Hamiltonian Path-Integral Methods

CLAUDE GARROD
University of California, Davis, California

A path-integral formulation of quantum mechanics is investigated which is closely related to that of Feynman. It differs from Feynman's formulation in that it involves the Hamiltonian function of the canonically conjugate coordinates and momenta. The classical limit yields the variational principle: $\delta\int(p\dot{q}-N)dt=0$. A path-integral formula is also obtained for the energy eigenstate projection operator associated with the time-independent Schrödinger equation. The classical limit of the projection operator formula yields a modified form of the well-known variational principle for the phase-space orbit of given energy. Relativistically covariant Hamiltonian variational principles are analyzed and lead naturally to a relativistic scalar wave equation which involves a proper time variable which is canonically conjugate to the mass in the same manner as the ordinary time variable is conjugate to the energy in nonrelativistic quantum theory.

I. INTRODUCTION

In 1948 the now well-known paper by R. Feynman[1] appeared, which presented a new formulation of nonrelativistic quantum mechanics. This formulation had three distinctive features.

(i) It concentrated attention on the propagator of the Schrödinger wave function rather than on the wave function itself, expressing this propagator (in a way to be described later) as an integral over all possible paths from one given point to another.

(ii) It introduced a new mathematical concept into quantum theory, namely the function space integral or "sum over paths."

(iii) It was directly related to Lagrangian rather than Hamiltonian classical dynamics. Thus the system to be quantized was not first described in terms of N coordinates and their canonical momenta.

The formulation presented in this paper is patterned after that of Feynman in that it retains the first two features mentioned above. It differs from Feynman's work in that it requires the description of the classical system in terms of canonical variables and in the classical limit reduces to a Hamiltonian variational principle. An advantage of the Hamiltonian formulation is that if the path integral is defined in the simplest and most obvious way there is no need to introduce special normalization constants to maintain the unitarity of the propagator. In the conversion from the Hamiltonian formulation to Feynman's formulation the correct normalization constants automatically appear. Besides the Hamiltonian path integral formulation for the time-dependent Schrödinger equation a path integral formula is also derived for the time-independent energy eigenvalue equation. The last section is devoted to the relativistic extension of the formulation; particularly to the derivation and interpretation of an equation for scalar particles which involves a proper time variable.

II. THE FEYNMAN METHOD

Since we are concerned with phase-space integrals throughout this paper it is convenient to choose units such that h (not $\hbar$), the natural unit of phase-space volume, is unity. The Schrödinger equation is then

$$-(8\pi^2m)^{-1}\nabla^2\psi+V(\mathbf{x})\psi=(i/2\pi)\,\partial_t\psi. \quad (2.1)$$

We define a propagator, $K(\mathbf{x}'', \mathbf{x}', t''-t')$ for the equation by stating that if $\psi(\mathbf{x}, t)$ is any solution (for all values of t) then

$$\psi(\mathbf{x}'', t'')=\int K(\mathbf{x}'', \mathbf{x}', t''-t')\psi(\mathbf{x}', t')\,d^3x' \quad (2.2)$$

for all t'' and t'. This is different from the more common practice of defining the propagator to be zero for negative values of $t''-t'$ but it proves more useful when discussing the time-independent Schrödinger equation. $K(\mathbf{x}'', \mathbf{x}', t)$ may be expressed in terms of the energy eigenfunctions by the equation

$$K(\mathbf{x}'', \mathbf{x}', t)=\sum_n \phi_n(x'')\phi_n^*(x')\,\exp(-2\pi iE_nt) \quad (2.3)$$

where the sum, which may actually include an integral if the spectrum is partly continuous, is over the complete set of normalized eigenstates. Using this formula we can see that

$$P(\mathbf{x}'', \mathbf{x}', E)=\int_{-\infty}^{\infty} K(\mathbf{x}'', \mathbf{x}', t)\exp(2\pi iEt)\,dt \quad (2.4)$$

$$=\sum_n \delta(E-E_n)\phi_n(\mathbf{x}'')\phi_n^*(\mathbf{x}')$$

is a projection operator for the time-independent states of energy E.

In his paper on nonrelativistic quantum mechanics, Feynman presented the following recipe for calculating $K(\mathbf{x}'', \mathbf{x}', t)$:

Let us denote by Ω the set of all continuous, piecewise differentiable functions $\mathbf{x}(t)$ which satisfy the conditions that $\mathbf{x}(t')=\mathbf{x}'$ and $\mathbf{x}(t'')=\mathbf{x}''$. For each function in Ω (which can obviously represent the trajectory of a particle moving from $\mathbf{x}'$ to $\mathbf{x}''$ in time interval t' to t'') we calculate the action integral

$$A[\mathbf{x}(t)]=\int_{t'}^{t''} L\,dt=\int_{t'}^{t''}\{\tfrac{1}{2}m\dot{x}^2(t)-V(\mathbf{x}(t))\}\,dt. \quad (2.5)$$

[1] R. Feynman, Rev. Mod. Phys. **20**, 367 (1948).

We define a method of integrating a functional over the set of functions Ω which allows us to write the propagator as

$$K(\mathbf{x}'', \mathbf{x}', t''-t') = \int_{\Omega} \exp\{2\pi i A[\mathbf{x}(t)]\}\, d[\mathbf{x}(t)]. \quad (2.6)$$

To define the function space integral let us consider a sequence of partitions of the interval $t' \leq t \leq t''$ such as used in defining the Riemann Integral. The Nth partition is given by choosing the $N+1$ points: $t_0 \equiv t' < t_1 < t_2 < \cdots < t_{N-1} < t_N \equiv t''$. For the Nth partition consider the piecewise linear function which has the values $\mathbf{x}_j$ at t_j. That is,

$$\mathbf{x}(t) = \mathbf{x}_{j-1}\left(\frac{t_j - t}{t_j - t_{j-1}}\right) + \mathbf{x}_j\left(\frac{t - t_{j-1}}{t_j - t_{j-1}}\right) \quad \text{for} \quad t_{j-1} \leq t \leq t_j, \quad (2.7)$$

where $\mathbf{x}_0 \equiv \mathbf{x}'$ and $\mathbf{x}_N \equiv \mathbf{x}''$.

In the Nth approximation to the functional integral we integrate over all such functions with a normalization constant which is chosen so that K_N approaches a unitary kernal. Such a normalization constant is

$$C_N = \prod_{j=1}^{N}\left(\frac{-im}{t_j - t_{j-1}}\right)^{\frac{3}{2}}.$$

Thus

$$K(\mathbf{x}'', \mathbf{x}', t''-t') = \lim_{N\to\infty}\left\{\prod_{j=1}^{N}\left(\frac{-im}{t_j - t_{j-1}}\right)^{\frac{3}{2}} \int d^3x_1 \cdots d^3x_{N-1} \exp\{2\pi i A[\mathbf{x}]\}\right\}. \quad (2.8)$$

In the classical limit, in which the potential changes only slightly over a DeBroglie wavelength of the particle, the only paths which contribute appreciably to the functional integral are those which are close to the classical path from $\mathbf{x}'$ to $\mathbf{x}''$ in time $t''-t'$. This is due to the fact that in this limit one may evaluate the integral by stationary phase and the trajectory which makes the phase stationary is the one for which $A[\mathbf{x}(t)]$ is stationary, but by Hamilton's Principle, this is just the classical path.

III. NONRELATIVISTIC HAMILTONIAN FORMULATION

The quantum mechanical path-integral formulas which we use are related to the following phase–space variational principles:

We consider a system described by N coordinates $q_1, \cdots, q_N$ and their canonical momenta $p_1, \cdots, p_N$. The system has a Hamiltonian $H(q, p)$. The first variational principle is an answer to the problem: Given that the total energy of the system is E and that at some unspecified times during its motion the coordinates of the system had the values $q_1', \cdots, q_N'$ and $q_1'', \cdots, q_N''$, what must have been the corresponding values of the canonical momenta, $p_1', \cdots, p_N'$ and $p_1'', \cdots, p_N''$ and what path through the system phase–space must have been taken in going from the point q', p' to q'', p''? At what rate the system point moved along the curve is not asked and is not obtained from the variational principle. The equation $H(q, p) = E$ defines a $(2N-1)$-dimensional energy surface. Consider the set of all curves on this energy surface which begin with coordinates q' and end with coordinates q''. For each such curve we calculate the integral:

$$I_0 = \int_{q'}^{q''} \sum p_j\, dq_j. \quad (3.1)$$

Any curve for which I_0 is an extremum is a solution to the problem posed.

The second variational principle is obviously related to the above but yields the explicit time dependence of the coordinates and momenta. Given that the coordinates of the system at times t' and t'' are, respectively, $q_1', \cdots, q_N'$ and $q_1'', \cdots, q_N''$ we consider the set of all phase–space trajectories $q(t)$, $p(t)$ which satisfy the given initial and final conditions with no restrictions on the energy or the initial and final values of the momenta. Among this set the path (or paths) which makes

$$I = \int_{t'}^{t''} \{\sum_j p_j \dot{q}_j - H(q(t), p(t))\}\, dt \quad (3.2)$$

an extremum is a solution of Hamilton's equations of motion with the specified initial and final conditions. We consider now the corresponding quantum mechanical principle.

We consider a system described by n coordinates $q_1, \cdots, q_n$ (denoted collectively as Q) and their canonical momenta $p_1, \cdots, p_n$ (denoted P). The probability that, at time t, the system has coordinates in the range d^nQ about Q is given by the absolute square of a complex valued amplitude $\psi(Q, t)$. $\psi(Q, t)$ satisfies an equation of the form:

$$\psi(Q', t') = \int K(Q', Q, t'-t)\psi(Q, t)\, d^nQ. \quad (3.3)$$

There may or may not exist a differential (or local) equation for ψ depending upon the form of the Hamiltonian function $H(Q, P)$. [The procedure of replacing p_j by $-i\hbar(\partial/\partial q_j)$ is reliable only in rectangular coordinates.] The essential element in the Hamiltonian formulation is the following prescription for calculating $K(Q', Q, t'-t)$.

We want to define an integral over the set of paths mentioned in the second Hamiltonian variational principle. In order to do this we consider a sequence of better and better approximations to any given path in terms of an ever increasing number of parameters. The Nth approximation is constructed by partitioning the time interval $t' \leq t \leq t''$ into N parts. The function $Q(t)$ is then approximated by a piecewise linear function

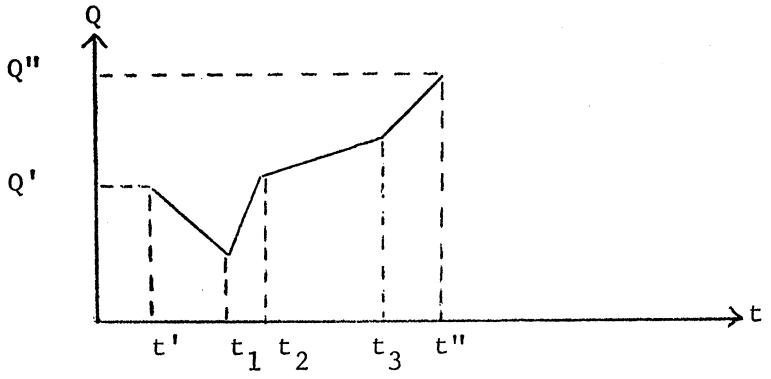
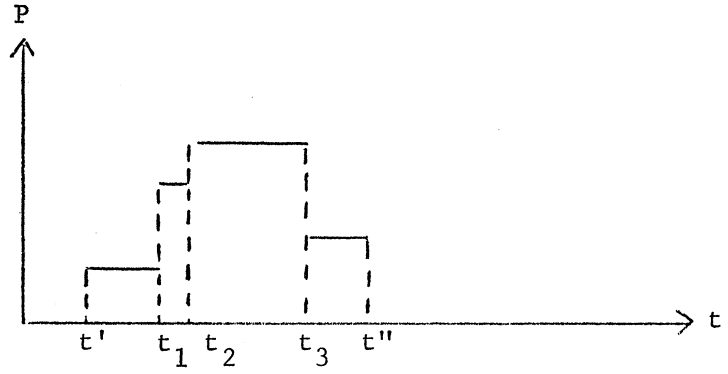


FIG. 1. Phase-space path, $Q(t)$, $P(t)$.



going from Q_{j-1} to Q_j in the interval t_{j-1} to t_j. (See Fig. 1.)

The function $P(t)$ is, in the same interval, approximated by a constant P_j. The piecewise constant but discontinuous approximation for the momentum is consistent with the fact that the velocity associated with out Nth approximation to $Q(t)$ is piecewise constant but discontinuous. For any functional $F[Q(t), P(t)]$ of the phase-space trajectory the integral over paths is defined as:

$$\int F[Q, P]d[Q(t)]d[P(t)]$$

$$\equiv \lim_{N\to\infty} \int F[Q, P]\, dQ_1 \cdots dQ_{N-1}\, dP_1 \cdots dP_N. \quad (3.4)$$

It will not be necessary to include any N-dependent normalization constant. With this definition the propagator $K(Q'', Q', t''-t')$ is given by

$$K(Q'', Q', t''-t') = \int \left\{\exp 2\pi i \left(\int_{Q'}^{Q''} P\, dQ - \int_{t'}^{t''} H\, dt\right)\right\} \times d[Q(t)]\, d[P(t)], \quad (3.5)$$

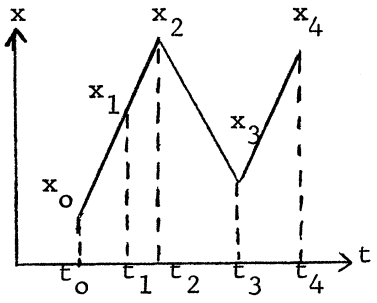
where

$$\int_{Q'}^{Q''} P\, dQ \equiv \int_{t'}^{t''} \left(\sum_{j=1}^{n} p_j \dot{q}_j\right) dt.$$

If Q is a rectangular coordinate and the Hamiltonian has the form.

$$H = (p^2/2m) + V(\mathbf{x}) \quad (3.6)$$

the above prescription may be readily converted into the Feynman formula. To do this we look at the Nth-order approximation for the integral over paths. The paths included are then as shown in Fig. 2 for $N=4$.



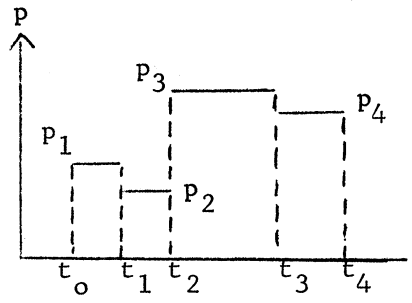


FIG. 2. Phase-space path, $x(t)$, $p(t)$.

Note: ($x_0=x'$, $x_N=x''$, $t_0=t'$, and $t_N=t''$). The action integral for such a path is

$$I=\int_{t'}^{t''}[\mathbf{p}\cdot\dot{\mathbf{x}}-(2m)^{-1}p^2-V(\mathbf{x})]\,dt$$

$$=\sum_{j=1}^{N}\left\{\mathbf{p}_j\cdot(\mathbf{x}_j-\mathbf{x}_{j-1})-(2m)^{-1}p_j^2(t_j-t_{j-1})-\int_{t_{j-1}}^{t_j}V\,dt\right\}. \tag{3.7}$$

We can now make use of the formulas:

$$\mathbf{p}_j\cdot\Delta\mathbf{x}_j-\frac{\Delta t_j}{2m}p_j^2=-\frac{\Delta t_j}{2m}\left(\mathbf{p}_j-\frac{m}{\Delta t_j}\Delta\mathbf{x}_j\right)^2+\tfrac{1}{2}m\frac{(\Delta\mathbf{x}_j)^2}{\Delta t_j} \tag{3.8}$$

and

$$\int_{-\infty}^{\infty}d^3q\exp\left[2\pi i\left(-\frac{\Delta t_j}{2m}q^2\right)\right]=\left(\frac{-im}{\Delta t_j}\right)^{\frac{3}{2}} \tag{3.9}$$

to obtain the relation

$$\int\left\{\exp\left[2\pi i\int(p\cdot\dot{x}-H)\,dt\right]\right\}d^3p_1\cdots d^3p_N\,d^3x_1\cdots d^3x_{N-1}$$

$$=\prod_{j=1}^{N}\left(\frac{-im}{t_j-t_{j-1}}\right)^{\frac{3}{2}}\int\left\{\exp\left[2\pi i\sum_{j=1}^{N}\left(\frac{m}{2}\frac{(\mathbf{x}_j-\mathbf{x}_{j-1})^2}{t_j-t_{j-1}}-\int_{t_{j-1}}^{t_j}V\,dt\right)\right]\right\}dx_1\cdots dx_{N-1}$$

$$=\prod_{j=1}^{N}\left(\frac{-im}{t_j-t_{j-1}}\right)^{\frac{3}{2}}\int\exp 2\pi i\left(A\left[x\right]\right)d^3x_1\cdots d^3x_{N-1}. \tag{3.10}$$

Thus the Nth order approximations to the functional integral in the two methods are identical. The derivation of the Schrödinger equation proceeds somewhat more easily if we do not first convert the Hamiltonian form of the path-integral formula into the Feynman formula since the latter contains terms involving negative powers of the time differences. In order to derive the equation we assume that when $\Delta t=t''-t'$ is sufficiently small we may approximate the functional integral by its first-order approximation. It is clear that such an assumption must be valid if the functional integrals are to converge at all as $N\to\infty$, which has been shown to be the case for a wide class of potentials. We can then write:

$$K(\mathbf{x}'',\mathbf{x}',\Delta t)=\int d^3p\exp 2\pi i(\mathbf{p}\cdot(\mathbf{x}''-\mathbf{x}')-(1/2m)p^2\,\Delta t-\bar{V}\,\Delta t), \tag{3.11}$$

where $\bar{V}$ is the average of $V(\mathbf{x})$ over the straight line connecting $\mathbf{x}'$ and $\mathbf{x}''$. Expanding the exponential to first order in Δt we obtain

$$K(\mathbf{x}'',\mathbf{x}',\Delta t)=\int d^3p(\exp 2\pi i\mathbf{p}\cdot(\mathbf{x}''-\mathbf{x}'))\times\left(1-2\pi i\,\Delta t\left[\frac{p^2}{2m}+\bar{V}\right]\right) \tag{3.12}$$

$$=\delta(\mathbf{x}''-\mathbf{x}')-2\pi i\,\Delta t[-(8\pi^2m)^{-1}\nabla^2\times\delta(\mathbf{x}''-\mathbf{x}')+V(\mathbf{x})\,\delta(\mathbf{x}''-\mathbf{x}')]$$

from which the Schrödinger equation easily follows.

IV. CANONICAL TRANSFORMATION

We begin our consideration of canonical transformations by looking at one of the simplest but most important examples, namely the transformation:

$$x\to p\qquad p\to -x.$$

With this transformation our equation for the propagator becomes

$$K(p'',p',t''-t')=\lim_{N\to\infty}\int dx_1\cdots dx_N\,dp_1\cdots dp_{N-1}\times\exp 2\pi i\left(-\int x\,dp-\int H\,dt\right). \tag{4.1}$$

If $H=(1/2m)p^2+V(x)$ and $\Delta t=t''-t'$ is small, then

$$K(p'',p',\Delta t)=\int dx\exp\left(-2\pi i\left[x(p''-p')+(2m)^{-1}\int p^2\,dt+\Delta tV(x)\right]\right). \tag{4.2}$$

If we assume that $V(x)$ has a power series expansion:

$$V(x)=V_0+V_1x+\cdots$$

we obtain

$$K(p'',p',\Delta t)=\left(1-2\pi i\,\Delta t\left[\frac{p^2}{2m}+V_0+V_1\left(\frac{i}{2\pi}\frac{\partial}{\partial p''}\right)+V_2\left(\frac{i}{2\pi}\frac{\partial}{\partial p''}\right)^2+\cdots\right]\right)\delta(p''-p') \tag{4.3}$$

which again yields the Schrödinger equation in its momentum representation. An interesting special case is that of a constant force, $V=-F_0x$, for in that case the first approximation is exact for finite time intervals yielding:

$$K(p'', p', t)$$
$$=\int dx \exp\left(-2\pi i\left[x(p''-p'-F_0t)+(2m)^{-1}\int p^2\,dt\right)\right]$$
$$=\delta(p''-p'-F_0t)\exp(-2\pi i)\int_0^t \frac{p^2}{2m}\,dt. \tag{4.4}$$

Before discussing nonlinear canonical transformations we should take note of the fact that, in contrast to the usual quantization techniques which involve the substitution of operators for classical scalar variables, the Hamiltonian path-integral formulation is completely unambiguous. The question therefore arises whether this procedure of quantization commutes with an arbitrary canonical transformation. That is, is one free to transform canonically the Hamiltonian into the most convenient form before calculating the sum over paths? If one is free to do so then we may take over into quantum theory the complete paraphernalia of canonical transformations including action-angle variables and the like. The question is at present unanswered. An example of the use of nonlinear canonical transformations is given in Appendix A where the harmonic oscillator is transformed to action-angle variables before quantization.

V. TIME-INDEPENDENT EQUATION

Using the relation between the propagator and the projection operator for the time-independent energy eigenstates it is rather easy to derive a path integral formula for the latter. We recall from Sec. II that

$$P(x'', x', E)=\int_{-\infty}^{\infty} K(x'', x', t)\exp(2\pi iEt)\,dt. \tag{5.1}$$

We may thus write an Nth-order approximation for P as:

$$P_N(x'', x', E)=\int dt\int dp_1\cdots dp_N\,dx_1\cdots dx_{N-1}$$
$$\times\exp 2\pi i\left[\sum p_j(x_j-x_{j-1})-\int_0^t H(x,p)\,dt'+Et\right]. \tag{5.2}$$

In order to extract explicitly the time dependence of

$$\int_0^t H(x,p)\,dt$$

let us choose for our Nth-order partition of the time interval a partition into N equal subintervals. We then describe the trajectory in terms of a parameter $\alpha\equiv t'/t$ which runs from zero to one. We also define:

$$\bar{H}\equiv\int_0^1 H(x,p)\,d\alpha \tag{5.3}$$

which is a function of $x_1, \cdots, x_{N-1}$ and $p_1, \cdots, p_N$ but not t. We may then use the fact that

$$\int_0^t H\,dt'=t\bar{H} \tag{5.4}$$

to evaluate the t integral and obtain

$$P_N(x'', x', E)=\int dp\cdots dp_N\,dx_1\cdots dx_{N-1}\,\delta(\bar{H}-E)$$
$$\times\exp 2\pi i\left[\int p\,dx\right]. \tag{5.5}$$

Thus P is obtained as an integral over those paths for which the average energy is E. This seems reminiscent of the first Hamiltonian variational principle of Sec. III but yet has a basic difference in that there we demanded that the trajectories remain always on the energy surface which is more stringent than the requirement that the average energy on the trajectory be E. We may now show that the usual form of the classical variational principle in which one considers only paths on the energy surface is unnecessarily restrictive and can be generalized so as to match our formula for the projection operator exactly.

We consider a set of phase–space paths described in terms of a parameter α running from zero to one. We demand that

$$x(0)=x', \quad x(1)=x'',$$

and

$$\int_0^1 H(x(\alpha), p(\alpha))\,d\alpha=E. \tag{5.6}$$

We then want to find the path for which the functional

$$A=\int p\,dx \tag{5.7}$$

is stationary with respect to variations within this set of paths. We do this by means of a Lagrange parameter, considering, instead of A, the functional

$$F_\lambda=\int_0^1 p(\alpha)\frac{dx}{d\alpha}\,d\alpha-\lambda\int_0^1 H(x(\alpha), p(\alpha))\,d\alpha. \tag{5.8}$$

We now drop the restriction involving E and set arbitrary variations (except for the end-point conditions) of F_λ equal to zero.

$$\delta F_\lambda=\int_0^1\left\{\delta p\frac{dx}{d\alpha}-\delta x\frac{dp}{d\alpha}-\lambda\frac{\partial H}{\partial x}\delta x-\lambda\frac{\partial H}{\partial p}\delta p\right\}d\alpha=0. \tag{5.9}$$

This yields

$$dx/d\alpha=\lambda(\partial H/\partial p)$$

and

$$dp/d\alpha=-\lambda(\partial H/\partial x) \tag{5.10}$$

as conditions for a stationary point. But we then see that, regardless of the value of λ, the rate of change of H along the path, namely,

$$\frac{dH}{d\alpha}=\frac{\partial H}{\partial x}\frac{dx}{d\alpha}+\frac{\partial H}{\partial p}\frac{dp}{d\alpha}=0. \tag{5.11}$$

Hence the variational principle for the classical phase-space orbit may be put into a form in which it bears the same relation to the quantum mechanical projection operator for the states of energy E as the classical variational principle for the time dependence of the phase point bears to the time-dependent quantum mechanical propagator.

As an example of the above formula (and the only simple example) we consider a one-dimensional free particle. The Nth-order approximation is then

$$P_N(x'', x', E)=\int dp_1\cdots dp_N\, dx_1\cdots dx_{N-1}\,\delta(E-\bar{H}_N) \times \exp 2\pi i\Big(\sum p_j(x_j-x_{j-1})\Big), \tag{5.12}$$

where

$$\bar{H}_N=N^{-1}\sum(p_j^2/2m).$$

By using the fact that

$$\sum p_j(x_j-x_{j-1})=p_Nx''-p_1x'+(p_1-p_2)x_1 +\cdots+(p_{N-1}-p_N)x_{N-1} \tag{5.13}$$

and doing the x_j integrations we obtain

$$\begin{aligned}P_N(x'', x', E)&=\int dp\;\delta\left(E-\frac{p^2}{2m}\right)\exp 2\pi ip(x''-x')\\&=[(2m)^{\frac{1}{2}}/E]\cos[2\pi(2mE)^{\frac{1}{2}}(x''-x')]\\&=\phi_p(x'')\phi_p^*(x')+\phi_{-p}(x'')\phi_{-p}^*(x'),\end{aligned} \tag{5.14}$$

where $\phi_p(x)$ is the properly normalized state of momentum $(2mE)^{\frac{1}{2}}$.

VI. RELATIVISTIC VARIATIONAL PRINCIPLES

Before we discuss the relativistic form of the Hamiltonian path integral formulation of quantum theory it would be useful to describe carefully the associated classical variational principles. In going from a nonrelativistic to a relativistic form of either the first or second Hamiltonian variational principles the most obvious thing to do is simply to modify the function $H(\mathbf{p}, \mathbf{x})$ so as to obtain the correct relativistic form of the equations of motion. For a particle of charge e in an electromagnetic field derived from the scalar and vector potentials A_0 and $\mathbf{A}$ a form of H which accomplishes the desired end is

$$H=[m^2+(\mathbf{p}-e\mathbf{A})^2]^{\frac{1}{2}}+eA_0. \tag{6.1}$$

Details of the calculation can be found in any text on relativistic mechanics. It has often been noted that when this Hamiltonian is employed in the variational principle:

$$\delta\int(\mathbf{p}\cdot d\mathbf{x}-H\,dt)=0. \tag{6.2}$$

The resulting theory is not explicitly covariant. It is possible to formulate this variational principle in a way which resembles the first Hamiltonian variational principle and which does have the property of being obviously covariant. In order to do this we first replace H by the symbol p_0 and solve Eq. (6.1) for the mass.

$$m=[(p_0-eA_0)^2-(\mathbf{p}-e\mathbf{A})^2]^{\frac{1}{2}}\equiv[(p-eA)^2]^{\frac{1}{2}}\equiv M(p, x). \tag{6.3}$$

The equation: $M(p, x)=m$ defines a surface in the eight-dimensional phase space of points (p, x). If we consider the set of phase-space trajectories which lie in that surface and also have the space-time points x' and x'' as end points then within this set the "invariant action integral"

$$A=\int_{x'}^{x''}(p\,dx) \tag{6.4}$$

is an extremum for the actual trajectory of a particle of mass m. Thus we see that the variational principle has the same form as the nonrelativistic one but has taken one step up in dimensionality while the three-dimensional scalar $H(\mathbf{p}, \mathbf{x})$ has been replaced by the four-dimensional scalar $M(p, x)$. We must notice however that it was the second Hamiltonian variational principle in three dimensions which was transformed into the first Hamiltonian variational principle in four dimensions. We presently discuss the hierarchy of first and second variational principles but first state and prove the four-dimensional form of the second Hamiltonian variational principle. We consider the set of phase-space trajectories described by a parameter τ which satisfy the endpoint conditions:

$x(\tau'')=x'$ and $x(\tau'')=x''$ where we assume x'' is in the future light cone of x'. For each such trajectory we calculate:

$$I=\int_{\tau'}^{\tau''}\left\{p(\tau)\frac{dx}{d\tau}-M(p(\tau), x(\tau))\right\}d\tau. \tag{6.5}$$

Unless $\tau''-\tau'>x_0''-x_0'$ there usually exists a trajectory for which I is an extremum. That trajectory satisfies the relativistic equations of motion for a particle of mass m where the value of m depends on the end-point

conditions imposed. Furthermore the path is such that the parameter τ is equal to the proper time along the path. To prove these assertions we set the variation of I equal to zero, obtaining:

$$\int_{\tau''}^{\tau'} \left\{ \delta p_0 \dot{x}_0 - \delta p_j \dot{x}_j - \delta x_0 \dot{p}_0 + \delta x_j \dot{p}_j - \frac{\partial M}{\partial p_0} \delta p_0 - \frac{\partial M}{\partial p_j} \delta p_j - \frac{\partial M}{\partial x_0} \delta x_0 - \frac{\partial M}{\partial x_j} \delta x_i \right\} d\tau = 0, \quad (6.6)$$

from which we get the following equations of motion:

$$\dot{x}_0 = \partial M/\partial p_0 \qquad \dot{x}_j = -(\partial M/\partial p_j)$$
$$\dot{p}_0 = -(\partial M/\partial x_0) \qquad \dot{p}_j = \partial M/\partial x_j \quad (6.7)$$

which immediately imply that

$$dM/d\tau = 0. \quad (6.8)$$

Thus M is a constant of the motion which we denote as m. For the specific form of M in the electromagnetic case; namely,

$$M = [(p - eA(x))^2]^{\frac{1}{2}} \quad (6.9)$$

the detailed equations of motion are

$$m\dot{x}_0 = p_0 - eA_0, \quad (6.10a)$$

$$m\dot{x}_j = p_j - eA_j, \quad (6.10b)$$

$$p_0 = \frac{e}{m}\left[(p_0 - eA_0) \frac{\partial A_0}{\partial x_0} - (p_j - eA_j) \frac{\partial A_j}{\partial x_0} \right], \quad (6.10c)$$

$$= e\dot{x}(\partial A/\partial x_0),$$

and

$$\dot{p}_j = e\dot{x}(\partial A/\partial x_j). \quad (6.10d)$$

These are the well-known equations of motion of a point charge of mass m which can easily be shown to imply that

$$d\tau = [(dx)^2]^{\frac{1}{2}}. \quad (6.11)$$

Let us return now to an analysis of the connections among this array of variational principles. In deriving the fact that M is a constant on the trajectory we made use of an assumption that the function M has no explicit dependence on the parameter τ. This assumption is all that is needed to derive the fact that M is a constant. Suppose, for given end-point conditions, we had deduced the value of the constant m. We would then be free to restrict our variational trajectories to those which lie totally within the surface $M = m$ secure in the knowledge that we were including the desired solution in our restricted set. This alone would not reduce our variational principle to one of the first kind because we would still have to maintain the end-point conditions. Now let us notice that within this restricted set, the functional we are minimizing can be written:

$$I = \int p\, dx - m(\tau'' - \tau'). \quad (6.12)$$

The second term, due to the fixed range of τ, is simply a constant, independent of the trajectory while the first term which must therefore be separately stationary is independent of the parameterization. We may therefore ignore the restrictions on the parameterization, drop the second term and demand that the first term be stationary within our restricted set. We thus obtain the four-dimensional form of the Hamiltonian variational principle of the first kind. Now it is easy to show that the four-dimensional form of the first Hamiltonian variational principle is identical with the three-dimensional form of the second Hamiltonian principle. One simply solves the equation $M = m$ for p_0 as a function of $\mathbf{p}$, $\mathbf{x}$, and x_0, containing m as a parameter. If one assumes that $M(p, x)$ and therefore $H(p, \mathbf{x}, \mathbf{x}_0)$ is independent of x_0 it is possible to carry the reduction in dimensionality one step further and in fact the process may be continued as long as cyclic coordinates can be found. The sequence of variational principles thus forms an overlapping hierarchy, the lower dimensional ones utilizing more and more constants of the motion and thus having smaller and smaller ranges of variation. We have gone into this long discussion of variational principles, not as an exercise in classical physics, but because we shall see that the quantum mechanical theory we obtain depends in an essential way upon the stage in this hierarchy at which we impose quantization. To be more explicit; if we extend the variational principle:

$$\delta \int (p\, dx - M\, d\tau) = 0 \quad (6.13)$$

to a formula for a quantum mechanical propagator, which as we shall see will describe particles of arbitrary mass, and then look at the mass eigenvalues of the theory we obtain an equation of motion different from that which we obtain if we directly quantize the fixed mass classical variational principle; namely,

$$\delta \int (\mathbf{p} \cdot d\mathbf{x} - H\, dt) = 0. \quad (6.14)$$

Before we attempt the quantization of these classical variational principles a few words are in order regarding the physical interpretation of quantum theories involving the proper time. Such theories have been discussed previously[2,3] but the introduction of τ into the theory has usually had a more formal character than it will here. We shall see that the variable τ is canonical to the mass which will be a dynamical variable rather than a scalar parameter in the theory. The relation between the proper time and the mass will be an almost exact analogy to the relation between the time and the energy in the nonrelativistic theory. To see how the proper time may be incorporated into the theory let

[2] R. Feynman, Phys. Rev. **80**, 440 (1950).
[3] Y. Nambu, Prog. Theoret. Phys. (Kyoto) **5**, 82 (1950).

us consider the following classical problem which is the daily concern of cosmic-ray physicists. Suppose we know that within a certain region of space–time a particle is created with a certain probability distribution in velocities. Assuming we know the electromagnetic fields present and assuming the particle is an unstable particle whose decay pattern we know we may ask what is the probability that the particle will decay in some other given region of space–time. In order to answer this question we construct an ensemble of particles which at $\tau=0$ has the given initial velocity distribution and follow this ensemble density $\rho(x,\tau)$ as τ progresses. The probability that the particle will decay in any given region of space–time is then proportional to the product of the amount of proper time it spends in that region and the probability that it reaches the region in question. The second factor depends upon the complete path which the particle takes in going from one region to the other but that fact introduces no difficulty into the classical analysis since the path (or paths) from one region to another are determined in detail. To determine the proper time of the particle if and when it reaches the region in question one simply integrates along the classical path from the initial region. There is no reason why one might not consider the same problem from a quantum mechanical viewpoint. (This is not to be confused with the field-theoretic calculations of decay processes where one calculates the decay pattern which is here assumed empirically known.) We assume a particle is created within some limited space–time region in a state $\psi(x)$ which within that region satisfies the Schrödinger equation. We may then use the Schrödinger equation to continue the wave function outside the region. We may therefore determine the probability that, if the particle were stable, it would be found at the space–time point x. (simply $|\psi(x)|^2$). But there is no satisfactory way from the Schrödinger equation point of view to assign a proper time value to the particle at x and this is essential for determining the decay probability. All knowledge of how the particle got from one point to the other has been lost. It seems to the author that one must approach this problem via a path-integral formulation. We now have the option of using a path integral formulation based on a variational principle which is not explicitly covariant and, for each path, calculating the proper time or using a path integral method which is related to the explicitly covariant variational principle which involves the proper time. That the two methods will not agree may be seen from the following considerations.

Up until now we have not carefully stated what our physical definition is of what we have been calling proper time. Although we specify the space–time location of a particle by a single point we shall always picture the particle as a dynamical system for which there is a variable which measures the internal evolution of the system. We shall call this variable the "particle time." If our particle were a miniature alarm clock this would be simply the reading on the face of the clock. Given a particular trajectory in space–time we may also define a "proper time" by taking the integral of $[(dx)^2]^{\frac{1}{2}}$ along the path. It is a principle of relativity that, at least for classical physics, these two definitions will coincide. We must realize, however, that in the path integral formula which is related to our variational principle there are finite contributions from nonclassical paths including those along which the parameter τ does not coincide with the integral of the proper time along the path. Thus, if we are to give any consistent physical interpretation to the parameter it seems most natural to interpret it as the dynamically defined particle time rather than the geometrically defined proper time. This would certainly agree with our use of the parameter in the unstable particle problem mentioned before.

We thus shall describe a particle by a wave function $\psi(x,\tau)$ which we interpret by stating that $|\psi(x,\tau)|^2$ is the probability that the particle will occupy space–time point x when the particle time is τ. We see that ψ satisfies an equation

$$(1/2\pi i)\,(\partial\psi/\partial\tau)=\hat{M}\psi, \tag{6.15}$$

where $\hat{M}$ is an integral operator which will shortly be derived from the path integral theory.

The extension of our classical variational principle to a quantum mechanical principle yields the propagator equation:

$$K(x'',x',\tau''-\tau')=\int\left\{\exp 2\pi i\left[-\int p\,dx+\int M\,d\tau\right]d[x(\tau)]\,d[p(\tau)]\right. . \tag{6.16}$$

(The sign in the exponential has been changed in order to ensure that the formula reduces to the nonrelativistic one in the appropriate limit.)

We consider this equation in some detail for the case of a free particle for which

$$M=[p_0^2-\mathbf{p}^2]^{\frac{1}{2}}.$$

Since we are integrating over all values of p we must carefully define the double-valued square root function. We do this by temporarily extending p_0 to a complex variable and then imposing the condition that, whenever the mass is real, it is positive. The p_0 integrations then take place over the contour shown in Fig. 3, where $p_0=u+iv$:

With this definition $M=i[\mathbf{p}^2-p_0^2]^{\frac{1}{2}}$ when $p_0^2<\mathbf{p}^2$. If in the Nth-order approximation to K we first do the integrals over $d^4x_1\cdots d^4x_{N-1}$ we find that due to the fact that M is not a function of x the Nth-order prop-

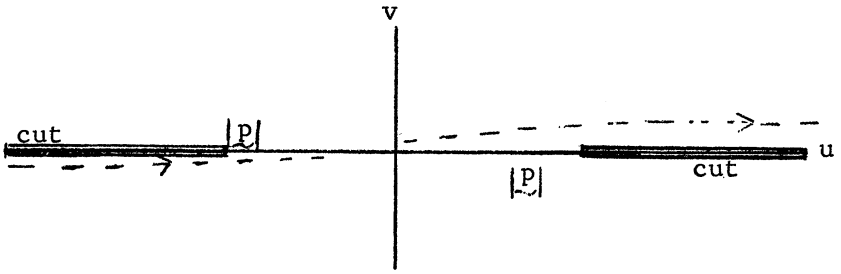


FIG. 3. Path of integration: $p_0=u+iv$.

agator reduces to the first-order propagator. Thus:

$$K(x'', x', \tau''-\tau') = \int d^4p \times \exp 2\pi i[-p(x''-x')+(\tau''-\tau')(p^2)^{\frac{1}{2}}]. \quad (6.17)$$

In contrast to the nonrelativistic case we are forced to consider only that propagator which is defined to be zero for negative $\tau''-\tau'$ since the space-like part of the momentum integration would strongly diverge for $\tau''-\tau'<0$. Let us now consider the plane wave solutions of the equation of motion:

$$\psi(x'', \tau'') = \int K(x'', x', \tau''-\tau')\psi(x', \tau')\, d^4x'. \quad (6.18)$$

If we assume ψ is of the form

$$\psi(x, \tau) = c \exp 2\pi i(-kx+m\tau) \quad (6.19)$$

we find that the equation is satisfied only if

$$m=(k^2)^{\frac{1}{2}}. \quad (6.20)$$

Consider now the behavior of a localized wave packet containing a small range of momentum components centered about the momentum k in the following three cases.

(i) $k_0>|\mathbf{k}|$. (Forward light cone) as τ increases the wave packet moves with a velocity $dx/d\tau=k/m$ where $m=(k^2)^{\frac{1}{2}}$.

(ii) $-k_0>|\mathbf{k}|$ (Backward light cone) as τ increases the wave packet moves with a velocity $dx/d\tau=k/m$ as before however since k_0 is negative we see that the time and the particle time increase in opposite directions along the trajectory. The physical interpretation of this phenomenon has been discussed in detail by Stückelberg[4] and R. Feynman.[2,5,6]

(iii) $|k_0|<|\mathbf{k}|$ (Space-like momentum) as τ increases the position in space–time of the wave packet does not change but its amplitude diminishes with decay constant $(k^2-k_0^2)^{\frac{1}{2}}$. Thus there do not exist propagating solutions of imaginary mass. By decomposing any solution of Eq. (6.15) into its plane wave components it can easily be seen that the integral operator $\hat{M}$ mentioned above has the kernal:

$$M(x, y) = \int d^4k\{[k^2]^{\frac{1}{2}} \exp 2\pi i(-k(x-y))\}. \quad (6.21)$$

The propagator $K(x, \tau)$ (where x is now written for $x''-x'$ and τ for $\tau''-\tau'$) satisfies an equation:

$$[(2\pi i)^{-1}(\partial/\partial\tau)-\hat{M}]K(x, \tau) = (2\pi i)^{-1}\,\delta(\tau)\,\delta(x), \quad (6.22)$$

where the $\delta(\tau)$ arises from the fact that K is defined to be zero for negative τ and the $\delta(x)$ is simply the limit of $K(x, \tau)$ as $\tau\to 0$.

In order to study the fixed mass theory we use the same method as was used to derive the time-independent theory from the time-dependent theory in the nonrelativistic case. Thus we define a function $G(x, m)$, where hereafter the dependence on m will not be explicitly indicated, by

$$G(x) = \int_0^\infty d\tau \exp[-2\pi i(m-i\epsilon)\tau]K(x, \tau) \qquad (\epsilon\to 0). \quad (6.23)$$

A small imaginary part has been added to the mass in order to make any purely oscillatory parts converge. As indicated the limit as $\epsilon\to 0$ is to be taken. Using this definition and the equation satisfied by K an equation for $G(x)$ may be derived in a straightforward manner. The equation is

$$MG-mG=-(1/2\pi i)\,\delta(x). \quad (6.24)$$

Thus we see that $G(x)$ is the propagator (or Green's function) for the mass eigenvalue equation. It is shown in Appendix B that, as might be expected, it is a propagator of the Feynman type. That is, for $x_0<0$ it has only negative energy components while for $x_0>0$ it has only positive energy components. We may develop a path-integral formula for $G(x)$ by a method identical to that used in the nonrelativistic case. The only significant difference is that here we integrate over the

[4] E. C. C. Stückelberg, Helv. Phys. Acta **15**, 23 (1942).
[5] R. Feynman, Phys. Rev. **76**, 749 (1949).
[6] R. Feynman, Phys. Rev. **76**, 769 (1949).

range $0<\tau<\infty$ rather than $-\infty<\tau<\infty$. We thus obtain the formula:

$$G_N(x)=\int d^4p_1\cdots d^4p_N\, d^4x_1\cdots d^4x_{N-1}$$

$$\times\left(\exp 2\pi i\left(-\int p\,dx\right)\Big/\bar{M}-(m-i\epsilon)\right), \quad (6.25)$$

where $\bar{M}$ is defined here in the same way as $\bar{H}$ of Eq. (5.3). The above integral is rather difficult to evaluate even in the free-particle case.

The eigenvalue equation:

$$(\hat{M}-m)\psi=0 \quad (6.26)$$

associated with the above Green's function is the eigenvalue equation for the operator M which is a sort of operator square root of the D'Alembertian operator. Thus the above equation is a square root of the Klein–Gordon equation:

$$\Box\psi=m^2\psi. \quad (6.27)$$

Square roots of this equation have been considered before but they have been attempts to eliminate the negative energies by first writing the equation in the form:

$$-\partial_0^2\psi=(-\nabla^2+m^2)\psi \quad (6.28)$$

and then taking the square root of the operators on both sides, obtaining

$$i\partial_0\psi=+(-\nabla^2+m^2)^{\frac{1}{2}}\psi. \quad (6.29)$$

Here we have, using the ideas of Stückelberg and Feynman, interpreted the negative energy states as antiparticles but have eliminated the negative masses. One might remark here that we could have avoided these square root operators by using a different form for the covariant Hamiltonian variational principle. One such form, found in the literature, is

$$\delta\int(-p\,dx+\mathfrak{K}\,d\tau)=0, \quad (6.30)$$

where

$$\mathfrak{K}=(p-eA)^2/2m.$$

However, the above variational principle is in direct conflict with an essential property of the covariant Hamiltonian principle which is that the mass is treated as an initial condition which is conserved only due to the fact that $M(p,x)$ is not explicit function of τ. The above variational principle would only yield a correct classical equation for special values of the end-point conditions and to use it as a starting point for path-integral quantization, where propagation along paths not on the mass shell is important, would be equivalent to using in the nonrelativistic theory the variational principle:

$$\delta\int(\mathbf{p}\cdot d\mathbf{x}-[H^2(\mathbf{p},\mathbf{x})/2E]\,dt)=0 \quad (6.31)$$

which would also yield the correct equations of motion for special values of the end-point conditions but would yield an entirely erroneous time-dependent Schrödinger equation which would contain the parameter E. Thus it seems that for scalar particles the Hamiltonian path integral method strongly suggests the validity of the particular form of square root equation used here. A much more important variation on what was done here is to take the Dirac square root of the Klein–Gordon equation; namely:

$$i\gamma^\nu\partial_\nu\psi=m\psi, \quad (6.32)$$

where

$$\gamma^\nu\gamma^\mu+\gamma^\mu\gamma^\nu=2g^{\nu\mu}.$$

This equation could be derived formally from a "classical" mass function:

$$M(p)=\gamma^\nu p_\nu \quad (6.33)$$

but the question of convergence of the path integral seems much more complicated than it was for the scalar equation due to the fact that γ^1, γ^2, and γ^3 are anti-Hermitian matrices. We shall not consider the Dirac equation here.

VII. ACKNOWLEDGMENTS AND FURTHER REFERENCES

The author gratefully acknowledges the support of the Crocker Nuclear Laboratory and the Atomic Energy Commission while this work was in progress. He also gladly thanks Professor J. Hurley for many suggestions, criticisms, and hours of invaluable conversation on the subjects of this paper.

After this work was completed the author came across a summary[7] of work done by R. Abe[8] on the calculation of the quantum mechanical partition function which is closely related to work presented here. The paper is especially valuable in that, after the transformation from temperature to time variable is made, it completely elucidates the connection between the concepts of the Hamiltonian path integral as the propagator for finite time and the Hamiltonian operator as the propagator for infinitesimal time. The nonrelativistic variational principles used here may be found presented clearly and in great detail by J. A. Wheeler.[9] A complete bibliography through 1960 can be found in the review article by S. G. Brush.[7]

APPENDIX A

We consider the special case of a harmonic oscillator with Hamiltonian

$$H=\tfrac{1}{2}(p^2+x^2). \quad (A1)$$

[7] S. G. Brush, Rev. Mod. Phys. **33**, 79 (1961).
[8] R. Abé, Busseiron Kenkyu **79**, 101 (1954).
[9] J. A. Wheeler, "Geometrodynamics and the Issue of the Final State," in *Relativity, Groups, and Topology*, edited by C. DeWitt (Gordon and Breach Science Publishers, Inc., New York, 1963).

The phase space for the system is the Euclidean x-p plane. Using the canonical transformation:

$$x=(2\rho)^{\frac{1}{2}}\cos\theta \qquad p=(2\rho)^{\frac{1}{2}}\sin\theta \tag{A2}$$

we may map all points in the plane except the origin onto the doubly connected region:

$$0<\rho<\infty \qquad 0\leq\theta\leq 2\pi \tag{A3}$$

where the points $(\rho, 0)$ are identified with the points $(\rho, 2\pi)$. The Hamiltonian in terms of these variables takes the trivial form

$$H=\rho. \tag{A4}$$

We want to derive an equation of motion for the wave function $\psi(\theta, t)$. In calculating the propagator for ψ we shall use the first-order approximation to the path integral since we are interested only in infinitesimal times. (Actually the result is good for all time.) In this approximation we are to take a straight-line path from θ' to θ'' but due to the double-connectedness there are an infinity of such straight-line paths. They must all be included in the first-order contribution. That this is necessary even for infinitesimal time can be seen from the fact that these paths would all make finite contributions to a stationary phase evaluation of the functional integral. Thus, if we denote by K_1 the first-order propagator, we have

$$K_1(\theta'', \theta', t)=\int_{\rho>0} d\rho \sum_{n=-\infty}^{\infty} \exp 2\pi i[\rho(\theta''-\theta'+2\pi n)-\rho t]. \tag{A5}$$

Using the relation:

$$\sum_{n=-\infty}^{\infty} \exp(2\pi i\alpha n)=\sum_{K=-\infty}^{\infty} \delta(\alpha-K) \tag{A6}$$

we obtain

$$K_1(\theta'', \theta', t)=(2\pi)^{-1}\sum_{K=1}^{\infty} \exp iK(\theta''-\theta'-t). \tag{A7}$$

This equation indicates that the values of $\psi(\theta, t)$ considered as mapped on a unit circle rotate with constant angular velocity in complete analogy with the classical phase-space density. The energy spectrum has the same structure as the well-known harmonic oscillator spectrum but is shifted by a constant. ($\frac{1}{2}\hbar\omega$ in the usual notation.) One could dismiss this shift with the observation that the addition of a constant to the classical Hamiltonian is itself a canonical transformation but it does seem to lessen the likelihood that a free use of canonical transformations before quantization is allowed. As $t\to 0$, $K(\theta'', \theta', t)$ does not approach $\delta(\theta''-\theta')$ but it does approach the unit operator on the subspace of positive energy states. That the first-order propagator in this case is exact for finite time can easily be seen from the fact that it satisfies the equation:

$$K_1(\theta'', \theta', t)=\int_0^{2\pi} d\theta K_1(\theta'', \theta, t-\tau)K_1(\theta, \theta', \tau).$$

APPENDIX B

Given the τ-dependent propagator:

$$K(x, \tau)=\int d^4p \exp 2\pi i[-px+\tau(p^2)^{\frac{1}{2}}].$$

We want to investigate the properties of the Green's function (of the mass eigenvalue equation) defined by

$$G(x, m)\equiv\int_0^{\infty} d\tau K(x, \tau) \exp[-2\pi i(m-i\epsilon)\tau].$$

Carrying out the τ integration and defining a variable $q\equiv|\mathbf{p}|$ we obtain:

$$G=\frac{i}{2\pi}\int d^3p \exp(2\pi i\mathbf{p}\cdot\mathbf{x})I(q)$$

where

$$I(q)=\int_C dp_0 \exp(-2\pi i x_0 p_0)/(p_0^2-q^2)^{\frac{1}{2}}-(m-i\epsilon)$$

and the contour C is shown in Fig. 4.

The denominator vanishes at two points infinitesimally inside the second sheet at which: $u=\pm(m^2+q^2)^{\frac{1}{2}}$.

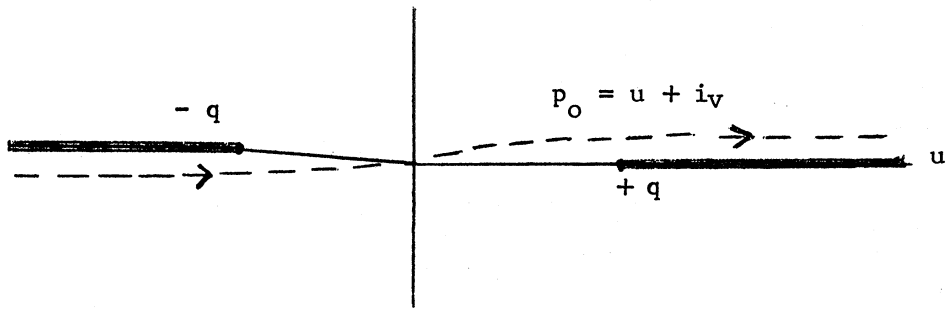


FIG. 4. Path of integration.

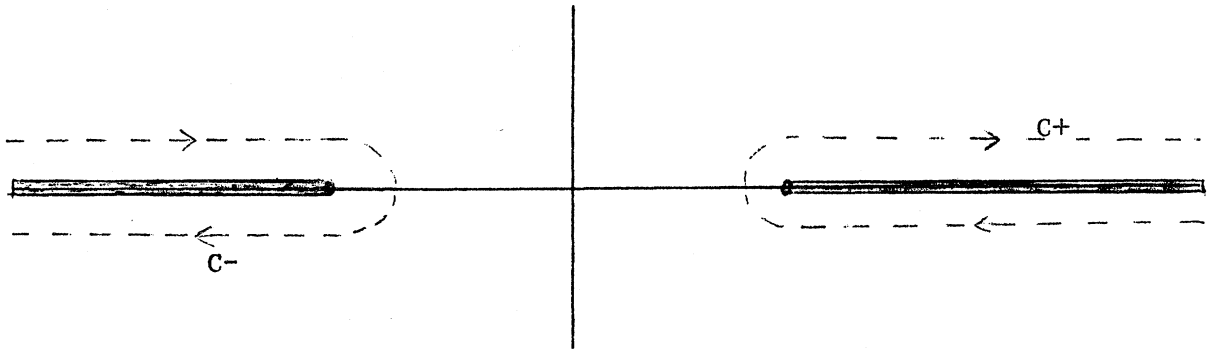


FIG. 5. Deformed paths of integration.

If $x_0>0$ the contour C may be deformed into the contour $C+$ shown in Fig. 5, while if $x_0<0$ the contour C_4^* may be deformed into the contour $C-$.

Thus, for $x_0>0$, the integral may be converted into one involving only positive frequencies, while for $x_0<0$ it may be converted into an integral over negative frequencies. The author has not succeeded in obtaining an explicit evaluation of the integrals.

Note added in proof. Since this article was submitted for publication, the author has become aware of some earlier work done on the subject. The earliest derivation of the Hamiltonian path-integral method can be found in Appendix B of R. Feynman, Phys. Rev. **84,** 108 (1951). See also: W. Tobocman, Nuovo Cimento **3,** 1213 (1956); and H. Davies, Proc. Cambridge Phil. Soc. **59,** 147 (1963).

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High-Frequency Conductivity of a Solid-State Plasma with Dynamically Screened Interactions. I. Thermal Equilibrium

D. E. McCUMBER
Bell Telephone Laboratories, Murray Hill, New Jersey

Dawson–Oberman high-frequency long-wavelength conductivities of solid-state plasmas in thermal equilibrium are derived using a variational formalism. These derivations are different from those of other authors and give additional insight into the nature of the approximations implicit in their results. The final expressions reflect screened carrier–carrier and carrier–phonon interactions. They extend published conductivity formulas principally in their inclusion of deformation-potential phonon–carrier coupling.

1. INTRODUCTION

It has been demonstrated by Perel' and Eliashberg[1] and by Dawson and Oberman[2] how dynamic electron screening of static ions affects the high-frequency conductivity of a classical one-component plasma. Their results have been generalized to multicomponent systems in which the scattering centers are themselves dynamic charge carriers,[3,4] to degenerate systems in which the carriers are described by quantum statistics,[5,6] and to solid systems in which the carriers are scattered by polar lattice vibrations (polar phonons).[7,8] In this paper we derive general conductivity expressions which include all of these effects simultaneously and which in addition include the effects of carrier scattering by phonons coupled to the carriers through a dynamically screened deformation potential. Our method of derivation, valid for quantum and for classical statistics, is different from that used by previous authors[1–8] and gives additional insight into the nature of the approxi-

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