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S-Matrix Theory of Electromagnetic Interactions*

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This paper studies the possibility of applying the principles of the S-matrix theory to electrodynamics. By using an approximation scheme different from that in the strong interactions, many results of quantum electrodynamics can indeed be reproduced. While the calculations of electrodynamical problems can be carried out in a more straightforward and less laborious way in the S-matrix theory, there still are some basic difficulties. However, as far as scattering processes and the anomalous magnetic moment are concerned, there is complete identity (up till fourth order) between the two methods. The bound-state problem, infrared divergence, and the many-particle problem remained unsettled.

I. INTRODUCTION

The perturbation method of field theory has been successfully applied to electromagnetic interactions. Numerous amazing experimental verifications have been obtained. The success of quantum electrodynamics depends on a particular feature, characteristic of the electromagnetic interaction, namely, the smallness of the coupling constant. This renders the perturbation method quite effective. Therefore, in spite of the existence of certain mathematical difficulties which still remain unsettled in the theory, quantum electrodynamics is a satisfactory theory as it stands right now. However, if the techniques of field theory are applied to strong interaction phenomena, they fail completely because either some of the theories are not renormalizable, or even worse, because one cannot construct perturbation solutions to the renormalized field equations. The coupling constant is too large to render the series convergent. During the past few years, physicists have turned their attention to the studies of S-matrix theory as a possible method for strong interaction physics. Although so far the S-matrix theory has no well-defined theoretical structure and is still incomplete, its predictions in some cases turned out to be in agreement with experiments, hence it can sensibly be regarded as a promising alternative.

This work investigates the possibility of applying the S-matrix theory to electromagnetic interactions. The equivalence of the usual form of quantum electrodynamics and the S-matrix theory of electromagnetic interactions will be demonstrated by displaying the identity of the results of calculations for various processes and diagrams. In Sec. II, a brief review of the general principles of the S-matrix theory of strong interactions will be given. Various assumptions will be analyzed so that they can be readily applied to electromagnetic interactions in Sec. III. In that section the unitarity condition will be used to generate a perturbation series in powers of the coupling constant; in contrast to the corresponding case of strong interactions where this is impossible. Using these principles—Lorentz invariance, unitarity, crossing symmetry, and analyticity, we can show by explicit calculation that in the S-matrix theory the finite contributions (which correspond to primitive divergent graphs in the usual theory) are the same as the renormalized results in the Feynman–Dyson theory. Various lowest-order scattering amplitudes will be constructed. Higher-order radiative corrections can be treated similarly; in certain cases simplifications will result if use is made of the Mandelstam representation and Cutkosky's rule. Despite the striking results given in Sec. IV, some unsolved problems, peculiar to electromagnetic interactions, remain. Section VI concludes the present work with a general discussion of these problems and the comparison of the S-matrix theory of electromagnetic interaction with the Feynman–Dyson's approach. Although at present, the S-matrix theory of electromagnetic interaction is not as satisfactory and self-contained as the usual form of quantum electrodynamics, it is certainly interesting to see that many significant results can be obtained via S-matrix theory without resorting to the concept of fields and operators.

II. REVIEW OF THE GENERAL PRINCIPLES OF THE S-MATRIX THEORY

A. General Observations

The traditional way to study quantum electrodynamics consists of the straightforward application of the ideas of quantum theory to the electromagnetic field. One of the many equivalent methods of quantization is applied to the classical electromagnetic field. The essential new feature compared to quantum mechanics is that the electromagnetic field, conceived as a dynamical system, possesses infinitely many degrees of freedom. It should be stressed that the electromagnetic interaction in this framework is given, either by giving the interaction Lagrangian or equivalently by giving the Heisenberg equations of motion of the field operators. The knowledge of this interaction comes directly from experiment (via Maxwell's

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equations). It can be *verified* that this interaction is both gauge invariant or relativistically invariant. With this given interaction, quantum electrodynamics proceeds in much the same way as ordinary quantum mechanics, the calculation of expectation values, transition rates, etc. This is the approach followed in most organized treatises.[1-3] It was noted early,[4,5] that most processes in quantum electrodynamics, could be described in terms of the S matrix—which (in principle, in any case) could be found once the interaction Hamiltonian is known. From the known interaction one can again verify that the S matrix is unitary and relativistically invariant. In the traditional form of quantum electrodynamics, the S matrix thus possesses the status of a *derived* notion. To be sure, it is a most significant notion, which allows the direct calculation of many experimental processes; but the physical principles are not stated directly in terms of the S matrix. In a pure S-matrix theory by contrast, it is precisely attempted to state the relevant physical principles directly and exclusively in S-matrix terms. Thus, in a pure S-matrix theory, the unitarity of the S matrix is not a property which can be derived by inspection of its explicit form, but rather, a condition to be imposed (on physical grounds) to help determine the S matrix. In a sense, one has here a rather curious inversion: The general properties of the S matrix were first obtained and inferred from the Lagrangian form of field theory. Now these properties, themselves, are made primary and assume the status of principles. The problem now is to deduce information about the electromagnetic S matrix from them. The main part of this discussion is concerned with the question of whether and to what way, these general principles determine the S matrix for specific interactions. This is precisely the question to be analyzed in this paper for the case of electromagnetic interactions. One would hope that, in addition to a reformulation of quantum electrodynamics, one would obtain a more flexible formalism which is easier to handle without the necessity of carrying out renormalizations.

To start the discussion, it is necessary to state the principles assumed in this S-matrix approach. Since the S-matrix theory is most commonly used in strong interaction physics, these principles must be adapted and modified so they can be used in the electromagnetic case. To show the close relationship between the axioms assumed, the generally accepted (more or less) principles of the strong interaction S-matrix theory are given here[6,7]; their specific adaptation to the electromagnetic case is given in Sec. III.

B. The Principles of the S-Matrix Theory

1. *Symmetry Properties*

Relativistic invariance. All theories constructed to date assume the unrestricted validity of relativistic invariance. This section contains the form which this requirement assumes in an S-matrix theory.

A general S-matrix element representing the transition from an initial state i to a final state f can be written as

$$\langle f \mid S \mid i \rangle = \delta_{fi} + i(2\pi)^4 \delta^{(4)}(P_f - P_i) \langle f \mid T \mid i \rangle. \quad (2.1)$$

Here $\langle f \mid T \mid i \rangle$ is referred to as the T-matrix element or the scattering amplitude. In this section we consider the scattering of spin 0 particles. The generalization needed to include charge and spin will be described later. The Feynman amplitude differs from the T-matrix element by a kinematic factor which depends on the normalization of particle wave functions in the states $\mid i \rangle$ and $\mid f \rangle$.

Relativistic invariance required, in the first instance, that $|\langle f \mid S \mid i \rangle|^2$ shall be invariant, or since a four-dimensional δ function is factored out, that $|\langle f \mid T \mid i \rangle|^2$ shall be invariant. Thus, the scattering amplitude shall be a function just of the invariants formed by the four-momenta of n incoming and outgoing particles, p_i $(i = 1 \cdots n)$. Since the states are physical states, all momenta are on the mass-shell, $p_i^2 = m_i^2$. It is well known that for such a process the number of independent invariants is $3n - 10$. For a two-in, two-out process, $p_1 + p_2 \to p_3 + p_4$, the number of independent invariants will be 2. They are, for example, the Mandelstam variables which can be defined as

$$s = (p_1 + p_2)^2,$$
$$t = (p_1 - p_3)^2,$$
$$u = (p_1 - p_4)^2.$$

They are related by

$$s + t + u = \sum_{i=1}^{4} m_i^2.$$

The functional dependence of the amplitude on s, t, u is in no way restricted by this invariance.

Discrete transformations. In strong interactions, the scattering amplitude is assumed to be invariant under P, C, and T transformations. This assumption again imposes restrictions on the amplitudes. Actually, these restrictions are always expressed as the absence of certain invariant scattering amplitudes which possess particular spin or isospin transformation properties.

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[1] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publ. Co., Inc., Reading, Mass., 1959), 2nd printing.

[2] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959).

[3] G. Källen, "Quantenelektrodynamik," in *Handbuch der Physik*, S. Flügge, Ed. (Springer-Verlag, Berlin, 1958), Band V, Teil. 1.

[4] W. Heisenberg, Z. Physik **120**, 513, 673 (1943).

[5] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

[6] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).

[7] S. Mandelstam, Rept. Progr. Phys. **25**, 99 (1962).

Consequently, the restrictions imposed by these invariances are best discussed in a formalism which explicitly exhibits the spin and isospin character of the amplitudes. This is shown in the Sec. IIIB4.

It is still interesting to note here that the C, P, and T invariance principles manifest themselves only in conditions on the invariant scattering amplitudes, and these invariance requirements separately do not put any restrictions on the unitarity condition (which serves as the fundamental dynamical principle of the S-matrix theory). However, CPT invariance has to be maintained throughout the discussion. The validity of the latter has not been questioned, in fact, we know that it is a property of all relativistically invariant local field theories.

Crossing symmetry. The substitution law first explicitly formulated in electrodynamics states that certain relations exist between the scattering amplitudes of various processes. This same idea was extended to strong interactions; for example, the scattering amplitudes for the following three processes will be related in the sense that they are each others analytic continuation. The three amplitudes corresponding to the three channels of any two-in, two-out reactions:

$$p_1+p_2\rightarrow p_3+p_4$$

$$p_1+(-\bar{p}_3)\rightarrow(-\bar{p}_2)+p_4$$

$$p_1+(-\bar{p}_4)\rightarrow p_3+(-\bar{p}_2)$$

are then to be the boundary values of one and the same analytic function. Energy-momentum conservation law is assumed for all processes. It should be noted that the analyticity requirements needed to carry out this continuation are conjectured to be true, as, for example, expressed by the Mandelstam representation. A further discussion is given in subsection 3. A particular case of this law provides us with the freedom of bending lines in Feynman diagrams and thus facilitates calculations.

2. *Unitarity*

The unitarity condition contains a good deal of the dynamics of the S-matrix theory. This combined with the principle of analyticity (which will be given below) forms the central part of the S-matrix theory. Together, these permit the calculations to be carried out.

To obtain the explicit and very useful form of the unitarity condition, one substitutes (2.1) into $SS^{+}=1$ and obtains

$$i^{-1}[\langle f\mid T\mid i\rangle-\langle f\mid T^{+}\mid i\rangle]$$
$$=(2\pi)^4\sum_n \langle f\mid T^{+}\mid n\rangle\langle n\mid T\mid i\rangle\delta^{(4)}(p_i-p_n). \tag{2.2}$$

One usually shows, using TP invariance (which asserts $\langle i\mid T\mid f\rangle=\langle f\mid T\mid i\rangle$), that the unitarity condition can be put in the well-known form:

$$\text{Im}\ \langle f\mid T\mid i\rangle=[(2\pi)^4/2]\sum_n\langle f\mid T^{+}\mid n\rangle$$
$$\times\ \langle n\mid T\mid i\rangle\delta^{(4)}(p_i-p_n). \tag{2.3}$$

If however, one assumes just unitarity and no discrete symmetries, just (2.2) is valid, not (2.3). Now, for further applications, one needs to determine the discontinuity across the cut along the positive real axis of the complex energy plane. This can be obtained from (2.3) but not (as it stands) from (2.2). However, by a careful re-examination of the unitarity condition in the context of field theory, Olive[8] has shown that if $T_{fi}\equiv\langle f\mid T\mid i\rangle$ is the boundary value of an analytic function of complex invariants, the limit of T_{fi} from above the real axis is equal and opposite to that of $T_{if}^{*}\equiv\langle f\mid T^{+}\mid i\rangle$ from below the real axis (in the complex energy plane). This result is independent of any special invariance principles, of crossing symmetry, of the type of process considered, but is just a consequence of the CPT theorem. With this result, the unitarity condition for the scattering amplitudes can be put in the elegant form:

$$\text{discontinuity of } T_{fi}=i(2\pi)^4\sum_n T_{fn}^{\pm}T_{ni}^{\mp}\delta^{(4)}(p_i-p_n), \tag{2.4}$$

where $+$ and $-$ denotes the boundary values from above and below the cut, respectively. In the S-matrix theory, this form of the unitarity condition is taken as an independent principle. For the case of boson–boson scattering, (2.4) will yield a relation between the imaginary part of the amplitude of a process and the product of scattering amplitudes of other processes. Note that the unitarity actually yields an infinite set of conditions, (2.4) holds for all f and i. If another set of relations could be found, (this is indeed the case, see below), we could in principle solve for the scattering amplitudes themselves. The nonlinearity of this infinite system (2.4) renders the exact solution almost impossible, however, approximate and perturbation solutions can be found.

3. *Analyticity*

Formulae (2.2) or (2.4) give *one* relation between the matrix elements of T (or the amplitudes). As noted, another relation is needed to provide a closed system. The type of relation envisaged might, for example, be a connection between the real and imaginary parts of a scattering amplitude. Mathematically such relations are well known in the theory of analytic functions. It is thus natural to make some assumptions regarding the analytic character of the amplitudes as a function of appropriate variables. Many more or less different assumptions can be made. For example, the scattering amplitudes can assume to be analytic

[8] D. I. Olive, Nuovo Cimento **26**, 73 (1962).

functions of the kinematic invariants in the space of these variables. One assumes, in addition, that there exists a "physical sheet" bounded by cuts and containing poles. This sheet contains a domain of analyticity of the scattering amplitude as a function of its variables, which includes among its boundary all physical points. Then, by means of Cauchy's theorem, we can find a relation connecting the imaginary to the real part of the amplitude (usually referred to as a dispersion relation). Once the analyticity of the scattering amplitude is assumed, the dispersion relation can then be written down. This, together with the unitarity condition (2.4), forms a set of basic equations for the scattering amplitude. It is this set of equations some physicists believe to be the dynamical equations for the strong interaction physics.

So far all the conjectured analytic properties of scattering amplitudes postulated are suggested by field theories. The dispersion relations which can rigorously be proven have been very limited in number. Landau and Cutkosky[9] have given general rules for locating the singularities of Feynman graphs in perturbation theory; it is hoped that, although the perturbation series diverges, we can still extract information about the analytic character from its series expansion. Alternatively, Chew has postulated the principle of maximal analyticity which states that the scattering amplitude can have only those singularities imposed on it by unitarity. In any case, in this "pure" S-matrix approach, some analyticity properties of the amplitudes *must* by postulated. One can still argue about the precise form needed. In this paper, analyticity requirements sufficient to allow the Mandelstam representation (thus, both single and double dispersion relations) will just be assumed. If, indeed, unitarity and analyticity can provide enough information to determine the theory,[10] the remaining problem will be to set up a workable approximate scheme, since an exact solution would certainly be impossible.

The unitarity condition (2.4), on the right-hand side, should include in the summation all the possible physical intermediate states that conserve energy, momentum, spin angular momentum and all other conserved quantum numbers. In strong interaction physics, in order to obtain a tractable system of equations, the infinite summation is replaced by just a finite number of terms. It can be seen that the location of the singularities in the amplitudes is determined by the total "masses" of the intermediate states. The higher the mass, the farther out from the origin the associated singularity occurs. It is assumed, in the approximation scheme of strong interaction physics, that lower the masses, the more important the role; thus, the contribution to the unitarity sum is mainly controlled by "near-by" singularities. The farther-away singularities would then have less importance. It is hoped that, in most cases, the inclusion of a few low mass states in the summation of the unitarity condition should already give a very good approximation. Thus, in this way, one can hope to construct a systematic approximation procedure. In actual calculations, one sometimes has to include a large number of terms. In that case, we can try to keep a reasonable number of terms and replace the neglected ones by one or more empirical constants. Doing this, one can not only save a lot of computational labor, but also fit the experimental results quite well; the effective range formula is an example. Of course, the resulting theory is no longer a "fundamental theory" in the sense of the S-matrix theory. This particular type of approximation will *not* be made in the application of the ideas of the S-matrix theory to electromagnetic phenomena. (At least in this paper.)

4. *Generalization to Include Charge and Spin*

When we deal with interacting particles with spin and charge, the scattering process can no longer be represented by just a single invariant amplitude; rather, it must be represented by a linear combination of a set of invariant operator functions, T_j, which can be formed from the four-momenta, p_i, polarization vectors, fermion spinors, γ matrices, etc. These functions T_j can also contain spin and isotopic spin operators β_i. A general scattering amplitude T can then be expressed as

$$T(s_i, p_i, \beta_i) = \sum_{j=1}^{n} A_j(s_i)\, T_j(p_i, \beta_i), \qquad (2.5)$$

where the A_j are scalar amplitudes formed by Lorentz scalars, e.g., the Mandelstam variables s, t, u, etc. (The index j has nothing to do with transformation properties.) We further require that A_j be free from kinematical singularities. A systematic method of producing invariants for any scattering processes such that the associated amplitudes are free from kinematical singularities was given by Hearn.[11]

The requirement that the scattering amplitudes be invariant under certain discrete symmetry transformations will limit the possible forms of the invariant operator functions T_j that can appear in the scattering amplitude. At the same time they impose reality conditions on the scalar amplitudes A_j. It can again be shown, using field-theoretic technique, that, if

$$T_{fi} = \sum_j A_j^{+}(s)\, T_j,$$

then

$$T_{if}^{*} = \sum_j A_j^{-}(s)\, T_j. \qquad (2.6)$$

where

$$A_j^{\pm}(s) = A_j(s \pm i\epsilon, t).$$

[9] R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).

[10] It has never been proved that unitarity and analyticity do provide a system of equations possessing a unique solution.

[11] A. C. Hearn, Nuovo Cimento 21, 333 (1961).

Thus the unitarity condition gives the imaginary part of the scalar amplitudes A_j across the cut in the complex energy plane. The scalar amplitudes themselves, are obtained by applying dispersion relations.

This concludes the sketch of the needed principles of S-matrix theory. In the next section, these same principles will be adapted to electrodynamics. The extent to which these ideas form a consistent and practical basis for quantum electrodynamic calculations will be examined in the succeeding sections.

III. ELECTRODYNAMICS AS AN S-MATRIX THEORY

In order to appreciate the elegance and simplicity of the S-matrix method, we present it against the background of the usual method of quantum electrodynamics. Thus, a very brief outline of the usual procedures is given so that the methods may easily be compared and contrasted.

A. The S Matrix of Conventional Electrodynamics

In spite of the successes recently achieved in constructing an axiomatic finite formulation of quantum field theory, the old-fashioned Feynman–Dyson approach to quantum electrodynamics remains a most useful tool in practical calculations. Quantum electrodynamics has, of course, been treated in many standard textbooks. Here we only recapitulate the needed ideas and results.

A Lagrangian density of electrons interacting with an electromagnetic field is postulated. The field equations are derived by using the action principle. The field equations in the Heisenberg pictures are:[12]

$$(i\boldsymbol{\partial}-m)\psi(x)=-e\mathbf{A}(x)\psi(x)$$
$$\partial_\mu\partial^\mu A_\alpha(x)=j_\alpha(x)=-e\bar{\psi}(x)\gamma_\alpha\psi(x). \qquad (3.1)$$

The action principle determines (at least, to a certain extent) the commutation relations for the free-field operators. The commutation relations for the coupled fields at arbitrarily space–time points can only be obtained once the field equations are solved. It is easy to check that, under C, P, T and gauge transformations, both the field equations and the commutation rules are invariant. The S-matrix operator in the Heisenberg picture is defined as the unitary operator satisfying

$$\phi_{\text{out}}(x)=S^{-1}\phi_{\text{in}}(x)\,S, \qquad (3.2)$$

where ϕ_{in} and ϕ_{out} are operators which satisfy the free-field equations and the free-field commutation rules. From a knowledge of S, most experimentally interesting quantities can be obtained. It can also be shown that the S matrix is invariant under C, P, T and gauge transformations. The agreement of the theoretical predictions and the experimental observations for various processes shows that both the field equations and the commutation rules are substantially correct; hence, the symmetry principles which are contained in these are established to the same degree of accuracy. At present, no indication of deviations of the basic laws has been seen. (Compare, however, Pipkin's experiment.[13])

It should be noted that both Eq. (3.1) and the definition of the S matrix (3.2) are given in terms of the Heisenberg picture. To find a useful expression for the S matrix, one must resort to a perturbation-type treatment. It is conceptually the easiest to carry out this approximation scheme[3] by substituting the following expressions for ψ and A_μ into (3.1):

$$\psi=\psi^{(0)}+e\psi^{(1)}+e^2\psi^{(2)}+\cdots$$
$$A_\mu=A_\mu{}^{(0)}+eA_\mu{}^{(1)}+e^2A_\mu{}^{(2)}+\cdots.$$

Here $\psi^{(0)}$ and $A_\mu{}^{(0)}$ satisfy free-field equations and free-field commutation rules. One can (at least in principle) determine $\psi^{(1)}$, $\psi^{(2)}\cdots$ and $A_\mu{}^{(1)}$, $A_\mu{}^{(2)}$, $\cdots$, by successive approximations. Utilization of (3.2) then gives S as a power series in e:

$$S=1+eS^{(1)}+\cdots. \qquad (3.3)$$

Although conceptually quite straightforward, this is *not* the way calculations are usually carried out. One instead, goes over to the interaction picture to obtain the familiar Feynman diagrams for each order. The expansion (3.3) still holds good in that case. The contribution $S^{(n)}$ contains a large number of diagrams. The general equivalence between these methods has been demonstrated by Källèn[14]—but the relation between them is quite involved. The Feynman diagram procedure is a good deal more tractable. In fact, the description of a physical scattering process can be carried in a much more natural manner in the interaction picture than in the Heisenberg picture.

It is well known that in the calculation of the various diagrams divergences appear. To obtain physically meaningful results, one has to, and one can carry out, a systematic renormalization, splitting of the divergent terms. The *renormalized* physical constants (charge and mass) are the ones to be identified with the experimental charge and mass. This leads to the well-known embarrassing fact that e and m in (3.1) are physically undefined objects.

As a side remark, one can note that the analytical structure of the various amplitudes can be obtained from their explicit forms as given by the Feynman integrals. Usually, these properties are obtained for scalar particles; generalization to spinor and vector particles is in principle straightforward, in practice, often tedious. In any case, the analytic properties are

[12] We adopt the same notation as used in Schweber's book (Ref. 19) except that here $\sigma_{\mu\nu}$ is defined as $\sigma_{\mu\nu}=\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu$, and we use spinors u, v in place of w, where $u^r=w^r$ and $v^r=w^{r+2}$ $(r=1, 2)$. $p_\mu\gamma^\mu$ is written as $\boldsymbol{p}$ (boldface italic).

[13] R. B. Blumenthal *et al.*, Phys. Rev. Letters **14**, 660 (1965).

[14] G. Källén, Arkiv Physik **2**, No. 37, 371 (1950).

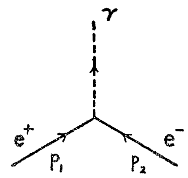


FIG. 1. Elementary interaction.

derived from the underlying theory—they are not postulated.

B. The S-Matrix Theory of the Electromagnetic Interactions

This section contains the adaptation of the principles of the S-matrix theory to quantum electrodynamics. It is the purpose of this discussion to provide a self-contained framework, which will follow the principles of the S-matrix theory as closely as possible; while at the same time, reproducing the results of quantum electrodynamics. In principle, it should be possible to develop all of quantum electrodynamics starting from the ideas and assumptions given here. Some of the postulates are, of course, peculiar to quantum electrodynamics, while others just express the ideas of general S-matrix theory. In the interest of clarity we will list *all* the assumptions here, discussing them as we go along.

(1) We are dealing with two types of particles, electrons (e^+ and e^-) and photons. Observations can be made on these *free* particles only—the totality of these free-particle observations can be described by the free Dirac and free Maxwell equations. The characterization of states is always in terms of free-particle wave functions, such as the spinors u, v, and plane waves for the photons.

(2) The interactions between the electrons and photons are described by the transition amplitudes $\langle b \mid S \mid a \rangle$ or $\langle b \mid T \mid a \rangle$, where $\mid b \rangle$ and $\mid a \rangle$ are arbitrary complicated free-particle states. The purpose of the further postulates is to provide a means of determining $\langle b \mid T \mid a \rangle$. It is clear that assumption (1) is peculiar to quantum electrodynamics, while (2) just expresses a typical S-matrix idea.

3. *Lorentz and Discrete Invariance*

We require the scattering amplitude $\langle b \mid T \mid a \rangle$ to be Lorentz invariant, and invariant under C, P, T, transformations. In electrodynamics, we are dealing with spinor particles and vector particles. Hence, the scattering amplitude can again be expressed as a linear combination of invariant operator functions T_j as mentioned in the last chapter. The requirement of invariance under any of the discrete symmetry transformations will limit the possible independent forms of the invariant operator functions that can appear in the combination.

4. *The Elementary Interaction in Electrodynamics*

We assume that the elementary interaction in electrodynamics involves three particles. This interaction then describes the vertex, corresponding to $e^+ + e^- \rightarrow \gamma$,[15]

$$\langle \gamma \mid T^{(1)} \mid e^+, e^- \rangle = [e/(2\pi)^{9/2}](m_1 m_2/E_1 E_2)^{1/2} \times [1/(2\omega)^{1/2}]\bar{v}(p_1)\gamma_\mu e^\mu(k) u(p_2), \quad (3.4)$$

where $\mathbf{p}_1$, $\mathbf{p}_2$, m_1, m_2, and E_1, E_2 are the momentum, mass, and energy of e^+, e^-, respectively. ω and $\mathbf{k}(=\mathbf{p}_1+\mathbf{p}_2)$ are energy and momentum of the photon. The numerical factor in front comes from normalization of wave functions. (Fig. 1). This is also the simplest possible interaction allowed, if each one of the particles is to participate in the elementary interaction. In principle, elementary interactions involving more particles could exist, but is explicitly assumed that they do not.

The justification of this assumption and just what it involves can be seen as follows. Consider any three-particle vertex of the electron–photon interaction then, just requiring Lorentz invariance, we can write down this vertex as a superposition of four linearly independent invariant operator functions:

$$\begin{aligned}\langle \gamma \mid T \mid e^+, e^- \rangle = {} & g_1 \bar{v}(p_1)\gamma_\mu e^\mu(p_1+p_2) u(p_2) \\ & + g_2 \bar{v}(p_1)\sigma_{\mu\nu} e^\mu(p_1+p_2)(p_1+p_2)^\nu u(p_2) \\ & + g_3 \bar{v}(p_1)\gamma_\mu\gamma_5 e^\mu(p_1+p_2) u(p_2) \\ & + g_4 \bar{v}(p_1)\sigma_{\mu\nu}\gamma_5 e^\mu(p_1+p_2)(p_1+p_2)^\nu u(p_2). \quad (3.5)\end{aligned}$$

Here, $g_1 \cdots g_4$ can be arbitrary functions of $(p_1+p_2)^2$; however, for elementary interaction, these are assumed to be constants. For nonelementary interactions, these are the form factors. It is interesting to compare this, term by term, with the interaction Lagrangian density in quantum electrodynamics:

$$\mathcal{L}_I = f_1 \bar{\psi}\gamma_\mu A^\mu \psi + f_2 \bar{\psi}\sigma_{\mu\nu} F^{\mu\nu}\psi + f_3 \bar{\psi}\gamma_\mu\gamma_5 A^\mu \psi + f_4 \bar{\psi}\sigma_{\mu\nu}\gamma_5 F^{\mu\nu}\psi. \quad (3.6)$$

The last two terms in both expressions are dropped by the requirement of P invariance; while the first two terms survive the requirements of C, P, (and gauge) invariance. The second term in the Lagrangian density is deleted in the usual theory because it is a derivative coupling type interaction which is unrenormalizable and leads to divergences in higher-order perturbation theory. The first term in (3.6) is the coupling of the

[15] To be sure, although this vertex is denoted by a T-matrix element, it does not represent a physical process—three external lines of this vertex cannot be put on the mass-shell simultaneously, if the particles involved are real ones. The basic question is precisely to build up to the scattering amplitudes for physical processes starting from this elementary vertex. The problem is that in the study of physical processes, such vertices will occur as parts of diagrams, where the particles are not all physical particles. The same form (3.4) will nonetheless be assumed. The detailed discussion of this point is given in Sec. IVA.

form $j_\mu A^\mu$ and it is the only term needed to account for the interaction of charged particles with photons; this choice is usually referred to as the principle of minimal electromagnetic interactions. That we choose only the first term in (3.5) as the elementary vertex is in fact, a reflection of this principle. (Actually if we also took the second term into our calculation, this would cause trouble. As its counterpart in field theory is an unrenormalizable interaction, in the S-matrix theory it will presumably bring in more and more subtraction constants in the dispersion relations as we go to higher order calculations.) By (3.4), one has effectively introduced an interaction without, however, the explicit use of interaction fields.

5. *Crossing Symmetry*

If in a reaction, certain particles participate and in another reaction the same particles or antiparticles participate, crossing symmetry establishes a relation between the amplitudes for these reactions. The only new feature added in quantum electrodynamics is that to relate incoming and outgoing electrons and photons, one has to make the following replacement:

$$\begin{aligned} &k' \text{ out} \leftrightarrow k \text{ in:} \quad && k' \leftrightarrow -k, \quad && e' \leftrightarrow e \\ &p' \text{ out} \leftrightarrow q \text{ in:} \quad && p' \leftrightarrow -q, \quad && u(p') \leftrightarrow v(q) \\ &q' \text{ out} \leftrightarrow p \text{ in:} \quad && q' \leftrightarrow -p, \quad && v(q') \leftrightarrow u(p). \end{aligned} \tag{3.7}$$

With this rule, all elementary interactions are defined, for example,

$$\langle e^- | T^{(1)} | e^-, \gamma \rangle = [e/(2\pi)^{9/2}](m_1 m_2/E_1 E_2)^{1/2} \times (2\omega)^{-1/2} \bar{u}(p_1) \gamma_\mu e^\mu(k) u(p_2). \tag{3.8}$$

(Although, of course, the rule is more general than that.)

6. *Gauge Invariance*

In the usual form of quantum electrodynamics, gauge invariance is considered as an independent principle. In terms of an S-matrix theory, the important quantities are the amplitude $T = \sum_j A_j T_j$. When dealing with processes where there are (external) photons, the invariant operator function T_j will always contain the photon polarization vectors e^μ, so that $T_j = e_\mu M_j^\mu$. Now gauge invariance means that $k_\mu M_j^\mu = 0$, where k is the momentum of the physical photon. This expresses gauge invariance directly in terms of the amplitudes. The elementary interaction is trivially gauge invariant and so is the interaction (3.5). Thus, gauge invariance will not let one eliminate the terms other than the first in (3.5).

Zwanziger[16] claims generally that for the purpose of constructing the necessary amplitudes and reducing the number of invariants, the principle of gauge invariance adds no new restrictions. Thus, for the construction of amplitudes, Lorentz invariance alone can be shown to be equivalent to the usual prescription which includes both gauge invariance and relativistic invariance. Since we have already assumed Lorentz invariance as one of the basic principles of the S-matrix theory, it is presumably no longer necessary to impose gauge invariance as a separate requirement.

7. *Unitarity, Analyticity, and the Approximation Scheme*

The general requirement of unitarity and analyticity of the S-matrix elements applies in the S-matrix theory of electromagnetic interactions without any change. Thus, again, we can write the unitarity condition as

$$\text{disc. } T_{fi} = i(2\pi)^4 \sum_n T_{fn}^{(+)} T_{ni}^{(-)} \delta^{(4)}(p_i - p_n). \tag{3.9}$$

where n denotes the set of on-the-mass-shell intermediate states that conserve energy, momentum, spin and charge, etc. Here and in the following, when the unitarity condition is used, it is always understood that the scattering amplitude T_{ab} be written in the form (2.5) as a linear combination of a set of invariant operator functions T_j, and the unitarity condition gives the imaginary part of the scalar amplitudes A_j across the cut in the complex energy plane as mentioned in Sec. II.

At this point, it should be noticed that the approximation scheme used in strong interaction physics cannot be applied here since photon has zero mass. In strong interactions the sum in (3.9) is replaced by just a few terms, corresponding to the lower mass intermediate states. Since the strong interactions are of short range and mediated by quanta of finite mass, this makes sense. In electrodynamics, the quanta has zero mass and produces a force of infinite range. Here the analytical structure is such that a pole is located at the beginning of infinitely many cuts superimposed on one another. To treat these infinitely many cuts at the same time would certainly be impossible; we have to resort to a different approximation scheme.

We recall that the usual formalism of quantum electrodynamics has been quite successful expanding the scattering amplitude in powers of e or α (the fine structure constant). As $\alpha = 1/137 \ll 1$, the perturbation series converges rapidly. Here we shall employ the same technique in the S-matrix theory by expanding both sides of the unitarity relation in powers of e. This is suggested by the expansion of S in powers of e as given by (3.3). Thus, the unitarity condition in the jth order becomes

$$\text{disc. } T_{fi}^{(j)} = i(2\pi)^4 \sum_{l=1}^{j-1} \sum_n T_{fn}^{(+)(l)} \times T_{ni}^{(-)(j-l)} \delta^{(4)}(p_i - p_n). \tag{3.10}$$

[16] D. Zwanziger, Phys. Rev. **133**, B1036 (1964).

If the scattering amplitudes are known to a given order in this perturbation scheme, then the discontinuity of the scattering amplitude across the cut in the next higher order can be determined using the unitarity condition. It was stated before, but should be mentioned at this point, that the necessary analyticity conditions to allow single and double dispersion relations will be assumed. The dispersion relation resulting from the assumed analyticity will then be used to fix the scattering amplitude. Thus, in principle, we can start out from the elementary interaction given in Sec. III B4 to construct scattering amplitudes of given initial and final states to arbitrary high orders by repeatedly using the unitarity condition and dispersion relations. Unfortunately, the analytical properties of scattering amplitudes in general higher than fourth order, have not yet been fully explored. Dispersion relations are hard to write down, at least they will be much more complicated than the Mandelstam representation, if initial or final states involve more than two particles. This is certainly a handicap in the present calculation scheme of the S-matrix theory; hopefully, it can be removed eventually. Actually, this does not constitute a very important restriction to electrodynamics, since higher-than-fourth-order effects are too small to be observed experimentally anyway. The corresponding calculations in the Feynman–Dyson method are also very tedious.

Before concluding this chapter, an important remark should be made about the photons. They appear in the S-matrix theory as massless vector particles (spin 1). As a result of their vanishing mass, the formalism should be gauge invariant. This has been taken into account by constructing gauge invariant scattering amplitudes to begin with. This was shown to be possible as a consequence of just Lorentz invariance (as was stated before). However, another consequence of the zero photon mass appears as the problem of infrared divergences in practical calculations. In all the calculations that we shall encounter in the next chapter, a small photon mass λ will be assigned whenever infrared divergences appear; λ will be made to approach zero as limit at the end of the calculation, so that no ambiguity would arise. Another question intimately connected with the infrared divergence problem is the possibility of emitting infinitely many soft photons in a scattering process. In the present scheme, it is obviously impossible to treat this problem adequately since we do not know as yet, how to tackle the problem of three, let alone, an infinite number of particles. We discuss this problem later in Sec. V.

IV. CALCULATIONS IN THE S-MATRIX THEORY OF THE ELECTROMAGNETIC INTERACTIONS

In this section, the amplitudes for the various scattering processes and the finite contributions of the higher order vertices will be obtained, using just the general ideas of the S-matrix theory outlined in Sec. III. These calculations can, and have, of course, been carried out using traditional field theory. It is precisely the purpose of the following calculations to show in detail just how these same results are obtained within an S-matrix framework. We shall start out with second order processes.

A. Second-Order Scattering Processes

1. *Compton Scattering*

The diagram for Compton scattering is shown in Fig. 2(a). The Mandelstam variables are then defined as

$$s=(p_1+q_2)^2,$$
$$t=(p_1-p_3)^2,$$
$$u=(p_1-q_4)^2.$$

We find the Compton scattering amplitude in the second order. As noted in the last section, a general scattering amplitude can be written as a linear combination of the product of scalar amplitudes $A_j\ (s, t, u)$ and invariant operator functions T_j [see Eq. (2.5)]. The T_j's for the case of Compton scattering can be constructed following the procedures given by Hearn as mentioned earlier; it can be demonstrated that there are six independent invariant operator functions for the Compton scattering process of any order. It is precisely our purpose to fix those (six) scalar amplitudes associated with each T_j for the case of second-order Compton scattering process using the S-matrix methods. Here, instead of writing down all the invariant operator functions explicitly and then proceeding the calculation, we adopt a slightly different approach which is physically more instructive.

The expanded version of the unitarity condition in second order takes the form

$$\text{disc. } \langle e_1^-, \gamma_2 \mid T^{(2)} \mid e_3^-, \gamma_4 \rangle = \sum_n i(2\pi)^4 \langle e_1^-, \gamma_2 \mid T^{+(1)} \mid n \rangle \langle n \mid T^{(1)} \mid e_3^-, \gamma \rangle\, \delta^{(4)}(p_n - p_{e_1} - p_{\gamma_2}). \quad (4.1)$$

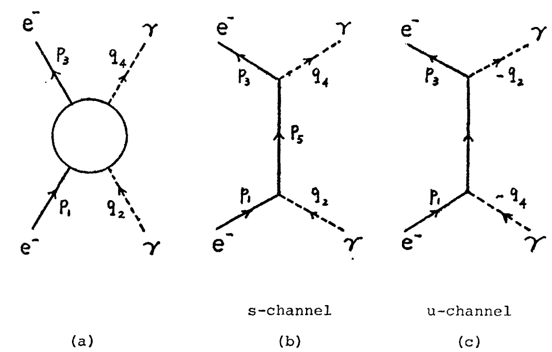


FIG. 2. Compton scattering.

This expression for disc $\langle e_1^-, \gamma_2 \mid T^{(2)} \mid e_3^-, \gamma_4 \rangle$ contains a sum of terms of matrix elements of the elementary interaction $T^{(1)}$ between the states $\mid e^-, \gamma \rangle$ and $\mid n \rangle$. By assumption, the only nonzero matrix element is $\langle e^-, \gamma \mid T^{(1)} \mid e^- \rangle$ and this is given by (3.8). Thus, the disc $\langle e_1^-, \gamma_2 \mid T^{(2)} \mid e_3^-, \gamma_4 \rangle$ can be evaluated by summing over the polarization states and integrating over the momenta of this one-particle (in fact, one-electron) intermediate state which is on the mass shell ($p_5^2 = m'^2$). Thus

$$\text{disc. } T^{(2)} [e^-(p_1) + \gamma(q_2) \to e^-(p_3) + \gamma(q_4), s \text{ channel}]$$
$$= \frac{ie^2}{(2\pi)^5} \frac{m}{(E_1 E_3)^{1/2}} \frac{1}{(2\omega_2 2\omega_4)^{1/2}} \bar{u}(p_3) e(q_4) (p_1 + q_2 + m') e(q_2) u(p_1) \delta(s - m'^2). \quad (4.2)$$

It was noted before that, although the elementary interaction (3.8) was written in terms of physical free-particle wave functions, the three lines of the elementary vertex cannot be on the mass shell simultaneously if the law of conservation of four-momentum holds. In other words, this vertex does not represent a physical process, if $p_1^2 = p_5^2 = m^2$ and $q_2^2 = 0$ in the lower part of Fig. 2(b). However, if the particle in the intermediate state were a "heavy" electron, i.e., we assume $p_5^2 = m'^2 > m^2$, then this *could* be a possible physical process which then is assumed to have a corresponding T-matrix element. Equation (4.2) was obtained exactly under this assumption.

Now (4.2) immediately suggests that one of the six invariant operator functions is

$$T_1 = \frac{m}{(E_1 E_2)^{1/2}} \frac{1}{(2\omega_2 2\omega_4)^{1/2}} \bar{u}(p_3) e(q_4) (p_1 + q_2 + m) e(q_2) u(p_1). \quad (4.3)$$

The numerical factor is the normalization constant for the wave functions, included for convenience. Then the Compton scattering amplitude in second order can be written as

$$T^{(2)} = A_1(s, t, u) T_1 + \sum_{j=2}^{6} A_j(s, t, u) T_j. \quad (4.4)$$

Using (2.6), (4.2) and the linear independence of the T_i, we get

$$\text{Im } A_1(s, t, u) = [e^2/2(2\pi)^5] \delta(s - m'^2), \quad \text{valid for } m' > m. \quad (4.5)$$

Now, imagine the unitarity condition (4.2) as a function of m' is analytically continued from $m'^2 > m^2$ to $m'^2 = m^2$. In that process, the intermediate state, though still on the mass shell, is no longer a physically possible state. It is assumed that this analytically continued unitarity condition is still valid; this is what is usually referred to as the "generalized unitarity condition." Whenever a one-(stable) particle intermediate state is involved, it is always necessary to carry this continuation out in order to construct the amplitude. This unitarity condition yields

$$\text{Im } A_1(s, t, u) = [e^2/2(2\pi)^5] \delta(s - m^2).$$

Now, applying a dispersion relation, we get

$$A_1(s, t, u) = \pi^{-1} \int_{-\infty}^{\infty} \frac{\text{Im } A_1(s', t, u)}{s' - s} ds' = (e^2/(2\pi)^6)(m^2 - s)^{-1}. \quad (4.6)$$

It is clear that $A_1(s, t, u)$ possesses a pole at $s = m^2$. Recall that A_1 was obtained starting from the contribution of Fig. 2(b) to the Compton scattering, using just unitarity and analyticity. The result for A_1 can be (and is usually) expressed by observing that a single-particle intermediate state in a given channel yields a *pole* for the amplitude at $s = m^2$, where m is the mass of the intermediate particle and s is the center of mass energy of that channel. We will use this terminology from here on out. It should be noted that, in the present, this represents merely a convenient way of description; no new assumptions are made, no new physics is thereby obtained.

Except for the first term on the right-hand side of (4.4), the other A_j and T_j remain so far undetermined. It should be noted at this point that, if only the first term in (4.4) were present, the scattering amplitude would not possess crossing symmetry. In fact, the crossing symmetry requires that the same $T^{(2)}$ given in (4.4) should also describe the u-channel process, i.e., $e^-(p_1) + \gamma(-q_4) \to e^-(p_3) + \gamma(-q_2)$, after the substitution [see (3.7)]:

$$q_2 \leftrightarrow -q_4,$$
$$e(q_2) \leftrightarrow e(q_4),$$

and

$$s \leftrightarrow u, \quad \text{with } t \text{ unchanged.}$$

Since crossing symmetry is one of our basic requirements, it must be imposed. This requires a term $A_2(s, t, u) T_2$ to be included in (4.4), where T_2 is obtained from T_1 by the above substitution, viz.,

$$T_2 = \frac{m}{(E_1 E_2)^{1/2}} \frac{1}{(2\omega_2 2\omega_4)^{1/2}} \times \bar{u}(p_3) e(q_2) (p_1 - q_4 + m) e(q_4) u(p_1),$$

and $A_2(s, t, u) = A_1(u, t, s)$, as a consequence of crossing. By (4.6), we immediately get

$$A_2(s, t, u) = [e^2/(2\pi)^6](m^2 - u)^{-1}. \quad (4.7)$$

Collecting all the results (4.4), (4.6), (4.7), we

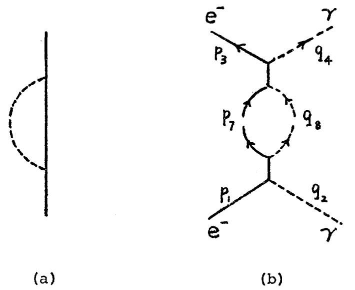


FIG. 3. Second-order electron self-energy.

obtain the second-order Compton scattering amplitude as

$$T^{(2)}[e^-(p_1)+\gamma(q_2)\rightarrow e^-(p_3)+\gamma(q_4)]$$

$$=-\frac{e^2}{(2\pi)^6}\frac{m}{(E_1E_3)^{1/2}}\frac{1}{(2\omega_2 2\omega_4)^{1/2}}$$

$$\times\left\{\frac{\bar{u}(p_3)\boldsymbol{e}(q_4)(\boldsymbol{p}_1+\boldsymbol{q}_2+m)\boldsymbol{e}(q_2)u(p_1)}{(p_1+q_2)^2-m^2}\right.$$

$$\left.+\frac{\bar{u}(p_3)\boldsymbol{e}(q_2)(\boldsymbol{p}_1-\boldsymbol{q}_4+m)\boldsymbol{e}(q_4)u(p_1)}{(p_1-q_4)^2-m^2}\right\},\quad(4.8)$$

in agreement with the usual result.

In passing, we note that in the t channel $(e^+ + e^- \rightarrow 2\gamma)$ the lowest-order contribution to the unitarity condition comes from the fourth order, hence its effect is neglected in this approximation. Other terms than $j=1$, 2 in (4.4) are therefore redundant to this order of approximation.

2. *Pair Annihilation and Pair-Production Processes*

Once the Compton scattering amplitude in second order is obtained, the scattering amplitudes for the two-photon free-pair annihilation process and the two-photon pair-production process in second order can readily be found by a direct application of crossing symmetry; one has to use the necessary substitutions as specified in (3.7).

3. *Møller Scattering and Bhabha Scattering*

Following the same method, we can get the second-order scattering amplitude of Møller scattering: $e^-(p_1)+e^-(p_2)\rightarrow e^-(p_3)+e^-(p_4)$. We note that in this case the scattering amplitude in second order can have a photon pole in both the t and the u channel. As noted earlier, the generalized unitarity condition has to be used; thus the photon in the intermediate state is treated as a massive vector meson—the mass is taken to be zero at the end of calculation. The scattering amplitude found in this way is identical with the usual result. The amplitude for Bhabha scattering can again be obtained by invoking crossing symmetry. Since these results are well known and the method used is a straightforward adaptation of the one just described for Compton scattering, this brief indication should suffice to demonstrate that the S-matrix method here is of comparable simplicity to the usual Feynman–Dyson method. For later purposes we record that for Møller scattering

$$T^{(2)}[e^-(p_1)+e^-(p_2)\rightarrow e^-(p_3)+e^-(p_4)]$$

$$=-\frac{1}{(2\pi)^6}\frac{m^2e^2}{(E_1E_2E_3E_4)^{1/2}}\left[\frac{\bar{u}(p_4)\gamma_\mu u(p_2)\bar{u}(p_3)\gamma^\mu u(p_1)}{(p_1-p_3)^2}\right.$$

$$\left.-\frac{\bar{u}(p_3)\gamma_\mu u(p_2)\bar{u}(p_4)\gamma^\mu u(p_1)}{(p_1-p_4)^2}\right],\quad(4.9)$$

and for Bhabha scattering

$$T^{(2)}[e^-(p_1)+e^+(p_2)\rightarrow e^-(p_3)+e^+(p_4)]$$

$$=-\frac{1}{(2\pi)^6}\frac{m^2e^2}{(E_1E_2E_3E_4)^{1/2}}\left[\frac{\bar{u}(p_3)\gamma_\mu u(p_1)\bar{v}(p_2)\gamma^\mu v(p_4)}{(p_1-p_3)^2}\right.$$

$$\left.-\frac{\bar{u}(p_3)\gamma_\mu v(p_4)\bar{v}(p_2)\gamma^\mu u(p_1)}{(p_1+p_2)^3}\right].\quad(4.10)$$

B. Electron Self-Energy Diagram of Second Order

It is well known that the electron self-energy diagram, Fig. 3(a) gives rise to a divergent integral in field theory. The method of renormalization was used to extract the finite, physically meaningful part from this infinite quantity. In the S-matrix theory we do not use the renormalization technique, instead the finite part of the electron self-energy is supposed to come out of the formalism automatically. This is an important advantage of the dispersion theory. To illustrate the technique, we consider the effect of the self-energy diagram in the radiative correction of the Compton scattering case, Fig. 3(b). We see that the finite part contributing to this diagram is exactly the same as that predicted by field theory. Although we use Compton scattering as an example, the same result would be obtained if other similar processes are considered.

Fourth-order Compton scattering gives rise to six diagrams in the s channel (see Fig. 10). In the present calculation just the contribution of Fig. 10(a), here given as Fig. 3(b), is considered. The others are discussed later.

The discontinuity of this particular scattering amplitude can now be found by invoking the unitarity condition to fourth order.

$$\text{disc. } T_a^{(4)}[p_1, q_2\to p_3, q_4]=i(2\pi)^4 \mathrm{S}\langle p_3, q_4 \mid T^{+(2)} \mid p_7, q_8\rangle\langle p_7, q_8 \mid T^{(2)} \mid p_1, q_2\rangle\delta^{(4)}(p_7+q_8-p_1-q_2). \quad (4.11)$$

Here, and in the following, S denotes the integration over the three-momenta and the summation over the spin and polarization states of the intermediate particles. The intermediate state in (4.11) is a state of one electron (p_7) and one photon (q_8). The choice of this intermediate state, and no others, corresponds precisely to the choice of a particular diagram. Inserting the direct terms in the second-order matrix elements as obtained from A (1) into the right-hand side of the unitarity condition, we get

$$\begin{aligned}
\text{disc. } T_a^{(4)}[p_1, q_2\to p_3, q_4]&=i(2\pi)^4\mathrm{S}\left[\frac{e^2}{(2\pi)^6}\right]^2 \frac{m^2}{[E_1E_3E_7{}^2 2\omega_2 2\omega_4(2\omega_8)^2]^{1/2}}\,\delta^{(4)}(p_1+q_2-p_7-q_8)\\
&\quad\times \frac{\bar u(p_3)\,\mathbf{e}(q_4)\,(\boldsymbol{p}_3+\boldsymbol{q}_4+m)\,\mathbf{e}(q_8)\,u(p_7)}{(p_3+q_4)^2-m^2}\cdot\frac{\bar u(p_7)\,\mathbf{e}(q_8)\,(\boldsymbol{p}_1+\boldsymbol{q}_2+m)\,\mathbf{e}(q_2)\,u(p_1)}{(p_1+q_2)^2-m^2}\\
&=\frac{-ie^4}{(2\pi)^8}\frac{m}{2(E_1E_3\omega_2\omega_4)^{1/2}}\,\bar u(p_3)\,\mathbf{e}(q_4)\int d^4p_7\int d^4q_8\,\delta(p_7{}^2-m^2)\theta(p_7)\\
&\quad\times\delta(q_8{}^2-\lambda^2)\theta(q_8)\,\delta^{(4)}(p_1+q_2-p_7-q_8)\,\frac{1}{(s-m^2)^2}\\
&\quad\times\left\{2m(s+m^2+\lambda^2)+\left[8m^2-\frac{(s+m^2)(s+m^2-\lambda^2)}{s}\right](\boldsymbol{p}_1+\boldsymbol{q}_2)\right\}\mathbf{e}(q_2)\,u(p_1), \quad (4.12)
\end{aligned}$$

where the sum on the spin states has been carried out by using

$$\sum u(p_7)\bar u(p_7)=(\boldsymbol{p}_7+m)/2m,$$

and that over the polarization states by

$$\sum\cdots\mathbf{e}(q_8)\cdots\mathbf{e}(q_8)\cdots=-\cdots\gamma_\mu\cdots\gamma^\mu\cdots.$$

s is defined as

$$s\equiv(p_1+q_2)^2.$$

The photon mass is assumed to be finite (λ) for purpose of this calculation. After we carry out the integration in (4.12), disc. $T_a^{(4)}(p_1, q_2\to p_3, q_4)$ as a function of s can be put in the form

$$\text{disc. } T_a^{(4)}(s)=-\frac{ie^4}{8(2\pi)^7}\frac{m}{(E_1E_3\omega_2\omega_4)^{1/2}}\,\bar u(p_3)\,\mathbf{e}(q_4)[B_1(s)(\boldsymbol{p}_1+\boldsymbol{q}_2+m)+\Sigma_1(s)]\mathbf{e}(q_2)\,u(p_1), \quad (4.13)$$

where

$$\begin{aligned}
B_1(s)&=\left[8m^2-\frac{(s+m^2)(s+m^2-\lambda^2)}{s}\right]\frac{\{[s-(m+\lambda)^2][s-(m-\lambda)^2]\}^{1/2}}{s(s-m^2)^2}\,\theta[s-(m+\lambda)^2],\\
\Sigma_1(s)&=\left\{2m(s+m^2+\lambda^2)-m\left[8m^2-\frac{(s+m^2)(s+m^2-\lambda^2)}{s}\right]\right\}\frac{\{[s-(m+\lambda)^2][s-(m-\lambda)^2]\}^{1/2}}{s(s-m^2)^2}\,\theta[s-(m+\lambda)^2].
\end{aligned} \quad (4.14)$$

Now write $T_a^{(4)}(s)$ as a linear combination of invariant amplitudes

$$\begin{aligned}
T_a^{(4)}&=B(s)[m/(E_1E_3)^{1/2}][1/2(\omega_2\omega_4)^{1/2}]\bar u(p_3)\,\mathbf{e}(q_4)\,(\boldsymbol{p}_1+\boldsymbol{q}_2+m)\,\mathbf{e}(q_2)\,u(p_1)\\
&\quad+\Sigma(s)[m/(E_1E_3)^{1/2}][1/2(\omega_2\omega_4)^{1/2}]\bar u(p_3)\,\mathbf{e}(q_4)\,\mathbf{e}(q_2)\,u(p_1)+\cdots. \quad (4.15)
\end{aligned}$$

(See Ref. 17.) Again other amplitudes are not listed since they do not occur here.

With the definition (4.15), (4.13) gives

$$\begin{aligned}
\operatorname{Im} B(s)&=[-e^2/(2\pi)^6]\tfrac{1}{4}\alpha B_1(s),\\
\operatorname{Im} \Sigma(s)&=[-e^2/(2\pi)^6]\tfrac{1}{4}\alpha\Sigma_1(s). \quad (4.16)
\end{aligned}$$

[17] Previously T has been written as $\Sigma_j A_j T_j$. In this case, A_1, A_2 are called $B(s)$ and $\Sigma(s)$, respectively. The choice of T_j is clearly suggested by (4.13).

If $B(s)$ and $\Sigma(s)$ again satisfy a dispersion relation, then they can be calculated as

$$B(s)=\pi^{-1}\int_{-\infty}^{\infty}\frac{\text{Im }B(s')}{s'-s}\,ds',$$

$$\Sigma(s)=\pi^{-1}\int_{-\infty}^{\infty}\frac{\text{Im }\Sigma(s')}{s'-s}\,ds'. \tag{4.17}$$

Since B_1 and Σ_1 are explicitly given, the integration can be carried out; it is quite straightforward but very tedious. Special attention should be paid to the term, contained in $B_1(s')$, which gives rise to the infrared divergence. It has the form:

$$B_1(s')=\cdots-[4/(s'-m^2)^2]\{[s'-(m+\lambda)^2][s'-(m-\lambda)^2]\}^{1/2}. \tag{4.18}$$

A photon mass λ has been included just for the purpose of carrying out this integral. The results of these calculations are

$$\Sigma(s)=\frac{-e^2}{(2\pi)^6}\frac{\alpha}{4\pi}[m(1-\rho)]^{-1}\left[1-\frac{2-3\rho}{1-\rho}\ln\rho\right] \tag{4.19}$$

$$B(s)=\frac{-e^2}{(2\pi)^6}\frac{\alpha}{4\pi}\left(-\frac{2}{m^2\rho}\right)\left\{[2(1-\rho)]^{-1}\left[2-\rho+\frac{\rho^2+4\rho-4}{1-\rho}\ln\rho\right]+1-2\int_{\lambda/m}^{1}\frac{dx}{x}\right\}, \tag{4.20}$$

where

$$\rho\equiv1-s/m^2,\qquad \alpha=e^2/4\pi.$$

The scattering amplitude of interest then becomes

$$T_a^{(4)}(s)=\frac{-e^2}{(2\pi)^6}\frac{m}{(E_1E_32\omega_22\omega_4)^{1/2}}\bar{u}(p_3)\,\boldsymbol{e}(q_4)\,\Sigma_f(p_1+q_1)\,\boldsymbol{e}(q_2)\,u(p_1), \tag{4.21}$$

where $\Sigma_f(p)$ stands for

$$\Sigma_f(p)=\frac{\alpha}{2\pi m}\left\{\frac{1}{2(1-\rho)}\left(1-\frac{2-3\rho}{1-\rho}\ln\rho\right)-\frac{p+m}{m\rho}\left[\frac{1}{2(1-\rho)}\left(2-\rho+\frac{\rho^2+4\rho-4}{1-\rho}\ln\rho\right)+1-2\int_{\lambda/m}^{1}\frac{dx}{x}\right]\right\}. \tag{4.22}$$

in complete agreement with the results of quantum electrodynamics. In passing, we note that the corresponding results obtained in Refs. 1 and 18, using Feynman–Dyson technique, are in error. It is observed that a wrong sign appearing in Eq. (9.24) of Ref. 1 has exactly caused this mistake.

It is still interesting to observe that no renormalization was necessary; the actual integrals encountered in this computation are definitely different from those occurring in the Feynman–Dyson theory, although the final result is identical. No question of crossing symmetry can be raised at this point, since what is calculated is only a *part* of a scattering amplitude.

C. Photon Self-Energy Diagram in Second Order

The contribution of the photon self-energy diagram in second order, Fig. 4(a), can be obtained in the S-matrix theory in much the same way as we did for the electron self-energy.

We have seen that in the lowest order of the scattering of a positron by an electron a pole occurs in the s channel, Fig. 4(b). If a two-particle intermediate state, such as an electron–positron pair, is included in the unitarity condition, Fig. 4(c), we would get the analog of the photon self-energy modification of the "photon propagator."

The second-order s-channel scattering amplitude of an electron by a positron is, as we mentioned earlier,

$$T_s^{(2)}[p_1,\bar{p}_2\rightarrow p_3,\bar{p}_4]=[1/(2\pi)^6][m^2e^2/(E_1E_2E_3E_4)^{1/2}][(p_1+p_2)^2]^{-1}\bar{u}(p_3)\gamma_\mu v(p_4)\,\bar{v}(p_2)\gamma^\mu u(p_1). \tag{4.23}$$

The discontinuity of the fourth-order scattering amplitude, corresponding to Fig. 4(c), is again given by the

[18] A. I. Akhiezer and V. B. Berestetskii, *Elements of Quantum Electrodynamics* (English Translation) (Oldbourn Press, London, 1959), 2nd revised edition.

unitarity condition,

$$\text{disc. } T_s^{(4)}[p_1,\bar{p}_2\to p_3,\bar{p}_4]=\frac{i}{(2\pi)^8}\frac{m^2e^4}{(E_1E_2E_3E_4)^{1/2}}\mathbf{S}d^3p_5d^3p_6\frac{m^2}{E_5E_6}\delta^{(4)}(p_5+p_6-p_1-p_2)$$

$$\times\frac{\bar{u}(p_3)\gamma_\mu v(p_4)\bar{v}(p_6)\gamma^\mu u(p_5)}{(p_5+p_6)^2}\cdot\frac{\bar{u}(p_5)\gamma_\nu v(p_6)\bar{v}(p_2)\gamma^\nu u(p_1)}{(p_1+p_2)^2}$$

$$=\frac{i}{(2\pi)^8}\frac{4e^4m^4}{(E_1E_2E_3E_4)^{1/2}}\int d^4p_5\,d^4p_6\delta(p_5^2-m^2)\theta(p_5)\delta(p_6^2-m^2)\theta(p_6)$$

$$\times\delta^{(4)}(p_5+p_6-p_1-p_2)\frac{\bar{u}(p_3)\gamma_\mu v(p_4)\bar{v}(p_2)\gamma^\nu u(p_1)}{(p_1+p_2)^2(p_5+p_6)^2}\frac{1}{m^2}[p_5{}^\mu p_6{}^\nu+p_5{}^\nu p_6{}^\mu-g^{\mu\nu}(p_5p_6+m^2)].\quad(4.24)$$

The spin sum has already been carried out in (4.24).

Following the previous pattern, define $A^{\mu\nu}$ by

$$T_s^{(4)}[p_1,\bar{p}_2\to p_3,\bar{p}_4]=\frac{m^2e^2}{(E_1E_2E_3E_4)^{1/2}}\frac{1}{(2\pi)^6}\frac{\bar{u}(p_3)\gamma_\mu v(p_4)\bar{v}(p_2)\gamma_\nu u(p_1)}{(p_1+p_2)^2}A^{\mu\nu},\quad(4.25)$$

here $s=(p_1+p_2)^2$, etc. Then (4.24) yields

$$\operatorname{Im}A^{\mu\nu}=\frac{e^2}{2(2\pi)^2}\int d^4p_5\,d^4p_6\delta(p_5^2-m^2)\theta(p_5)\delta(p_6^2-m^2)\theta(p_6)\delta^{(4)}(p_5+p_6-p_1-p_2)\cdot\frac{2}{q^2}(q^\mu q^\nu-g^{\mu\nu}q^2-4Q^\mu Q^\nu),\quad(4.26)$$

where we introduced the new variables

$$q=p_5+p_6=p_1+p_2,$$
$$Q=\tfrac{1}{2}(p_6-p_5).\quad(4.27)$$

Thus the integration in (4.26) is facilitated by observing that $Q^\mu Q^\nu$ is a tensor. The number of tensors which can be constructed from $g^{\mu\nu}$ and q^μ, q^ν, is very restricted. In the integrand $Q^\mu Q^\nu$ can be replaced by:

$$Q^\mu Q^\nu\overset{\text{eff}}{=}\tfrac{1}{3}(m^2-\tfrac{1}{4}q^2)[g^{\mu\nu}-(q^\mu q^\nu/q^2)].$$

Thus

$$\operatorname{Im}A^{\mu\nu}=\frac{e^2}{8\pi^2}\int d^4q\,d^4Q\delta[(\tfrac{1}{2}q+Q)^2-m^2]\delta[(\tfrac{1}{2}q-Q)^2-m^2]$$

$$\times\theta(\tfrac{1}{2}q+Q)\theta(\tfrac{1}{2}q-Q)\delta^{(4)}(q-p_1-p_2)(4/3q^2)[1+(2m^2/q^2)](q^\mu q^\nu-q^2g^{\mu\nu}).\quad(4.28)$$

We then get after integration

$$\operatorname{Im}A^{\mu\nu}(s)=-\frac{e^2}{12\pi}\frac{s-4m^2}{s}\left(1+\frac{2m^2}{s}\right)\left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)\theta(s-4m^2)$$

$$\equiv\operatorname{Im}A(s)[g^{\mu\nu}-(q^\mu q^\nu/q^2)].\quad(4.29)$$

This defines the Im $A(s)$ explicitly.

Note that as $s\to\infty$, Im $A(s)\to 1$, hence if $A(s)$ satisfies a dispersion relation at all, it must be a subtracted dispersion relation. Assuming this, $A^{\mu\nu}(s)$ can be found as

$$A(s)=\pi^{-1}\int_{-\infty}^{\infty}[(s'-s)^{-1}-s^{-1}]\operatorname{Im}A(s')\,ds'+A(0)$$

$$=\frac{e^2}{4\pi^2}\left[\frac{1}{9}-\frac{1+2\sin^2\theta}{3\sin^2\theta}\left(1-\frac{\theta}{\tan\theta}\right)\right],\quad(4.30)$$

where we defined $\sin^2\theta=s/4m^2$, and set $A(0)=0$. Thus

$$A^{\mu\nu}(s)=\frac{\alpha}{\pi}\left[\frac{1}{9}-\frac{1+2\sin^2\theta}{3\sin^2\theta}\left(1-\frac{\theta}{\tan\theta}\right)\right]\left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right).\quad(4.31)$$

This is exactly the renormalized polarization tensor which can also be obtained by field theory. (Compare, for example, Reference 19, p. 553.) It would seem

[19] S. S. Schweber, *Introduction to Relativistic Quantum Field Theory* (Row and Peterson, Evanston, Ill., 1961).

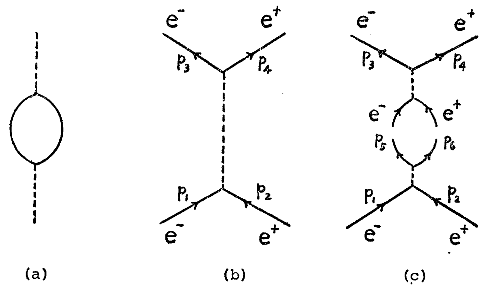


FIG. 4. Photon self-energy diagram.

that this follows in quite a straightforward manner from the S-matrix postulates.

The separation of $A^{\mu\nu}$ into the factors $A(s)$ and $g^{\mu\nu}-(q^\mu q^\nu/q^2)$ corresponds to a particular decomposition of T into scalar amplitude and invariant operator function. If one were to write instead

$$\mathrm{Im}\, A^{\mu\nu}\equiv \mathrm{Im}\,(A(s)/q^2)(g^{\mu\nu}q^2-q^\mu q^\nu), \qquad (4.32)$$

which the invariance arguments certainly allow, one would assume an unsubtracted dispersion relation for $A(s)/q^2=A(s)/s$, which would give the same result.

If (4.31) is substituted into (4.25), the scattering amplitude corresponding to Fig. 4(c) is obtained:

$$T_s^{(4)}[p_1,\bar{p}_2\rightarrow p_3,\bar{p}_4]=T_s^{(2)}[p_1,\bar{p}_2\rightarrow p_3,\bar{p}_4]$$
$$\times\frac{\alpha}{\pi}\left[\frac{1}{9}-\frac{1+2\sin^2\theta}{3\sin^2\theta}\left(1-\frac{\theta}{\tan\theta}\right)\right], \qquad (4.33)$$

where $\sin^2\theta=(p_1+p_2)^2/4m^2$.

D. The Third Order Vertex Part With External Electron Lines (Electron Form Factors)

We now consider the determination of the third-order vertex part using dispersion techniques. A general vertex corresponding to the annihilation of an electron–positron pair is shown in Fig. 5(a).

In the center of mass system of the electron–positron pair, the vertex part can be expressed by a function, $F(s)$, of one variable. This function is the matrix element $\langle\gamma\mid T\mid e^+,e^-\rangle$ between a virtual photon of mass $s^{1/2}$ and a physical state consisting of an electron–positron pair with total energy $s^{1/2}$, where $q^2=s=(p_1+p_2)^2$.

Just from invariance arguments (Lorentz and C, P, T, invariance), one can write for $\langle\gamma\mid T\mid e^+,e^-\rangle$

$$\langle\gamma\mid T\mid e^+,e^-\rangle=F(s)$$
$$=e[m/(E_1E_2)^{1/2}][1/(2\pi)^3]\bar{v}(p_2)$$
$$\times[F_1(q^2)\gamma_\mu+F_2(q^2)\sigma_{\mu\nu}q^\nu]u(p_1)a^\mu. \qquad (4.34)$$

(For a transverse photon, $a^\mu=(2\pi)^{-3/2}(2\omega)^{-1/2}e^\mu$.) F_1 and F_2 are usually referred to as the charge and magnetic form factors of the electron. When $s=0$, the photon becomes a real one with zero energy; the matrix element $\langle\gamma\mid T\mid e^+,e^-\rangle$, or equivalently $F(s)$, will then become the elementary vertex introduced in Sec. III, Eq. (3.4):

$$\langle\gamma\mid T\mid e^+,e^-\rangle_{s=0}=e\frac{m}{(E_1E_2)^{1/2}}\frac{1}{(2\omega)^{1/2}}\frac{1}{(2\pi)^{1/2}}$$
$$\times\bar{v}(p_2)\gamma_\mu u(p_1)e^\mu. \qquad (4.35)$$

In field theory this condition is used to renormalize the divergent vertex part. It will be used in a similar vein in the S-matrix theory to fix the undetermined constant in the once subtracted dispersion relation for the charge form factor of the electron. By insisting on (4.35) in field theory, one actually fixes the charge e. In dispersion theory, this same condition fixes a subtraction constant. These constants govern the asymptotic behavior (for high energies) of matrix elements. Thus there is some similarity between the physical significance of the use of (4.35) in both approaches—but they do not appear to be rigorously the same.

Following the by now usual procedure, the imaginary part of $F_i(s)$ or the discontinuity of $\langle\gamma\mid T\mid e^+,e^-\rangle$ in third order can be determined by the unitarity condition:

$$\text{disc.}\,\langle q\mid T^{(3)}\mid p_1,\bar{p}_2\rangle=i(2\pi)^4\sum\langle q\mid T^{(1)+}\mid n\rangle\langle n\mid T^{(2)}\mid p_1,\bar{p}_2\rangle\delta^{(4)}(p_n-p_1-p_2), \qquad (4.36)$$

where n is a set of physical intermediate states which conserve energy-momentum and all other quantum numbers. It is seen that the only possible choice for n is the state of one electron–positron pair, with momenta q_1 and q_2. Hence

$$\text{disc.}\,\langle q\mid T^{(3)}\mid p_1,\bar{p}_2\rangle=i(2\pi)^4\frac{e^3}{(2\pi)^9}\int d^3q_1\,d^3q_2\,\frac{m^3}{(E_1E_2)^{1/2}\epsilon_1\epsilon_2}\,\delta^{(4)}(p_1+p_2-q_1-q_2)\bar{v}(q_2)\gamma_\mu u(q_1)$$
$$\times a^\mu\left[-\frac{\bar{u}(q_1)\gamma_\nu u(p_1)\bar{v}(p_2)\gamma^\nu v(q_2)}{(p_1-q_2)^2}+\frac{\bar{u}(q_1)\gamma_\nu v(q_2)\bar{v}(p_2)\gamma^\nu u(p_1)}{(p_1+p_2)^2}\right], \qquad (4.37)$$

where use has been made of (4.10). The two terms in (4.37) correspond to diagrams 5(b) and 5(c). For $T^{(1)}$ one uses

$$\langle q\mid T^{(1)}\mid q_1,\bar{q}_2\rangle=e[m/(\epsilon_1\epsilon_2)^{1/2}][1/(2\pi)^3]\bar{v}(q_2)\gamma_\mu u(q_1)a^\mu. \qquad (4.38)$$

The spin sums in (4.37) can be carried out; there results

$$\text{disc. } \langle q \mid T^{(3)} \mid p_1, \bar{p}_2 \rangle = \frac{ie^3}{(2\pi)^5} \frac{m}{(E_1E_2)^{1/2}} \int d^4q_1\, d^4q_2 \delta(q_1^2-m^2)\,\delta(q_2^2-m^2)\theta(q_1)\theta(q_2)\,\delta^{(4)}(p_1+p_2-q_1-q_2)$$

$$\times a^\mu(q) \left[-\frac{\bar{v}(p_2)\gamma^\nu(q_2-m)\gamma_\mu(q_1+m)\gamma_\nu u(p_1)}{(p_1-q_1)^2} + \frac{\bar{v}(p_2)\gamma^\nu u(p_1)\ \mathrm{Tr}\left[(q_1+m)\gamma_\nu(q_2-m)\gamma_\mu\right]}{(p_1+p_2)^2} \right]$$

$$= -\frac{4ie^3}{(2\pi)^5} \frac{m}{(E_1E_2)^{1/2}} \int d^4Q \delta[(\tfrac{1}{2}q+Q)^2-m^2]\delta[(\tfrac{1}{2}q-Q)^2-m^2]\theta(\tfrac{1}{2}q+Q)\theta(\tfrac{1}{2}q-Q)\, a^\mu(q)$$

$$\times \bar{v}(p_2) \left\{ \left(1+\frac{\frac{1}{4}q^2-Q^2}{(P-Q)^2}\right)\gamma_\mu + \frac{2mQ_\mu}{(P-Q)^2} + QQ_\mu\left[\frac{1}{(P-Q)^2}+\frac{2}{q^2}\right] + \frac{QP_\mu}{(P-Q)^2} \right\} u(p_1), \quad (4.39)$$

where we have defined the new variables:

$$P=\tfrac{1}{2}(p_2-p_1),$$
$$Q=\tfrac{1}{2}(q_2-q_1).$$

Comparing (4.34) and (4.39), we get

$$\mathrm{Im}\ F_1(q^2)\gamma_\mu+\mathrm{Im}\ F_2(q^2)\sigma_{\mu\nu}q^\nu = -\frac{e^2}{2\pi^2}\int d^4Q\delta[(\tfrac{1}{2}q+Q)^2-m^2]\delta[(\tfrac{1}{2}q-Q)^2-m^2]\theta(\tfrac{1}{2}q+Q)\theta(\tfrac{1}{2}q-Q)$$

$$\times \left\{ \left[1+\frac{\frac{1}{4}q^2-Q^2}{(P-Q)^2}\right]\gamma_\mu + \frac{2mQ_\mu}{(P-Q)^2} + QQ_\mu\left[\frac{1}{(P-Q)^2}+\frac{2}{q^2}\right] + \frac{QP_\mu}{(P-Q)^2} \right\}. \quad (4.40)$$

It is to be understood that this equation holds as an equality for matrix elements taken between $\bar{v}(p_2)\cdots u(p_1)$, where v and u are free-particle states. Simplifications result if use is made of the following relations:

$$\bar{v}(p_2)\, p u(p_1) = -\bar{v}(p_2)\, mu(p_1)$$
$$\bar{v}(p_2)\, qu(p_1) = 0.$$

Using these we can show, after some straightforward but lengthy calculations, that (4.40) can be written as

$$\mathrm{Im}\ F_1(q^2)\gamma_\mu+\mathrm{Im}\ F_2(q^2)\sigma_{\mu\nu}q^\nu = -\frac{e^2}{2\pi^2}\int d^4Q\delta[(\tfrac{1}{2}q+Q)^2-m^2]\delta[(\tfrac{1}{2}q-Q)^2-m^2]\theta(\tfrac{1}{2}q+Q)\theta(\tfrac{1}{2}q-Q)$$

$$\times \left\{ \left[1+\frac{\frac{1}{4}q^2-Q^2}{(P-Q)^2-\lambda^2} + \left(\frac{1}{(P-Q)^2}+\frac{2}{q_2}\right)\frac{P^2Q^2-(P\cdot Q)^2}{2P^2}\right]\gamma_\mu \right.$$

$$\left. + \left[\frac{P\cdot Q}{P^2(P-Q)^2} + \left(\frac{1}{(P-Q)^2}+\frac{2}{q^2}\right)\frac{P^2Q^2-3(P\cdot Q)^2}{2P^4}\right] mP_\mu \right\}. \quad (4.41)$$

In the above equation, the term $[(\frac{1}{4}q^2-Q^2)/(P-Q)^2]\gamma_\mu$ in the integrand will cause an infrared divergence, if the integration over Q is carried out. Drell and Zachariasen[20] have calculated the charge form factors using dispersion technique, however, the infrared divergence has not been treated sufficiently carefully. To study the infrared divergence more carefully here, a photon mass λ is introduced. Now the integration over Q can be performed unambiguously with the result:

$$\mathrm{Im}\ F_1(q^2)\gamma_\mu+\mathrm{Im}\ F_2(q^2)\sigma_{\mu\nu}q^\nu = \frac{e^2}{4\pi}\left(\frac{q^2-4m^2}{q^2}\right)^{1/2} \left\{ \frac{1}{2}\frac{q^2-2m^2}{q^2-4m^2}\int_{\lambda 2/2}^{2[(q^2/4)-m^2]}\frac{dx}{x}\gamma_\mu - \frac{13}{12}\gamma_\mu - \frac{2m^2}{3q^2}\gamma_\mu \right.$$

$$\left. - \frac{m^2}{q^2-4m^2}\gamma_\mu + \tfrac{1}{4}m\frac{\sigma_{\mu\nu}q^\nu}{q^2-4m^2} \right\} \theta(s-4m^2). \quad (4.42)$$

Denote $s=q^2$. Im F_1 and Im F_2 can be written as functions of s.

$$\mathrm{Im}\ F_1(s) = \frac{e^2}{4\pi}\frac{s-4m^2}{s}\theta(s-4m^2)\left\{\frac{1}{2}\frac{s-2m^2}{s-4m^2}\ln\frac{s-4m^2}{\lambda^2} - \frac{13}{12} - \frac{2m^2}{3s} - \frac{m^2}{s-4m^2}\right\}$$

$$\mathrm{Im}\ F_2(s) = (e^2/4\pi)\tfrac{1}{4}m[s(s-4m^2)]^{1/2}\theta(s-4m^2). \quad (4.43)$$

[20] S. D. Drell and F. Zachariasen, Phys. Rev. **111**, 1727 (1958).

$F_2(s)$ and $F_1(s)$ themselves, can then be found by using dispersion relations. For $F_2(s)$, we have an unsubtracted dispersion relation since Im F_2 behaves as $1/s$ for $s\rightarrow\infty$,

$$F_2(s)=\pi^{-1}\int\frac{\text{Im } F_2(s')}{s'-s}\,ds'$$
$$=(e^2/4\pi)(8m\pi)^{-1}(2\theta/\sin 2\theta), \tag{4.44}$$

where we defined $\sin^2\theta=s/4m^2$.

Since the anomalous magnetic moment of the electron is defined as the magnetic form factor of the electron at zero momentum transfer, except for a factor $4m$, we immediately get the anomalous magnetic moment as

$$\mu=4mF_2(0)=\alpha/2\pi. \tag{4.45}$$

To get $F_1(s)$, we observe that Im $F_1(s)$ approaches a constant as s goes to infinity. This forces us to use a subtracted dispersion relation. Hence

$$F_1(s)-F_1(0)=\pi^{-1}s^{-1}\int\frac{\text{Im } F_1(s')}{s'(s'-s)}\,ds'$$
$$=\frac{\alpha}{2\pi}\left\{2\left(\ln\frac{2m}{\lambda}-1\right)\left(1-\frac{2\theta}{\tan 2\theta}\right)+\theta\tan\theta+\frac{2}{\tan 2\theta}\left[\int_1^\infty\frac{\ln(z^2-1)}{z^2+\tan^2\theta}\,dz\tan\theta-\tan 2\theta\ln 2\right]\right\}$$
$$+\frac{\alpha}{\pi}\left\{\frac{1}{9}-\frac{1+2\sin^2\theta}{3\sin^2\theta}\left(1-\frac{\theta}{\tan\theta}\right)\right\}. \tag{4.46}$$

$F_1(0)$ is, of course, arbitrary; however we will put $F_1(0)$ equal to zero. This is actually necessary for we require, as stated at the beginning, that [see (4.35)]

$$\langle\gamma\mid T\mid e^+,e^-\rangle_{s=0}=e[m/(E_1E_2)^{1/2}][1/(2\pi)^3]\bar{v}(p_2)\gamma_\mu u(p_1)a^\mu.$$

Up to the third order

$$\langle\gamma\mid T\mid e^+,e^-\rangle=\langle\gamma\mid T^{(1)}+T^{(3)}\mid e^+,e^-\rangle$$
$$=e[m/(E_1E_2)^{1/2}][1/(2\pi)^3]\bar{v}(p_2)\{[1+F_1(s)]\gamma_\mu+F_2(s)\sigma_{\mu\nu}q^\nu\}u(p_1)a^\mu.$$

Hence, for $q=0$ or $s=0$, we must have

$$F_1(0)=0 \tag{4.47}$$

and the subtraction constant is in fact, determined by (4.35). The result (4.46) can be compared with that obtained from field theory, e.g., Ref. 19, pp. 543, 553. They indeed agree, since we can prove the following (nontrivial) equality by a series expansion

$$\tan\theta\int_1^\infty\frac{\ln(z^2-1)}{z^2+\tan^2\theta}\,dz-\ln 2\tan 2\theta=2\int_0^\theta\chi\tan\chi\,d\chi-\ln 2\tan 2\theta\left(1-\frac{2\theta}{\tan 2\theta}\right).$$

E. Magnetic Moment of the Electron

We have seen in Sec. IVD how the electron form factors can be obtained by using dispersion techniques. These techniques can, in fact, be used to compute the form factors to higher orders by including in the right-hand side of (4.36) other intermediate states. While it is simple in principle, the actual computation would be rather tedious and difficult. However, for the calculation of the anomalous magnetic moment of the electron, simplifications result since the dispersion relation requires no subtractions and, moreover, just the magnetic form factor at *zero* momentum transfer is of interest. A calculation in fourth order in e has been carried out by Terent'ev[21] using this technique. The result obtained is in complete agreement with the previous ones, e.g., by Petermann,[22] using quantum electrodynamics. The advantage of the dispersion method in this case consists in the fact that the problem can be reduced without much labor to double integrations over rational functions and logarithms. Hence, the computational labor is somewhat less than in the usual method using Feynman techniques.

In the following we shall, instead of presenting all the details of the calculations, give the contributions from various dispersion graphs (Fig. 6). These will be compared with the corresponding contributions of Feynman diagrams.

The anomalous magnetic moment of the electron in fourth order of e can be separated into a sum of

[21] M. V. Terent'ev, Zh. Eksperim. i Teor. Fiz. **43**, 619 (1962) [English transl. Soviet Phys.—JETP **16**, 444 (1963)].

[22] A. Petermann, Fortschr. Physik **6**, 505 (1958).

terms, each corresponding to a particular dispersion graph:

$$\mu=\mu^{(2)}+\mu^{(3)}, \tag{4.48}$$

where

$$\mu^{(2)}=\sum_{k=1}^{6}\mu_k{}^{(2)}, \quad \text{and} \quad \mu^{(3)}=2\sum_{k=1}^{4}\mu_k{}^{(3)}.$$

The superscripts indicate the number of particles appearing in the intermediate state of the unitarity condition, and the kth terms in the sums correspond to the graph M_k in Fig. 6. Those lines which contain a cross, represent the intermediate particle states which should be put on the mass shell according to unitarity. Since to each diagram $M_k{}^{(3)}$, there exists a corresponding symmetrical counterpart, a factor of two appears in $\mu^{(3)}$.

The results of dispersion calculation are listed below:

$$\begin{aligned}
\mu_1{}^{(2)}&=(\alpha^2/\pi^2)(\tfrac{119}{36}-\tfrac{1}{3}\pi^2)\\
\mu_2{}^{(2)}&=(\alpha^2/4\pi^2)[\tfrac{11}{2}\zeta(3)-\pi^2\ln 2+\tfrac{1}{2}\pi^2-7-\ln\lambda(3-\tfrac{1}{3}\pi^2)]\\
\mu_3{}^{(2)}&=\mu_2{}^{(2)}\\
\mu_4{}^{(2)}&=(\alpha^2/\pi^2)(\pi^2/32)\\
\mu_5{}^{(2)}&=(\alpha^2/4\pi^2)[\tfrac{15}{4}\zeta(3)-(9\pi^2/2)\ln 2+\tfrac{79}{42}\pi^2-\tfrac{1}{4}+2\ln\lambda(1+\tfrac{1}{4}\pi^2)]\\
\mu_6{}^{(2)}&=(\alpha^2/4\pi^2)[\tfrac{17}{3}\zeta(3)-\tfrac{22}{3}\pi^2\ln 2+\tfrac{53}{9}\pi^2-\tfrac{446}{27}+2\ln\lambda(\tfrac{2}{3}\pi^2-3)]\\
\mu_1{}^{(3)}&=(\alpha^2/2\pi^2)[-(\pi^2/18)+\tfrac{24}{11}+\ln\lambda]\\
\mu_2{}^{(3)}&=(\alpha^2/2\pi^2)[-\tfrac{15}{16}\zeta(3)-\tfrac{115}{144}\pi^2+\tfrac{9}{8}\pi^2\ln 2+\tfrac{7}{24}-\tfrac{1}{2}\ln\lambda(1+\tfrac{1}{4}\pi^2)]\\
\mu_3{}^{(3)}&=(\alpha^2/2\pi^2)[-\tfrac{13}{4}\zeta(3)+(5\pi^2/6)\ln 2-(7\pi^2/36)+\tfrac{17}{24}-\tfrac{1}{2}\ln\lambda(\tfrac{1}{3}\pi^2-1)]\\
\mu_4{}^{(3)}&=(\alpha^2/2\pi^2)[-\tfrac{1}{6}\zeta(3)+\pi^2\ln 2-(10\pi^2/9)+\tfrac{116}{27}-\tfrac{1}{2}\ln\lambda(\tfrac{2}{3}\pi^2-3)].
\end{aligned} \tag{4.49}$$

Summing up, we get

$$\begin{aligned}
\mu&=\sum_{k=1}^{6}\mu_k{}^{(2)}+2\sum_{k=1}^{4}\mu_k{}^{(3)}\\
&=(\alpha^2/\pi^2)[\tfrac{197}{144}+(\pi^2/12)+\tfrac{3}{4}\zeta(3)-\tfrac{1}{2}\pi\ln 2],
\end{aligned} \tag{4.50}$$

where λ is the photon mass and $\zeta(x)$ is the Riemann function.

Using Feynman techniques, Petermann obtained the contributions μ_i from each graph M_i in Fig. 7.

$$\begin{aligned}
\mu_{\mathrm{I}}&=(\alpha^2/\pi^2)(\tfrac{1}{6}+\tfrac{13}{36}\pi^2+\tfrac{5}{4}\zeta(3)-\tfrac{5}{6}\pi^2\ln 2)\\
\mu_{\mathrm{IIa}}&=(\alpha^2/\pi^2)[\tfrac{11}{48}+(\pi^2/18)]\\
\mu_{\mathrm{IIc}}&=(\alpha^2/\pi^2)[-\tfrac{67}{24}+(\pi^2/18)-\tfrac{1}{2}\zeta(3)+\tfrac{1}{3}\pi^2\ln 2-\tfrac{1}{2}\ln(\lambda^2/m^2)]\\
\mu_{\mathrm{IId}}&=(\alpha^2/\pi^2)[\tfrac{11}{24}-(\pi^2/18)+\tfrac{1}{2}\ln(\lambda^2/m^2)]\\
\mu_{\mathrm{IIe}}&=(\alpha^2/\pi^2)(\tfrac{119}{36}-\tfrac{1}{3}\pi^2).
\end{aligned} \tag{4.51}$$

The sum of these five terms coincides exactly with the dispersion results.

It is easily seen by inspection that various terms in both calculations are indeed related:

$$\begin{aligned}
\mu_{\mathrm{I}}&=\mu_6{}^{(2)}+2\mu_4{}^{(3)},\\
\mu_{\mathrm{IIa}}&=\mu_4{}^{(2)}+\mu_5{}^{(2)}+2\mu_2{}^{(3)},\\
\mu_{\mathrm{IIc}}&=2\mu_3{}^{(3)}+\mu_2{}^{(2)}+\mu_3{}^{(2)},\\
\mu_{\mathrm{IId}}&=2\mu_1{}^{(3)},\\
\mu_{\mathrm{IIe}}&=\mu_1{}^{(2)}.
\end{aligned} \tag{4.52}$$

These relations can be understood immediately by tracing "their background" to the diagrams. A given Feynman diagram can be evaluated by using the dispersion relations and unitarity. The unitarity condition requires that all possible intermediate states be summed over; thus, to a given order, it is equivalent with the procedure of cutting the diagram into two halves with the lower half representing a physical scattering process and the upper half is then a vertex. The lines being cut now representing physical particles must be on the mass shell. Hence, we have the set of relations (4.52).

It is perhaps pertinent to note that in the actual integrations, important differences occur. In the dispersion theory, all the intermediate integrations are on the mass shell—hence more easily performed than those in the Feynman–Dyson theory, which are off the mass shell.

F. The Third-Order Vertex Part for a Single External Electron Line

Consider next, the vertex with one photon line (q) and one electron line (p) both regarded as external or free, while the other electron line $(p+q)$ may or may not be free. We shall again compute the contribution of this kind of vertex using the S-matrix theory. The dispersion diagram for this case is shown in Fig. 8 where the intermediate particles (photon with momentum k, and electron with momentum p_1) have to be on the mass shell.

Following the same procedures as before, we can find the discontinuity of the matrix element, $\langle p+q \mid T \mid p, q\rangle$, representing the vertex under consideration.

The unitarity condition gives us, to third order in e,

$$\text{disc. } \langle p+q \mid T^{(3)} \mid p, q\rangle = i(2\pi)^4 \mathrm{S} \langle p+q \mid T^{(1)+} \mid k, p_1\rangle \langle k, p_1 \mid T^{(2)} \mid p, q\rangle \delta^{(4)}(k+p_1-p-q), \tag{4.53}$$

where we can use

$$\langle p+q \mid T^{(1)+} \mid k, p_1\rangle = [e/(2\pi)^3](m/E_1 2\omega)^{1/2} \bar{\psi}(p+q)\, \mathbf{e}(k) u(p_1)$$

as the first-order vertex. The appropriate term for $T^{(2)}$ is one of the terms in the Compton scattering amplitude (4.8), namely,

$$\langle k, p_1 \mid T^{(2)} \mid p, q\rangle = -\frac{e^2}{(2\pi)^6} \frac{m}{(EE_1)^{1/2}} \frac{1}{(2\omega 2\omega_1)^{1/2}} \cdot \frac{\bar{u}(p_1)\, \mathbf{e}(q)(\mathbf{p}-\mathbf{k}+m)\, \mathbf{e}(k) u(p)}{(p-k)^2-m^2}.$$

The summation over spin and polarization can be carried out as usual; hence, we have, after integrating over p_1,

$$\begin{aligned}\text{disc. } \langle p+q \mid T^{(3)} \mid p, q\rangle = &\frac{ie^3}{(2\pi)^5} \left(\frac{m}{E}\right)^{1/2} \frac{1}{(2\omega)^{1/2}} \bar{\psi}(p+q) \int d^4k\, \delta(k^2-\lambda^2)\theta(k)\, \delta[(p+q-k)^2-m^2] \\ &\times \theta(p+q-k)[(p-k)^2-m^2]^{-1}([2\mathbf{e}(\lambda^2+2pq-m\mathbf{q})-4(p, e)\, \mathbf{q}+4m(p\cdot e)] \\ &+k_\mu\{2\mathbf{e}[-2(p+q)^\mu+\mathbf{q}\gamma^\mu]+4(p\cdot e)\gamma^\mu+4\mathbf{q}e^\mu-4me^\mu\}-4e^\mu\gamma^\nu k_\mu k_\nu)u(p). \end{aligned} \tag{4.54}$$

The integrations can be performed, thought they are quite lengthy; the results can be put into the following form:

$$\text{disc. } \langle p+q \mid T^{(3)} \mid p, q\rangle = \frac{ie^3}{(2\pi)^5} \left(\frac{m}{2\omega E}\right)^{1/2} \bar{\psi}(p+q)[A_1\mathbf{e}+B_1\mathbf{e}\mathbf{q}+C_1(p\cdot e)+D_1(p\cdot e)\, \mathbf{q}]u(p), \tag{4.55}$$

where

$$\begin{aligned} A_1 = &-\pi\theta[s-(m+\lambda)^2] \left(\frac{s+m^2-\lambda^2}{2s(s-m^2)} \{[s-(m+\lambda)^2][s-(m-\lambda)^2]\}^{1/2}\right. \\ &\left. -\frac{m^2-\lambda^2}{s-m^2} \ln \frac{s+m^2-\lambda^2+\{[s-(m+\lambda)^2][s-(m-\lambda)^2]\}^{1/2}}{s+m^2-\lambda^2-\{[s-(m+\lambda)^2][s-(m-\lambda)^2]\}^{1/2}}\right) \\ B_1 = &(\pi m/s)\theta(s-m^2) \\ C_1 = &-2m\pi[(3/2s)-(m^2/2s^2)]\theta(s-m^2) \\ D_1 = &\left\{\frac{\pi m^2}{s^2}+2\pi\left[-\frac{m^2}{(s-m^2)^2} \ln \frac{s}{m^2}+\frac{1}{s-m^2}\right]\right\} \theta(s-m^2). \end{aligned} \tag{4.56}$$

If one writes $\langle p+q \mid T^{(3)} \mid p, q\rangle$ as a linear combination of invariant amplitudes:

$$\langle p+q \mid T^{(3)} \mid p, q\rangle = \frac{e}{(2\pi)^3} \frac{\alpha}{2\pi} \left(\frac{m}{E}\right)^{1/2} \frac{1}{(2\omega)^{1/2}} \bar{\psi}(p+q)[A\mathbf{e}+B\mathbf{e}\mathbf{q}+C(p, e)+D(p, e)\, \mathbf{q}]u(p), \tag{4.57}$$

then A, B, etc., as a function of s, can be determined again through dispersion relations. For A, again a once-subtracted dispersion relation is required, as for the other vertex part discussed in Sec. IVD. A straightforward and tedious calculation gives,

$$\begin{aligned} A(s) &= \pi^{-1} \int \frac{A_1(s')\, ds'}{s'-s} - \pi^{-1} \int \frac{A_1(s')\, ds'}{s'-m^2} \\ &= 1+\ln(\lambda/m)-\tfrac{1}{2}[(2-\rho)/(1-\rho)] \ln \rho+I(\rho, \lambda), \end{aligned}$$

where

$$I(\rho,\lambda)\equiv\int_{(m+\lambda)^2}^{\infty}ds'\left(\frac{1}{s'-s}-\frac{1}{s'-m_2}\right)\left(-\frac{m^2-\lambda^2}{s'-m^2}\right)\ln\frac{s'+m^2-\lambda^2+\{[s'-(m+\lambda)^2][s'-(m-\lambda)^2]\}^{1/2}}{s'+m^2-\lambda^2-\{[s'-(m+\lambda)^2][s'-(m-\lambda)^2]\}^{1/2}},$$

$$B(s)=m^{-1}(\rho-1)^{-1}\ln\rho,$$

$$C(s)=-m^{-1}\{-(\rho-1)^{-1}+[(3\rho-2)/(\rho-1)^2]\ln\rho\},$$

$$D(s)=-\frac{1}{m^2}\left\{-\frac{1}{\rho-1}+\frac{2}{\rho}-\frac{(\rho-2)(2\rho-1)}{\rho(1-\rho)^2}\ln\rho-\frac{2}{\rho^2}[F(\rho-1)-F(-1)]\right\},\tag{4.58}$$

where

$$F(z)\equiv\int_0^z[\ln(1+u)/u]\,du$$

and

$$\rho=(m^2-s)/m^2.$$

The corresponding results are obtainable from field theory; the only place these results are given is by Akhiezer and Berestetskii.[18] The two results do agree except in $A(s)$ where it remains to be shown that the integral $I(\rho,\lambda)$ is equal to $1+\ln(\lambda/m)-\rho^{-1}[F(\rho-1)-F(-1)]$. This is presumably true, because if λ is set to be 0 before the integration is carried out, one does get $-\rho^{-1}[F(\rho-1)-F(-1)]$. However, keeping finite renders the integration quite difficult, and the identity of the two expressions remains to be shown. This is really of some importance, for this is the first instance in which the S-matrix theory and field theory do not give manifestly identical results.

G. Photon–Photon Scattering

We present in this section the procedures involved in deriving the scattering amplitude for photon–photon scattering from the S-matrix theory.[23] In the usual Feynman–Dyson theory, the calculation was quite cumbersome, since it is a fourth-order process. The corresponding calculations in the S-matrix theory can be carried out by following the same prescription we have been using. Knowing the amplitudes for pair creation and pair annihilation, the imaginary part of the photon–photon scattering amplitude can be obtained through the use of the unitarity condition.

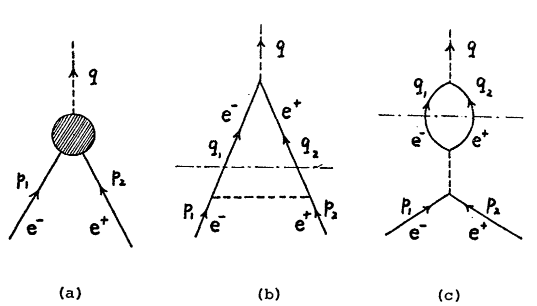


FIG. 5. Third-order vertex part for external electron lines.

[23] B. De Tollis, Nuovo Cimento **32**, 757 (1964).

By using dispersion relation, the calculation would follow the usual pattern. However, in view of the symmetry that this particular process possesses, the calculation can be carried out in a much simpler way in the S-matrix theory by using the Mandelstam representation and the Cutkosky rules, although the standard procedure would certainly work.

The Feynman diagram for the process is shown in Fig. 9, the corresponding amplitude is denoted by $M^{(1)}$. This is one of the six possible diagrams. Two others [with amplitudes denoted by $M^{(2)}$ and $M^{(3)}$] are obtained by the interchange $2\leftrightarrow4$, $3\leftrightarrow4$. The other three differ from these only by reversing the arrow direction in the closed loop; therefore they add nothing new, except a factor of two to the total amplitude.

The photons have four-momenta k_i, satisfying the conservation law:

$$\sum_i k_i=0\ (i=1,2,3,4).$$

The Mandelstam variables in this case are defined as

$$s=(k_1+k_2)^2/4,$$
$$t=(k_1+k_3)^2/4,$$
$$u=(k_1+k_4)^2/4,\tag{4.59}$$

where the factor $\frac{1}{4}$ is introduced for convenience, and s, t, u are connected by $s+t+u=0$. The total scattering amplitude M can be written as

$$M=M^{(1)}(s,t)+M^{(2)}(s,t)+M^{(3)}(s,t),$$

$M^{(2)}$ and $M^{(3)}$ are the contributions of the crossed diagrams, or equivalently

$$M=M^{(1)}(s,t)+M^{(1)}(u,t)+M^{(1)}(s,u).\tag{4.60}$$

Thus it is clear that M is totally symmetrical with respect to s, t, and u. $M^{(1)}$, in the Feynman–Dyson theory, is given by the following expression (apart from a constant):

$$M^{(1)}=i\int d^4p \,\mathrm{Tr}\left\{e_1\frac{p+m}{p^2-m^2}e_2\frac{(p-k_2)+m}{(p-k_2)^2-m^2}\right.$$
$$\left.\times e_3\frac{(p+k_1+k_3)+m}{(p+k_1+k_3)^2-m^2}e_4\frac{(p+k_1)+m}{(p+k_1)^2-m^2}\right\}. \quad (4.61)$$

This integration is difficult to perform.

In the following, for the sole purpose of illustrating the technique of the S-matrix theory, we assume the scattering takes place in a plane and the polarization vectors all point perpendicular to the scattering plane.

The whole problem is, as always, the determination of the amplitudes. The analyticity condition allows integral representations of these amplitudes. The integrands contain a special function whose determination now becomes the central problem. Cutkosky[9] has found general rules giving these spectral functions. For the special case of the square diagram, which is the one needed here, the rule just asserts that each one of the four lines shall be simultaneously put on the mass shell. In other words, the Feynman propagators in (4.61) must just be replaced by δ functions. Thus the Mandelstam double spectral function $A^{(1)}$ which occurs in $M^{(1)}$ can readily be written down as

$$A^{(1)}(s,t)=\int d^4p\,\mathrm{Tr}\,\{e(p+m)e(p-k_2+m)e(p+k_1+k_3+m)e(p+k_1+m)\}$$
$$\times\delta(p^2-m^2)\delta[(p-k_2)^2-m^2]\delta[(p+k_1+k_3)^2-m^2]\delta[(p+k_1)^2-m^2]. \quad (4.62)$$

This integral is easily performed with the result

$$A^{(1)}(s,t)=\tfrac{1}{2}[st(st-s-t)]^{-1/2}\left[-st+2\left(1-\frac{st}{s+t}\right)^2\right]. \quad (4.63)$$

Thus, the Mandelstam representation gives (with this value of $A^{(1)}(s,t)$

$$M^{(1)}(s,t)=\iint\frac{A^{(1)}(s',t')}{(s'-s)(t'-t)}ds'\,dt', \quad (4.64)$$

where the region of integration is defined by $st-s-t\geq 0$. Since (4.64) diverges, we are forced to use a once subtracted dispersion relation, i.e.,

$$M^{(1)}=C+s\int\frac{ds'}{s'}\frac{f(s')}{s'-s}+t\int\frac{dt'}{t'}\frac{f(t')}{t'-t}+st\iint\frac{A^{(1)}(s',t')}{s't'(s'-s)(t'-t)}ds'\,dt'. \quad (4.65)$$

The subtraction constant, C, is zero, for if s and t are both zero, one must expect that there is no scattering, hence M should vanish. The single spectral function f has the same form for both dispersion integrals as a consequence of the symmetry of M with respect to s and t. It follows from (4.65) that as $t\rightarrow 0$ one obtains the single spectral function $f(s)$:

$$f(s)=\pi^{-1}\,\mathrm{Im}\,M^{(1)}(s,t=0)$$
$$=\frac{2}{\pi}\int d^4p\frac{\frac{1}{4}\mathrm{Tr}[\cdots]}{(p^2-m^2)^2}\delta[(p+k_1)^2-m^2]\delta[(p-k_2)^2-m^2]. \quad (4.66)$$

This integral can again be performed without any difficulty, yielding

$$f(s)=\tfrac{1}{2}[1+(4/s)](1-s^{-1})^{1/2}-[1+(2/s)]\cosh^{-1}(s)^{1/2}. \quad (4.67)$$

With (4.63) and (4.67), $M^{(1)}$ can indeed be put into the same form as the results obtained by Karplus and Neuman[24] using perturbation theory. This is by no means surprising, since we know that the fourth order scattering amplitude satisfies the Mandelstam representation. Yet in the S-matrix theory approach, the computations are much simpler and more straightforward.

[24] R. Karplus and M. Neuman, Phys. Rev. **80**, 380 (1950); **83**, 776 (1951).

Incidently, we remark that although the transcendental functions defined in the De Tollis' paper appear formally different from the corresponding one in Karplus and Neuman's paper, they are in fact identical as can be shown by partial integrations.

H. Higher-Order Radiative Corrections to Scattering Processes

We have obtained the scattering amplitudes for various lowest-order scattering processes using only the basic principles cited in the last chapter. The higher-order scattering amplitudes can be generated from the lower-order ones by repeatedly applying unitarity and dispersion relations. So, leaving out the difficulties associated with having three or more partices in in or out states which renders the dispersion relation quite complicated (or practically impossible), we can, using our present techniques, calculate at least for any two-in two-out process to fourth-order without any difficulties. Thus, the fourth-order radiative corrections to any of the scattering processes discussed in Sec. IVA can be worked out. We pick out the Compton scattering as an example to calculate its fourth-order radiative corrections.

The dispersion diagrams for fourth-order Compton scattering in the s channel are given in Fig. 10.

The complete scattering amplitude in fourth order can again be obtained following the usual procedure. The unitarity condition asserts that

$$\begin{aligned}\text{disc. } \langle p_3, q_4 \mid T^{(4)} \mid p_1, q_2\rangle &= i(2\pi)^4\{\mathsf{S}\langle p_3, q_4 \mid T^{+(2)} \mid p_7, q_8\rangle \\ &\times\langle p_7, q_8 \mid T^{(2)} \mid p_1, q_2\rangle\delta^{(4)}(p_7+q_8-p_1-q_2) \\ &+\mathsf{S}\langle p_3, q_4 \mid T^{+(3)} \mid p_9\rangle\langle p_9 \mid T^{(1)} \mid p_1, q_2\rangle\delta^{(4)}(p_1+q_2-p_9) \\ &+\mathsf{S}\langle p_3, q_4 \mid T^{+(1)} \mid p_9\rangle\langle p_9 \mid T^{(3)} \mid p_1, q_2\rangle\delta^{(4)}(p_1+q_2+p_9)\}. \end{aligned} \tag{4.68}$$

Now, if the second-order Compton scattering amplitude, (4.8), which contains the "direct" and the "exchange" terms, is substituted into the first term on the right-hand side of the above equation, one gets four terms out of it. The "direct–direct" term corresponds to Fig. 10(a), the "direct–exchange" term to Fig. 10(d), the "exchange–direct" term to Fig. 10(c) and the "exchange–exchange" term to Fig. 10(f). The second and third terms on the right-hand side of (4.68) arise from the one-particle intermediate state, for which the generalized unitarity condition has to be used. The second term thus corresponds to Fig. 10(b), and the third term to Fig. 10(e). After the imaginary parts of the scattering amplitude corresponding to each diagram in Fig. 10 are obtained, the amplitudes themselves can again be obtained by a direct application of the dispersion relation, a procedure which is by now a routine one.

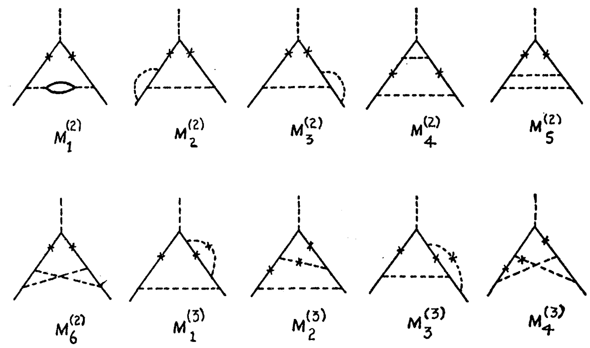


FIG. 6. Dispersion graphs in magnetic moment calculations.

The scattering amplitude for Fig. 10(a) has been obtained in Sec. IVB, (4.21); no more discussion is necessary. Figure 10(f) can be calculated in essentially the same way as Fig. 10(a), only a little bit more complicated. Actually Fig. 10(f) is a square diagram; one can again use, as in Sec. IVG, the Mandelstam representation and Cutkosky rules to simplify the calculation. One gets the same result as in perturbation field theory. Therefore, we do not present the calculation here.

Figures 10(b)–(e) are symmetrical. We demonstrate how the contributions from (b) and (c) can be added up to give the same result as in Feynman–Dyson theory.

The discontinuities of the scattering amplitude corresponding to Fig. 10(d) and (e) are denoted by disc $\langle p_3, q_4 \mid T_d^{(4)} \mid p_1, q_2\rangle$ and disc $\langle p_3, q_4 \mid T_e^{(4)} \mid p_1, q_2\rangle$, respectively.

disc $\langle p_3, q_4 \mid T_d^{(4)} \mid p_1, q_2\rangle$, which can be obtained from the unitarity condition using a two-particle intermediate state, has essentially the same form as (4,55) in Sec. IVF, except for the replacement $p\rightarrow p_1$, $q\rightarrow q_2$, and

$$\psi(p+q)\rightarrow -\frac{e}{(2\pi)^3}\left(\frac{m}{E_3}\right)^{1/2}\frac{1}{(2\omega_4)^{1/2}}\frac{\bar{u}(p_3)\mathbf{e}'(q_4)(\mathbf{p}_1+\mathbf{q}_2+m)}{(p_1+q_2)^2-m^2}.$$

Thus we have

$$\begin{aligned}\text{disc. } \langle p_3, q_4 \mid T_d^{(4)} \mid p_1, q_2\rangle = -\frac{ie^4}{(2\pi)^8}\frac{m}{(E_1E_3)^{1/2}}\frac{1}{(2\omega_2 2\omega_4)^{1/2}}\frac{1}{s-m^2}\bar{u}(p_3)\mathbf{e}'(q_4)(\mathbf{p}_1+\mathbf{q}_2+m) \\ \times[A_1(s)\mathbf{e}+B_1(s)\mathbf{e}\mathbf{q}_2+C_1(s)(p_1\cdot e)+D_1(s)(p_1\cdot e)\mathbf{q}_2]u(p_1). \end{aligned} \tag{4.69}$$

Disc. $\langle p_3, q_4 \mid T_e^{(4)} \mid p_1, q_2\rangle$ can be obtained from the "generalized unitarity condition" since it has a one-particle intermediate state. Hence,

$$\begin{aligned}\text{disc. } \langle p_3, q_4 \mid T_e^{(4)} \mid p_1, q_2\rangle &= i(2\pi)^4\mathsf{S}\langle p_3, q_4 \mid T^{+(1)} \mid p_q\rangle\langle p_q \mid T^{(3)} \mid p_1, q_2\rangle\delta^{(4)}(p_1+q_2-p_q) \\ &= \frac{ie^4}{2(2\pi)^7}\frac{m}{(E_1E_3)^{1/2}}\frac{1}{(2\omega_2 2\omega_4)^{1/2}}\delta(s-m^2)\bar{u}(p_3)\mathbf{e}'(q_4)(\mathbf{p}_1+\mathbf{q}_2+m) \\ &\times[A(s)\mathbf{e}+B(s)\mathbf{e}\mathbf{q}_2+C(s)(p_1\cdot e)+D(s)(p_1\cdot e)\mathbf{q}_2]u(p_1), \end{aligned} \tag{4.70}$$

where use is made of (4.68) and (4.57), and the summation and integration can be performed as usual.

Now, applying the dispersion relation to the scalar functions in (4.70), we immediately get,

$$\langle p_3, q_4 | T_e^{(4)} | p_1, q_2 \rangle = \frac{e^4}{2(2\pi)^8} \frac{m}{(E_1E_3)^{1/2}} \frac{1}{(2\omega_2 2\omega_4)^{1/2}} \frac{1}{m^2-s} \bar{u}(p_3) \mathbf{e}'(q_4)(\mathbf{p}_1+\mathbf{q}_2+m)$$
$$\times[A(m^2)\mathbf{e}+B(m^2)\mathbf{e}\mathbf{q}_2+C(m^2)(p_1\cdot e)+D(m^2)(p_1\cdot e)\mathbf{q}_2]u(p_1), \quad (4.71)$$

where $A(m^2)=0$, according to Sec. IVF.

If the once-subtracted dispersion relations are applied to the scalar functions in (4.69), we note that,

$$\int \frac{A_1(s')}{s'-m^2} \frac{ds'}{s'-s} = \frac{1}{s-m^2}\left[\int \frac{A_1(s')}{s'-s} ds' - \int \frac{A_1(s')}{s'-m^2} ds'\right]$$
$$= \frac{A(s)}{s-m^2}$$

and

$$\int \frac{B_1(s')}{s'-m^2} \frac{ds'}{s'-s} = \frac{1}{s-m^2}\left[\int \frac{B_1(s')}{s'-s} ds' - \int \frac{B_1(s')}{s'-m^2} ds'\right]$$
$$= \frac{1}{s-m^2}[B(s)-B(m^2)].$$

Similarly,

$$\int \frac{C_1(s')}{s'-m^2} \frac{ds'}{s'-s} = \frac{1}{s-m^2}[C(s)-C(m^2)]$$

and

$$\int \frac{D_1(s')}{s'-m^2} \frac{ds'}{s'-s} = \frac{1}{s-m^2}[D(s)-D(m^2)].$$

Hence

$$\langle p_3, q_4 | T_d^{(4)} | p_1, q_2 \rangle = -\frac{e^4}{2(2\pi)^8} \frac{m}{(E_1E_3)^{1/2}} \frac{1}{(2\omega_2 2\omega_4)^{1/2}} \frac{1}{s-m^2} \bar{u}(p_3) \mathbf{e}'(q_4)(\mathbf{p}_1+\mathbf{q}_2+m)$$
$$\times[A(s)\mathbf{e}+B(s)\mathbf{e}\mathbf{q}_2+C(s)(p_1\cdot e)+D(s)(p_1\cdot e)\mathbf{q}_2-B(m^2)\mathbf{e}\mathbf{q}_2-C(m^2)(p_1\cdot e)-D(m^2)(p_1\cdot e)\mathbf{q}_2]u(p_1). \quad (4.72)$$

Thus the contributions from Figs. 10(e) and (d) are

$$\langle p_3, q_4 | T_e^{(4)} + T_d^{(4)} | p_1, q_2 \rangle = -\frac{e^4}{2(2\pi)^8} \frac{m}{(E_1E_3)^{1/2}} \frac{1}{(2\omega_2 2\omega_4)^{1/2}} \frac{1}{s-m^2} \bar{u}(p_3) \mathbf{e}'(q_4)(\mathbf{p}_1+\mathbf{q}_2+m)$$
$$\times[A(s)\mathbf{e}+B(s)\mathbf{e}\mathbf{q}_2+C(s)(p_1\cdot e)+D(s)(p_1\cdot e)\mathbf{q}_2]u(p_1). \quad (4.73)$$

This is exactly the result of the Feynman–Dyson theory contributed by a Feynman diagram which looks like Fig. 10(d) or (e).

In exactly the same manner, one can get the contributions from Figs. 10(b)(c).

Now, if one adds up all the scattering amplitudes arising from Fig. 10, one would get Compton scattering amplitude in the fourth order only for the "direct" term. Again, it does not satisfy the crossing symmetry as we have seen in Sec. IVA1. To get the complete scattering amplitude, we have to add to the final result a term obtained from it by the replacement $e'\leftrightarrow e$ and $q_2\leftrightarrow -q_4$. Then one would get the complete Compton scattering amplitude in the fourth order which is in agreement with the field-theoretic perturbation results. (Compare Reference 1, p. 243.) However, it is again clear that, in the S-matrix theory, the calculations are straightforward, without renormalizations. And since we are dealing with physical in-and-out particles throughout, we do not have to (in fact are not allowed to) consider those diagrams corresponding to "radiative corrections of external lines." The physics seems to be more transparent in the S-matrix theory than the usual field-theoretic approach.

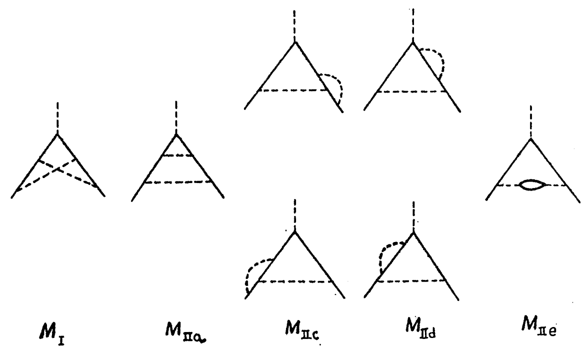
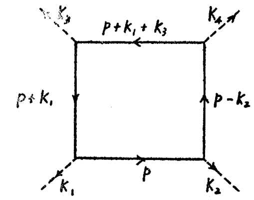


Fig. 7. Feynman diagrams contributing to the magnetic moment in second order in α.

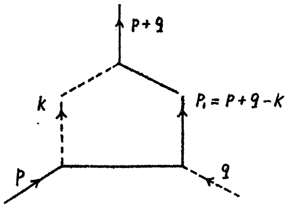
Fig. 9. Feynman diagram for photon–photon scattering.



V. THE COMPARISON OF THE S-MATRIX THEORY OF ELECTROMAGNETIC INTERACTIONS AND THE FEYNMAN–DYSON THEORY, DIFFICULTIES AND CONCLUSIONS

We have presented in the preceding sections various results which one can obtain from the S-matrix theory of the electromagnetic interactions. Although the starting points, or the basic principles, differ significantly from the usual formalism of quantum electrodynamics, all the results we obtained are in agreement. Here, in the S-matrix approach, we forsake altogether those notions used in the field-theoretic approach, such as the existence of state vectors and field operators, the concept of bare and dressed particles, as well as the program of renormalization. Recently, new formalisms[25,26] using field-theoretical approach without renormalization have been suggested to avoid some of these defects; comparison of these theories with the S-matrix theory is not immediately obvious. On the other hand, the S-matrix approach which we have presented bears a great similarity to the Feynman–Dyson approach. The agreement of results for lower order processes in both approaches is by no means accidental; in effect, the basic principles of the S-matrix theory have been extracted or suggested by field theory. Here, in the S-matrix approach, though no Lagrangian or Hamiltonian has been introduced, the interaction was effectively brought in by postulating the elementary interaction. Unitarity and analyticity principles, which the dynamics of the S-matrix theory is based on, again emerge from field theory. In fact, the usual theory is indeed both unitary and analytic in each order of e. Since we have expanded the scattering matrix element in power series of e, we can almost expect that the same results would emerge from the S-matrix theory as from the usual theory. However, in this approach, there are some merits. The calculations are in most cases, more simple and straightforward. In the S-matrix theory, we deal only with physical particles with their mass, charge, and other quantum numbers given. Thus, no renormalization is necessary in this program, although a similar, but certainly not identical, method, viz., the subtraction technique, is used to regulate the asymptotic behavior of certain functions. The subtraction technique is simple in concept, and easy to work with. In field theory renormalization can be classified as mass, charge, and wave function renormalizations. In the S-matrix theory, all these are absent as such, hence its physical basis is more straightforward. As far as low-order calculations (not higher than fourth order) are concerned, the S-matrix theory technique is more convenient (as can be seen from the various examples given in Sec. IV); this stems from the simple analytic structure of the lower-order Feynman diagrams.

Fig. 8. The third-order vertex part for a single external electron line.



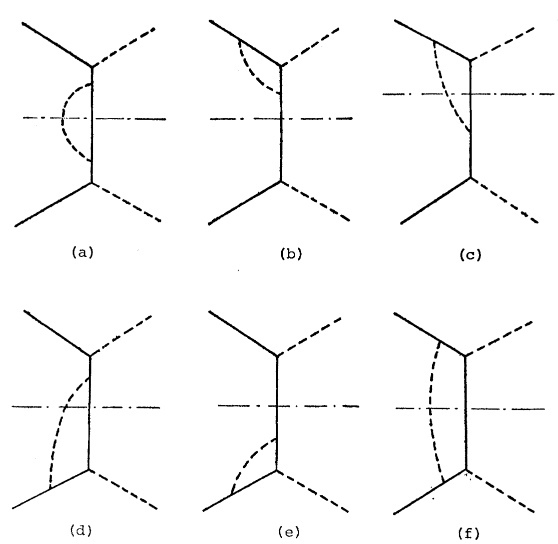


Fig. 10. Dispersion graphs for fourth-order Compton scattering in the s channel.

[25] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento **1**, 205 (1955); **6**, 319 (1957).
[26] R. E. Pugh, Ann. Phys. (N.Y.) **23**, 335 (1963).

It is undeniable that at the present stage the S-matrix theory of electromagnetic interactions has some difficulties. Unless these could be resolved, the theory cannot be properly regarded as "rigorous" or "complete."

(a) We require the scattering amplitudes to be analytic in the complex plane of kinematical variables. This is apparently an empty statement if the analytic properties are not *given* by the theory. For lower-order processes, these are known from perturbation theory by studying the Feynman diagrams corresponding to each order. We can therefore write down single or double dispersion relations satisfied by the scattering amplitudes. This is not true for higher-order processes, so the needed analyticity properties are not even known.

(b) Closely associated with the objections mentioned above, is the problem of handling many particle processes. Although this is a problem with the S-matrix theory in general, it does not plague strong interaction physics very much. There, in general, the two-in two-out processes are of primary importance; many-particle intermediate states are generally less important (at least this is assumed) in the approximation scheme used. In electromagnetic interactions, however, production processes for photons are important even at low energies, therefore, a practical way to handle these processes should be investigated.

It is certainly true that higher-order calculations in the Feynman–Dyson theory are more tedious and lengthier than the lower orders, but there is never any question of principle involved. However, the dispersion relations for more complicated processes are certainly profoundly different from those of two particles in and out. In fact, nobody really has constructed the analogue of the Mandelstam representation for two particles in and three particles out, let alone more complicated processes. This would not appear to be a mere increase in complexity—matters of principle are, by contrast to field theoretic situations, most likely involved.

(c) The problem of infrared divergences and soft photons. The problem of the infrared divergences is characteristic of the electromagnetic interaction. It arises from the fact that the photon has zero rest mass. In any finite resolution experiment, it is always energetically possible for an arbitrarily large number of low-energy photons to be emitted without being detected. This has been a problem in quantum electrodynamics for quite a long time. The essential difficulty associated with this problem comes from the fact that the probability of soft photon emission does not decrease sufficiently fast with an increasing number of photons. Thus, strictly speaking, an approach using successive approximation does not work for soft photons. However, some progress has been made using field-theoretic methods in recent years. One can indeed prove (Ref. 1, p. 390) that when the appropriate terms in the iteration solution are combined, the infrared divergences completely disappear for any process and any order. More recently, Yennie *et al.*[27] and Chung[28] have intensively studied this problem. They were able to factor the infrared singularities from the remaining expression. This is quite useful in estimating radiative corrections due to the emission of soft photons.

In the S-matrix theory of electromagnetic interactions, we encounter the same old problem of infrared divergence again as well as the emission of infinitely many soft photons. In all of our previous calculations, wherever an infrared divergence appeared, a small photon mass, λ, was included; the limit $\lambda \to 0$ was taken at the end of the calculation. However, we did not (and could not) show that the infrared divergence does in fact cancel for any process and to any order. Moreover, we do not have any idea of how to handle the soft photons. Yet the field-theoretic result does provide us with some clues as to how to solve the problem. It is likely that we can show that the factorization of the infrared singularity is also possible in the S-matrix theory. We hope to return to these questions including the bound-state problem in a later paper.

In conclusion, the S-matrix theory of electromagnetic interactions at the present is still far from satisfactory; however, because of its simplicity in carrying out practical calculations in lower orders and its well-defined conceptual basis, it is certainly profitable to pursue further studies. Even though the S-matrix approach can probably not replace the already well-established theory of quantum electrodynamics, it at least provides an alternate and physically different way of looking at the same phenomena.

[27] D. R. Yennie, S. C. Frautschi, and H. Surra, Ann. Phys. (N.Y.) **13**, 379 (1961).
[28] V. Chung, Phys. Rev. **140**, B1110 (1965).