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A Proposed Solution of the Measurement Problem in Quantum Mechanics by a Hidden Variable Theory

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The measurement problem in quantum mechanics is re-examined and it is shown that it cannot really be solved in a satisfactory way, within the framework of the usual interpretation of the theory. We then discuss von Neumann's attempts to prove that quantum mechanics is incompatible with the introduction of hidden variables, and develop a more detailed form of Bell's argument, showing that von Neumann's analysis is invalid. Using certain ideas that are implicit in the "differential-space" theory of Wiener and Siegel, we go on to propose a new deterministic equation of motion, describing a kind of coupling of the measuring instrument to the observed system that explains in detail how the wave packet is "reduced" during a measurement in a continuous and causally determined way. By averaging over the hidden parameters, we then recover the usual statistical results of quantum mechanics as a special case. However, a more detailed analysis of the theory shows that new experimental and theoretical questions can now be raised, which go outside the framework of the quantum theory as it is now formulated. These questions are examined briefly.

1. INTRODUCTORY REMARKS

In recent years a number of articles have been published indicating a revival of interest in the problem of the physical interpretation of quantum mechanics. In these articles it becomes evident that the whole question is far from being settled in a clear and simple way. It therefore seems worthwhile to go into this subject as carefully as possible. We shall attempt to clarify the nature of these questions and problems, as well as to suggest some tentative steps towards answers and solutions.

Now, in one of the most widely accepted interpretations of the quantum theory, i.e., that of the Copenhagen school,[1,2] the physical state of a system is assumed to be completely specified by its wave function which, however, defines only the probabilities of results that can be obtained in a statistical ensemble of similar measurements. The possibility that there exist further dynamical variables determining the actual behavior of each individual system at the quantum level is rejected. In support of this orthodox doctrine there is the well-known proof of von Neumann[3] that the assumption of such "hidden variables" is incompatible with the established results of quantum mechanics.

On the other hand, there are theoretical as well as practical reasons for regarding the present form of the quantum theory as unsatisfactory. The now classic Einstein–Podolsky–Rosen paradox[4] suggests that the theory is incomplete (in spite of refutations by Bohr and others). The results of experiments involving very high energies (of the order of 10^9 eV or more) or very short distances (of the order of 10^{-13} cm or less) are becoming increasingly more difficult to explain without ad hoc extensions to the theory. In addition, many authors have suggested that the reasoning involved in von Neumann's proof is circular and that the conclusion is tacitly assumed in the premises on which the argument is based. Other authors (the most recent of whom are Jauch and Piron[5]) deny that the conclusion depends on a circular argument. Still others[6,7] fully accept neither the Copenhagen interpretation nor von Neumann's point of view and yet, at least by implication, agree that the usual interpretation is in essence the only possible one. Finally, it has recently been pointed out by Bell[8] that von Neumann's proof is based on certain unnecessarily restrictive assumptions and that when these are not made the proof breaks down.

It seems, then, that the question of the possibility of hidden variables underlying the quantum theory is still problematic. Besides, as we shall make clear, if the claims based on von Neumann's theorem are accepted as valid then it would follow, from the facts confirming the current quantum theory, that a different general structure of concepts is impossible. Thus, it is made to appear that the linguistic structure of quantum mechanics prevents even the assertion of the *possibility* that the basic postulates underlying the theory may be false. In effect, this would mean that certain features of the basic postulates of the current theory are absolute truths that can never be falsified, or shown to be valid only as approximations or limiting cases. This

[1] N. Bohr, article in *Albert Einstein: Philosopher-Scientist*, P. A. Schilpp, Ed. (Library of Living Philosophers, Inc., Evanston, Ill., 1949).
[2] W. Heisenberg, article in *Niels Bohr and the Development of Physics*, W. Pauli, Ed. (Pergamon Press, Ltd., London, 1955).
[3] J. von Neumann, *The Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, N.J., 1955).
[4] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[5] J. M. Jauch and C. Piron, Helv. Phys. Acta **36**, 827 (1963).
[6] H. Margenau, Ann. Phys. (N.Y.) **23**, 469 (1963).
[7] H. Margenau, Phil. Sci. **30**, 1, 138 (1963).
[8] J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966).

kind of unfalsifiability would be almost as dangerous in any theory as is the claim to unfalsifiability on *a priori* grounds.

There have been many attempts to develop other interpretations of quantum mechanics and a number of hidden variable theories have been proposed,[2,9–11] which supply explicit refutations of von Neumann's proof. Nevertheless, these theories have all suffered from various inadequacies and, for this reason, are largely ignored in current research. From the outset, however, this whole controversy has been plagued by tacit assumptions, very often of a philosophical rather than a physical character, leading to the kind of difficulties referred to above concerning the relationship of hidden variable theories to the experimental content of quantum mechanics and the relevance of von Neumann's proof. In addition, the theory of the process of measurement involves a great many unclear features and unresolved problems, arising mainly because the role of the measuring instrument in the phenomenon of the "collapse" of the wave packet in a quantum mechanical measurement process is obscure. This has been referred to as the *measurement problem* in quantum mechanics.

In this article, as has been indicated, it is hoped to present the controversy in a more coherent and unified manner and to investigate and propose a solution to the measurement problem. After a review of the basic principles of quantum mechanics with the aim of clarifying the problem, we shall consider the Copenhagen interpretation, along with one of its modern variants,[12] which latter aims to give a more detailed mathematical treatment of the role of the observing apparatus than Bohr does. These attempts to treat the "collapse" within the framework of quantum mechanics will be seen to avoid the real problem, because they do not, in fact, propose a scheme which adequately incorporates the role of the measuring instrument into the dynamics of a theory that includes the measurement process. The idea that the quantum mechanical description of phenomena is incomplete, and that there are additional variables (at present "hidden") which must be taken into account, seems therefore to provide a very natural basis for a proper understanding of the situation. Before proceeding with the development of such a theory, however, it is of course necessary to consider the claim that a hidden variable theory underlying quantum mechanics is impossible. In this connection, we shall discuss the assumptions underlying von Neumann's proof[3] and expose their inadequacy. (Besides this proof, there has been a more recent attempt by Jauch and Piron[5] to establish the impossibility of a hidden variable quantum mechanics. This will be discussed in a subsequent paper.)

We present a hidden variable theory which is an extension of the "differential space" theory of Wiener and Siegel[11,13,14] and develop certain ideas which are only tacitly implied in their work. A deterministic equation of motion is introduced which couples the measuring instrument to the observed system and so explicitly incorporates the role of the measuring instrument into the dynamics of the theory. Since the apparatus may be regarded, from a physical point of view, as merely a particular case of the large-scale environment of the system, this suggests a multi-level theory in which the general movement of a system is determined by an equation of motion coupling the large scale to the small scale. A measurement process is then interpreted as a special case of this movement, to which each system is always subject.

An important feature of any such hidden variable theory is its potential contradiction with quantum mechanics. The latter is, of course, recovered in a certain special case. This theory therefore opens up new experimental possibilities, which are briefly examined.

It is definitely not proposed that the theory developed here is likely to be a "right" one. Rather, the main aim of this theory is to provide a language and a set of concepts in which it is possible to show clearly the meaning behind the different interpretations of quantum mechanics and to indicate what it means to assert or deny the existence of hidden variables. However, the discussion of the questions outlined above would almost certainly be relevant in any new hidden variable theory which might be developed, even if the form of the latter were very different from the theory proposed here. Thus, while this theory serves mainly as an example for making the discussion more concrete, at the same time it also suggests certain new kinds of experimental and theoretical questions whose relevance transcends the particular example of the theory in which they are raised.

2. REVIEW OF THE BASIC PRINCIPLES OF QUANTUM MECHANICS

The basic principles of quantum mechanics have been expressed in many different ways in various publications. It is, however, necessary to discuss briefly here certain essential features of the quantum description, with the aim of clarifying what the assumptions behind the usual interpretation of quantum mechanics actually are and how these may be changed in the development of a theory of hidden variables. In addition, we shall attempt to distinguish clearly between those assumptions confirmed in experiment and those based on largely tacit notions which are essentially philosophical

[9] D. Bohm, Phys. Rev. **85,** 166, 180 (1952).
[10] L. de Broglie, *The Current Interpretation of Wave Mechanics* (Elsevier Publ. Co., Inc., Paris, 1964).
[11] N. Wiener and A. Siegel, Nuovo Cimento Suppl. **2,** Ser. X, 304 (1955).
[12] A. Daneri, A. Loinger, and G. M. Prosperi, Nucl. Phys. **33,** 297 (1962).
[13] A. Siegel and N. Wiener, Phys. Rev. **101,** 429 (1956).
[14] N. Wiener and S. Siegel, Phys. Rev. **91,** 1551 (1953).

rather than physical. It will become evident in this discussion that some of these philosophical assumptions underlying the usual interpretation are misleadingly vague and unclear, especially on the question of the relation between the properties of individual systems and those of statistical ensembles.

Two of the basic postulates of quantum mechanics are the following:

(1) The state of a quantum mechanical system is defined by a continuous, single-valued wave function, $\Psi(\mathbf{x}, t)$, which obeys a deterministic equation of motion, Schrödinger's equation:

$$i\hbar(\partial\Psi/\partial t)=H\Psi. \tag{2.1}$$

(2) The wave function determines the probabilities of the possible results of any measurement on the system. Equivalently, the average or expectation value for an ensemble of measurements of any observable R is derived from Ψ through the algorithm:

$$\bar{R}=(\Psi, \mathbf{R}\Psi)=\int \Psi^*(\mathbf{x})R(\mathbf{x}, -i\hbar\nabla)\Psi(\mathbf{x})\, d\mathbf{x}, \tag{2.2}$$

where $\mathbf{R}$ is the operator corresponding to the observable R.

These two postulates, which are fundamental in any conventional formulation of quantum mechanics, incorporate the notion of probability into the theory in an inherent way. Yet it is not immediately clear how the ensembles, to which these probabilities refer, are formed and what their individual elements are. For the very terminology of quantum mechanics contains an unusual and significant feature, in that what is called the *physical state* of an *individual* quantum mechanical system is assumed to manifest itself only in an *ensemble of systems*.

Because it is indeed mysterious for an individual system thus to manifest itself only in a statistical ensemble, there is a tendency among physicists tacitly to avoid this problem by interpreting the wave function as referring directly to an ensemble rather than to an individual system. In other words, it is quite often supposed that a wave function is not to be associated with an individual system, but *only* with an ensemble of similar systems. This interpretation is in fact untenable, as can be seen by the analysis of a simple hypothetical experiment.

Consider a two-slit diffraction experiment with electrons, which is set up with a movie camera replacing the single photographic plate in the usual arrangement. If the film moves at the rate of one frame per second and the electrons arrive at the slit system one at a time at intervals of one second, then each frame will record a different individual track. If each electron is similarly prepared, then their wave packets are similar in shape and will each be represented by the same function Ψ. For example, suppose that the electron passes through a slit system and a time gate which localizes it to within a region of size Δr. According to Schrödinger's equation, there will be a slowly expanding wave packet, initially of this width, which moves through the slit system to the plate. From this wave packet, something is known about the individual system; viz., *that the electron is somewhere in the region where the wave function is appreciable.* Likewise, the momentum of the electron is somewhere in the region in which the Fourier coefficient of the wave function is appreciable. Indeed, this is just how quantum mechanics approaches the definiteness of the classical concepts of momentum and position of an individual electron, which are, however, limited in their degree of simultaneous definition by the uncertainty principle. Therefore, if one considers the movement of the packet representing an individual electron as it passes through the slit system and develops a set of fringes, according to Schrödinger's equation, one can predict that it will strike a certain region of one frame of the film in which the transformed wave function Ψ is appreciable. After the experiment, a superposition of all the frames will yield the diffraction pattern determined by $|\Psi|^2$, where Ψ is a *typical* wave function. Although they are all represented formally by the same function of $\mathbf{x}$, each electron has a wave function that depends on the time t in its own way (such that only one electron is in the system at the time). The packets are similar in shape, but different in their times of going through the slit system.

Therefore, the wave function refers primarily to an individual electron and the statistical ensembles refers to a set of electrons having different wave functions of similar shapes. Nevertheless, because the probability interpretation is the only physical meaning given for the precise shape of the wave function (as opposed to its general extension in $\mathbf{x}$ or $\mathbf{p}$ space), it cannot give a model of how the electron moves in the slit system, which is more detailed than the limits of applicability of classical concepts to individual particles, as determined by the uncertainty principle. To bring out in more detail what this means, we note that the wave function of the electron at a point $\mathbf{x}$ to the right of the slits is

$$\Psi(\mathbf{x})=\Psi_A(\mathbf{x})+\Psi_B(\mathbf{x}), \tag{2.3}$$

where $\Psi_A(\mathbf{x})$ represents that part of the wave reaching the point $\mathbf{x}$ that has come from slit A and $\Psi_B(\mathbf{x})$ represents that part which has come from slit B. If only slit A were open, then the probability of an electron reaching the point $\mathbf{x}$ would be equal to $P_A(\mathbf{x})=|\Psi_A(\mathbf{x})|^2$, while if only slit B were open this probability would be $P_B(\mathbf{x})=|\Psi_B(\mathbf{x})|^2$. When both slits are open, however, the probability is

$$\begin{aligned} P(\mathbf{x}) &= |\Psi_A(\mathbf{x})+\Psi_B(\mathbf{x})|^2 \\ &= P_A(\mathbf{x})+P_B(\mathbf{x})+\Psi_A^*(\mathbf{x})\Psi_B(\mathbf{x})+\Psi_B^*(\mathbf{x})\Psi_A(\mathbf{x}). \end{aligned} \tag{2.4}$$

In addition to the "classical" probability terms P_A+P_B, P contains the interference terms, $\Psi_A^*\Psi_B+\Psi_B^*\Psi_A$, which would not be present if the experiment involved a probability distribution of classical particles, coming either through slit A or slit B. (These terms may, for example, cause Ψ to vanish at certain points where there would have been a nonvanishing probability with only one slit open.) Therefore the particle model fails as a description of the details of the motion of an individual electron within the slit system. On the other hand, since the electron acts like an individual particle in producing a track on one frame of the film, the wave model also fails to describe the order and structure of this motion adequately.

At the risk of laboring an obvious point, we repeat: Each individual quantum system is associated with a certain wave function. All information about the system is assumed to be contained in the wave function through the probabilities that can be deduced from it. But this function provides no representation of the detailed movement of an individual electron. To call Ψ the *state function* of such an individual electron therefore leads to confusion, because the significance of this function is in general manifested physically only in an ensemble of systems having wave functions of a similar form. On the other hand, because these functions actually all refer in a certain way to different systems, it is also wrong to say that the wave function belongs *only* to an ensemble.

It seems therefore that there is no clear *physical* concept of the detailed state of movement of the individual electron. At best, then, the quantum theory can be regarded as an elaborate system of algorithms for computing the probabilities of experimental results. Nevertheless, most of those physicists who follow the usual probabilistic interpretation of the theory are not content to do this, so that they are ultimately faced with the inability to provide a satisfactory resolution of the question of how the individual and the ensemble are related, as they try to interpret the algorithms with the aid of physical concepts. It will be our intention to show that this difficulty can be removed by extending the concepts of the theory, in a manner equivalent to the introduction of *some* kind of hidden variables.

Before continuing with a third, and yet more problematic, postulate of the usual theory, we shall digress here briefly in order to clarify the distinction between pure and mixed ensembles in quantum mechanics.

A *pure* (or homogeneous) ensemble consists of individuals (e.g., electrons) which all have similar wave functions, although they are generally ordered in different places or times. In a pure ensemble, therefore, the typical wave function Ψ defines the probability distribution for the ensemble, so that the expectation value of an observable R is given by the relation:

$$\bar{R}=\int \Psi^*(\mathbf{x})R(\mathbf{x}, -i\hbar\nabla)\Psi(\mathbf{x})\, d\mathbf{x}. \tag{2.5}$$

A *mixed* ensemble is composed of individuals whose wave packets are not all similar; e.g., electrons which boil out of a filament have a distribution of energies, hence a distribution of wave functions. If, in a given representation, the coefficients of the expansion of a wave function Ψ into an orthonormal (discrete, for simplicity) set of basis functions $S_i(\mathbf{x})$ are ψ_i, then the expectation value determined by this wave function for an observable R may be expressed as:

$$\bar{R}=\sum_{ij}\psi_i^*R_{ij}\psi_j, \tag{2.6}$$

where R_{ij} is the matrix $(S_i, \mathbf{R}S_j)$. For a mixed ensemble, the expectation value is determined by averaging over all the wave functions present, i.e.,

$$\begin{aligned}\langle\bar{R}\rangle_{\text{Av}}&=\sum_{ij}\langle\psi_i^*\psi_j\rangle_{\text{Av}}R_{ij}\\&=\sum_{ij}\rho_{ji}R_{ij},\end{aligned} \tag{2.7}$$

where $\rho_{ji}=\langle\psi_i^*\psi_j\rangle_{\text{Av}}$ is the statistical matrix (or density matrix), so that:

$$\langle R\rangle_{\text{Av}}=\text{Tr}\,(f\cdot\mathbf{R}). \tag{2.8}$$

From the above assumptions, it follows that f is Hermitian and $\langle\bar{R}\rangle_{\text{Av}}$ is real.

We come now to postulate (3):

(3) After a precise measurement of the observable represented by the operator $\mathbf{R}$ yielding the eigenvalue R_i, the wave function $\Psi(\mathbf{x})=\sum_i\psi_iR_i(\mathbf{x})$ becomes $R_i(\mathbf{x})$.

This phenomenon is known as the "collapse" or "reduction" of the wave packet and the postulate in question is sometimes referred to as the "projection postulate", because the vector in Hilbert space representing Ψ is projected after the measurement onto one of the basis vectors of the representation in which the matrix of the operator representing the observable measured is diagonal. The postulate is necessary to ensure the *reproducibility* of measurement results: after measuring R and finding the result R_i, an immediate subsequent measurement should yield the same value R_i, or the theory would clearly be inadequate. It is, however, incompatible with Schrödinger's equation, since an ensemble of measurements on systems similarly prepared with wave functions Ψ, i.e., a *pure* ensemble, will yield a *mixed* ensemble represented by a probability distribution of eigenfunctions, $R_i(\mathbf{x})$, each with weight $p_i=|\psi_i|^2$. This represents an irreversible change involving a nonunitary transformation of Ψ. In contrast to the classical situation, where every dynamical process is, in principle, covered by Hamilton's equations, the measurement process in quantum mechanics is placed in a category on its own.

It may perhaps clarify the situation to point out here that, according to the usual interpretation of quantum mechanics, the process of collapse is equivalent to assuming that a given wave function is replaced

by a second wave function after measurement, but for no assignable reason: the second wave function cannot be derived as a solution of the Schrödinger equation of motion for the system initially described by the first wave function. If we consider a pure ensemble of similar first wave functions, as in the cine camera experiment, then this ensemble is associated with an ensemble of second wave functions which will not in general be pure, i.e., the individual members will have different wave functions. The relative frequencies of these different second wave functions are given by the usual rules of the quantum theory.

It is not easy to avoid the feeling that such a sudden break in the theory (i.e., the replacement, unaccounted for in the theory, of one wave function by another when an individual system undergoes a measurement) is rather arbitrary. Of course, this means the renunciation of a deterministic treatment of physical processes, so that the statistics of quantum mechanics becomes *irreducible* (whereas in classical statistical mechanics it is a simplification—in principle more detailed predictions are possible with more information). More significant still, a precise conceptual description of the process of measurement in quantum mechanics becomes in principle impossible, so that one also gives up all hope of ever being able to obtain a clear conception of the *nature* of the individual electron or of how it moves. The physicist may have a rough picture of a wave packet, which supplies a limited kind of physical intuition, but this does not in any sense complete the picture of what the electron is and of what it does.

The deeper significance of the lack of such a clear concept of the electron and its motion can be brought out more clearly by considering the fact that, in an ensemble of electrons with similar first wave functions, the differences among the second wave functions cannot be referred to differences that exist in the individual members of the first "pure" ensemble. Does it not seem natural, however, to suppose that differences in the later states of an ensemble of systems should be related to corresponding differences in the earlier states of the systems? The assumption that differences appear in time which are totally unrelated to *any* differences whatsoever in the earlier states of the systems implies a kind of breakdown of physical law that should not be accepted lightly without some clear proof that an assumption of this nature is unavoidable. For, once it is accepted, we shall tend very strongly to cease to be interested in looking for such relationships, so that even if they should exist, we are unlikely to find them.

At this point, the idea seems quite naturally to suggest itself that the differences in the second wave functions (after a measurement) could be referred to differences, before the measurement, in the values of some new kinds of variables, at present "hidden" (but in principle ultimately observable with the aid of suitable new methods of observation may later be suggested by thinking in terms of such variables). Since the behavior of each system would depend on these hidden variables, as well as on the wave function, we can understand how electrons with initially similar wave functions can display different properties when they are observed, through the differences of the hidden variables before measurement. The quantum mechanical distributions of observed results could then be recovered by averaging over a suitable ensemble of hidden variables in the first (pure) ensemble.

Such a theory would, in certain ways, be rather similar to classical statistical mechanics, in which the large-scale thermodynamic properties, relationships of P, V, T, etc., are explained in terms of statistical distributions of atomic variables. Relative to the large-scale level, these variables are at first "hidden," but (as has been suggested with regard to hidden variables of quantum mechanics) they are ultimately observable with new kinds of instruments—Geiger counters or cloud chambers instead of thermometers and pressure gauges—which have in fact been developed as a result of the new ideas suggested by thinking in terms of atomic concepts.

3. INADEQUACY OF THE COPENHAGEN INTERPRETATION AND OTHER RELATED THEORIES OF MEASUREMENT

There are various reasons for developing such a hidden variable theory. As we have indicated, the Copenhagen interpretation of quantum mechanics, in which the collapse of the wave packet is accepted as a fundamental and irreducible phenomenon (an ultimate fact of nature which is incapable of any further analysis), entails the renunication of any conception of the order and structure of movement of a microsystem in favor of a set of rules for the prediction of the results of specific experiments. Of course, a system of calculation of experimental probabilities is useful. Nevertheless, science is surely more than *merely* a set of algorithms for an engineer's handbook. Science also aims at an *understanding* of the over-all structure and order of movement of matter from the atom to the galaxies. While science also has the aim of prediction and useful application, these in themselves cannot correctly be identified with the whole of the act of understanding, in which one grasps the order the structure of a complex process in a unified coherent set of concepts. From these latter, one can abstract relationships of the parts, which make predictions possible. However, the difference between understanding and *mere* prediction can be illustrated by the example of trying to find one's way through a city that one does not know. One may be given directions, which in effect constitute predictions of the order of streets and buildings that one will discover if one walks in certain ways. But a map of the city gives a unified coherent understanding of its over-all order and structure, from which useful predictions can be taken by reading it. Quantum

mechanics at present yields a coherent structure of *mathematics*. However, its *physical concepts* resemble the giving of directions to find one's way through a city. Thus, if electrons enter a slit system, quantum mechanics predicts the distribution with which they will leave. But this is not deduced from a concept of the over-all order and structure of movement of the electrons. Rather, as has been indicated, it has the character of a mathematical algorithm, equivalent to a set of "directions." This algorithm is surely a reflection of the whole of natural law. But to assume that *all* theories must henceforth take the form of such algorithms will evidently limit understanding and channel research into a study of the statistical properties of observation processes involving such "collapses."

This point can perhaps be further clarified by considering the fact that there is a very important methodological justification for the consideration of hidden variable theories, even those which are not necessarily seriously regarded as "right" ones. For the language of quantum mechanics in the usual interpretation of the formalism makes it virtually impossible even to assert linguistically what is to be meant by the detailed order and structure of this process of collapse, because the denial that this process has any meaning is incorporated into the very structure of the language. In this way the theory seems to provide a justification for an inherently statistical concept of an individual system.

This, by itself, is a very serious fault in the present formulation of quantum mechanics. As Popper[15,16] has indicated, it is a basic requirement of any scientific theory that it should be expressed in terms that make it *falsifiable*. But current quantum mechanics has been given a linguistic form that prevents even the hypothetical assertion of the contrary to any of the basic postulates, because this would apparently entail a change in the experimental facts on which the theory is based. (As will be shown later, it was the aim of von Neumann's theorem to prove this.) Since to deny the experimental content of the theory is evidently absurd, and since we can neither say nor think anything contrary to the basic principles of quantum mechanics, it seems that the latter are inescapable and absolute truths, proved by a tremendous number of experiments.

It is not generally realized that in the present language we can only ask questions and consider experiments that do not go outside the structure of this language. As long as we use this language, no experiment is *ever* likely to be devised that could conceivably refute the basic postulates of quantum mechanics and thus in principle provide a test of these postulates. We shall always search for energy levels, scattering probabilities, magnetic moments, etc., and if they are not as predicted we can always suppose that a change of the forces between atoms, the introduction of particles with new properties, new groups such as SU_3, SU_6, $\tilde{U}_{12}$ and other such orders, etc., will eventually bring experiments back into agreement with the theory, without changing the *basic* postulates given above, except in the way of enriching and adding to them: we need never take a step that contradicts them. Of course, in a way, every theory can be adapted by means of enough assumptions to accommodate any kind of data whatsoever. But the current linguistic form of quantum mechanics has the further special feature that it asserts the logical impossibility of any other scheme that would fit the facts as they are known today. It is, therefore, necessary to *challenge* such a point of view, for otherwise it might lead to our being trapped in a given set of concepts, without our even realizing that we are thus trapped. As we have already indicated, we shall do this by changing the language of quantum mechanics, so that questions may be asked which would show what it could mean if the basic postulates of the current theory were false. Even if the theory in terms of which the new language is embodied should not be correct in detail, such a questioning on a theoretical basis can be useful, because it may ultimately lead to the proposal of experiments that test these postulates.

A number of authors, notably Ludwig,[17] Green,[18] and the group Daneri, Loinger, and Prosperi[12] have proposed theories of measurement which take into account the macroscopic nature of the measuring instrument. This point of view differs from that of Bohr and the Copenhagen school, since the macroscopic measuring instrument is in fact treated as a quantum mechanical system, with certain added conditions which enable "macroscopically distinguishable states" of the system to be defined in some way. These macrostates are, of course, related to probability distributions over microstates of the system, so that the measurement process is treated as a problem of quantum statistical mechanics. Before the measurement interaction, the apparatus is assumed to be in a thermodynamically metastable state, such that a very small perturbation makes it evolve (irreversibly) towards a thermodynamically stable state, dependent on the state of the microsystem.

The most sophisticated and general analysis along these lines has been given by Daneri, Loinger, and Prosperi and many prominent physicists, including Rosenfeld,[19] have favored this apparent solution to the problem. The argument is essentially that the collapse of the wave packet in a measurement process is not really a problem, because interference is destroyed in

[15] K. R. Popper, *The Logic of Scientific Discovery* (Hutchinson, London, 1959).

[16] K. R. Popper, *Conjectures and Refutations* (Routledge and Kegan Paul, London, 1962).

[17] G. Ludwig, article in *Werner Heisenberg und die Physik Unserer Zeit* (Friedrich Vieweg und Sohn, Braunschweig, 1961).

[18] H. S. Green, Nuovo Cimento **9**, 880 (1958).

[19] L. Rosenfeld, "The Measuring Process in Quantum Mechanics," preprint (May 1965).

the process of *amplification* involved in the detection of a microsystem by a macroinstrument, so that probability amplitudes become ordinary probabilities in the irreversible transition of the amplifying apparatus to a condition of stable equilibrium. Hence what "collapses" is only our *knowledge* of the system and not some property of the system itself. However, in the cine camera experiment, we have seen that, according to the basic principles of quantum mechanics, the wave function always refers to an individual system. The fact that interference between different parts of the wave function, corresponding to different values for the observable measured, is effectively destroyed in the amplification process does not explain why the wave function of the *individual total* system "condenses" onto one and only one of the component noninterfering wave packets that are produced in the interaction with the measuring apparatus. Thus, the question of the behavior of an individual total system is avoided.

In order to clarify this criticism, we can also refer to the example of the measurement of the z component of the angular momentum of an atom with angular momentum $\hbar/2$ by means of a Stern–Gerlach experiment (Ref. 20, p. 593). In this experiment, an atom is passed through an inhomogeneous magnetic field which gives it a momentum that is directed up or down, according to whether the spin is up or down. The resulting z motion of the atom after it leaves the field carries it to a height that depends on the spin and, in this way, a rather rough observation of the position determines whether the spin was up or down.

It can be shown that, after the measurement interaction, the wave function of the total system of atom plus measuring instrument takes the form of two localized wave packets which no longer interfere significantly. But this does not alter the fact that only *one* of these packets can be "actual" if the total system is to be in a certain definite state after the measurement. In terms of the cine camera experiment, if an individual electron is sent through this apparatus, until it arrives at the photographic film it is represented by a wave packet spread over a large region of space. When this packet arrives at the film it begins to interact with the wave functions of the atoms of the film. In the interaction it is "broken up" into many very small packets, which cease to interfere coherently. What is still unexplained, however, is that only one of these packets contains the electron: there is no possibility within the theory of understanding how and why the individual electron suddenly "condenses" into one of the packets. In other words, there is no *logical connection* between the first wave function, which may be regarded as "broken up" into noninterfering localized packets, and the second wave function, which represents a definite result.

[20] D. Bohm, *Quantum Theory* (Prentice-Hall, Inc., Englewood Cliffs, N.J., 1951).

To be sure, if we now average over the resulting wave functions in an ensemble of measurements on similarly prepared systems, then the pure ensemble of similar first wave functions is connected statistically to the ensemble of second wave functions, i.e., the probability distribution of second wave functions is determined by the algorithm of quantum mechanics. But this averaging process can only be understood as an *empirical* average over the wave functions which are actually present after an ensemble of measurements: there is no corresponding distribution of any parameter over the ensemble of first wave functions which can be averaged over to produce the resulting distribution of second wave functions. This amounts to a confusion between the individual and the statistical interpretation of the meaning of the wave function. For, the attempt to regard the different components of the "broken up" first wave function as nothing more than probabilities would make sense if the wave function were given *only* a statistical interpretation. The purely statistical aspects of the treatment are in themselves therefore perfectly clear, and would be still clearer, if it made no claims at all (beyond limits determined by the uncertainty principle) to discuss the details of the movement of individual electrons. But then, it would also be very obvious that no explanation has been given for how an individual electron arrives in *one* of the packets. The assertion that the individual has in fact been adequately treated is just what leads to the confusion, in which the one word "electron" is required at the same time to refer to individual and statistical ensemble. And this assertion is based on the tacit assumption that the current interpretation of quantum mechanics is complete.

Thus, although the analyses of Daneri, Loinger, and Prosperi, and other such treatments do help to clarify the measurement process to a certain extent by emphasizing the existence of an amplifying stage in the process, the basic problem is still unresolved. A thermodynamic averaging operation is performed over an ensemble and reasonable probability functions are obtained, but it remains unclear exactly what the statistical mechanical parameters are. The question of the behavior of an individual system, for example the process of movement of an individual electron through the apparatus of the movie camera experiment, is not answered. The analysis of Daneri, Loinger, and Prosperi does not alter the fact that the extended wave model is needed to describe the movement of the electron through the slits, while after the packet reaches the film and interacts with it there is a set of noninterfering localized packets, only one of which contains the electron.

We conclude that the attempts to develop a theory of measurement within the quantum theory do not succeed in solving the real problem, and that the order and structure of the process by which a microsystem

such as an electron is measured cannot be conceived within the formalism of quantum mechanics alone. Before proceeding with the development of a hidden variable theory, however, we must first dispose of von Neumann's proof that any attempt to explain the probability distributions of quantum mechanics in this way would be impossible, because it would contradict the experimental facts on which the theory is based.

4. VON NEUMANN'S PROOF

Von Neumann's proof[21] that hidden variables can be excluded in quantum mechanics, contained in Chap. IV of the "Mathematical Foundations of Quantum Mechanics," is based on the following assumptions:

(i) There corresponds to each observable of a quantum mechanical system a unique hypermaximal Hermitian operator in Hilbert space. This correspondence is assumed to be one-one, i.e., each such operator corresponds to an observable. (Page 313.) (In place of the term "observable", von Neumann uses the neutral phrase "physical quantity," but with the same connotation.)

(ii) If the observable R has the operator $\mathbf{R}$, then the observable $f(R)$ has the operator $f(\mathbf{R})$. (I, page 313.)

(iii) If the observables $R, S, \cdots$ have the operators $\mathbf{R}, \mathbf{S}, \cdots$, then the observable $R+S+\cdots$ has the operator $\mathbf{R}+\mathbf{S}+\cdots$. (The simultaneous measurability of $R, S, \cdots$ is not assumed.) (II, page 314.)

(iv) If the observable R is by nature a nonnegative quantity, for example, if it is the square of another quantity S, then $\bar{R}\geq 0$, where $\bar{R}$ is the expectation value of R for an ensemble of measurements. (A′, page 311.)

(v) If $R, S, \cdots$ are arbitrary observables and $a, b, \cdots$ real numbers, then $\langle aR+bS+\cdots\rangle_{\text{Av}}=a\bar{R}+b\bar{S}+\cdots$. (B′, page 311.)

Two definitions are also relevant:

(a) An ensemble is *dispersion-free* if $(\bar{R})^2=\langle R^2\rangle_{\text{Av}}$ for every observable R. (α', page 312.)

(b) An ensemble is *homogeneous* or *pure* if it cannot be split into subensembles with different statistical properties. In other words, if the ensemble is split into any two subensembles, so that $\bar{R}=\bar{R}'+\bar{R}''$ for every observable R, then $\bar{R}'=c'\bar{R}$, $\bar{R}''=c''\bar{R}$ (where c', c'' are constants $c'+c''=1$; $c', c''>0$).

On the basis of these assumptions, von Neumann demonstrated that there exists a linear, semidefinite, Hermitian matrix U_{mn}, such that for any observable R:

$$\bar{R}=\sum_{mn} U_{nm}R_{mn}=\text{Tr}\,(\mathbf{UR}). \tag{4.1}$$

Thus, every ensemble in quantum mechanics is characterized by a certain (density) matrix (or statistical operator), from which the average of any observable can be deduced according to a certain algorithm.

Since there is no physically meaningful density matrix which gives zero dispersion for all observables R, von Neumann's first conclusion was that there are no dispersion free ensembles. In addition, there are density matrices which represent homogeneous ensembles, which means that it is not always possible to split an ensemble into subensembles with different statistical properties. The interpretation of this result for the question of hidden variables is that the statistics of the homogeneous ensemble cannot result from averaging over the hidden variables because, firstly, the homogeneous ensemble could then be represented as a mixture of two different ensembles and, secondly, because the dispersion free subensembles consisting of one or more individual systems in the same precisely defined state do not exist.

The argument here follows the analogy with classical statistical mechanics. A classical observable is defined as a function of the coordinates and momenta (hidden variables) which determine the microstates of the system:

$$R=R(q_1, q_2, \cdots; p_1, p_2, \cdots), \tag{4.2}$$

which is abbreviated symbolically to $R(q_1p)$. If the "state" of the system is only defined by certain thermodynamic (i.e., large-scale) variables, then the expectation value of R for such a system is computed as the average over a suitably defined ensemble of systems:

$$\bar{R}=\int \rho(T, q, p)R(q, p)\, dq\, dp=\bar{R}(T), \tag{4.3}$$

where the symbol T is used to represent the thermodynamic variables which define the "state" of the system macroscopically and ρ is a probability density. In the equilibrium case the statistical or thermodynamic properties of the system may be derived from the "normal" Boltzmann distribution function $C\exp(-E/kT)$:

$$\bar{R}=\int C\exp(-E(q, p)/kT)R(q, p)\, dq\, dp, \tag{4.4}$$

where T is the temperature. In general we may consider any distribution function $\rho(T, q, p)$, in particular the extreme case of a δ-function, representing a dispersion free ensemble in which the values q, p are precisely defined and determine a value for every observable $R(q, p)$ without dispersion.

Similarly, in a statistical explanation of quantum mechanics by hidden variables, there should be "nonnormal" distributions of these parameters possible as well as a "normal" distribution which would reproduce the results of quantum mechanics. The extreme case of such a "nonnormal" distribution would be the dispersion free ensemble, in which the values of all hidden parameters are precisely defined and hence determine

[21] Reference 19, Chap. IV, Parts 1 and 2 (pages 295–328).

a particular value for every observable. Von Neumann demonstrated that the existence of such "nonnormal" ensembles, with statistical properties violating those of quantum mechanics (which is certainly implicit in any hidden variable theory), is impossible. He concluded this proof with the remark that:

> "... we need not go any further into the mechanism of the 'hidden parameters' since we now know that the established results of quantum mechanics can never be re-derived with their help"[22]

and he even stressed that:

> "... it is therefore not, as is often assumed, a question of a re-interpretation of quantum mechanics,—the present system of quantum mechanics would have to be objectively false, in order that another description of the elementary processes than the statistical one be possible."[22]

Thus, since the concept of hidden variables is incompatible with von Neumann's assumptions, and since the denial of these assumptions would seem to entail the denial of the postulates of quantum mechanics as a special case, it appears that any hidden variable theory would contradict the experimental results confirming quantum mechanics.

If von Neumann's assumptions are accepted, then his conclusion is indeed inescapable. However, these assumptions, in spite of their apparent innocence, are unnecessarily restrictive. If the dispersion in the measured values of an observable R is assumed to be due to a distribution in the values of certain hidden parameters over the ensemble of systems measured, then the expectation value of R, $\bar{R}$, is clearly to be regarded as an average over this distribution of hidden variables. Hence, the relation

$$\bar{R}=\sum_{mn} U_{nm}R_{mn}$$

should be expressed as:

$$\bar{R}=\sum_{mn} \bar{U}_{nm}R_{mn}, \tag{4.5}$$

where $\bar{U}_{mn}$, the matrix defining the statistics of a quantum mechanical ensemble, is computed as an average over the distribution of hidden variables in the ensemble. Explicitly:

$$\bar{U}_{mn}=\int U_{mn}(\Psi,\lambda)\rho(\lambda)\,d\lambda, \tag{4.6}$$

so that:

$$\bar{R}=\sum_{mn}\int U_{nm}(\Psi,\lambda)R_{mn}\rho(\lambda)\,d\lambda, \tag{4.7}$$

where the symbol λ is used generically to denote the set of hidden variables $\lambda_1, \lambda_2, \cdots$ and $\rho(\lambda)=\rho(\lambda_1, \lambda_2, \cdots)$ is a probability distribution function defined over the hidden variables. [The above expression is, of course, only true for a pure ensemble. In the case of a mixed ensemble, there is a certain probability, $P(\Psi)$, of any particular quantum state. Hence R must be averaged over the distribution of quantum states as well as the distribution of hidden variables:

$$\langle\bar{R}\rangle_{\text{Av}}=\sum_{mn}\int \bar{U}_{nm}R_{mn}\rho(\lambda)\,d\lambda, \tag{4.8}$$

where $\bar{U}_{mn}$ is averaged over the distribution of wave functions, not over λ.]

This relation for $\bar{R}$ implies that, for particular values of Ψ and λ (i.e., the dispersion free case), the value of R is determined by a relation of the form:

$$R'=\sum_{mn} U_{nm}(\Psi',\lambda')R_{mn}, \tag{4.9}$$

where the $'$ is used to denote a particular value of the relevant parameter. This is a linear relation between the value of the observable R and its associated matrix R_{mn}. Now, *a priori*, there is no reason why the value of R should not be determined by some *nonlinear* function of Ψ, the hidden variables and the matrix R_{mn}:

$$R=F(\Psi,\lambda,R_{mn}). \tag{4.10}$$

In an ensemble:

$$\bar{R}=\int F(\Psi,\lambda,R_{mn})\rho(\lambda)\,d\lambda \tag{4.11}$$

and we would expect to obtain the statistical results of quantum mechanics from a "normal" ensemble specified by some $\rho_N(\lambda)$.

Von Neumann's implicit choice of a particular linear form for F as a function of the R_{mn}, i.e.,

$$F=\sum_{mn} U_{nm}(\Psi,\lambda)R_{mn}, \tag{4.12}$$

depended essentially on the linearity assumption (v), i.e., for any set of observables $R, S, \cdots$, whether simultaneously measurable or not, $\langle aR+bS+\cdots\rangle_{\text{Av}}=a\bar{R}+b\bar{S}+\cdots$. Clearly (v) follows immediately from the assumption of linearity for the function F:

$$\begin{aligned}\langle R+S\rangle_{\text{Av}}&=\sum_{mn}\int U_{nm}(\Psi,\lambda)\,(R+S)_{mn}\rho(\lambda)\,d\lambda\\&=\sum_{mn}\int U_{nm}(\Psi,\lambda)\,(R_{mn}+S_{mn})\rho(\lambda)\,d\lambda\\&=\sum_{mn}\int U_{nm}(\Psi,\lambda)R_{mn}\rho(\lambda)\,d\lambda\\&\quad+\sum_{mn}\int U_{nm}(\Psi,\lambda)S_{mn}\rho(\lambda)\,d\lambda=\bar{R}+\bar{S}.\end{aligned}$$

[22] Reference 19, pages 324, 325.

Von Neumann justified the assumption (v) by the remark that it is always true in quantum mechanics, i.e., for any quantum state Ψ:

$$(\Psi, (R+S)\Psi) = (\Psi, R\Psi) + (\Psi, S\Psi).$$

Hence (v) is true for every pure or mixed ensemble. But this relationship could conceivably be reproduced for *normal* ensembles, specified by some $\rho_N(\lambda)$ without rejecting the possibility of a nonlinear function F. This would imply that the assumption (v) is in general false and is valid only in a certain special case which includes the ensembles of quantum mechanics. (This has been pointed out by Bell.[8])

Without the assumption of linearity

$$\bar{R} = \sum_{mn} \int U_{nm}(\Psi, \lambda) R_{mn} \rho(\lambda)\, d\lambda$$

tacitly contained in von Neumann's assumption (v), the proof fails, because it is based on an analysis of the statistical properties of the density matrix

$$\bar{U}_{mn} = \int U_{mn}(\Psi, \lambda) \rho(\lambda)\, d\lambda,$$

which is now no longer characteristic of all conceivable ensembles.

This proof discouraged further work on hidden variable theories for some time. In the light of the above analysis it seems that the proof rejects only those hidden variable theories which depend on a linear law of the form (4.7). We shall therefore turn our attention to theories which contain the more general assumption (4.11):

$$\bar{R} = \int F(\Psi, \lambda, R_{mn}) \rho(\lambda)\, d\lambda.$$

As we have demonstrated, the assumption (v) is now no longer valid and von Neumann's proof breaks down. A more recent impossibility proof has been proposed by Jauch and Piron, in which they come to the same conclusion as von Neumann without using the linearity assumption (4.7) or (v). We shall demonstrate the inadequacy of this proof in a subsequent article.

A theory which embodied the possibility of such a nonlinear assumption for R was first proposed by one of us in 1951.[9] However, that theory was suggested merely in order to demonstrate, by a counter-example to von Neumann's proof, the possibility of a hidden variable theory. It was not seriously envisaged as an ideal theory and suffered from a lack of simplicity and elegance of structure. We now wish to propose a hidden variable theory, an extension of certain ideas of Wiener and Siegel, which we consider is at least a step toward what could seriously be considered as a tentative theory, insofar as it suggests certain questions which we feel to be relevant.

5. THE THEORY

We propose, firstly, that one of the basic variables of the theory is the wave function Ψ. In fact, as indicated in Sec. 2 ("Review of the Basic Principles of Quantum Mechanics") we regard the Ψ as playing a role somewhat analogous to the thermodynamic variables P, V, T, etc. in classical statistical mechanics. As P, V, T, can be known from measurements on a classical thermodynamic system done with the aid of suitable instruments, so the measurement of a "complete set of observables" enables us to know the wave function of a quantum mechanical system (by finding the eigenfunctions of the corresponding operators). Thus, after a measurement, the wave function can be known. At this stage, this is a basic postulate of the usual theory, but we shall later derive it from deeper assumptions concerning the hidden variables.

In order to simplify the presentation in the following we shall consider the problem of a system whose wave function can be represented as a vector in a two-dimensional Hilbert space, for example a spin-$\frac{1}{2}$ particle without translational motion. Using the Dirac notation to represent vectors in Hilbert space, the vector $|\Psi\rangle$ may be expressed in the form:

$$|\Psi\rangle = \psi_1 |S_1\rangle + \psi_2 |S_2\rangle, \tag{5.1}$$

where $|S_1\rangle$, $|S_2\rangle$ is the basis in which the operator of the spin observable S is diagonal. We assume that $|\Psi\rangle$ is normalized ($|\psi_1|^2 + |\psi_2|^2 = 1$).

We postulate a *dual* Hilbert space, a typical vector of which is represented with respect to the same basis by:

$$\langle\xi| = \xi_1 \langle S_1| + \xi_2 \langle S_2|. \tag{5.2}$$

This dual vector space is *different* to the original space and, whereas the vector $|\Psi\rangle$ satisfies Schrödinger's equation, the dual vector $\langle\xi|$ obeys an entirely different equation of motion, which we shall discuss presently. We shall use the components of the vector $\langle\xi|$ to represent the hidden variables corresponding to the λ in Eqs. (4.10), (4.11).

We propose also that for the "normal" quantum mechanical ensemble these hidden variables are randomly distributed on the hypersphere of unit radius in Hilbert space defined by $\sum_i |\xi_i|^2 = 1$. With the aid of certain further assumptions, which in effect define the function F of Eqs. (4.10), (4.11), we shall demonstrate that this distribution reproduces the usual quantum mechanical averages for all observables. Nevertheless, the theory also allows subensembles which have less dispersion than the "normal" ones and even, in the limiting case, dispersion free ensembles, in which the ξ_i are all precisely defined.

We define the two ratios:

$$R_1 = |\psi_1|^2/|\xi_1|^2 = J_1/|\xi_1|^2$$
$$R_2 = |\psi_2|^2/|\xi_2|^2 = J_2/|\xi_2|^2 \tag{5.3}$$

and postulate that, in addition to the change in $|\Psi\rangle$ defined by the Schrödinger equation, the following deterministic equations operate during a measurement process of the observable S:

$$d\psi_1/dt=\gamma(R_1-R_2)\psi_1 J_2$$
$$d\psi_2/dt=\gamma(R_2-R_1)\psi_2 J_1, \tag{5.4}$$

where γ is a suitable quantity that remains nearly constant during a measurement and negligible before and after. (We assume that γ is always positive.)

(For simplicity, we are considering an *impulsive* measurement, where the interaction with the apparatus is so great that the effects of the Hamiltonian of the undisturbed system can be neglected during the interaction. The generalization to include arbitrary measurements is straightforward and does not essentially alter the argument.)

From these equations it follows that:

$$dJ_1/dt=2\gamma(R_1-R_2)J_1J_2$$
$$dJ_2/dt=2\gamma(R_2-R_1)J_2J_1. \tag{5.5}$$

Hence $d(J_1+J_2)\ dt=0$, so that $|\Psi\rangle$ remains normalized during the measurement process.

Also:

$$d(\log J_1)\ dt=2\gamma(R_1-R_2)J_2$$
$$d(\log J_2)\ dt=2\gamma(R_2-R_1)J_1. \tag{5.6}$$

(We assume for the present that the ξ_i are constants, at least during a measurement process. They may be changing, but so slowly that this change can be neglected during the measurement interaction. Later, in fact, we shall discuss such relatively slow changes in the ξ_i.)

If $R_1>R_2$ and $J_2\neq 0$, then J_1 must always increase and J_2 must always decrease. Since $J_1+J_2=1$, this process will continue until $J_1=1$ and $J_2=0$; so $|\Psi\rangle\rightarrow \exp(i\phi_1)\ |S_1\rangle$. Similarly, if $R_2>R_1$ initially and $J_1\neq 0$, then $|\Psi\rangle\rightarrow \exp(i\phi_2)\ |S_2\rangle$. ($\phi_1$ and ϕ_2 are phase factors.)

Thus, the motion described by these equations produces the eigenstate $|S_1\rangle$ or $|S_2\rangle$ after a measurement of S, representing a spin value of $+\hbar/2$ or $-\hbar/2$, respectively. In an ensemble of measurements, this results in the mixture represented by the statistical operator $f=|\psi_1|^2\,|S_1\rangle\langle S_1|+|\psi_2|^2\,|S_2\rangle\langle S_2|$. We see that the result of a measurement of the spin is determined both by the Hilbert space vector of the system $|\Psi\rangle$ and by the dual vector $\langle\xi|$, in wuch a way that if $R_1>R_2$, or $|\psi_1|^2/|\xi_1|^2>1$, then the result is $+\hbar/2$, while if $R_2>R_1$, or $|\psi_2|^2/|\xi_2|^2>1$, then the result is $-\hbar/2$.

These two possibilities are mutually exclusive and between them they cover every possible set of values of the components of the dual vector ξ_1, ξ_2 whenever the quantum state, i.e., ψ_1, ψ_2, is determined. For, if $R_1>R_2$, $|\psi_1|^2>|\xi_1|^2$ (and $|\psi_2|^2<|\xi_2|^2$), while if $R_2>R_1$, $|\psi_2|^2>|\xi_2|^2$ (and $|\psi_1|^2<|\xi_1|^2$). This leaves only the case: $|\psi_1|^2=|\xi_1|^2$ and $|\psi_2|^2=|\xi_2|^2$, a set of measure zero which has no effect on the statistics: whatever we assume for this case it will have no significance for physical measurements, which are never *perfectly* precise.

Now, in general, the hidden parameters ξ_1, ξ_2 could have any conceivable statistical distribution. As indicated earlier, we demonstrate that a certain "normal" distribution, with probability function ρ_N, determines the usual results of quantum mechanics.

We write:

$$\xi_1=\rho_1\exp(i\theta_1)=a_1+ib_1$$
$$\xi_2=\rho_2\exp(i\theta_2)=a_2+ib_2. \tag{5.7}$$

Substituting:

$$\rho_1=\rho\cos\phi$$
$$\rho_2=\rho\sin\phi \tag{5.8}$$

we transform to spherical coordinates. The element of volume in the space of hidden parameters is

$$d\Omega=da_1\,db_1\,da_2\,db_2=\rho_1\rho_2\,d\rho_1\,d\rho_2\,d\theta_1\,d\theta_2$$
$$=\rho^3\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta_1\,d\theta_2$$
$$=\tfrac{1}{2}\rho^3\,d(\sin^2\phi)\,d\rho\,d\theta_1\,d\theta_2. \tag{5.9}$$

We assume now that the dual vector is also normalized, i.e., $\rho=1$, so that the dual point is on the surface of the three-dimensional hypersphere of unit radius in the four-dimensional space—i.e., the complex two-dimensional space. Hence we ignore $d\rho$ hereafter; the element of volume of interest (with $\rho=1$) is

$$d\Omega=\tfrac{1}{2}\,d(\sin^2\phi)\,d\theta_1\,d\theta_2. \tag{5.10}$$

Our basic assumption is that $\rho_N=$ constant over this hyperspherical "shell," i.e., a random distribution for ξ_1, ξ_2.

The integral over all points in the dual space for which the following conditions are satisfied:

(a) $|\psi_1|^2/|\xi_1|^2>|\psi_2|^2/|\xi_2|^2$, i.e., $|\psi_1|^2/\rho_1^2>|\psi_2|^2/\rho_2^2$

(b) $|\psi_1|^2+|\psi_2|^2=1$

(c) $|\xi_1|^2+|\xi_2|^2=1$, i.e., $\rho_1^2+\rho_2^2=1$ or $\rho^2=1$

will give the probability of the result corresponding to the state $|S_1\rangle$.

We have:

$$|\psi_1|^2\,\rho_2^2>|\psi_2|^2\,\rho_1^2$$
$$|\psi_1|^2\,\rho_2^2>|\psi_2|^2(1-\rho_1^2)$$
$$\rho_2^2>|\psi_2|^2$$
$$\sin^2\phi>|\psi_2|^2. \tag{5.11}$$

Integration over the range of $\sin^2\phi$ gives (with the correct normalization for the surface area of the hypersphere):

$$\frac{2}{(2\pi)^2}\int_{|\psi_2|2}^{1}\tfrac{1}{2}\,d(\sin^2\phi)=\frac{1}{(2\pi)^2}\,(1-|\,\psi_2\,|^2)=\frac{1}{(2\pi)^2}\,|\,\psi_1\,|^2. \tag{5.12}$$

Integration over θ_1, θ_2 (from 0 to 2π) then gives the measure or probability of the set of points (on the unit "shell") for which $R_1>R_2$ as:

$$P_1=|\,\psi_1\,|^2. \tag{5.13}$$

For $R_2>R_1$, a similar calculation gives the result $|\,\psi_2\,|^2$.

The form of the deterministic equations guarantees that the Hilbert space vector representing the wave function of the system after a measurement of the spin S is $|\,S_1\rangle$ if $R_1>R_2$ and $|\,S_2\rangle$ if $R_2>R_1$. The inequality $R_1>R_2$ or $R_2>R_1$ is maintained after a measurement, since either $J_1=|\,\psi_1\,|^2$ or $J_2=|\,\psi_2\,|^2$ becomes and remains unity in this representation while the other becomes and remains zero. So, if $R_1>R_2$ *before* the measurement then, since $R_1=J_1/|\,\xi_1\,|^2$ increases (or remains the same) during the measurement, while $R_2=J_2/|\,\xi_2\,|^2$ decreases to zero (or remains the same), this inequality is satisfied *after* the measurement as well. Hence the result of a measurement will be reproduced if the same observable is measured immediately afterwards on the same system. This conclusion is not altered if the ξ_i should undergo a change after completion of the measurement, because if, for example, $J_1=1$, $J_2=0$ we always have $R_1=|\,\psi_1\,|^2/|\,\xi_1\,|^2=1/|\,\xi_1\,|^2>1$ (and $R_2=|\,\psi_2\,|^2/|\,\xi_2\,|^2=0$) for any values of ξ_1, ξ_2 (except $|\,\xi_1\,|^2=1$ and $|\,\xi_2\,|^2=0$, a set of measure zero which can be neglected.)

It is clear that we have developed here a theory of hidden variables that reproduces the usual probabilities of quantum mechanics as well as the feature of the "collapse" of the wave packet. However, the probabilities are now the result of a random distribution of "hidden" variables (and not "irreducible") and the "collapse" is due to a deterministic process that satisfies a law that could in principle be studied with regard to its order and structure of movement. Of course, the theory is similar to the usual quantum theory in that it also makes statistical assumptions of a certain kind. But it is different in the crucial respect that these assumptions permit a more general kind of statistics, containing the usual quantum statistics as one possible limit, and a "delta function" distribution of the hidden variables as another limit (in which the behavior is in fact completely specified and determined). In addition, the theory opens up the possibility of a further study, aimed at demonstrating that our statistical assumptions might be justified (as is done in classical statistical mechanics) by showing that the equations of motion of complex systems imply a quasi-ergodic character to the solutions. Evidently, such questions would have no meaning in terms of the usual linguistic structure of quantum mechanics.

Before investigating these consequences of the theory further, we shall outline briefly the generalization to a Hilbert space of N dimensions (where, in principle, $N\to\infty$).

The generalized deterministic equations describing the change in $|\,\Psi\rangle$ during a measurement process are:

$$d\psi_i/dt=\gamma\psi_i\sum_j J_j(R_i-R_j)\qquad(i=1,2,\cdots,N). \tag{5.14}$$

If $\sum_i|\,\psi_i\,|^2=\sum_i J_i=1$ initially, a simple calculation shows that $d(\sum_i J_i)/dt=0$ and the normalization of $|\,\Psi\rangle$ is therefore unaltered. Further, it is easily demonstrated that if R_s is a maximum then J_s will increase faster than any other J_i, so that eventually $J_s\to1$ and all the other $J_i\to0$. Thus, $|\,\Psi\rangle$ is projected onto the vector corresponding to the result s. If we assume that the dual vector $\langle\xi\,|$ is randomly distributed over the hyperspherical surface $\sum_i|\,\xi_i\,|^2=1$, then a similar calculation gives for the probability of the result s: $P_s=|\,\psi_s\,|^2$, in agreement with quantum mechanics. A remeasurement of the same observable will always yield the same result.

6. SOME CONSEQUENCES OF THE THEORY

It may now be of interest to re-examine von Neumann's proof in the light of this theory. As already indicated, the theory goes beyond the assumption of linearity in the proof: $\langle aR+bS+\cdots\rangle_{Av}=a\bar{R}+b\bar{S}+\cdots$. For example, in the spin case, let:

$$\mathbf{R}=\mathfrak{d}_x,\qquad \mathbf{S}=\mathfrak{d}_y,\qquad \mathbf{T}=\mathfrak{d}_z$$

$$a\mathbf{R}+b\mathbf{S}+c\mathbf{T}=\mathfrak{d}=\cos\alpha\mathfrak{d}_x+\cos\beta\mathfrak{d}_y+\cos\gamma\mathfrak{d}_y. \tag{6.1}$$

$\mathfrak{d}_x$, $\mathfrak{d}_y$, and $\mathfrak{d}_z$ are operators representing the components of the spin in three orthogonal directions. The operator $\mathfrak{d}$ represents the component of the spin in some arbitrary direction inclined to these axes. Consider the special case of a dispersion free ensemble, in which the hidden variables ξ_1, ξ_2, the components of the vector $\langle\xi\,|$ in the dual space, are precisely defined. These parameters determine the result of every possible measurement. Now, every measurement gives either $+1$ or -1 for σ_x, σ_y, σ_z, and σ. Hence, since the distribution is now a delta function of the hidden variables, we have

$$\bar{\sigma}=\sigma=\cos\alpha\sigma_x+\cos\alpha\sigma_y+\cos\alpha\sigma_z$$
$$=\cos\alpha(\pm1)+\cos\beta(\pm1)+\cos\gamma(\pm1). \tag{6.2}$$

On the other hand, we also have:

$$\sigma=\pm1.$$

Therefore, in general, it is not true that $\bar{\sigma}=\cos\alpha\bar{\sigma}_x+\cos\beta\bar{\sigma}_y+\cos\gamma\bar{\sigma}_z$. So the linearity assumption is not

satisfied by this model and von Neumann's proof has no relevance here.

The deterministic equations which we have postulated in addition to the movement defined by Schrödinger's equation have a form dependent on the representation in which the matrix of the operator representing the observable measured is diagonal:

$$\dot{\psi}_i=\gamma\psi_i\sum_j J_j(R_i-R_j).$$

Physically, we postulate that the representation chosen depends on the effect of the measuring apparatus. Thus, an apparatus that measures σ_x creates a different motion in the Hilbert space of the wave function than an apparatus that measures σ_y. These two motions are incompatible: a movement projecting the vector representative of Ψ onto one or another of a particular set of axes in Hilbert space cannot be carried out at the same time as a movement onto another set of axes. This provides a simple explanation of the impossibility of measuring simultaneously two "observables" whose operators do not commute. The two "observables" are associated with two different measuring instruments, represented in the theory by two different orthonormal bases in Hilbert space. From the point of view of the system alone, these "observables" represent certain "potentialities" only, i.e., the wave function of the system, represented as a vector in Hilbert space, determines only a set of potential results for each "observable". By defining a particular set of axes in the space, the formal effect of each apparatus is to choose a particular set of potentialities and the hidden variables, represented by the dual vector, then determine which result in this set is actually realized.

For this reason the term "observable" as used in ordinary quantum mechanics seems to be highly inappropriate. In our theory, the state of a system is determined by the wave function as well as the values of a certain set of hidden variables, i.e., these parameters are sufficient to determine the dynamical behavior of the system. These additional variables are regarded as being in principle observable or measurable, as in the statistical explanation of the macroscopic properties of matter we assume that the atomic variables are measurable. On the other hand, the quantum "observable" is no longer identified with any physical quantity or measurable property of the system in the usual (classical) sense of these terms and is not a dynamical variable of the system *alone*. Instead, each quantum observable is to be associated with a specific process of interaction between the system and a certain "apparatus", i.e., another system with certain well-defined characteristics which differentiate it both from the original system and from other apparatus systems. The hidden variables which define the state of the system in the sub-quantum level determine the outcome of this process and it is this result which is labelled with a specific value of the quantum observable. The process of interaction is said to constitute a "measurement" of the "observable."

Physically, the measuring apparatus is merely part of the large-scale environment of the system and it would be completely artificial to postulate a particular process for "measurement interactions" which would not also be characteristic of all interactions between a quantum system and its large-scale environment. Instead of Schrödinger's equation, therefore, we suggest that the equation of motion of the system is in general of the form:

$$\dot{\psi}_i=\gamma\psi_i\sum_j J_j(R_i-R_j)-(i/\hbar)\sum_j H_{ij}\psi_j \quad (i=1, 2,\cdots, n) \tag{6.3}$$

where the additional nonunitary term represents the effect of the large-scale environment on the system.

We may consider the possibility that the motion of the wave function depends to some extent on all levels of its environment, out to the cosmological scale. There may, for example, be a "natural" set of variables on the large-scale, say space and time, determined in some as yet unknown way by the relationship of the atomic to the cosmological level. Einstein's notion in the general theory of relativity, that the metric depends on the large-scale distribution of matter in the universe, does suggest a deep relation between the large-scale and small-scale levels. It is interesting to consider such an extension of his idea to include the "natural" observables.

Since our theory explicitly couples the large-scale and the small-scale levels together, all macro-quantities can not be calculated completely from the micro-laws. In fact, according to the Copenhagen interpretation, this was always implicit in quantum mechanics, but with the unacceptable qualification that such a multi-level theory dealing with the totality, in which quantum and subquantum movements depend on the large-scale movement and vice versa, is in principle incapable of being developed.

Lastly, it is clear that the idea of a measurement *disturbing* a system is completely inappropriate in this model. We have proposed that the wave function and Schrödinger equation do not provide adequate terms of description of the whole physical process, and we have extended these terms to include the dual space, the effects of the large-scale level (which includes the measuring instrument as a special case) and the equation coupling the wave function, the dual vector and the large-scale level. The motion described by Eq. (6.3) is in fact the usual motion of the system and a measurement process is only a special and simple case of this motion which leads to results that are easy to interpret. In practice, it is realized by an arrangement of matter for which γ is appreciable for long enough to carry

Ψ over to one of the eigenfunctions of the representation. If γ were not great enough during the time of the interaction, no *measurement* would be possible, but there would still be a complex change in Ψ that would be difficult to interpret. Thus, when a measurement takes place, the motion of the wave function is only *quantitively* different from otherwise, not *qualitatively*.

7. THE POSSIBILITY OF AN EXPERIMENTAL TEST OF THE THEORY

The theory has certain interesting consequences which suggest situations in which it might be possible to test these against the predictions of quantum mechanics.

In the two-dimensional example, which we have analyzed in detail, suppose a measurement of the spin is made in a certain direction. Now, in order to reproduce the usual quantum mechanical probabilities it was necessary to assume that the dual vector $\langle\xi|$ was randomly distributed on the surface of the unit hypersphere. However, after the measurement, this vector will no longer be completely random, since from the result of the measurement it is possible to deduce that the components of $\langle\xi|$ satisfy one or other of the relations:

$$|\psi_1|^2/|\xi_1|^2>1 \qquad (\text{and } |\psi_2|^2/|\xi_2|^2<1) \qquad (7.1)$$

or

$$|\psi_1|^2/|\xi_1|^2<1 \qquad (\text{and } |\psi_2|^2/|\xi_2|^2>1). \qquad (7.2)$$

Therefore, the quantum mechanical probabilities will not in general be reproduced for a subsequent measurement of the spin component in some other direction, unless there are further equations of motion which define a quasi-ergodic movement of the ξ_i, leading to an eventual random distribution of these variables over the unit "shell." These equations may, for example, be due to a coupling of the ξ_i with the ψ_i which is negligible during a measurement. As we have shown, such a movement in the ξ_i after measurement will not affect the inequalities $R_1>R_2$ or $R_2>R_1$, and so will not alter the requirement of reproducibility of the measurement result. However, in a long enough time after a measurement the ξ_i may be completely randomized so that the theory will imply the usual quantum mechanical probabilities for the results of a subsequent measurement.

This raises the interesting question of the amount of time needed to randomize the dual vector. This is, of course, unknown, but some plausible suggestions can be made at this stage.

(i) Since most systems are either in thermal equilibrium or have emerged from a source in equilibrium, it seems plausible to assume a random process resulting from this. The characteristic unit of time of thermal processes in relation to quantum mechanics is: $\tau=\hbar/kT\simeq10^{-13}$ sec, for room temperatures. Typical measurements generally involve longer times, so that the usual results of quantum mechanics are to be expected.

(ii) The time for randomization might be related to the lifetime τ_0 of the quantum state in the conventional sense of the word. This would require that the equations of motion for the dual vector couple it to the wave function in additional ways that would make the randomization process depend on what is at present called the "quantum state."

The double Stern–Gerlach experiment, in which two consecutive measurements are made of the spin components in different directions, might provide a possible test of the theory. Such experiments have not so far been done for short enough times with the required accuracy in statistics (about 1%).

Otherwise, phenomena of superconductivity, superfluidity, or low-temperature measurements seem to offer the best possibilities of tests.

Besides investigating the dependence of the statistics of successive measurements on the time between them, one could investigate the time needed to "complete" a measurement, i.e., to bring ψ_s to exp $(i\phi_s)$ and all the other ψ_i to 0 (the "collapse" of the wave packet).

A discrepancy with the usual predictions of quantum mechanics would overthrow the whole conceptual structure and logical foundation of orthodox quantum mechanics and open up the possibility of investigating the order and structure in the process of measurement.

8. MEASUREMENT OF THE "HIDDEN VARIABLES"

As indicated, the ψ_i are "observable." At present, the ξ_i are *unobserved*, since no experiments have so far been designed to test for their effects. This is, of course, understandable since the very existence of such variables has generally been tacitly and explicitly assumed to be in conflict with the experimental facts of quantum mechanics. However, they are not intrinsically unobservable. Thus, in the spin example, after the measurement of the spin component in a certain direction, if there has been no time for randomization, $|\Psi\rangle$ is either $|S_1\rangle$ or $|S_2\rangle$ and hence the variables ξ_1, ξ_2 must be such as to satisfy either the inequality $R_1>1$, or $R_2>1$. This is a *rough* observation of the ξ_i. A subsequent measurement of the spin component in a different direction would restrict the possible range of values for the ξ_i still further. This process can be continued until a fairly accurate measurement has been made. It is clear that the "hidden" variables are in principle as observable as the ψ_i, but they must be observed with the aid of techniques based on new relationships arising in a time sequence of successive measurements relationships that cannot even be stated or thought about in terms of the linguistic structure of the usual interpretation of the mathematical formalism of quantum mechanics.

9. GENERALIZATION TO A RELATIVISTIC THEORY

It may be as well to mention here that the equation of motion:

$$\dot{\psi}_i=\gamma\psi_i\sum_j J_j(R_i-R_j)-(i/\hbar)\sum_j H_{ij}\psi_j \quad (i=1,2,\cdots,n) \tag{6.3}$$

is both nonlocal and nonlinear. The change in the component ψ_i of the state vector depends in a complicated way on the values of all the other components. The analogous equation for the continuous case:

$$\dot{\psi}_x=\gamma\psi_x\int J_y(R_x-R_y)\,dy-(i/\hbar)H\psi_x \tag{9.1}$$

would therefore imply that the change in the wave function at a particular point in space depends on the values of the wave function at every other point in space. Clearly there is no possibility of generalizing this theory to include relativistic phenomena if the equation of motion is retained in its present form, which incorporates an explicit causal mechanism whereby any effect in one part of space is instantaneously transmitted to another part. The theory is therefore limited to nonrelativistic phenomena.

Another limitation of the theory, connected with this nonlocal character of the equation of motion, is that only *complete* measurements are now feasible. For example, in the case of a composite system consisting of two interacting particles, the theory only deals with a process of measurement in which the state vector of the composite system "collapses" onto an eigenvector of an observable of the composite system. The possibility of carrying out a measurement on one particle alone does not exist: this can be interpreted now only in the sense that a complete measurement has been made and certain information ignored. Consequently, the paradox of Einstein, Podolski, and Rosen does not arise in this theory, because a composite system must always be regarded as an indivisible totality which in principle cannot be subdivided into independently existing units. (This is really the essence of Bohr's refutation of the paradox.) Of course, this is a fault and not an advantage of the theory, which has been formulated in terms which exclude the framing of the paradox. If the paradox is accepted as a real inadequacy in our present theoretical description of certain phenomena, then the problem of finding a theory which satisfactorily resolves the paradox remains.

As we have repeatedly emphasized, we regard the present nonrelativistic theory merely as a step towards a more elaborate theory, which should include relativistic phenomena. The Einstein–Podolski–Rosen paradox seems relevant in this connection and suggests a possible line of research, i.e., we require a theory which resolves the paradox in the terms in which it is presented. Nevertheless, in spite of its shortcomings, the nonrelativistic theory does provide a theoretical structure which makes it possible to discuss relationships which go beyond those of formal quantum mechanics, relationships which we believe are relevant to the understanding of the measurement problem.

10. IRREVERSIBILITY OF THE FUNDAMENTAL EQUATIONS OF MOTION

It is a well-known fact that the equations of motion of both classical and quantum mechanics are time-reversible, in the sense that to every solution of the equations representing a forward motion there is a corresponding solution representing a reversed motion. The irreversible behavior of macro-systems is explained in a statistical sense as the average or probable movement of a system from one imprecisely defined state to another. Boltzmann's H theorem is applicable to such systems and the irreversible transition to a condition of macroscopic or thermodynamic equilibrium is characterized as a transition to a condition corresponding to a minimum value of H, i.e., a maximum value of the entropy, $-H$. In formal quantum mechanics, an entropy can be defined which is zero for pure ensembles and positive for mixed ensembles so that the collapse of the wave packet leads to an increase in entropy, i.e., the process of measurement itself introduces an irreversible change in the state of the system measured. However, as we have seen, the measurement process in quantum mechanics is rather obscure and even an analysis which treats the measuring instrument as a macroscopic system cannot avoid certain inherent problems. Nevertheless, it does seem that the irreversibility associated with a measurement process, in the sense of an interaction between a micro-system and a macro-system, may be very significant for the whole question of irreversibility and the "arrow of time" in general which, in view of certain recent experiments indicating a breakdown of the CPT theorem, is still a controversial subject.

In the theory developed here, the fundamental equations of motion are clearly irreversible. The terms $\gamma\psi_i\sum_j J_j(R_i-R_j)$ in the equations (6.3):

$$\dot{\psi}_i=\gamma\psi_i\sum_j J_j(R_i-R_j)-(i/\hbar)\sum_j H_{ij}\psi_j \quad (i=1,2,\cdots,n)$$

describe an exponential-type decay of $(n-1)$ components ψ_j of the wave function in a particular representation and a simultaneous exponential-type growth of one component ψ_s to a limiting absolute value of 1. This process is intrinsically different in character from the reverse process [an exponential-type growth of $(n-1)$ components of the wave function and a simultaneous exponential-type decay of one component], which could not, therefore, be described by equations

of the same form. Hence there is no reverse motion of the system which would develop the initial past wave function from the final eigenfunction produced by the movement described by these equations.

As well as being reversible, the equations of motion of classical mechanics are time-symmetrical with repect to the *prediction* of the results of future measurements from a given initial state and the *retrodiction* of the results of past measurements from a given final state. Thus, Hamilton's equations can be used to retrodict the past, by following the solution of the equations backwards in time, in the same way that they can be used to predict the future, by following the solution forwards in time. On the other hand, in quantum mechanics, whereas the equation of motion for the wave function is formally time-reversible, the probabilistic interpretation introduces an asymmetry with respect to prediction and retrodiction. For example, an initial pure ensemble of systems, each in the same quantum state, will yield a mixed ensemble after measurement. From a knowledge of the initial quantum state, it is possible to predict the relative frequencies with which the different states will actually be found in the measurement. However, if one uses Schrödinger's equation to calculate what the wave function must have been before the measurement, this will not in general give a correct statistical retrodiction of the earlier states, from which the system came (assuming, for example, that someone else had already made such a set of measurements and that the data were available).

It seems to be characteristic of all statistical theories, and not necessarily a feature of quantum mechanics alone, that the predictive and retrodictive situations are symmetrical only under certain special circumstances. Watanabe[23] has argued that phenomenological one-way-ness has its origin in the very notion of probability, i.e., the temporal asymmetry is related to irretrodictability and has nothing to do with the structure of the dynamical laws. He formulates general criteria for retrodictability and predictability and shows that quantum mechanics is irretrodictable, microscopically as well as macroscopically, whether or not it is invariant with respect to time-reversal in the usual sense.

Aharanov, Bergmann, and Lebowitz[24] have also investigated time-symmetrical situations in quantum mechanics. They demonstrate that if both the initial and final states are fixed, then the predictive and retrodictive formulae, from the initial and final state, respectively, to an intermediate state (later in time than the initial state, but earlier in time than the final state) are symmetrical. If only one state is fixed, i.e., the initial or final state, the predictive and retrodictive formulae are asymmetrical. The explanation suggested for this asymmetry is that, while the future behavior of a micro-system depends on the present (macroscopic) boundary conditions, which define its quantum state, we do not believe that the past behavior of the system can be affected by the boundary conditions which we choose to establish now.

In our theory, the equations of motion (6.3) can be used to predict, for a certain future time, the result of the measurement of the "observable" defining the representation, if the hidden variables are known, or the probabilities of different results if the hidden variables are not known. They make no sense retrodictively. Thus, the asymmetry of prediction and retrodiction is associated with the time irreversibility of the fundamental equations.

In order to solve the problem of measurement in quantum mechanics, we have proposed a hidden variable theory which incorporates a basic irreversibility into the dynamics of an interaction coupling micro- and macro-levels. This is a new concept; it provides a fresh insight into the asymmetry of prediction and retrodiction in quantum mechanics and its development in a more sophisticated theory might conceivably clarify the relationship between macro- and micro-physics and the nature and origin of irreversibility in general.

11. CONCLUSION

We have demonstrated the falsity of von Neumann's proof, which rejects the possibility of hidden variables underlying the statistics of quantum mechanics, both by exposing the inadequacy of the assumptions upon which the proof is based and by explicitly developing a hidden variable theory.

The current formulation of quantum mechanics must be regarded merely as a statistical algorithm, which provides no conceptual structure in terms of which the movement of individual systems can be understood. In order to be able to raise questions about the behavior of individual systems, it seems to be necessary to extend the linguistic structure of the theory by the concept of hidden variables. The problem of measurement is at present a problem only because of an inadequate set of concepts and the confusion which arises with the attempt to interpret this set as complete. There is, therefore, no solution to the measurement problem within the framework of quantum mechanics.

A careful analysis of the reasons for the inadequacy of the von Neumann proof suggests that, in order to succeed, a hidden variable theory should incorporate a nonlinear relation between the value of a quantum mechanical observable and the precise state of the system. In the theory proposed here, a completely specified physical state is defined by two vectors: the Hilbert space vector of the system and the dual vector.

[23] S. Watanabe, Suppl. Progr. Theoret Phys. (Kyoto) Extra Number (Commemorative Issue for the 30th Anniversary of the Meson Theory by Dr. H. Yukawa), p. 135 (1965).

[24] Y. Aharanov, P. G. Bergmann, and J. L. Lebowitz, Phys. Rev. **134**, B1410 (1964).

From this point of view, the quantum mechanical specification of the state of a system by a wave function only, represented by a vector in Hilbert space, is incomplete and we recover the statistical results of quantum mechanics from an ensemble of systems with the same $|\Psi\rangle$ vector but with a random distribution for the dual vector $\langle\xi|$ representing the hidden variables. The "collapse" of the wave packet in a measurement process is described, by a nonlinear deterministic equation of motion, as a real dynamical process which takes a certain amount of time to complete. The measuring apparatus is regarded as a part of the large-scale environment of the system and its influence is reflected in the above equation of motion by the representation chosen. This suggests a general modification to Schrödinger's equation and an interpretation of a physical process as a coupling of the large-scale and small-scale levels. The measurement process is then merely a particular case of this process. This theory has some interesting consequences, which may be used to provide an experimental comparison with the predictions of quantum mechanics. In particular, the equation of motion introduces an inherent irreversibility into all physical processes, which becomes especially significant in the case of a measurement process.

The theory, in its present form, suffers from a number of inadequacies, but it does provide a new conceptual structure in which certain questions can be considered, which cannot even be formulated within the framework of quantum mechanics. Thus, to describe the same facts in a new linguistic structure may be significant. The rather prevalent idea that differences of language for saying the same thing are *always* unimportant is false. In some cases a different language opens up a different structure of thinking and thus leads to new kinds of actions in relationship to nature. What is needed now is a hypothetical tentative approach, to attempt both by theory and by experiment to inquire into the conditions in which quantum mechanics might break down, to reveal a new structure of physical law and a new order in physical movement. Experiments devised in order to study questions raised in such an inquiry could, in principle, falsify the basic principles of quantum mechanics and show the need for new ones.