^*_l(r_n)$$
$$\langle I_AM_A\mid\xi\rangle=\chi_{I_AM_A}(\xi)^*.$$

We find easily

$$\langle I_BM_B\mid\xi\mathbf{r}_n\rangle=\sum_{M_A\mu_n}K_1$$  $$u^*_l(r_n)\chi^*_{I_AM_A}(\xi)\langle S_n\mu_n\mid. \qquad (6)$$

By the integration of (6) over ξ internal variables we get

$$\int\chi^*_{I_BM_B}(\xi\mathbf{r}_nS_n)\chi_{I_AM_A}(\xi)\,d\xi=\sum_{\mu_n}K_1u^*_l(r_n)\Theta^*_{jl}\langle S_n\mu_n\mid$$ 

When a spin–orbit potential is introduced in the entrance and exit channels, we obtain in the same way the wave functions in these channels. Thus the outcome is

$$I(\mu_p\mu_dM_BM_A)=4\pi g\sum_{ljL_pL_dJ_pJ_d}K\Theta^*_{jl}R_{L_pL_dJ_pJ_d}G(9).$$

(a) g is a factor introduced by the choice of the interacting potential

$$V_{np}(|\mathbf{r}_p-\mathbf{r}_n|)\phi_d(\mathbf{r}_n-\mathbf{r}_p)=g\,\delta(\mathbf{r}_n-\mathbf{r}_p).$$

[17] D. Robson, Nucl. Phys. 22, 47 (1961).

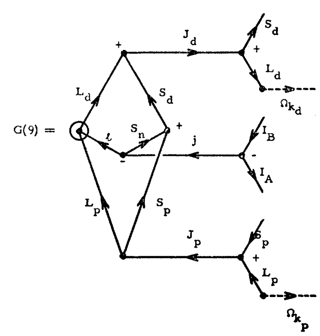
(b) K is the so-called K factor, the value of which is found to be

$$K=\hat{I}_B\hat{J}_p^2\hat{J}_d^2\hat{j}\hat{S}_d\hat{L}_d^2\hat{L}_p\hat{l}(-)^{S_p+S_d+L_p+I_B+I_A}(-)^{S_p-\mu_p+I_B-M_B}.$$

(c) Θ^*_{jl} is the reduced width obtained by integration over internal coordinates ξ.
(d) $R_{L_pL_dJ_pJ_d}$ is the radial integral

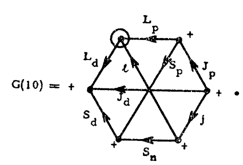
$$\int f_{J_pL_p}(k_pr)u^*_l(r)f_{L_dJ_d}(k_dr)r^2\,dr.$$

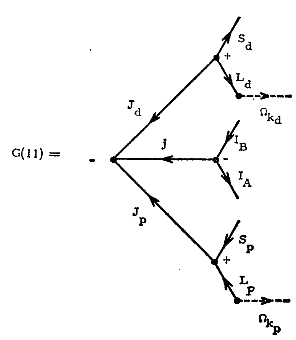
(e) $G(9)$ is the geometrical part represented by the diagram



After lines $(J_d jJ_p)$ have been directed in the same way [giving a phase $(-)^{2J_d}$] we cut $G(9)$ on these lines, getting $G(10)$ and $G(11)$

$$G(9)=(-)^{2J_d}G(10)G(11)$$





So we get

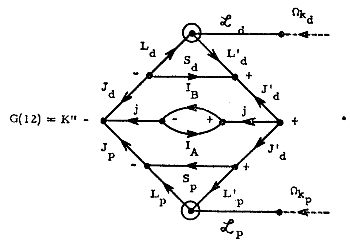
$$I(\mu_p\mu_dM_BM_A)=4\pi g\sum_{L_pL_dJ_pJ_d}\Theta^*_{jl}R_{L_pL_dJ_pJ_d}K(-)^{2J_d}G(10)G(11)$$

$$\sigma(\theta)\propto(4\pi)^2g^2\sum_{L_pL'_pJ_pJ'_pJ_dJ'_d}\Theta^*_{jl}\Theta_{j'l}R_{L_pL_dJ_pJ_d}R^*_{L'_pL'_dJ'_pJ'_d}$$

$$\times(-)^{2J_d+2J'_d}KK''\sum_{\mu_p\mu_dM_BM_A}G(10)G(11)G(10')^*G(11')^*.$$

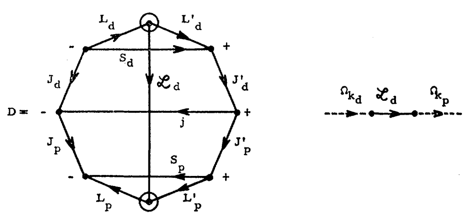
$G(10)$ and $G(10')^*$ being independent of magnetic moments we can extract them from the sum and put:

$$G(12)=\sum_{\mu_p\mu_dM_BM_A} G(11)G(11')^*,$$



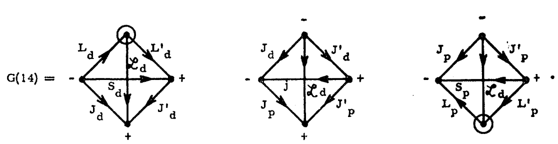
$$G(12)=(-)^{2I_B}\{I_AI_Bj\}G(13),$$

$$G(13)=K''(-)^{\mathcal{L}_d+2J_p+2J'_d}\,\delta_{\mathcal{L}_p\mathcal{L}_d}\cdot D,$$



We get thus

$$G(13)=(-)^{2J_p+2J'_d+\mathcal{L}_d}K''P_{\mathcal{L}_d}(\cos\,(\hat{k}_d,\hat{k}_p))G(14)$$



We obtain then

$$\sigma(\theta)\propto g^2\sum_{\substack{jlL_pL_p'L_dL_d'\\ J_pJ_dJ_p'J_d'}}\Theta^*{}_{jl}\Theta_{jl}R_{L_pL_dJ_pJ_d}R^*{}_{L'_pL'_dJ'_pJ'_d}\hat{I}_B^2\hat{J}_p^2\hat{J}'^2_p\hat{j}^2\hat{J}_d^2\hat{J}'^2_d\hat{L}_d^4\hat{L}'^4_d\hat{l}^2\hat{\mathcal{L}}_d^2\hat{L}_p^3\hat{L}'^3_p(-)^{L'_d+L_p-j+I_B+I_A}$$

$$\times\begin{pmatrix}L_d & L'_d & \mathcal{L}_d\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}L_p & L'_p & \mathcal{L}_d\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}L_p & l & L_d\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}L'_p & l & L'_d\\ 0 & 0 & 0\end{pmatrix}\{I_AjI_B\}$$

$$\times\begin{Bmatrix}L_d & J_d & S_d\\ J'_d & L'_d & \mathcal{L}_d\end{Bmatrix}\begin{Bmatrix}J_d & J_p & j\\ J'_p & J'_d & \mathcal{L}_d\end{Bmatrix}\begin{Bmatrix}J_p & L_p & S_p\\ L'_p & J'_p & \mathcal{L}_d\end{Bmatrix}\begin{Bmatrix}L_d & L_p & l\\ S_d & S_p & S_n\\ J_d & J_p & j\end{Bmatrix}\begin{Bmatrix}L'_d & L'_p & l\\ S_d & S_p & S_n\\ J'_d & J'_p & j\end{Bmatrix}P_{\mathcal{L}_d}(\cos(\hat{k}_p\hat{k}_d)).$$

We are indebted to Professor Sarazin for the opportunity of this work and B. Castel for very helpful discussions.