Group Theory and the Hydrogen Atom (II)

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A previous work dealing with bound states in a Coulomb potential is extended to the case of scattering states. The symmetry of the problem under the Lorentz group 0(1, 3) is used to construct wave functions. Harmonic analysis on two-sheeted hyperboloids is briefly discussed in arbitrary dimension. The set of scattering states for both an attractive and a repulsive potential is shown to provide a unitary representation of the group 0(1, 4).

I. INTRODUCTION

In a previous article, hereafter denoted by I, we have reviewed the symmetries of the Schrödinger equation for a Coulomb potential and discussed the use of a noncompact group isomorphic to the pseudo-orthogonal group 0(1, 4) to relate the bound state levels.[1] It is of some interest to pursue the analysis further to the scattering states. This is the goal of this work. The "hidden symmetry" is now an invariance under the homogeneous Lorentz group 0(1, 3) and the space of scattering states can be written as a direct integral of infinite dimensional Hilbert spaces, which are carrier spaces of unitary representations of the Lorentz group. This is a frequent occurrence when dealing with a noncompact group, the simplest example being Fourier analysis for the group of translations in one dimension. We shall follow the usual device of introducing nonnormalizable scattering states in order to achieve the decomposition. Physically, of course, the direct integral is related to the continuous spectrum of the Hamiltonian. As in the bound state case, there exists a larger group, isomorphic to 0(1, 4) which connects the scattering states, but we have to introduce wave functions for both attractive and repulsive potentials. We shall show that the representation of this group, obtained in this way, is equivalent with the one previously discussed in I. This representation is in fact not unique and the ambiguity which arises is the same as the one already encountered.

We shall slightly generalize the discussion by taking an arbitrary dimension f for the configuration space. In order to construct the wave functions it will be necessary to use harmonic analysis on a two-sheeted hyperboloid. This has been developed in four dimensions in a series of papers by Dolginov and collabrators[2] but we shall briefly recall the main features including orthogonality and completeness of the "spherical functions." This part can perhaps be used for other purposes in a different context.

In an appendix we have performed the necessary transformations in order to show that the wave functions coincide with their ordinary expressions in configuration space, both for scattering and bound states.

We will have to use repeatedly the following notations:

S_p: unit sphere in a p-dimensional real Euclidian space, the measure on the sphere being $d^{p-1}\Omega$ with

$$\int_{S_p} d^{p-1}\Omega = \omega_p = \frac{2\pi^{p/2}}{\Gamma(p/2)}.$$

$\delta_{\rm sph}(n_1, n_2)$ will stand for the "δ function" on S_p, i.e.,

$$\int_{S_p} d^{p-1}\Omega(n_1)\, g(n_1)\, \delta_{\rm sph}(n_1, n_2) = g(n_2).$$

T_p: unit hyperboloid in a p-dimensional real Euclidian space,

$$u: u_0^2 - \sum_{i=1}^{p-1} u_i^2 = 1.$$

$T_p^{\pm}$ will denote, respectively, the upper ($u_0 \geq 1$) or lower ($u_0 \leq -1$) sheet of this hyperboloid. Our constant parameterization of T_p^+ will be: $u_0 = \cosh\theta$, $u_i = \sinh\theta n_i$, with $\theta \geq 0$ and n_i on S_{p-1}. The measure on T_p^+ will be written as $d^{p-1}\mu$ with $d^{p-1}\mu = \sinh\theta^{p-2} d\theta d^{p-2}\Omega$. The measure on T_p^- will be related to the one on T_p^+ by the transformation $u \epsilon T_p^- \rightarrow u\epsilon T_p^+$. Finally, $\delta_{\rm hyp}(u_1, u_2)$ will stand for the "δ function" on T_p^+.

II. THE SYMMETRY GROUP

A. Infinitesimal Method

We study the scattering states in a Coulomb potential. As in I, we introduce the following two vector operators:

The angular momentum,

$$\mathbf{L} = \tfrac{1}{2}(\mathbf{r}\times\mathbf{p} - \mathbf{p}\times\mathbf{r})$$

The Runge–Lenz vector,

$$\mathbf{M} = (1/2\mu)(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}) - k\hat{r} \qquad (1)$$

with $\hat{r}$ standing for $\mathbf{r}/|\mathbf{r}|$; $\mathbf{p}$ is the linear momentum,

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[1] M. Bander and C. Itzykson, Rev. Mod. Phys. **38**, 330 (1966), preceding paper.

[2] A. Z. Dolginov, Zh. Eksperim. i Teor. Fiz. **30**, 746 (1956) [English transl.: Soviet Phys.—JETP **3**, 589 (1956)]; A. Z. Dolginov and I. N. Toptygin, Zh. Eksperim. i Teor. Fiz. **37**, 1441 (1959) [English transl.: Soviet Phys.—JETP **10**, 1022 (1960)]; A. Z. Dolginov and A. N. Moskalev, Zh. Eksperim. i Teor. Fiz. **37**, 1967 (1959)] [English transl.: Soviet Phys.—JETP **10**, 1202 (1960)].

μ the reduced mass; k denotes the strength of the potential.

The Hamiltonian

$$H=(\mathbf{p}^2/2\mu)-(k/|\mathbf{r}|)$$

commutes with both these vectors. They also satisfy the following commutation relations:

$$[L_i, L_j]=i\hbar\epsilon_{ijk}L_k, \qquad [M_i, M_j]=-(2i\hbar H/\mu)\epsilon_{ijk}L_k$$
$$[L_i, M_j]=i\hbar\epsilon_{ijk}M_k. \tag{2}$$

Consider the subspace corresponding to the positive spectrum of H. In this subspace let $\tilde{M}_i=(\mu/2H)^{\frac{1}{2}}M_i$, where the square root of H is defined by the condition of positiveness. Then L_i and $\tilde{M}_i$ build up the Lie algebra of the homogeneous Lorentz group $0(1, 3)$ [our notation implies that $0(p, q)$ is the pseudo-orthogonal real group leaving the metric with p plus signs and q minus signs invariant]. In terms of L and $\tilde{M}$ the Hamiltonian reads:

$$H=-(k^2\mu/2)[L^2-\tilde{M}^2+\hbar^2]^{-1}.$$

The requirement that H be Hermitian restricts L and $\tilde{M}$ to be likewise Hermitian, and hence we are interested in unitary representations of the Lorentz group.[3] These representations are labeled by two numbers (l_0, c) with l_0 a nonnegative integer and c pure imaginary, $c=i\rho$ (the principal series) or $l_0=0$, $0\leq c\leq 1$ (the supplementary series). $(L^2-\tilde{M}^2)/\hbar^2$ is a Casimir operator equal for each representation to $l_0^2+c^2-1$. From Eq. (1) it follows that we have one further relation among L and $\tilde{M}$, namely,

$$L\tilde{M}+\tilde{M}L=0=2il_0c. \tag{3}$$

Equation (3) thus restricts us to the representations with $l_0=0$. Thus

$$E=-(k^2\mu/2\hbar^2)(1/c^2), \tag{4}$$

where E is the energy; and as we have assumed H to be positive $c=i\rho$ and we are only concerned with representations of the principal series.

B. Global Method—The Fock Transformation

As mentioned above, we shall now generalize the problem to an arbitrary dimension $f\geq 2$. The Schrödinger equation in momentum space takes the form

$$(p^2-2\mu E)\Phi(\mathbf{p})=\frac{2\mu k}{\pi\omega_{f-1}\hbar}\int d^fq\,\frac{\Phi(\mathbf{q})}{|\mathbf{p}-\mathbf{q}|^{f-1}} \tag{5}$$

with $E>0$. Changing variables from $\mathbf{p}$ to $(2\mu E)^{-\frac{1}{2}}\mathbf{p}$ and letting

$$\Phi(\mathbf{p})=\Psi[(2\mu E)^{-\frac{1}{2}}\mathbf{p}]$$

[3] M. A. Naimark, *Linear Representations of the Lorentz Group*, American Mathematical Society Translations (American Mathematical Society, Providence, R.I., 1957), Series 2, Vol. 6.

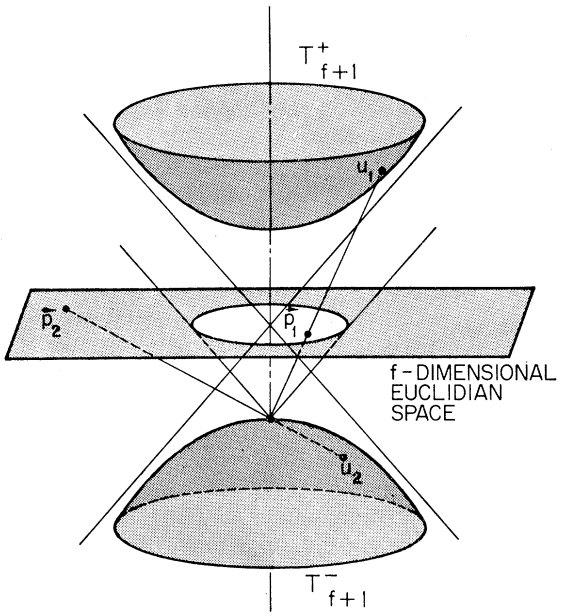

FIG. 1. Stereographic projection of an f-dimensional space to the unit hyperboloid in $f+1$ dimensions.

Eq. (5) becomes

$$(p^2-1)\Psi(\mathbf{p})=\frac{k(2\mu)^{\frac{1}{2}}}{\pi\omega_{f-1}\hbar E^{\frac{1}{2}}}\int d^fq\,\frac{\Psi(\mathbf{q})}{|\mathbf{p}-\mathbf{q}|^{f-1}}. \tag{6}$$

We now imbed the f-dimensional space into one of $f+1$ dimensions and perform a projection of the original momentum space onto a two-sheeted hyperboloid, T_{f+1} (the Fock transformation, see Fig. 1). Let u be an arbitrary point in the $(f+1)$-dimensional space with component u_0 along the $f+1$ direction and $\mathbf{u}$ its ordinary projection in the original space. We introduce a Minkowski metric into this space, i.e., $u^2=u_0^2-\mathbf{u}^2$. The hyperboloid is given by the equation $u^2=1$ and the point $\mathbf{p}$ corresponds to a point on this hyperboloid with coordinates:

$$\mathbf{p}\to u=\left(\frac{1+p^2}{1-p^2}, \frac{2\mathbf{p}}{1-p^2}\right). \tag{7}$$

The region $p^2<1$ is mapped on the upper sheet while $p^2>1$ is mapped on the lower one. We shall need the following relations. If $\mathbf{p}$ and $\mathbf{q}$ correspond to u and v, respectively, then

$$|\mathbf{p}-\mathbf{q}|^2=-\frac{(u-v)^2}{(1+u_0)(1+v_0)}=\frac{|(u-v)|^2}{|1+u_0||1+v_0|}$$

$$d^fp=\frac{2\delta(u^2-1)d^{f+1}u}{|1+u_0|^f}=\frac{d^f\mu(u)}{|1+u_0|^f},$$

where we remind ourselves that all scalar products involving u and v must be taken with the Minkowski

metric. For u given by Eq. (7) and

$$\hat{\Phi}(u) = |1+u_0|^{-\frac{1}{2}(f+1)}\Psi(\mathbf{p}) \tag{8}$$

we obtain the following equation for $\hat{\Phi}$:

$$\hat{\Phi}(u) = \frac{k(2\mu)^{\frac{1}{2}}\epsilon(u_0)}{2\pi\omega_{f-1}\hbar E^{\frac{1}{2}}}\int_{T_{f+1}} d^f\mu(v)\,\frac{\hat{\Phi}(v)}{|(u-v)^2|^{\frac{1}{2}(f-1)}} \tag{9}$$

with $\epsilon(u_0)=+1$ if $u_0>1$, $\epsilon(u_0)=-1$ if $u_0\leq -1$. The above equation exhibits explicitly the invariance of the problem under the group of homogeneous, metric preserving, transformations in a $(f+1)$-dimensional Minkowski space, i.e., under the group $0(1,f)$.

Equation (9) is of the same type as the one obtained in I for bound states which was solved using the properties of spherical harmonics on the sphere. It will turn out that the solution, in the present case, can also be obtained by introducing a set of "spherical functions" on the hyperboloid. (The terminology is somehow misleading and hyperbolic functions would seem more appropriate; however, we stick to this name which is apparently of general use.) However, while the spherical harmonics are well known, the corresponding functions for the "Lorentz group" $0(1,f)$ enjoy less popularity. As mentioned in the introduction, they were studied in particular for $f=3$ by Dolginov and collaborators.[2] The case of $f=2$ was also used in the context of Regge poles but dates back in the mathematical literature to Mehler. We shall for the moment interrupt our discussion of the Coulomb problem to give a description of these functions in order to apply them to the solution of Eq. (9). However, they certainly deserve some study for their own sake and, while exhibiting some results with lots of "δ-functions," we shall be careful to present the proofs in such a way that they can, hopefully, be made rigorous.

C. Definition of Spherical Functions on Hyperboloids

Let T_{f+1}^+ be the upper sheet, $u_0\geq 1$, of the hyperboloid T_{f+1},

$$u: u_0^2 - \sum_{i=1}^{f} u_i^2 = 1.$$

Given two points on T_{f+1}^+, u_i and u_2, there always exists a transformation $\Lambda\in 0^{(+,+)}(1,f)$ such that $u_1=\Lambda u_2$. $0^{(+,+)}(1,f)$ is the component of the identity of $0(1,f)$. The set of transformations which leave a point invariant is isomorphic to a proper rotation group $0^{(+)}(f)$ so that $T_{f+1}^+\approx 0^{(+,+)}(1,f)/0^{(+)}(f)$. The measure of $d^f\mu$ is invariant under $0^{(+,+)}(1,f)$. Hence the Hilbert space $\mathfrak{H}_{f+1}$ of square integrable functions defined on T_{f+1}^+,

$$\mathfrak{H}_{f+1} = \left\{g: \int_{T_{f+1}^+} |g|^2\, d^f\mu < \infty\right\},$$

is the carrier space of a unitary representation of the noncompact group $0^{(+,+)}(1,f)$

$$g\to U_\Lambda g \quad \text{with} \quad (U_\Lambda g)(u) = g(\Lambda^{-1}u).$$

This representation is not irreducible and decomposes into a direct integral of irreducible ones:

$$U_\Lambda = \int_0^\infty dN\, U_\Lambda^N.$$

Each U_Λ^N is itself a unitary irreducible representation of the group but again the noncompactness results in the fact that it is infinite dimensional. Different N corresponds to inequivalent representations. Correspondingly, the space $\mathfrak{H}_{f+1}$ will be split as a direct integral of infinite dimensional Hilbert spaces and any function in $\mathfrak{H}_{f+1}$ will have a representation

$$g(u) = \int_0^\infty dN \sum_\nu g_{N,\nu}(u), \tag{10}$$

where ν is a discrete index which distinguishes the components in the space of the representation U_Λ^N. The function $g_{N,\nu}(u)$ will not belong to $\mathfrak{H}_{f+1}$ but will satisfy a certain partial differential equation in terms of the Casimir operator of the group. Having properly chosen the indices, $g_{N,\nu}$ will be proportional to a spherical function. We shall now derive the equation satisfied by these functions. Let $u\equiv(\cosh\theta, \sinh\theta\,\mathbf{n})$, we shall investigate functions of the type

$$H_{N,\alpha,\beta}^{(f+1)}(u) = Z_{N,\alpha}^{(f)}(\theta)\, Y_{\alpha,\beta}^{(f)}(\mathbf{n}), \tag{11}$$

where $Y_{\alpha,\beta}^{(f)}(\mathbf{n})$ is a spherical harmonic on the sphere S_f. The discrete index α takes the values $\alpha=0, 1, 2, \cdots$. If $\mathbf{x}=r\mathbf{n}$ is a point in an f-dimensional Euclidian space, $r^\alpha Y_{\alpha,\beta}^{(f)}(\mathbf{n})$ is a homogeneous polynomial of degree α in the components of $\mathbf{x}$ and

$$\Delta_f[(r^\alpha Y_{\alpha,\beta}^{(f)})(\mathbf{n})] = 0$$

with the Laplace operator given in polar coordinates by

$$\Delta_f \equiv \frac{\partial^2}{\partial r^2} + \frac{f-1}{r}\frac{\partial}{\partial r} - \frac{\mathfrak{L}^2}{r^2}$$

and $\mathfrak{L}^2$ is the generalization of the angular momentum and operates only on the angular variables. Hence

$$\mathfrak{L}^2 Y_{\alpha,\beta}^{(f)}(\mathbf{n}) = \alpha(\alpha+f-2)\, Y_{\alpha,\beta}^{(f)}(\mathbf{n}).$$

The index β distinguishes the finitely many, linearly independent, solutions of this equation. Once normalized, the spherical harmonics satisfy:

$$\sum_{\alpha,\beta} Y_{\alpha,\beta}^{(f)}(n_1)\,(Y_{\alpha,\beta}^{(f)}(n_2))^* = \delta_{\text{sph}}(n_1, n_2) \tag{12}$$

$$\int_{S_f} d^{f-1}\Omega(n)\, Y_{\alpha_1,\beta_1}^{(f)}(n)\,(Y_{\alpha_2,\beta_2}^{(f)}(n))^* = \delta_{\alpha_1,\alpha_2}\,\delta_{\beta_1,\beta_2}. \tag{13}$$

In $f+1$ dimensions we write the wave equation in the

neighborhood of T_{f+1}^+:

$$\Box_{f+1}\equiv\frac{\partial^2}{\partial w_0^2}-\sum_{i=1}^{f}\frac{\partial^2}{\partial w_i^2}=\frac{\partial^2}{\partial\rho^2}+\frac{f}{\rho}\frac{\partial}{\partial\rho}-\frac{\mathfrak{C}}{\rho^2}, \tag{14}$$

where $\mathfrak{C}$ is the Casimir operator on the hyperboloid and $w_0=\rho\cosh\theta$, $w_i=\rho\sinh\theta n_i$, ρ close to $+1$. Then $H_{N,\alpha,\beta}(u)$ is required to be an eigenfunction of $\mathfrak{C}$, or $\rho^\lambda H_{N,\alpha,\beta}(u)$ defined in the neighborhood of T_{f+1}^+ to be an homogeneous function of $w=\rho u$, of degree λ, with $\lambda=-\frac{1}{2}(f-1)+iN$ (N will turn to be real) such that

$$\Box_{f+1}[\rho^\lambda H_{N,\alpha,\beta}(u)]=0.$$

The condition on N will be obtained by requiring that $H_{N,\alpha,\beta}^{(f+1)}(u)$ be a continuous function of T_{f+1}^+ with the smallest possible growth at infinity. The above conditions provide for $Z_{N,\alpha}(\theta)$ the following equation:

$$\left\{\sinh^2\theta\frac{d^2}{d\cosh\theta^2}+f\cosh\theta\frac{d}{d\cosh\theta}-\frac{\alpha(\alpha+f-2)}{\sinh^2\theta}+N^2+\left(\frac{f-1}{2}\right)^2\right\}Z_{N,\alpha}^{(f)}(\theta)=0. \tag{15}$$

The boundary conditions just mentioned require N to be real and the relevant solutions of Eq. (15) are for our purpose—including a normalization to be discussed later:

For $f_{f\geq 3}$ odd;

$$Z_{N,\alpha}^{(f)}(\theta)=\{\tfrac{1}{2}\pi N^2(N^2+1^2)\cdots[N^2+(f-1+\alpha)^2]\}^{-\frac{1}{2}}\times\sinh\theta^\alpha\left(\frac{d}{d\cosh\theta}\right)^{\frac{1}{2}(f-1)+\alpha}\cos N\theta, \tag{16}$$

For $f_{f\geq 2}$ even;

$$Z_{N,\alpha}^{(f)}(\theta)=\left[\frac{[N^2+(\frac{1}{2})^2][N^2+(\frac{3}{2})^2]\cdots\{N^2+\frac{1}{2}[(f-2)+\alpha-\frac{1}{2}]^2\}}{N\tanh\pi N}\right]^{-\frac{1}{2}}\times\sinh\theta^\alpha\left(\frac{d}{d\cosh\theta}\right)^{\frac{1}{2}(f-2)+\alpha}P_{iN-\frac{1}{2}}(\cosh\theta), \tag{17}$$

where it is understood in Eq. (17) that the factor on the right-hand side reduces to $(N\tanh\pi N)^{\frac{1}{2}}$ for $f=2$; $P_{iN-\frac{1}{2}}(\cosh\theta)$ is the Legendre function, and we shall follow throughout and use Chap. III of Ref. 4 for these functions which in this context are called conical functions. In both cases we have an even function of N; in the following it is assumed that N is positive. With $Z_{N,\alpha}^{(f)}(\theta)$ given by Eq. (16) or (17), we have the following basic three relations for spherical functions:

Orthogonality relations, N_1 and N_2 are positive:

$$\int_{T_{f+1+}}d^f\mu(u)H_{N_1,\alpha_1,\beta_1}^{(f+1)}(u)\bar{H}_{N_2,\alpha_2,\beta_2}^{(f+1)}(u)=\delta_{\alpha_1,\alpha_2}\delta_{\beta_1,\beta_2}\,\delta(N_1-N_2), \tag{18}$$

completeness relations in $\mathfrak{IC}_{f+1}$:

$$\int_0^\infty dN\sum_{\alpha,\beta}H_{N,\alpha,\beta}^{(f+1)}(u_1)H_{N,\alpha,\beta}^{(f+1)}(u_2)=\delta_{\text{hyp}}(u_1,u_2), \tag{19}$$

integral equation

$$\int_{T_{f+1+}}d^f\mu(v)\frac{H_{N,\alpha,\beta}^{(f+1)}(v)}{(1+t^2+2tuv)^{\frac{1}{2}(f-1)}}=\frac{2\pi^{\frac{1}{2}(f+1)}}{\Gamma\frac{1}{2}(f-1)}\times\frac{t^{-\frac{1}{2}(f-1)}\cos(N\log t)}{N\sinh\pi N}H_{N,\alpha,\beta}^{(f+1)}(u). \tag{20}$$

In Eq. (20) t is a complex variable in a cut plane from $-\infty$ to 0; $\log t$ is real for t real positive and the argument of the expression $(1+t^2+2tuv)$ which appears in the integrand is 0 for t real positive. As is apparent from their definition [Eqs. (16) and (17)], the spherical functions are somehow different according to whether f is even or odd. This again reflects the fact that the kernel in Eq. (20) has a square root singularity for even f. Hence, we shall distinguish between the two cases to prove our basic three relations.

D. Proof of the Three Relations in the Case f Odd

We start with the orthogonality relations. According to Eqs. (11) and (13) one has

$$\int_{T_{f+1+}}d^f\mu(u)H_{N_1,\alpha_1,\beta_1}^{(f+1)}(u)H_{N_2,\alpha_2,\beta_2}^{(f+1)}(u)=\delta_{\alpha_1,\alpha_2}\delta_{\beta_1\beta_2}\times\int_0^\infty\sinh\theta^{f-1}Z_{N_1,\alpha_1}^{(f)}(\theta)Z_{N_2,\alpha_1}^{(f)}(\theta)\,d\theta.$$

Using the definition of $Z_{N,\alpha}^{(f)}$ [Eq. (16)], this reduces to the study of the integral:

$$\int_0^\infty d\theta\sinh\theta^{f-1+2\alpha_1}\left[\left(\frac{d}{d\cosh\theta}\right)^{\frac{1}{2}(f-1)+\alpha_1}\cos N_1\theta\right]\times\left[\left(\frac{d}{d\cosh\theta}\right)^{\frac{1}{2}(f-1)+\alpha_1}\cos N_2\theta\right],$$

Let us introduce the notation

$$q_p(N,\Omega)=\sinh\theta^p(d/d\cosh\theta)^p\cos N\theta.$$

We want to evaluate

$$\int_0^\infty d\theta q_p(N_1,\theta)q_p(N_2,\theta).$$

It is understood that N_1 and N_2 are positive. The auxiliary functions $q_p(N, \theta)$ enjoy the following properties which are easily eatablished:

$$q_{p+1}(N,\theta)=[(d/d\theta)-p\coth\theta]q_p(N,\theta)$$

$$[N^2+(p-1)^2]q_{p-1}(N,\theta)$$
$$=-[(d/d\theta)+(p-1)\coth\theta]q_p(N,\theta)$$

$$\left\{\frac{d^2}{d\theta^2}+N^2-\frac{p(p-1)}{\sinh^2\theta}\right\}q_p(N,\theta)=0.$$

$q_p(N, 0)=0$ if $p\geq 1$ and $q_p(N, \theta)$ has an oscillatory behavior at infinity. Using the recurrence relations given above we find by using integration by parts that for $p\geq 1$:

$$\int_0^\infty d\theta q_p(N_1,\theta)q_p(N_2,\theta)$$

$$=\int_0^\infty d\theta q_p(N_1,\theta)\left(\frac{d}{d\theta}-(p-1)\coth\theta\right)q_{p-1}(N_2,\theta)$$

$$=\int_0^\infty d\theta q_{p-1}(N_2,\theta)\left(-\frac{d}{d\theta}-(p-1)\coth\theta\right)q_p(N_1,\theta)$$

$$=[N^2+(p-1)^2]\int_0^\infty d\theta q_{p-1}(N_1,\theta)q_{p-1}(N_2,\theta).$$

Since for $p=0$, $q_p(N, \theta)$ reduces to $\cos N\theta$ and

$$\int_0^\infty d\theta\cos N_1\theta\cos N_2\theta=\tfrac{1}{2}(\pi)\,\delta(N_1-N_2)$$

we find for $p\geq 1$

$$\int_0^\infty d\theta q_p(N_1,\theta)q_p(N_2,\theta)$$
$$=[N_1^2(N_1^2+1^2)\cdots(N_1^2+(p-1)^2]\tfrac{1}{2}(\pi)\,\delta(N_1-N_2).$$

Let now $p=\alpha_1+\frac{1}{2}(f-1)$ and introduce the proper normalization for $Z_{N,\alpha}{}^{(f)}$. One sees that we have just proved Eq. (18).

We turn to the completeness relation. For that purpose we use the following addition theorem (see Ref. 4):

$$\sum_{\alpha,\beta}H_{N,\alpha,\beta}{}^{(f+1)}(u_1)\,(H_{N,\alpha,\beta}{}^{(f+1)}(u_2))^*$$
$$=\frac{(-1)^{\frac{1}{2}(f-1)}}{(2\pi)^{\frac{1}{2}(f-1)}\pi}\left(\frac{d}{d\cosh\theta}\right)^{\frac{1}{2}(f-1)}\cos N\theta, \quad (21)$$

f off ≥ 3, $u_1u_2=\cosh\theta$.

Integrating this result over N requires a little attention, since in interpreting the result as $\delta_{\rm hyp}(u_1, u_2)$ we shall naturally want to adapt the coordinate system on the hyperboloid in order that, say, u_1 be the point $(1, 0, 0, \cdots, 0)$. Integration of Eq. (21) over N will introduce "δ functions" of θ. However, with this special system of coordinates, $\theta=0$ will be an end point of the integration interval in θ. Hence we use the following procedure. We multiply Eq. (21) by $\cos(N\epsilon)$, with $\epsilon>0$, ϵ will be allowed to go to zero at the end of the calculation (the same device will later be used without comment in the case of f even). With this in mind we find:

$$\int_0^\infty dN\sum_{\alpha,\beta}H_{N,\alpha,\beta}{}^{(f+1)}(u_1)\,(H_{N,\alpha,\beta}{}^{(f+1)}(u_2))^*$$
$$=\lim_{\epsilon\to+0}\frac{(-1)^{\frac{1}{2}(f-1)}}{(2\pi)^{\frac{1}{2}(f-1)}}\left(\frac{d}{d\cosh\theta_{12}}\right)^{\frac{1}{2}(f-1)}\delta(\theta_{12}-\epsilon).$$

The right-hand side does not appear at first sight to be equal to $\delta_{\rm hyp}(u_1, u_2)$, but is indeed equal to it. To see that, recall that $d^f\mu=\sinh\theta^{f-1}d\theta d^{f-1}\Omega$. Adapting the coordinate system as explained above, we compute with a test function ψ:

$$\lim_{\epsilon\to+0}\frac{(-1)^{\frac{1}{2}(f-1)}}{(2\pi)^{\frac{1}{2}(f-1)}2}\int_0^\infty d\theta\int_{S_f}d^{f-1}\Omega(n)\sinh\theta^{f-1}\psi(\theta,\mathbf{n})\left(\frac{d}{d\cosh\theta}\right)^{\frac{1}{2}(f-1)}\delta(\theta-\epsilon)$$
$$=\lim_{\epsilon\to+0}\frac{1}{2(2\pi)^{\frac{1}{2}(f-1)}}\int_0^\infty d\theta\int_{S_f}d^{f-1}\Omega(n)\delta(\theta-\epsilon)\sinh\theta\left(\frac{d}{d\cosh\theta}\right)^{\frac{1}{2}(f-1)}\sinh\theta^{f-2}\psi(\theta,\mathbf{n}).$$

We now use the fact that $\sinh\theta(d/d\cosh\theta)^q\sinh\theta^{f-2}$ goes to zero for $\theta\to+0$ as long as $q<\frac{1}{2}(f-1)$ (recall that f is odd and greater or equal to 3), while

$$\lim_{\theta\to+0}\sinh\theta\left(\frac{d}{d\cosh\theta}\right)^{\frac{1}{2}(f-1)}\sinh\theta^{f-2}=\frac{\Gamma(f-1)}{2^{\frac{1}{2}(f-1)-1}\Gamma\frac{1}{2}(f-1)}.$$

Our integral is thus $\psi(0)$ (which stands for $\psi(0, \mathbf{n})$, $\mathbf{n}$ arbitrary, the value being the same for all $(\mathbf{n})$ times $\Gamma(f-1)/(4\pi)^{\frac{1}{2}(f-1)}\Gamma[\frac{1}{2}(f-1)]\omega_f$ which is in fact equal to 1, using the area of the sphere given in the introduction $2\pi^{f/2}/\Gamma(f/2)$. In short, Eq. (19) is proved.

It remains to prove Eq. (20). Let t be for the moment a real positive variable with $\log t$ real and compute the following absolutely convergent integral:

$$t^{-1}\int_0^\infty\frac{dN}{\sinh\pi N}\cos(N\log t)\frac{\sin N\theta}{\sin\theta}=(2\sinh\theta t)^{-1}$$
$$\times\int_0^\infty dN\left(\frac{\sin N(\theta+\log t)}{\sinh\pi N}+\frac{\sin N(\theta-\log t)}{\sinh\pi N}\right).$$

It is easy to show that

$$\int_{-\infty}^{+\infty}dx\frac{\sin yx}{\sinh x}=\pi\tanh\pi\tfrac{1}{2}y.$$

Hence our integral is equal to

$$\frac{1}{4t\sinh\theta}\times\left\{\tanh\frac{\theta+\log t}{2}+\tanh\frac{\theta-\log t}{2}\right\}=\frac{1}{1+t^2+2t\cosh\theta}.$$

The integral is absolutely convergent for complex t as long as t varies in a plane cut from $-\infty$ to 0, such that $\log t$ is real for real positive t. For these values of t we get by analytic continuation:

$$\frac{1}{1+t^2+2t\cosh\theta}=\frac{-1}{t}\int_0^\infty\frac{dN}{N\sinh\pi N}\cos(N\log t)\frac{d}{d\cosh\theta}(\cos N\theta). \quad (22)$$

The addition theorem (21) together with Eq. (22) yields with the same restrictions on t

$$\frac{1}{(1+t^2+2tu_1u_2)^{\frac{1}{2}(f-1)}}=\frac{2\pi^{\frac{1}{2}(f+1)}}{[\frac{1}{2}(f-1)-1]!}\frac{1}{t^{\frac{1}{2}(f-1)}}\int_0^\infty\frac{dN}{N\sinh\pi N}\cos(N\log t)\times\sum_{\alpha,\beta}H_{N,\alpha,\beta}^{(f+1)}(u_1)\,(H_{N,\alpha,\beta}^{(f+1)}(u_2))^*.$$

Hence, using the orthogonality relations Eq. (18), we get the desired result Eq. (20)

E. Proof of the Three Relations in the Case of f Even

We first must give some properties of the Legendre functions which will be used in the following, principally the Mehler's transformation formulas. $P_{iN-\frac{1}{2}}(z)$ is the solution of the Legendre equation:

$$[(d/dz)(z^2-1)(d/dz)+N^2+\tfrac{1}{4}]P_{iN-\frac{1}{2}}(z)=0 \quad (23)$$

with the property that it is regular at the point $+1$ where it takes the value $+1$. For z real greater than one, N real, which is the domain in which we are interested, $P_{iN-\frac{1}{2}}$ is real and is even in N. For θ real positive, the following two integral representations hold[4]:

$$P_{iN-\frac{1}{2}}(\cosh\theta)=\frac{\sqrt{2}}{\pi\tanh\pi N}\int_\theta^\infty\frac{\sin N\psi d\psi}{(\cosh\psi-\cosh\theta)^{\frac{1}{2}}}=\frac{\sqrt{2}}{\pi}\int_0^\theta\frac{\cos N\psi}{(\cosh\theta-\cosh\psi)^{\frac{1}{2}}}d\psi. \quad (24)$$

These two integrals are reminiscent of the theory of Abel's integral equation. We propose to show that it leads us to the theory of Mehler's transforms. Let $g(\psi)$ be a function defined for $\psi\geq 0$ vanishing for $\psi=0$, square integrable between 0 and ∞ and sufficiently regular for the following integrals to be well defined. One has the couple of equations:

$$g(\psi)=\frac{2}{\pi}\int_0^\infty\sin N\psi G(N)\,dN$$

$$G(N)=\int_0^\infty\sin N\psi g(\psi)\,d\psi. \quad (25)$$

Consider the linear transformation:

$$g(\psi)\to f(\theta)=\int_0^\infty\frac{g(\psi)}{(\cosh\psi-\cosh\theta)^{\frac{1}{2}}}d\psi, \quad (26)$$

where $\theta\geq 0$. The inverse formula of this Abel equation is defined, if say $g(\psi)<A\psi$ for small ψ, and gives us

$$g(\psi)=-\frac{1}{\pi}\frac{d}{d\psi}\int_\psi^\infty\frac{f(\theta)\sinh\theta d\theta}{(\cosh\psi-\cosh\theta)^{\frac{1}{2}}}. \quad (27)$$

Combining Eqs. (26) and (25) we find

$$f(\theta)=\frac{2}{\pi}\int_0^\infty dN\,G(N)\int_\theta^\infty\frac{\sin N\psi}{(\cosh\psi-\cosh\theta)^{\frac{1}{2}}}d\psi=\int_0^\infty dN\,P_{iN-\frac{1}{2}}(\cosh\theta)\sqrt{2}(\tanh\pi N)G(N) \quad (28)$$

where we have used the first integral representation in Eq. (24) for the Legendre function. Similarly combining Eqs. (27) and (25) we find:

$$G(N)=\int_0^\infty d\psi(\sin N\psi)g(\psi)=-\int_0^\infty d\psi\sin N\psi\pi^{-1}\frac{d}{d\psi}\int_\psi^\infty\frac{f(\theta)\sinh\theta}{(\cosh\theta-\cosh\psi)^{\frac{1}{2}}}d\theta$$

$$=\left[-\frac{\sin\psi}{\pi}\int_\psi^\infty\frac{f(\theta)\sinh\theta}{(\cosh\theta-\cosh\psi)^{\frac{1}{2}}}d\theta\right]_0^\infty+\pi^{-1}\int_0^\infty d\psi N\cos N\psi\int_\psi^\infty\frac{f(\theta)\sinh\theta}{(\cosh\theta-\cosh\psi)^{\frac{1}{2}}}d\theta.$$

Assuming that $g(\psi)$ is the derivative of a function which vanishes at infinity, we can drop the integrated term and we find, using the second integral representation in Eq. (24):

$$G(N)=\frac{N}{2^{\frac{1}{2}}}\int_0^\infty\sinh\theta d\theta f(\theta)P_{iN-\frac{1}{2}}(\cosh\theta)\,d\theta. \quad (29)$$

If we call $F(N)=G(N)\sqrt{2}\tanh\pi N$ we deduce from Eqs. (28) and (29) the couple of Mehler transforms:

$$f(\theta)=\int_0^\infty F(N)P_{iN-\frac{1}{2}}(\cosh\theta)\,dN$$

$$F(N)=N\tanh\pi N\int_0^\infty d\theta\sinh\theta f(\theta)P_{iN-\frac{1}{2}}(\cosh\theta). \quad (30)$$

We can express Eq. (30) in a different language with

[4] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1.

x_1, x_2 real and greater than one, N_1 and N_2 positive;

$$\int_0^\infty dN P_{iN-\frac{1}{2}}(x_1) P_{i-N\frac{1}{2}}(x_2) N \tanh \pi N = \delta(x_1 - x_2)$$

$$\int_1^\infty dx P_{iN_1-\frac{1}{2}}(x) P_{iN_2-\frac{1}{2}}(x) = \frac{\delta(N_1 - N_2)}{N_1 \tanh \pi N_1}. \tag{31}$$

We are now in a position to derive formulas (18), (19), and (20) in the case of f even ≥ 2. From Eqs. (11) and (13) we get as before

$$\int_{T_{f+1}+} d^f\mu(u) H_{N_1,\alpha_1,\beta_1}(u) \left(H_{N_2,\alpha_2,\beta_2}(u)\right)^* = \delta_{\alpha_1,\alpha_2}\, \delta_{\beta_1,\beta_2} \times \int_0^\infty \sinh \theta^{f-1} d\theta Z_{N_1,\alpha_1}{}^{(f)}(\theta) Z_{N_2,\alpha_1}{}^{(f)}(\theta).$$

Using the definition of $Z_{N,\alpha}{}^{(f)}$, Eq. (17), leads us to study the integral:

$$\int_0^\infty r_{\frac{1}{2}(f-2)+\alpha_1}(N_1, \theta) r_{\frac{1}{2}(f-2)+\alpha_1}(N_2, \theta) \sinh \theta d\theta$$

where the auxiliary function $r_p(N, \theta)$ (equal up to a factor to the associated Legendre function) is defined through:

$$r_p(N, \theta) = (\sinh \theta)^p (d/d \cosh \theta)^p P_{iN-\frac{1}{2}}(\cosh \theta).$$

This function satisfies the following relations deduced from the analogous ones valid for the Legendre functions:

$$r_{p+1}(N, \theta) = [(d/d\theta) - p \coth \theta] r_p(N, \theta),$$

$$[N^2 + (p-\tfrac{1}{2})^2] r_{p-1}(N, \theta) = -[(d/d\theta) + p \coth \theta] r_p(N, \theta),$$

$$\left[\sinh^2 \theta \left(\frac{d}{d \cosh \theta}\right)^2 + 2 \cosh \theta \frac{d}{d \cosh \theta} + N^2 + \tfrac{1}{4} - \frac{p^2}{\sinh^2 \theta}\right] r_p(N, \theta) = 0.$$

Hence, by integration by parts using the fact that $r_p(N, \theta)$ is bounded at infinity by $A(\cosh \theta)^{-\frac{1}{2}}$:

$$\int_0^\infty d\theta \sinh \theta r_p(N_1, \theta) r_p(N_2, \theta) = \int_0^\infty d\theta \sinh \theta r_p(N_1, \theta) \left(\frac{d}{d\theta} - (p-1) \coth \theta\right) r_{p-1}(N_2, \theta)$$

$$= -\int_0^\infty d\theta \sinh \theta r_{p-1}(N_2, \theta) \left(\frac{d}{d\theta} + p \coth \theta\right) r_p(N_1, \theta)$$

$$= [N^2 + (p-\tfrac{1}{2})^2] \int_0^\infty d\theta \sinh \theta r_{p-1}(N_1, \theta) r_{p-1}(N_2, \theta).$$

Since for $p=0$, $r_p(N, \theta)$ reduces to the Legendre function for which Eq. (31) holds, we get

$$\int_0^\infty d\theta \sinh \theta r_p(N_1, \theta) r_p(N_2, \theta) = \{[N_1^2 + (p-\tfrac{1}{2})^2] \cdots (N^2 + \tfrac{1}{4})\} (N_1 \tanh \pi N_1)^{-1} \delta(N_1 - N_2).$$

The coefficient in front of the δ-function is to be understood as $(N_1 \tanh \pi N_1)^{-1}$ if $p=0$. Comparing this result with the normalization of $Z_{N,\alpha}{}^{(f)}(\theta)$ we see that we have proved Eq. (18).

For the completeness relations we use the addition theorem[4]:

$$\sum_{\alpha,\beta} H_{N,\alpha,\beta}{}^{(f+1)}(u_1) \left(H_{N,\alpha,\beta}{}^{(f+1)}(u_2)\right)^* = \frac{n \tanh \pi N}{(2\pi)^{f/2}} (-1)^{\frac{1}{2}(f-2)} \left(\frac{d}{d \cosh \theta}\right)^{\frac{1}{2}(f-2)} P_{iN-\frac{1}{2}}(\cosh \theta). \tag{32}$$

$$\cosh \theta = \mu_1 u^2$$

To prove Eq. (19) it remains to integrate Eq. (32) over N. With the same trick as in the previous section, we find with the help of Eq. (31):

$$\int_0^\infty dN \sum_{\alpha,\beta} H_{N,\alpha,\beta}{}^{(f+1)}(u_1) \left(H_{N,\alpha,\beta}{}^{(f+1)}(u_2)\right)^* = \lim_{\epsilon \to +0} \frac{(-1)^{\frac{1}{2}(f-2)}}{(2\pi)^{f/2}} \left(\frac{d}{d \cosh \theta}\right)^{\frac{1}{2}(f-2)} \delta(\cosh \theta - 1 - \epsilon).$$

Again, this is indeed $\delta_{\text{hyp}}(u_1, u_2)$ since with the test function ψ

$$\lim_{\epsilon \to +0} \int_0^\infty d^f\mu(u) \psi(u) \frac{(-1)^{\frac{1}{2}(f-2)}}{(2\pi)^{f/2}} \left(\frac{d}{d \cosh \theta}\right)^{\frac{1}{2}(f-2)} \delta(\cosh \theta - 1 - \epsilon)$$

$$= \lim_{\epsilon \to +0} \int_0^\infty \sinh \theta d\theta \delta(\cosh \theta - 1 - \epsilon) \frac{1}{(2\pi)^{f/2}} \left(\frac{d}{d \cosh \theta}\right)^{\frac{1}{2}(f-2)} \sinh \theta^{\frac{1}{2}(f-2)} \int_{S_f} \psi(\theta, \mathbf{n})^{f-1} d\Omega(n).$$

Again

$$\lim_{\theta\to+0}(d/d\cosh\theta)^q\sinh\theta^{f-2}=\begin{cases}0 & \text{if } q<\frac{1}{2}(f-2)\\ \frac{1}{2}(f-2)!2^{\frac{1}{2}(f-2)} & \text{if } q=\frac{1}{2}(f-2)\end{cases}$$

so that the integral reduces by Leibniz rule to $\psi(0)$ times $\omega_f[(f-2)/2]!2^{(f-2)/2}(2\pi)^{f/2}=1$. The completeness relations, Eq. (19), are thus proved.

It remains to obtain the integral equation (20). For this we use the alternative integral representation of $P_{iN-\frac{1}{2}}(\cosh\theta)$:

$$P_{iN-\frac{1}{2}}(\cosh\theta)=\frac{\cosh\pi N}{N}\int_0^\infty\frac{t^{-\frac{1}{2}-iN}}{(1+2t\cosh\theta+t^2)^{\frac{1}{2}}}\,dt. \tag{33}$$

By Fourier transformation Eq. (33) gives

$$t^{-\frac{1}{2}}\int_0^\infty\cos(N\log t)P_{iN-\frac{1}{2}}(\cosh\theta)\frac{dN}{\cosh\pi N}=\frac{1}{(1+2t\cosh\theta+t^2)^{\frac{1}{2}}}. \tag{34}$$

This result can again be continued analytically in the complex t plane cut from $-\infty$ to 0, with the principal determination of the logarithm in the left-hand side and the square root on the right-hand side such that it will be positive for real positive t.

Combining Eqs. (32) and (34) we get

$$\int_0^\infty\frac{dN}{N\sinh\pi N}\cos(N\log t)\sum_{\alpha,\beta}H_{N,\alpha,\beta}^{(f+1)}(u_1)\,(H_{N,\alpha,\beta}^{(f+1)}(u_2))^*=\frac{t^{\frac{1}{2}(f-1)}}{(2\pi)^{f/2}}\frac{1\cdot3\cdots(f-3)}{(1+2tu_1\cdot u_2+t^2)^{\frac{1}{2}(f-1)}} \tag{35}$$

with $1\cdot3\cdots(f-3)$ replaced by 1 if $f=2$ and the same restrictions on t as before. With the help of the orthogonality relation, Eq. (18), we deduce from Eq. (35) the desired integral equation (20).

F. Solution of the Integral Equation for the Coulomb Potential

Having now the required tools, we compare Eqs. (9) and (20). If we let t go to 1 in Eq. (20) the expression is well defined. If we approach -1, either by the upper or lower imaginary plane, we again find a unique limit. Using these facts, let $g(u)$ be a function defined on the whole hyperboloid $T_{f+1}=T_{f+1}^+ + T_{f+1}^-$, by

$$u\in T_{f+1}^+,\qquad g(u)=N_{N,\alpha,\beta}(u)$$
$$u\in T_{f+1}^-,\qquad g(u)=aH_{N,\alpha,\beta}(-u).$$

One deduces from Eq. (20) that

$$\int_{T_{f+1}}d^f\mu(v)\frac{g(v)}{|(u-v)^2|^{(f-1)}}=\frac{2\pi^{\frac{1}{2}(f+1)}}{\Gamma\frac{1}{2}(f-1)N\sinh\pi N}\times g(u)\begin{cases}a+\cosh N\pi & \text{if } u\in T_{f+1}^+\\ a^{-1}+\cosh N\pi & \text{if } u\in T_{f+1}^-.\end{cases}$$

In order that $g(u)$ be a solution of Eq. (9), it is thus necessary that $(a+\cosh\pi N)=-(a^{-1}+\cosh\pi N)$, in which case $a=-e^{\mp\pi N}$. Correspondingly we have two solutions (up to a normalization):

$$\hat\Phi_{N,\alpha,\beta}^{(\pm)}(u)=\begin{cases}H_{N,\alpha,\beta}(u) & \text{if } u\in T_{f+1}^+\\ -e^{\mp\pi N}H_{N,\alpha,\beta}(-u) & \text{if } u\in T_{f+1}^-\end{cases} \tag{36}$$

which satisfy

$$\hat\Phi_{N,\alpha,\beta}^{(\pm)}(u)=\pm N\frac{\Gamma^{\frac{1}{2}(f-1)}}{2\pi^{\frac{1}{2}(f+1)}}\in(u)\times\int_{T_{f+1}}d^f\mu(v)\frac{\hat\Phi_{N,\alpha,\beta}^{\pm}(v)}{|(u-v)^2|^{\frac{1}{2}(f-1)}}. \tag{37}$$

Because of the completeness relations, Eq. (19), one can convince oneself that the functions, Eq. (36), exhaust the solutions of Eq. (9). The fact that for given N, α, β there exist two solutions is a natural consequence from the fact that choosing N positive was a matter of indifference; indeed $\hat\Phi_{N,\alpha,\beta}^{(+)}(u)=\hat\Phi_{-N,\alpha,\beta}^{(-)}(u)$. On the other hand, this double solution reflects the fact that we can as well treat attractive or repulsive potentials. $\hat\Phi^{(+)}$ corresponds to the attractive case ($k>0$), $\hat\Phi^{(-)}$ to the repulsive case ($k<0$). Comparing Eqs. (9) and (37) we get the following relation between the energy E and N:

$$E=(\mu k^2/2\hbar^2)(1/N^2) \tag{38}$$

independently of the dimension f. Equation (38) is the analytic continuation in the index λ of the equation corresponding to bound states [Eq. (19) in I] from real integer values to complex values of the form $\lambda=-\frac{1}{2}(f-1)+iN$. So, in fact, are the eigenfunctions. We observe also in the case $f=3$, from the Lie-algebra

analysis, that the eigenvalue c which characterizes our representations of $0(1,3)$ is equal to iN.

Each set of functions $H_{N,\alpha,\beta}{}^{(f+1)}(u)$ for fixed N provides a basis for an irreducible unitary representation of $0^{(+,+)}(1,f)$ through

$$H_{N,\alpha,\beta}{}^{(f+1)}(\Lambda^{-1}u)=D_{\alpha',\beta';\alpha,\beta}{}^{N}(\Lambda)H_{N,\alpha',\beta'}{}^{(f+1)}(u). \quad (39)$$

This is clear since the equation satisfied by this function for fixed N is invariant under the group. We shall not exhibit explicitly the matrix elements of these representations; it would require too lengthy calculations. We return to this question in the following section.

III. TRANSFORMATION GROUP

We shall discuss briefly the introduction of a larger group of transformations. The motivation is the same as in I: We want to find a group which relates the various scattering states corresponding to different energies but we have to mix solutions for attractive and repulsive potentials. Consider the space $\mathfrak{K}_{f+1}$ of square integrable functions on T_{f+1} with the measure $d^f\mu$. It is spanned by our scattering states $\hat{\Phi}_{N,\alpha,\beta}{}^{(\pm)}(u)$. We are looking for a group which contains $0(1,f)$ as subgroup and has a unitary representation in $\mathfrak{K}_{f+1}$ which reduces to the ones described above when restricted to $0(1,f)$. Again, the answer is in terms of projective transformations.

We recall the construction of this group. Let us denote for the moment by u an arbitrary point in Euclidian $(f+1)$-dimensional space. Introduce the quantity $z=u^2$ (where u^2 denotes the Minkowski square). In the (z,u) space we are restricted to the previous "paraboloid." Consider the projective transformations which leave this paraboloid invariant. Introducing the homogenity variable t we find the homogeneous group, leaving

$$zt-u^2\equiv\tfrac{1}{2}(z+t)^2-\tfrac{1}{2}(z-t)^2-u_0^2+\sum_{i=1}^{f}u_i^2$$

invariant. That is the "conformal Lorentz group" $0(2,f+1)$. Going back to our $f+1$, Euclidian space we ask for the subgroup of $0(2,f+1)$ which leaves the initial hyperboloid T_{f+1} invariant. The condition is $z-t=0$. Hence the required group is $G\equiv 0(1,f+1)$. The result turns out to be the same as for the bound state case. This will appear clear at the end of this section. Going through the previous transformations, the action of G on the hyperboloid is found to be

$$u\to\Lambda u=u'; \qquad u,u'\in T_{f+1}$$
$$u'_\alpha=(\sum_\beta a_{\alpha\beta}u_\beta+a_{\alpha t})/(\sum_\beta a_{t\beta}u_\beta+a_{tt}). \quad (40)$$

The Greek indices run from 0 to f, the Latin ones from 1 to f. The real matrix Λ belongs to G; that is

$$\Lambda=\begin{pmatrix} a_{\alpha\beta} & a_{\alpha t} \\ a_{t\alpha} & a_{tt}\end{pmatrix}$$

$$\gamma=\begin{bmatrix} 1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & \cdot & & \\ & & & & -1 & \\ & & & & & -1 \end{bmatrix},$$

$$\Lambda^T\gamma\Lambda=\gamma. \quad (41)$$

The hyperboloid is clearly invariant since one finds

$$u'^2-1=(u^2-1)/(\sum_\alpha a_{t\alpha}u_\alpha+a_{tt})^2=0. \quad (42)$$

It is to be remarked that in Eqs. (40) and (42) the denominator may vanish for certain transformations. This means in fact that in order to consider the action of G on T_{f+1} we have to add to the hyperboloid extra points "at infinity." In other words, T_{f+1} has been compactified by the adjunction of a surface at infinity.

We build the following unitary representations of G in $\mathfrak{K}_{f+1}$ which depend on the index ρ:

$$\Phi(u)\to T_\rho{}^\Lambda\Phi(u)$$
$$=\frac{\Phi(\Lambda^{-1}u)}{|\sum a_{t\alpha}(\Lambda^{-1})u_\alpha+a_{tt}(\Lambda^{-1})|^{(f/2)+i\rho}}. \quad (43)$$

Using Eq. (42) one checks the unitarity of these representations:

$$\int_{T_{f+1}}d^f\mu\bar{\Phi}\psi=\int_{T_{f+1}}d^f\mu([T_\rho{}^\Lambda\Phi])^*[T_\rho{}^\Lambda\psi]. \quad (44)$$

We have studied in I similar representations of $0(1,f+1)$ realized in the Hilbert space of square integrable functions on the sphere S_{f+1}; denote it by $\mathfrak{K}_{f+1}$. It may be interesting to know whether we have constructed equivalent or inequivalent representations of the same group. The remainder of this section will be devoted to the proof of the unitary equivalence of the two sets of representations. All the properties investigated before, such as irreducibility, will thus hold true in the present case.

For that purpose we first define a transformation which maps the sphere

$$S_{f+1}=\{v: v_0^2+\sum_{i=1}^{f}u_i^2=1\}$$

on the hyperboloid

$$T_{f+1}=\{u: u_0^2-\sum_{i=1}^{f}u_1^2=1\}.$$

It is the following mapping [see Fig. 2(a)]

$$v\in S_{f+1}\rightarrow u=\tau v\in T_{f+1}; \qquad u=[v_0^{-1},\ (-v_i/v_0)]$$
$$u\in T_{f+1}\rightarrow v=\tau^{-1}u\in S_{f+1}; \qquad v=[u_0^{-1},\ (-u_i/u_0)]. \quad (45)$$

Let $v\rightarrow\Lambda v=v'$ be a conformal transformation of S_{f+1}; $\Lambda\in 0(1,f+1)$ and

$$v'_0=(a_{00}v_0+\sum_j a_{0j}v_j+a_{0t})/(a_{t0}v_0+\sum_j a_{tj}v_j+a_{tt}),$$
$$v'_i=(a_{i0}v_0+\sum_j a_{ij}v_j+a_{it})/(a_{t0}v_0+\sum_j a_{tj}v_j+a_{tt}), \quad (46)$$

with

$$\Lambda=\begin{pmatrix} a_{tt} & a_{t\beta} \\ a_{\alpha t} & a_{\alpha\beta}\end{pmatrix}; \qquad \Lambda^T\gamma\Lambda=\gamma, \quad (47)$$

γ being as in Eq. (41). Performing the projection τ we find that the corresponding point on T_{f+1} undergoes the following transformation:

$$u\rightarrow\Lambda_\tau u=u'$$
$$u_0=(a_{tt}u_0-\sum_j a_{tj}u_j+a_{t0})/(a_{0t}u_0-\sum_j a_{0j}u_j+a_{00}),$$
$$u_i=(-a_{it}u_0+\sum_j a_{ij}u_j-a_{i0})/(a_{0t}u_0-\sum_j a_{0j}u_j+a_{00}). \quad (48)$$

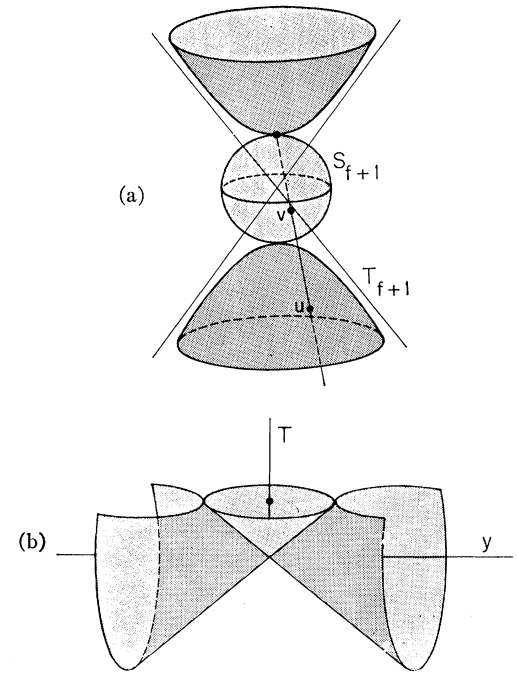

FIG. 2. Transformations between a sphere and a hyperboloid.

That is, with

$$\Lambda_\tau=\begin{pmatrix}1&0&0\\0&0&-I_f\\0&1&0\end{pmatrix} \qquad \Lambda_0^{-1}=\begin{pmatrix}1&0&0\\0&0&1\\0&-I_f&0\end{pmatrix}$$

one has

$$\Lambda_0\Lambda\Lambda_0^{-1}=\begin{pmatrix} a_{tt} & -a_{tj} & a_{t0} \\ -a_{it} & a_{ij} & -a_{i0} \\ a_{0t} & -a_{0j} & a_{00}\end{pmatrix}. \quad (49)$$

I_f is a unit $f\times f$ matrix. The correspondence $\Lambda\rightarrow\Lambda_\tau$ is an inner automorphism of $0(1,f+1)$ and one even remarks that Λ_0 always belongs to the component of the identity.

If $u=\tau v$ one also finds, using Eq. (45), that

$$d^f\Omega(v) \quad d^f\mu(u)/|\,u_0\,|^f; \qquad u=\tau v. \quad (50)$$

Hence, if $\psi(v)$ is a function defined on S_{f+1} and belongs to $\mathcal{K}_{f+1}$;

$$\int_{S_{f+1}}|\,\psi(v)\,|^2\,d^f\Omega(v)=\int_{T_{f+1}}\frac{|\,\psi(\tau^{-1}u)\,|^2}{|\,u_0\,|^{f/2}}\,d^f\mu(u). \quad (51)$$

Using Eq. (51) we define a unitary mapping U_ρ from $\mathcal{K}_{f+1}$ to $\mathcal{H}_{f+1}$ by

$$\psi(v)\in\mathcal{K}_{f+1}\rightarrow[U_\rho\psi](u)=[\psi(\tau^{-1}u)/|\,u_0\,|^{(f/2)+i\rho}]\in\mathcal{H}_{f+1}$$
$$\Phi(u)\in\mathcal{H}_{f+1}\rightarrow[U_\rho^{-1}\Phi](v)=[\Phi(\tau v)/|\,v_0\,|^{(f/2)+i\rho}]\in\mathcal{K}_{f+1}. \quad (52)$$

The unitary representation of $0(1, f+1)$ that we investigated in I for $\mathcal{H}_{f+1}$ was defined through

$$\psi(v)\rightarrow[R_\rho^\Lambda\psi](v)=\frac{\psi(\Lambda^{-1}v)}{|\sum_\alpha a_{\epsilon\alpha}(\Lambda^{-1})v_\alpha+a_{tt}(\Lambda^{-1})\,|^{(f/2)+i\rho}} \quad (53)$$

with Λv given by Eq. (46). Similarly, Eq. (43) defines a unitary representation T_ρ^Λ of the same group in $\mathcal{H}_{f+1}$. We will now prove that

$$U_\rho R_\rho^\Lambda=T_\rho^{\Lambda_\tau}U_\rho. \quad (54)$$

Since in virtue of Eq. (49)

$$T_\rho^{\Lambda_\tau}=T_\rho^{\Lambda_0}T_\rho^\Lambda(T_\rho^{\Lambda_0})^{-1} \quad (55)$$

and all operators are unitary, Eq. (54) is indeed a statement of unitary equivalence. The proof of this equality is rather straightforward. Consider for instance $[U_\rho R_\rho^\Lambda\psi](u)$

$$=\frac{\psi(\Lambda^{-1}\tau^{-1}u)}{|\,a_{tt}(\Lambda^{-1})u_0-\sum_j a_{tj}(\Lambda^{-1})u_j+a_{t0}(\Lambda^{-1})\,|^{(f/2)+i\rho}};$$

on the other hand,

$$[T_\rho^{\Lambda_\tau} U_\rho \psi](u)$$
$$= \frac{\psi(\tau^{-1}\Lambda_\tau^{-1}u)}{|\, a_{00}(\Lambda^{-1})u_0 - \sum_j a_{0j}(\Lambda^{-1})u_j + a_{0t}(\Lambda^{-1})\,|^{(f/2)+i\rho}}$$
$$\times \left| \frac{a_{00}(\Lambda^{-1})u_0 - \sum_j a_{0j}(\Lambda^{-1})u_j + a_{0t}(\Lambda^{-1})}{a_{tt}(\Lambda^{-1})u_0 - \sum_j a_{tj}(\Lambda^{-1})u_j + a_{t0}(\Lambda^{-1})} \right|^{(f/2)+i\rho},$$

and since by definition of the automorphism t, $\tau^{-1}\Lambda_\tau^{-1} = \Lambda_\tau^{-1}\tau^{-1}$, Eq. (54) is easily obtained and the unitary equivalence established. The geometric transformation (45) which was at the basis of our proof is best understood by adding an homogeneous coordinate such that $u_0 = Y/T$, $u_i = X_i/T$ and the same for v. The sphere and the hyperboloid appear as cones centered respectively around the T and the Y axis; the abstract group $0(1, f+1)$ is realized in two different ways as homogeneous linear group which leave one of these cones invariant. A rotation of 180 degrees around the line $X_i = 0$, $Y = T$ which is tangent to both cones is the substitute for the transformation τ. This is pictured in Fig. 2(b).

As a matter of fact, one can prove that the unitary representations of $0(1, f)$ constructed in Sec. II with the help of spherical functions and those obtained in the section by means of conformal transformations on a sphere or an hyperboloid are equivalent. The proof is an extension of the one given in the case of $f=3$ in the third paper of Ref. 2.

Before concluding this section we shall briefly mention the case of zero energy. From the commutation relations, Eq. (2), we note that $[M_i, M_j]=0$ and the operators $\mathbf{L}$, $\mathbf{M}$, build the algebra of the Euclidian group in three dimensions, $E(3)$. In f dimensions this may likewise be realized globally. We project the f-dimensional momentum space onto a paraboloid in $f+1$ dimensions, P_{f+1}. If u is a point on the paraboloid $u_0 = \mathbf{u}^2/2$, with u_0 in the $f+1$ direction and $\mathbf{u}$ in the original f-dimensional subspace, we let $p \to u = [1/(2p^2), \mathbf{p}/p^2]$. Performing this transformation on Eq. (5) we obtain, with $\hat{\Phi}(u) = \Phi(\mathbf{p}/p^2)1/p^{f+1}$.

$$\hat{\Phi}(u) = \frac{2uk}{\pi\omega_{f-1}\hbar} \int \frac{d^{f+1}f\delta(v_0 - \frac{1}{2}\mathbf{v}^2)\hat{\Phi}(v)}{|\, \mathbf{u}-\mathbf{v}\,|^{f-1}},$$

which exhibits the invariance of the problem under the Euclidian group in f dimensions, E_f. It may also be noted that the group of conformal transformations in $f+1$ dimensions which leaves P_{f+1} invariant is again $0(1, f+1)$. The action of the group on functions defined on P_{f+1} may be obtained from its action on S_{f+1} or T_{f+1} by noting that in Fig. 2(b), if we make a rotation of 45° in the (T, Y) plane, we transform S_{f+1} and T_{f+1} into P_{f+1}. It is amusing to note that the large group of transformations in all cases is $0(1, f+1)$ and the subgroups which are symmetries of the problem in the cases $E<0$, $E=0$, $E>0$ are $0(f+1)$, $E(f)$, $0(1, f)$ respectively, which are the little groups of the Poincaré group in an $(1, f+1)$-dimensional Minkowski space.

IV. CONCLUSION

The hydrogen atom illustrates several aspects of the use of group theory, and especially noncompact groups, in quantum mechanics. The theory of infinite dimensional representations can be of interest in various problems. We hope to have shown in this work that it can be used beyond the realm of Lie algebras.

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APPENDIX

In Sec. II, Eq. (36), and Ref. 1, Eqs. (12) and (40), we obtained representations of the wave function in momentum spaces, in the scattering and bound-state cases, respectively. We shall explicitly transform back to configuration space and show that we obtain the usual expressions. We set, of course, $f=3$.

1. Bound States

Using the results of I, we find for the radial wave function with principal quantum number n and angular momentum l:

$$\psi_{n,l}(r) = \frac{(-1)^l 8 p_0^{\frac{3}{2}}}{\pi\hbar^{\frac{3}{2}}}$$
$$\times [(n^2-1^2)\cdots(n^2-l^2)]^{-\frac{1}{2}} \int_0^\infty \frac{q^2 dq}{(1+q^2)^2}$$
$$\times j_l\left(q\frac{rp_0}{\hbar}\right) \sin\delta^l \left(\frac{d}{d\cos\delta}\right)^l \frac{\sin n\delta}{\sin\delta}$$
$$q = \tan\delta/2, \qquad \text{(A1)}$$

where $p_0 = (-2\mu E)^{\frac{1}{2}}$. We want to evaluate the last integral; denote it by $I_{n,l}(rp_0/\hbar)$. It is clear that $I_{n,l}(y)_{y\to 0} = 0(y^l)$. We consider the following sum for $|\, t\,| < 1$

$$\sum_1^\infty n I_{n,l}(y) t^n = (1-t^2) t^{l+1} 2^{2l} (l+1)\,!$$
$$\times \int_0^\infty dq j_l(qy) \left[\frac{q}{q^2(1+t)^2 + (1-t)^2}\right]^{l+2}.$$

The integrand is even; we can extend the integration from $-\infty$ to $+\infty$ and use

$$j_l(z)=(1/2i)\{h_l^{(+)}(z)-h_l^{(-)}(z)\}.$$

The Hankel functions satisfy:[5]

$$h_l^{(+)}(-z)=(-1)^{l+1}h^{(-l)}(z), \qquad h_l^{(\pm)}(z)_{z\to 0}=0(z^{-l-1}).$$

Hence we can write

$$\sum nI_{n,l}(y)t^n$$

$$=\frac{(1-t^2)}{2i}t^{l+1}2^{2l}(l+1)!$$

$$\times\int_{-\infty}^{+\infty}dqh^{(+)}(qy)\left(\frac{q}{q^2(1+t)^2+(1-t)^2}\right)^{l+2}$$

$$=\frac{(1-t^2)}{2i}t^{l+1}2^{2l}\frac{2\pi i}{(1+t)^{l+2}}\left(\frac{d}{dq}\right)^{l+1}$$

$$\times\left\{h_l^{(+)}(qy)\left(\frac{q}{q(1+t)+i(1-t)}\right)^{l+2}\right\}_{q=i[(1-t)/(1+t)]}$$

where the integral was performed by closing the contour in the upper half q plane. Since the sum behaves like y^l for $y\to 0$ we can drop in the derivative all the terms which are of smaller power in y. We use explicit form of the Hankel function[5] and obtain:

$$\sum_1^\infty nI_{n,l}(y)t^n=\pi t^{l+1}2^{l2}y^le^{-y}\frac{e^{y[2t/(1+t)]}}{(1+t)^{2l+2}}.$$

On the right-hand side we recognize the generating function of associated Laguerre polynomials:

$$\frac{e^{y[t/(1+t)]}}{(1+t)^{1+k}}=\sum_{p=0}^{\infty}\frac{(-1)^p t^p}{(p+k)!}L_p^k(y) \qquad |t|<1$$

and thus get

$$I_{n,l}(y)$$

$$=\int_0^\infty\frac{q^2dq}{(1+q^2)}j_l(qy)\sin\delta^l\left(\frac{d}{d\cos\delta}\right)^l\left(\frac{\sin n\delta}{\sin\delta}\right)_{\delta=2\arctan}$$

$$=\frac{(-1)^{n-l-1}}{n(n+l)!}\pi 2^{l-2}y^le^{-y}L_{n-l-1}^{2l+1}(2y). \qquad \text{(A2)}$$

Introducing the Bohr radius

$$a=\hbar^2/k\mu, \qquad \text{(A3)}$$

we obtain by combining Eqs. (A1) and (A2) the radial wave function:

$$\psi_{n,l}(r)=\frac{(-1)^{n-1}}{a^{\frac{3}{2}}}\frac{2}{n^2}\left[\frac{(n-l-1)!}{[(n+l)!]^3}\right]^{\frac{1}{2}}\rho^le^{-\rho/2}L_{n-l-1}^{2l+1}(\rho);$$

$$\rho=(2r/na). \qquad \text{(A4)}$$

2. Scattering Case

With the necessary adjustments for normalization we find for the radial wave function with angular momentum l and energy

$$\psi_{N,l}(r)=\frac{4\pi i^l}{(2\pi)^{\frac{3}{2}}}\left\{\frac{N}{(1-e^{-2\pi N})N^2(N^2+1^2)(N^2+l^2)}\right\}^{\frac{1}{2}}$$

$$\times\left[\int_0 dqq^2j_l(q\rho)\left(\frac{2}{1-q^2}\right)^2\sinh\theta_-{}^l\left(\frac{d}{d\cosh\theta_-}\right)^{l+1}\cos N\theta_- - e^{-\pi N}\int_1^\infty dqq^2j_l(q\rho)\left(\frac{2}{1-q^2}\right)^2\sinh\theta_+{}^l\left(\frac{d}{d\cosh\theta_+}\right)^{l+1}\cos N\theta_+\right], \qquad \text{(A5)}$$

$$N=k(\mu/2\hbar^2E),\ \rho=[(2\mu E)^{\frac{1}{2}}/\hbar]r,\ \tanh(\theta_-/2)=q \quad \text{for} \quad 0<q<1,\ \tanh(\theta_+/2)=(1/q) \quad \text{for} \quad q>1.$$

The positive (negative) values of N correspond to attractive (repulsive) potentials. Of course we must give some prescription to deal with the singular point $q=1$. We introduce

$$F_l(N,\theta)=-\frac{1}{\sinh\pi N}\left(\frac{d}{d\cosh\theta}\right)^{l+1}\frac{\cos N\theta}{\sinh\theta},$$

$$\tilde{F}_l(t,\theta)=\int_{-\infty}^{+\infty}dNe^{iNt}F_l(N,\theta)=\frac{i\sinh t(-1)^l(l+1)!}{(\cosh\theta+\cosh t)^{l+2}},$$

$$\psi_{N,l}=\frac{4\pi i^{l-1}}{(2\pi)^{\frac{3}{2}}}\sinh\pi N\left[\frac{N}{(1-e^{-2\pi N})N^2(N^2+1^2)\cdots(N^2+l^2)}\right]^{\frac{1}{2}}(-1)^l(l+1)!2^{l+1}$$

$$\times\int_{-\infty}^{+\infty}dt\exp(-iNt)\sinh t\int_{-\infty}^{+\infty}dqj_l(q\rho)\left(\frac{q}{q^2[1-\cosh(t-i\epsilon)]+[1+\cosh(t-i\epsilon)]}\right)^{l+2}.$$

[5] A. Messiah, *Quantum Mechanics* (North-Holland Publ. Co., Amsterdam, 1964), Vol. I.

The prescription $\cosh(t-i\epsilon)$ takes care of the singularity at $q=1$. We replace again j_l by $(1/2i)(h_l^{(+)}-h_l^{(-)})$, close the contour in the upper half of the q plane and use the explicit form of the Hankel function to calculate the derivative which occurs at the point $q=\coth[(t-i\epsilon)/2]$. The result is

$$\psi_{N,l}=\frac{8\pi^2 i^l}{(2\pi)^{\frac{5}{2}}}\sinh\pi N \times\left[\frac{N}{(1-e^{-2\pi N})N^2(N^2+1^2)\cdots(N^2+l^2)}\right]^{\frac{1}{2}}2^{-l-2}\rho^l \times\int_{-\infty}^{+\infty}\frac{dt}{[\sinh^2(t/2)]^{l+1}}\exp[i\cosh(t/2)\rho-iNt].$$

We take $x=\cosh(t/2)$ as a variable, the region of integration being the real axis except the segment $(-1, +1)$. We can close the contour in the upper half-plane (ρ is positive) and move it to $(-1, +1)$. Taking into account changes of arguments in the integrand we get

$$\psi_{N,l}=\sinh\pi N\left(\frac{N}{2\pi 1-e^{-2\pi N}}\right)^{\frac{1}{2}}\left(\frac{-i\rho}{2}\right)^l\frac{e^{\pi N}}{[N^2\cdots(N^2+l^2)]^{\frac{1}{2}}} \times\int_{-1}^{+1}dx e^{i\rho x}(1-x)^{l-iN}(x+1)^{l+iN}$$

with $\operatorname{Arg}(x+1)=\operatorname{Arg}(1-x)=0$. Apart from a proportionality factor and an exponential, the integrand is a classical representation of the hypergeometric function. Taking this factor into account and

$$[\pi N^2(N^2+1)\cdots(N^2+l^2)]^{\frac{1}{2}}=(N\sinh\pi N)^{\frac{1}{2}}\,|\,\Gamma(l+iN+1)\,|$$

our result reads

$$\psi_{N,l}(r)=\exp(\pi N/2)\frac{|\,\Gamma(l+iN+1)\,|}{(2l+1)!}\left(\frac{2\rho}{i}\right)^l e^{-i\rho} \times F(l+iN+1\,|\,2l+2\,|\,2i\rho) \quad \text{(A6)}$$

with ρ as in (A5).