A Refutation of the Proof by Jauch and Piron that Hidden Variables Can Be Excluded in Quantum Mechanics

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In this paper, we examine an argument of Jauch and Piron, which aims to prove the impossibility of hidden variables underlying the quantum theory, on the basis of certain assumptions that are weaker than those used by von Neumann for the same purpose. We show that, while the assumptions of Jauch and Piron are in fact weaker than those of von Neumann, the net result is that they actually prove nothing new at all. The conclusions of Jauch and Piron concerning the nonexistence of hidden variables are indeed seen to follow from a false assumption; i.e., that the impossibility of propositions that describe simultaneously the results of measurements of two noncommuting observables is an "empirical fact." Actually, it is shown that this assumption follows if and only if one first assumes what the authors set out to prove; i.e., that the current linguistic structure of quantum mechanics is the only one that can be used correctly to describe the empirical facts underlying the theory.

INTRODUCTION

In a previous paper[1] we analyzed the classic von Neumann proof of the impossibility of a "hidden variable" theory underlying quantum mechanics and found that the argument excluded only a certain very restricted class of hidden variable theories. A more general theory, which went beyond the assumptions of von Neumann, was developed, and it was shown that the usual results of quantum mechanics could be recovered in certain limiting cases. In particular, we showed that one need not make the linearity assumption of von Neumann, i.e., that for any real linear combination of observables, whether simultaneously measurable or not, the expectation value of the sum in an ensemble of measurements is equal to the sum of the expectation values. This property follows as a theorem in quantum mechanics, but it is quite possible to construct theories in which this relation does not hold in general but is true only in the case of specially constructed ensembles.

A more recent impossibility proof has been proposed by Jauch and Piron,[2] in which they come to the same conclusion as von Neumann without using this assumption. We examine their proof here.

SUMMARY OF THE ARGUMENT

The argument by which Jauch and Piron attempt to prove that the structure of the quantum theory is incompatible with the assumption of hidden variables is based on an analysis of the type of experimental question that can be asked in the theory. Thus, they consider those observables of a physical system which are associated with only two alternatives or possibilities, which may be designated by 1 or 0, yes or no, true or false, and are represented in quantum mechanics by the projection operators. The results of such yes–no experiments, for example, that a certain observable P has the value P', or that the value of P is positive, etc., are referred to as the *propositions* of the system, and these are classified as *compatible* or *incompatible*. depending on whether or not the corresponding measurements can be performed simultaneously. Jauch and Piron refer to a paper by Birkhoff and von Neumann[3] on "The Logic of Quantum Mechanics," in which these authors formally develop a logical calculus of propositions for quantum mechanics, based on the relation between the propositions of a quantum mechanical system and the associated projection operators. In contrast to the usual propositional calculus of ordinary logic, applicable to classical mechanics, this system is extended by the concept of "incompatibility" characteristic of quantum mechanics. The contribution of Jauch and Piron is to prove as a theorem in this calculus that if a propositional system admits hidden variables then all propositions are compatible. This is the conclusion of Theorem I and Theorem II (proved under slightly weaker assumptions) in their paper. Since the propositions of a quantum mechanical system are not all compatible, because there are certain measurements which cannot be carried out on the system simultaneously, the possibility of hidden variables underlying the quantum theory is rejected. This rejection does not make use of von Neumann's linearity assumption (see Ref. 1, Sec. 4), which has been shown by various authors to be unnecessarily restrictive.

Now, at first sight, it is not at all clear why the authors should wish to use the terminology of logic to describe the experimental questions of a physical theory, unless they intend to propose a deep connection between all human reasoning and the theory. If this is the case,

[1] D. Bohm and J. Bub, Rev. Mod. Phys. **38**, 453 (1966).
[2] J. M. Jauch and C. Piron, Helv. Phys. Acta **36**, 827 (1963).
[3] G. Birkhoff and J. von Neumann, Ann. Math. **37**, 823 (1936).

then the impossibility of developing a hidden variable theory for microsystems could be traced to a limitation in our thought processes. Unfortunately, the treatment of this question is rather vague, especially with regard to the significance of "incompatible propositions" in the general sense, and a careful analysis of the argument will be necessary in order to clarify what is in fact implied.

As we see, however, such an analysis indicates that Jauch and Piron are really only making assumptions about physics and that the suggested relationship with logic is both artificial and misleading. If this is so, then it is important to discover what these physical assumptions are and to consider whether they are well founded. In this connection, a key point for consideration is the claim that the existence of incompatible propositions is an "empirical fact." Such incompatible propositions could be taken as necessary inferences from experiment, however, only if it could be established that no other propositions besides those of quantum mechanics are applicable to a microsystem, from which it would follow that the only relevant experimental questions are those of quantum mechanics. But this is in essence just what Jauch and Piron have set out to prove in their efforts to exclude hidden variables from the quantum theory. For if there are such hidden variables, then (as we see in more detail in the present paper) they do make possible a language in terms of which relevant experimental questions do not involve any *incompatible propositions*, but only incompatible processes of measurement, in which the actions needed in making one kind of measurement may interfere physically with those needed for making another kind of measurement. In this way, it becomes clear that the basic assumption of Jauch and Piron is that the conceptual structure of all the laws of physics is the same as that of the laws of current quantum mechanics and nothing else. But it has long been fairly clear that without extending or otherwise altering the basic conceptual structure of the laws of quantum theory, no hidden variables can be introduced into the theory. So in fact, Jauch and Piron have actually proved essentially nothing new at all. This fact may, however, be obscured by the use of a logical terminology for the description of the properties of physical systems, which might perhaps at first sight cause the old conceptual structure to appear as if it had a new meaning.

We now proceed to a more detailed examination of the argument.

THE LOGIC OF QUANTUM MECHANICS

As we have indicated, Jauch and Piron propose that the propositions of a quantum mechanical system are represented in the formalism by projection operators. Each projection operator $\mathbf{P}_a$ defines a certain closed subspace V_a of the Hilbert space $\mathcal{H}$, spanned by the eigenvectors of V_a with eigenvalue 1. The (normalized) vectors of V_a all represent states of the system for which the proposition a is true with certainty, whereas for those states represented by the vectors orthogonal to the members of V_a the proposition a is certainly false. If $\mathbf{P}_a$, V_a belong to the proposition a, then it follows that $\mathbf{1}-\mathbf{P}_a$, $\mathcal{H}-V_a$ belong to the denial of a, the proposition "not a," denoted by a'. If the proposition a is always true, then $\mathbf{P}_a=\mathbf{1}$, $V_a=\mathcal{H}$. If the proposition a is never true, i.e., impossible, then $\mathbf{P}_a=\mathbf{0}$, $V_a=0$. The trivially true proposition is denoted by I and the absurd or trivially false proposition by ϕ.

If the projection operators $\mathbf{P}_a\cdot\mathbf{P}_b$ of the propositions a, b commute, then the additional propositions "a and b" and "a or b" can be formed. The proposition "a and b," denoted by $a\cap b$, is true if the propositions a and b are both true, and it is false if one of them is false. The corresponding projection operator is $\mathbf{P}_a\cdot\mathbf{P}_b$, and the closed subspace is the set of vectors common to both V_a and V_b. "a or b," denoted by $a\cup b$, can be expressed as "not [(not a) and (not b)]," or $(a'\cap b')'$, and its operator is therefore, $\mathbf{1}-(\mathbf{1}-\mathbf{P}_a)(\mathbf{1}-\mathbf{P}_b)=\mathbf{P}_a+\mathbf{P}_b-\mathbf{P}_a\cdot\mathbf{P}_b$, which is also a projection operator. The associated closed subspace is defined by the set of all vectors which are linear combinations of a vector from V_a and a vector from V_b.

As long as all operators commute, then the use of projection operators to represent the answers to yes–no questions that can be "asked" in an experiment is merely a formal representation of Boolean algebra of ordinary logic in terms of quantum mechanical operator theory. However, if the field of operators is extended to include noncommuting projection operators, then new features are introduced. In terms of the usual quantum mechanical language, one says that when two operators do not commute, then the corresponding measurements cannot both give rise to precise and unambiguous results simultaneously. Jauch and Piron transcribe this notion into their own language by asserting that the *propositions* corresponding to the operators in question are *incompatible.*

Here it is necessary to consider carefully whether this change in the language in fact implies a change in the physical assumptions underlying the theory, or whether what has been introduced is merely a new terminology for the treatment of the same assumptions. If what is involved is merely a change in terminology, then, of course, the authors cannot really have gone beyond the result of von Neumann, the inadequacy of which has been demonstrated in our previous article. On the other hand, if Jausch and Piron have actually introduced new physical assumptions going beyond those of the usual (probabilistic) interpretation of the quantum theory, incorporated in the formulation of von Neumann, then the question arises as to whether or not these assumptions are justified in the context of the proof.

In explaining the meaning of their terminology, Jauch and Piron define the concept of compatibility, characterizing the propositions corresponding to commuting projection operators, as a certain formal relationship between pairs of propositions. Two propositions a and b are said to be compatible if they satisfy the relation,

$$(a\cap b')\cup b=(b\cap a')\cup a, \tag{1}$$

which is equivalent to the property that the projection operators $\mathbf{P}_a$, $\mathbf{P}_b$ commute. It is clear that in the case of incompatible propositions, corresponding to noncommuting projection operators, the intersection $a\cap b$ and union $a\cup b$ of propositions will lead almost always to the absurd proposition ϕ and the trivially true proposition I, respectively. If $\mathbf{P}_a$ and $\mathbf{P}_b$ do not commute, then, although there may exist simultaneous eigenvectors of both projection operators, such vectors can not form a complete set. In general, the subspace $V_{a\cap b}$ will be empty and $V_{a\cup b}$ will be the whole space, so that,

$$a\cap b=a'\cap b=a\cap b'=a'\cap b'=\phi$$

and

$$a\cup b=I. \tag{2}$$

Now, it is demonstrated in the paper by Birkhoff and von Neumann that the propositions of a physical system form an ortho-complemented lattice—i.e., essentially a set with a partial ordering relation, a greatest lower bound (and least upper bound) for every nonempty set of propositions and an operation with the formal properties of the negation in the propositional calculus of ordinary logic. The partial ordering relation is defined as the relation of implication,

$$a\subseteq b \quad \text{if, whenever } a \text{ is true, } b \text{ is also true.} \tag{3}$$

However, although the propositions of classical systems (represented by functions on the phase space with the two values 1 and 0) obey the distributive law:

$$a\cap(b\cup c)=(a\cap b)\cup(a\cap c)$$

and

$$a\cup(b\cap c)=(a\cup b)\cap(a\cup c), \tag{4}$$

the lattice of quantum propositions is nondistributive. Birkhoff and von Neumann quote the following example to demonstrate this point:

If a denotes the experimental observation of a wave packet ψ on one side of a plane in ordinary space, a' correspondingly the observation of ψ on the other side and b the observation of ψ in a state symmetric about the plane, then

$$\begin{aligned} b\cap(a'\cup a)=b\cap I=b\supset\phi=(b\cap a)=(b\cap a') \\ =(b\cap a)\cup(b\cap a') \end{aligned} \tag{5}$$

(where the notation $b\supset\phi$ means $b\supseteq\phi$ but not $b=\phi$). The fact that the lattice of quantum propositions is nondistributive (which is evidently a direct consequence of the existence of propositions which are "incompatible") means that, whereas the propositional calculus of classical mechanics forms a Boolean algebra, the propositional calculus of quantum mechanics does not.

At this point, one must be careful to avoid a kind of confusion that tends to result from the replacement of the usual term "experimental questions" by the term "proposition." The fact that different experimental questions cannot all be answered simultaneously is, of course, characteristic of the quantum theory (whereas in the classical theory the corresponding questions can all be answered together). But by using the term "proposition" instead of "question" and by asserting that propositions corresponding to noncommuting operators are "incompatible," the authors would seem, at least at first sight, to be suggesting that their basic assumptions refer, not to particular physical theories such as classical or quantum mechanics, but rather to the structure of the *logic* that applies in the whole field of human reasoning. If there is in fact such a perfect correspondence between the possible structures of all logical reasoning and the structures of these two particular physical theories, then to establish this as true would indeed be a very revolutionary discovery, in all probability far transcending both classical and quantum mechanics in its ultimate significance. Unfortunately, the authors do not make it very clear what their intentions are with regard to this point. Are they proposing, for example, that in ordinary reasoning one could extend the operations of intersection and union to "incompatible" propositions, similar to those represented by noncommuting operators? As we have remarked, this would lead almost always to the absurd or trivially true proposition; such a proposal does not, therefore, appear to be particularly meaningful. It is surely significant here that Jauch and Piron and Birkhoff and von Neumann continue to employ the conventional "classical" logic in their own reasoning and make no attempt to persuade the reader by a non-Boolean argument. From this fact, it seems fairly safe to infer that the authors do not actually wish to suggest that their assumptions about "propositions" should apply generally in the whole field of human logic, but rather, they would seem to be tacitly restricting such assumptions to particular physical theories.

If this is so, then the replacement of the term "experimental question" by the term "proposition" can in itself add nothing new to the structure of the theory and has the disadvantage of tending to lead the reader into a kind of confusion between physical assumptions and the structure of all logic.

THE ASSUMPTIONS OF JAUCH AND PIRON

Accepting that Jauch and Piron do not wish to make new assumptions about the whole field of human reasoning, we come then to the question of what new assumptions, if any, are actually involved in their proof. In this connection, we find a possible clue by noting that they interpret the example of Birkhoff and von Neumann "empirically." In other words, they regard the nondistributivity of what they call "quantum propositions" (but what we would prefer to call "experimental questions that can be raised in the quantum theory") as a fundamental and factually established property of the structure of the objectively existing systems themselves, and not merely as a feature of the *theory* used to describe these systems. Thus, Jauch and Piron prove as a theorem that a lattice is distributive if and only if any two propositions are compatible, in the sense defined, and conclude that, since the distributivity law is "empirically false" for quantum systems, it is "empirically established" that for such systems there always exist propositions which are not compatible. As they remark: "This important point will be essential in the argument to be presented establishing the impossibility of hidden variables." (Ref. 2, p. 831.)

In the thesis of Piron, to which the authors also refer, it is shown that the compatibility relation is equivalent to the relation,

$$(a\cap b)\cup(a'\cap b)\cup(a\cap b')\cup(a'\cap b')=I. \tag{6}$$

Using this relation, Piron constructs another example by which he attempts to prove that: "The existence of systems of propositions not all compatible amongst themselves, that is to say of non-distributive systems of propositions, is a fact of experience." (Ref. 4, p. 467, note 9.) He considers a beam of light crossing a crystal of tourmaline. If the incident beam contains only a single photon, the emergent beam will contain either a whole photon of energy equal to the original photon or no photon at all. Thus the crystal constitutes a measuring apparatus and the corresponding proposition may be denoted by a_0. The proposition corresponding to the same crystal turned through an angle α is denoted by a_α. The ortho-complementary proposition to a_0, defined by the interchange of the yes and the no, corresponds to the crystal turned through $\frac{1}{2}\pi$. Piron demonstrates that a_0 and a_α constitute a pair of incompatible propositions if $\alpha\neq\frac{1}{2}n\pi$ (where n is an integer).

"It suffices to verify:

$$x=(a_0\cap a_\alpha)\cup(a_{\pi/2}\cap a_\alpha)\cup(a_0\cap a_{\alpha+\pi/2})\cup(a_{\pi/2}\cap a_{\alpha+\pi/2})\neq I.$$

Now $a_\alpha\cap a_\beta=\phi$ for $\alpha-\beta\neq n\pi$, for by definition $a_\alpha\cap a_\beta$ is true if and only if a_α and a_β are both true. Now there exists no photon which can with certainty traverse each of the two crystals if their optic axes form an angle different from zero. Thus $x=\phi$." (Ref. 4, p. 467, note 9.)

It seems clear then that Jauch and Piron regard the necessity for describing quantum mechanical experiments in terms of "incompatible propositions" as a physically demonstrated fact. But actually, it is only when one is restricted to the language of quantum theory that the experiment described above necessarily leads to "incompatible propositions" (i.e., when the state of the photon is assumed to be completely defined by a vector in Hilbert space). Indeed, as has been shown in our previous article, an entirely different structure of theoretical ideas can be proposed for describing this experiment, in terms of a set of hidden variables which do in fact determine the future behavior of the photon completely (so that, for example, there do exist photons which can *with certainty* traverse both crystals). Therefore, the need to describe this experiment in terms of incompatible propositions would be a "fact" only if it could be proved that the language of quantum mechanics is necessarily universally applicable to all possible physical phenomena and is the only one that is of this nature. To *begin* with such an assumption, without experimental or theoretical justification, would however evidently amount to circular reasoning, because this is just what the argument of Jauch and Piron aims to prove. Since no justification for an assumption of the kind described above is in fact offered, it seems clear that the proof of Jauch and Piron breaks down.

THE PROOF OF THE THEOREM

To see in more detail how this happens, we note that their object is to establish that, if a propositional system admits hidden variables, then all propositions are compatible. This is proved in Theorem II of their paper. (The conclusion of Theorem I is the same, but the proof depends on slightly stronger assumptions.)

"This theorem permits the reduction of the question concerning hidden variables to an empirical one, viz., whether there exist propositions which are not compatible. Since the lattice operations have a physical interpretation which is accessible to an empirical verification, we can decide the question by examining the actual behavior of specific propositions under observation. To rule out hidden variables, it suffices to exhibit two propositions of a physical system which are not compatible. It turns out that this is quite easy. In fact, the occurrence of incompatible propositions leads to gross macroscopic effects which can easily be verified. With this result the possible existence of hidden variables is decided in the negative." (Ref. 2, pp. 836, 837.)

In spite of the unfortunate wording—it is difficult to see how propositions, "incompatible" or otherwise, could lead to gross macroscopic effects—the error in the above reasoning is clear. The authors refer to the thesis of Piron and the article of Birkhoff and von Neumann for examples, and we have already indicated that these examples are not really "empirical" because they are based on tacitly assuming just what the authors wish to prove, i.e., that the language of propositions corresponding to projection operators in Hilbert space is the only one that can be used to describe the actual experimental facts about the examples in question.

Of course, if quantum mechanics is assumed from the outset as the theory to which all experimental

[4] C. Piron, Helv. Phys. Acta **37**, 439 (1964).

propositions must necessarily refer, then it is possible to introduce the concept of compatibility of propositions as a formal reflection of the commutativity of projection operators. But then the term *experimental proposition* is little more than another name for *projection operator*, and the *incompatibility of propositions* means nothing more than the *noncommutativity of projection operators*.

Jauch and Piron define the "state" of a physical system by a probability function $\omega(a)$ on the set of all propositions $a \in \mathfrak{L}$, where $\mathfrak{L}$ is the lattice in question, i.e., the state of a system is determined if for each proposition of the system we know the probability of obtaining the answer yes during a measurement. Two states are different if there exists a proposition a such that $\omega_1(a) \neq \omega_2(a)$. If ω_1 and ω_2 are two different states, then

$$\omega(a) = \lambda_1\omega_1(a) + \lambda_2\omega_2(a),$$

with

$$\lambda_1 > 0, \qquad \lambda_2 > 0, \qquad \lambda_1 + \lambda_2 = 1, \tag{7}$$

defines a new state which is different from either one. Such a state, which can be represented with two different states, is called a *mixture* and distinguished from the *pure* or *homogeneous* state. A system of propositions $\mathfrak{L}$ which admits hidden variables is defined as a lattice on which every state is a mixture of dispersion-free states, so that every state can be expressed in the form:

$$\omega(a) = \sum_i \lambda_i \omega_i(a)$$

with

$$\lambda_i > 0, \qquad \sum_i \lambda_i = 1, \tag{8}$$

where $\omega_i(a)$ is dispersion-free.

Whether or not one agrees with the manner in which the term "state" is introduced, quite obviously, if quantum mechanics is assumed, there are no "dispersion-free states," i.e., states which give zero dispersion for all the observables or propositions represented by all projection operators in Hilbert space. The conclusion of Jauch and Piron is therefore completely trivial.

This becomes immediately obvious with a closer examination of the definition of the function $\omega(a)$. $\omega(a)$ is defined as a function with the following properties:

(1) $0 \leq \omega(a) \leq 1$;

(2) $\omega(\phi) = 0, \qquad \omega(I) = 1$;

(3) if a is compatible with b, then $\omega(a) + \omega(b) = \omega(a \cap b) + \omega(a \cup b)$;

(4) if $\omega(a_i) = 1$, then

$$\omega(\bigcap_i a_i) = 1;$$

(5) if $a = \phi$, then there exists a state ω such that $\omega(a) \neq 0$.

In the proof of Theorem II, (4) is replaced by the weaker assumption (4°)

$$\omega(a) = \omega(b) = 1 \quad \text{implies} \quad \omega(a \cap b) = 1.$$

Now if a and b are incompatible, i.e., if the corresponding projection operators do not commute and have no common eigenvectors, then, as we have already pointed out, $a \cap b = a' \cap b = a \cap b' = a' \cap b' = \phi$. Since $\omega(\phi) = 0$, by the requirement (2), $\omega(a)$ and $\omega(b)$ cannot both be equal to 1. Thus, their definition of an acceptable state $\omega(a)$ rules out states in which the result of the measurement of a and b can be predicted with certainty if the projection operators corresponding to a and b do not commute. On the other hand, the very essence of a hidden variable theory is that, for a completely specified state of the system, in which the values of the hidden variables are all determined, the result of a measurement of any observable can be predicted with certainty. This has been pointed out by Bell.

"QUANTUM PROPOSITIONS" IN TERMS OF A HIDDEN VARIABLE THEORY

As we have shown in our previous article, by explicitly developing a hidden variable theory, it is possible to go beyond the assumptions of quantum mechanics and, with the aid of dynamical variables (at present "hidden") which are not represented by projection operators in Hilbert space, to define a "dispersion-free state" in the sense that all results of all possible measurements on the system are determined. In this theory, the measurement process of an observable P, represented by the Hermitian operator $\mathbf{P}$ with eigenvalues P_i and eigenvectors $|P_i\rangle$, is described by a set of equations of the form:

$$d\psi_i/dt = \gamma\psi_i \sum_i |\psi_j|^2 (R_i - R_j) \qquad (i = 1, 2, \cdots, n). \tag{9}$$

The ψ_i are the components of the Hilbert space vector of the system—the "state" vector in quantum mechanics—in the representation in which the matrix of $\mathbf{P}$ is diagonal:

$$|\Psi\rangle = \sum_i \psi_i |P_i\rangle. \tag{10}$$

The hidden variables are represented by the components of a normalized vector in the dual space:

$$\langle\xi| = \sum_j \xi_j \langle P_j| \tag{11}$$

and appear in the terms R_i, R_j, which are defined as:

$$R_i = |\xi_i|^2 / |\psi_i|^2, \qquad R_j = |\xi_j|^2 / |\psi_j|^2. \tag{12}$$

γ is a constant representing the strength of the interaction between the apparatus and the system measured. If R_s is the greatest such term, then these equations describe a dynamical process in which ψ_s increases to 1 and all the other ψ_i decrease to 0, assuming the vector

$|\Psi\rangle$ is normalized, so that the result of the measurement is the eigenvalue P_s and the system is left in the "quantum state" $|P_s\rangle$. The result of the measurement is therefore determined by the vector $|\Psi\rangle$, the hidden variables and the apparatus, which we characterize as a macroscopic system defining the representation.

Thus, a completely specified physical state is defined in this theory by two vectors: the Hilbert space vector of the system and the dual vector. From our point of view, the quantum mechanical specification of the state of a system by a wave function only, represented by a vector in Hilbert space, is incomplete, and we recover the statistical results of quantum mechanics from an ensemble of systems with the same $|\Psi\rangle$ vector but with a random distribution for the dual vector $\langle\xi|$. The propositions of this theory are statements about the Hilbert space vector of the system, the hidden variables represented in the dual space, which are also in principle measurable, and the measuring apparatus. There does not seem to be any sense in which two such statements can be considered to be "incompatible." However, each quantum mechanical observable is associated with a specific process of interaction between the system and a suitable measuring apparatus, and if we consider two different processes for the measurement of two observables P, Q whose operators do not commute, then these *processes* are incompatible in the obvious sense that their actions interfere. If

$$|\Psi\rangle=\sum_i \psi_i\,|P_i\rangle=\sum_j \varphi_j\,|Q_j\rangle, \tag{13}$$

then the process tending to develop ψ_s to 1 and ψ_i $(i\neq s)$ to 0 in the measurement of P conflicts with the process developing φ_t to 1 and φ_j $(j\neq t)$ to 0 in the measurement of Q. In this theory, two quantum propositions a and b, represented by noncommuting projection operators, can both be "true" with certainty in the sense that they would each be verified as true by the corresponding processes, whereas there is clearly no process for the proposition $a\cap b$. For example, if two systems are prepared in the same completely specified physical state, i.e., with the same $|\Psi\rangle$ vectors and the same $\langle\xi|$ vector, then the one system could be used to verify the proposition a and the other to verify the proposition b. If the state is defined in such a way that the results of both measurements can be predicted as positive with certainty, then for this state we would have

$$\omega(a)=\omega(b)=1 \quad \text{but} \quad \omega(a\cap b)=0, \tag{14}$$

since the proposition $a\cap b$ is certainly false. As to the example of Piron, it is clear that if the state of the photon is completely defined, by specifying the hidden variables as well as the "quantum state," then it is possible to predict with certainty whether or not the photon will pass through both crystals from the equations of measurement.

In our theory, every (quantum) proposition is either true or false with certainty, if the values of the hidden variables are known. However, these propositions refer to the results of experiments on the system, i.e., physical processes involving the system in interaction with a measuring apparatus, and to *realize* the truth or falsity of experimental propositions representing projection operators which do not commute involves *incompatible processes*. All quantum propositions are compatible as potentialities referring to the outcomes of various dynamical processes for the system. Although the concept of incompatibility of propositions seems to be meaningless, except in the sense of a rather inappropriate synonym for the noncommutativity of projection operators, the concept of incompatibility of processes is quite clear: two processes are incompatible if their actions interfere in the sense that the one process implies an order of movement which conflicts with that needed for the other to take place.

To sum up, the argument of Jauch and Piron is circular since they are really assuming quantum mechanics, whereas von Neumann's proof does rule out a certain restricted class of hidden variable theories. They are led to the idea of incompatible propositions only because they tacitly suppose that all experiments must be analyzed in terms of the usual terminology of quantum mechanics, which we have suggested provides an inadequate set of propositions for the problem.