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Dynamical Symmetry in Particle Physics*

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A review is given of recent applications of symmetry arguments in particle physics. It deals with the developments which start with $SU(6)$.

I. INTRODUCTION

The purpose of this paper is to give an account of recent applications of symmetry arguments in particle physics. It deals with the developments which start with $SU(6)$. The properties of $SU(3)$ [G 1] and those of Pauli and Dirac spinors and matrices are supposed known. It has been attempted to make this paper fairly self-contained otherwise, though it is certainly not complete in many mathematical details.

In order to explain the plan of the paper we begin by giving the table of contents.

Section III can be skipped by readers familiar with the algebra of unitary tensors. In that section, Lie groups and Lie algebras are only touched on lightly, as their treatment is well accessible elsewhere [B 1, H 1].

During the period covered in this paper, intense work went on concerning three main and distinct problems. (The Babylonian confusion of this period was largely due to the fact that their distinct nature was not always stated.)

(A) Investigations of approximate dynamical symmetries such as static $SU(6)$ and of larger ones, in which the latter is contained.

(B) Explorations of dynamical equations which yield $SU(6)$ or related groups as a symmetry of some of their approximate solutions.

(C) Limitations on approximate kinematic symmetries imposed by the rules of quantum theory and relativity theory.

It seems possible at this time, and it is tried in this paper, to assess the success and limitations of work related to (A); to state what preliminary steps have

* The present paper contains an enlarged version of lectures given by the author in Erice and in Dubna in the fall of 1965. It will be attempted to outline the main ideas in this field up till December 1965.

been taken and what questions have arisen in regard to (B), which thus far is an open problem; and to illustrate the nature of theorems that bear on (C).

The purpose of Sec. II is to make the distinction between kinematical and dynamical symmetry. Section IV deals with the properties of static $SU(6)$. In Sec. V, relativistic explorations are discussed which fall mainly under the heading of problem (A). Section VI contains a brief account of what is known about current algebras; here one deals with questions of type (B). Some results concerning (C) are summarized in Sec. VII.

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II. SOME PHYSICAL APPLICATIONS OF UNITARY GROUPS

A. A Few Definitions

Notation. Matrices will be written as $A_\beta{}^\alpha$ where the upper and lower indices denote rows and columns, respectively. x^α, $\alpha = 1, \cdots, N$ denotes a complex vector, represented as a one-column matrix x. The vector with complex conjugate components is conveniently written as $x_\alpha{}^*$ and as such it is represented as a one-row matrix $x^\dagger$. († stands for Hermitian conjugate).

Consider a transformation

$$x' = Ax \quad \text{or} \quad x'^\alpha = A_\beta{}^\alpha x^\beta. \tag{2.1}$$

The complex conjugate of x transforms as

$$x^{\dagger\prime} = x^\dagger A^\dagger \quad \text{or} \quad x_\alpha{}'^* = x_\beta{}^* A_\beta{}^{*\alpha} = x_\beta{}^*(A^\dagger)_\alpha{}^\beta. \tag{2.2}$$

(Summations over β are understood.) The unitary length of x is defined by

$$x^\dagger x = x_\alpha{}^* x^\alpha. \tag{2.3}$$

The unitary group $U(N)$ may be defined as the group of all transformations (2.1) which leave $x^\dagger x$ invariant. This implies that the matrices A satisfy

$$A^\dagger A = 1. \tag{2.4}$$

The further restriction

$$\det A = 1 \tag{2.5}$$

corresponds to the unimodular subgroup $SU(N)$ of $U(N)$.

The vector x is a particular representation of $U(N)$, or $SU(N)$, called the defining representation and often denoted by its dimension: N. The complex conjugate vector $x^\dagger$ gives a representation with the same dimension but which is in general inequivalent to N [for the special case $N=2$ see the discussion of Eq. (3.6) below]. We denote it by N^* and call it the representation conjugate to N.

The set of all A acting on an x generates an N-dimensional complex vector space. In this space we can choose a base of N linearly independent vectors x_j, $j=1, \cdots, N$. Thus $x_j{}^\alpha$ is the αth component of the jth base vector. In physical language, a particular choice of the x_j corresponds to the choice of "axes of quantization." We read the transformation (2.1) on x_j as

$$x_j{}'^\alpha = A_\beta{}^\alpha x_j{}^\beta, \tag{2.6}$$

that is, we let A act on the *components* for fixed axes of quantization, so that A changes the direction of a vector in a fixed coordinate system. Cf. the discussion in [F 1]. In what follows, x^α will mean the αth component of a vector, regardless of choice of axes. In the later discussion of boost matrices (Sec. V), the more fully specified notation used in Eq. (2.6) will be needed again.

B. Kinematical and Dynamical Unitary Symmetries

There are a variety of groups $SU(N)$ which play a role in physics. As an introduction to our subject, it is useful to give some examples of these physically interesting cases. They have in common that conserved quantities for physical systems are defined in terms of the generators of the group at hand. They are distinct in the way the region of validity of the conservation is defined.

(I) Strict kinematical group $SU(2)$: angular momentum. We shall call this strict symmetry kinematical to indicate that the corresponding conservation laws are valid independently of what may be the detailed dynamics of the system. The defining representation *2* is "the spinor."

(II) Approximate kinematic group $SU(2)$: isospin. The group applies in an approximate way, namely, only in the limit $e\to 0$. This restriction is expressed in terms of a parameter which is independent of the dynamical variables (positions, momenta, etc.) of the system. For this reason the approximate group will be called an approximate kinematic group. The defining representation *2* is the nucleon.

Of course one can use isospin in practice also when $e\neq 0$, for example when we say: isospin works well in hadron–hadron scattering for energies large compared with the isomultiplet splittings and for angles large compared with the region of Coulomb interference. However, we like to think that the size of these "bad" regions tends to zero as $e\to 0$. This is probably not a very physical way of putting things. The only reason this $e\to 0$ limit is discussed here is to contrast the approximate nature of isospin, as it is thought of presently, with approximate symmetries of a different kind.

(III) Approximate dynamical group $SU(2)$: normal coupling in atomic spectra [C 1]. Write the Hamiltonian for an atom as

$$H=H_0+H' \tag{2.7}$$

$$H_0=\sum \frac{p_i^2}{2m}-\sum \frac{Ze^2}{r_i}+\sum_{i>j} \frac{e^2}{r_{ij}}. \tag{2.8}$$

H' contains all nonstatic effects, for example, spin-orbit coupling. Whenever it is legitimate to neglect H' to a good approximation, the orbital angular momentum L and the spin angular momentum S are separately conserved and we have S-multiplets $^{2S+1}L_J$ (normal or Russell–Saunders coupling). The group $SU(2)$ is now the spin group, the defining representation *2* is the electron with spin.

Take the ^3P and ^1P levels in helium as an example. Even though (2.8) is explicitly spin independent, a ^3P–^1P split arises due to the exclusion principle, (Austausch effect). On the other hand, we have a "supermultiplet" in the H_0 approximation, as $^3P_{2,1,0}$ are degenerate. This degeneracy is lifted by spin-orbit coupling.

The spin-orbit coupling is proportional to e and also proportional to v/c via the momentum. Yet we cannot define the H_0 approximation by $e\rightarrow 0$, for then we lose the atom. The normal coupling approximation is thus to be defined as the neglect of v/c effects, that is in terms of the small expectation value in specific states of certain dynamical variables.

Normal coupling may be better in one part of the spectrum than in another; for heavy atoms it is bad for the Roentgen part of the spectrum, it is better in the outer shells provided that there is not too much excitation. All these effects which make the spin group $SU(2)$ to an approximate one are not (as for isospin) expressable in terms of a nondynamical parameter. Thus we speak of an approximate dynamical group.

With the Hamiltonian (2.7) in hand, the dynamical nature of the approximation is of course obvious from the start, and there is no need to discuss trouble with $SU(2)$. There would be trouble indeed if one would think of the spin group as approximately kinematical in nature. This is so because taking a limit of the kinematical type "coupling constant"$\rightarrow 0$ *is itself a relativistically invariant procedure.* In the zero limit we would then have a strict symmetry in which spin and total angular momentum would be separately strictly conserved. We would then have a theory of finite multiplets with more integrals of the motion than the Poincaré group allows which is absurd. This is at the root of many no-go theorems for certain approximate kinematical symmetries [W 1]. These theorems will be briefly reviewed in Sec. VII.

(IV) Approximate group $SU(3)$: strong interaction symmetry [G 1]. It is not impossible, as far as is known, that this is an approximate kinematical group, that there is a symmetry breaking interaction which is characterized by one (or more) nondynamical parameters which in their zero limit define a strict $SU(3)$. However, some bootstrap ideas favor a more dynamical view (which could perhaps also be applied to isospin.) The defining representation *3* is the triplet.

Regarding the experimental status of $SU(3)$, it seems that the predictions are best for mass relations and for semileptonic and electromagnetic vertices. On the other hand, from the analysis of scattering and production amplitudes alone one would not be very clear about this symmetry [H 2, A 1].

(V) Approximate dynamical group $SU(4)$: nuclear supermultiplets [W 2]. This symmetry was suggested to apply as a dynamical approximation for low-lying nuclear levels only [W 2]. In that region the theory works well [F 2]. The defining representation *4* is the nucleon with spin. This $SU(4)$ has much in common with the $SU(2)$ of (III). But of course we do not know as much about the Hamiltonian in this case as we do for (2.7). All the success of $SU(4)$ really says about the fundamental dynamics is that there is a regime in which a spin- and isospin-independent H_0 is a a good approximation. It is not part of the $SU(4)$ theory per se to *derive* an H_0 and an H' from first dynamical principles, as in (2.7, 8).

This brief discussion of some aspects of the foregoing unitary symmetries is given in order to set the stage for the main topic of this paper.

(VI) Approximate dynamical group $SU(6)$ in particle physics [S 1, G 3, P 1, G 4]. The fundamental representation *6* is the sextet, an $SU(3)$ triplet of spin $\frac{1}{2}$ particles. Thus (as will be explained more fully in Sec. IV) this group contains the unitary spin group $SU(3)$ and the ordinary spin group $SU(2)$. By the argument given for the case of Russell–Saunders coupling, $SU(6)$ must be an approximate dynamical group because it *contains* the spin group. Indeed, what is often referred to as the difficulties of $SU(6)$ has nothing to do with the fact that $SU(6)$ contains the internal symmetry group $SU(3)$. It is instructive to read the pertinent literature as if it had reference to atomic Russell–Saunders coupling, where the internal symmetry group is shrunk to the identity.

For the nuclear $SU(4)$, it was mentioned how one may refer back from the validity of that supermultiplet theory to the structure of a leading part of the Hamiltonian. The analog for $SU(6)$ would be to ask for the "inner" dynamics of a baryon (for example) built up out of sextets (see Sec. IV J). But this is only one facet of what has been attempted with $SU(6)$ and its descendants. Beyond that, there arise questions in particle physics which have had no analogous treatments elsewhere. For example, can one say anything new about the scattering of two particles each of which is assigned to an $SU(6)$ supermultiplet? The analog would be the scattering of two nuclei on each other where the state of each one separately is (approximately) described in terms of the nuclear $SU(4)$. This clearly must involve questions of recoupling with orbital angular momentum (except for S-wave scattering see Sec. IV G).

There is another type of problem which arises in the context of $SU(6)$, namely the off-the mass shell problems as they appear for example in the study of vertices. In this context belong also many questions related to the possible existence of a field theory. Attempts in this direction must go well beyond the phenomenological level as it was sketched for $SU(4)$. These problems, which seem particularly pressing as the principal encouragements for $SU(6)$ theory come from vertices, are taken up again in Sec. IV I, V, and VI.

III. SOME MATHEMATICAL TOOLS

A. Representations of *SU(N)*

(1) Following a procedure familiar from the ordinary rotation group, one can obtain general representations from the defining representation by the construction of tensors. A tensor $T_{\beta_1\cdots\beta_n}{}^{\alpha_1\cdots\alpha_m}$ is defined by

$$T_{\beta_1\cdots\beta_n}{}^{\alpha_1\cdots\alpha_m}\sim x^{\alpha_1}(1)\cdots x^{\alpha_m}(m)x_{\beta_1}(1')\cdots x_{\beta_n}(n'), \tag{3.1}$$

where $\sim$ means: transforms like. This definition of a tensor holds for $U(N)$ and $SU(N)$ alike. Thus unitary tensors have two kinds of indices, upper and lower ones. Apart from the component labels the x's are marked with " configuration labels" (i), or (i') to distinguish one unitary vector from another. x^α transforms with A, x_β with $A^\dagger$, as in Eqs. (2.1) and (2.2), respectively.

For the case of $SU(N)$ only one can, without loss of generality, consider tensors which have only upper (or only lower) indices by introducing the Levi–Civita symbols $\epsilon^{\lambda_1\cdots\lambda_N}$. ϵ is totally antisymmetric in its N indices and $\epsilon^{123\cdots N}=1$; $\epsilon_{\lambda_1\cdots\lambda_N}$ has similar properties. Under a transformation of $U(N)$, these symbols transform as follows:

$$\epsilon'^{\lambda_1\cdots\lambda_N}=D^{-1}a_{\mu_1}{}^{\lambda_1}\cdots a_{\mu_N}{}^{\lambda_N}\epsilon^{\mu_1\cdots\mu_N}, \tag{3.2}$$

$$\epsilon'_{\lambda_1\cdots\lambda_N}=D^{*-1}a_{\lambda_1}{}^{\dagger\mu_1}\cdots a_{\lambda_N}{}^{\dagger\mu_N}\epsilon_{\mu_1\cdots\mu_N}, \tag{3.3}$$

$$D=\det\|a\|,\qquad D^*=\det\|a^*\|. \tag{3.4}$$

The transformations (3.2, 3) preserve the total antisymmetry, while also $\epsilon'^{1\cdots N}=\epsilon'_{1\cdots N}=1$, as is easily checked. It is in order to preserve these unit values that the factors D^{-1} (or D^{*-1}) had to be introduced. Thus the ϵ symbols do *not* behave as tensors under $U(N)$. But for $SU(N)$ where $D=D^*=1$, the ϵ's are true tensors. One may call the ϵ's pseudo-tensors under $U(N)$, in straight generalization of the terminology for the rotation group, where ϵ^{ijk} is a constant tensor under $SO(3)$ but changes sign under the reflections included in $O(3)$.

Confining ourselves now to $SU(N)$, we can raise all lower tensor indices to upper ones. Thus to x_i we can associate a totally antisymmetric tensor y:

$$x_i=\epsilon_{il_2\cdots l_N}y^{l_2\cdots l_N},$$
$$y^{l_2\cdots l_N}=[1/(N-1)!]\epsilon^{il_2\cdots l_N}x_i. \tag{3.5}$$

Note the familiar case $N=2$:

$$x_i=\epsilon_{ij}y^j,\qquad y^j=\epsilon^{ij}x_j, \tag{3.6}$$

where one may associate a spinor y^j to a conjugate spinor x_i.

Let us return to the tensor T of Eq. (3.1) with both upper and lower indices. This T corresponds in general to a reducible representation, for two reasons.

(a) If its trace with respect to α_i and β_j is unequal to zero, then the original T contains a nonvanishing tensor of lower rank which is itself a representation [as $x^\alpha(i)x_\alpha(i')$ is a scalar].

(b) Consider

$$T^{\alpha_1\alpha_2}=\tfrac{1}{2}(T^{\alpha_1\alpha_2}+T^{\alpha_2\alpha_1})+\tfrac{1}{2}(T^{\alpha_1\alpha_2}-T^{\alpha_2\alpha_1}).$$

Each of the two terms in brackets separately form a representation of $SU(N)$: the unitary transformations commute with permutations on the indices, they preserve the (anti-)symmetry. It is true in general for tensors of any rank that if there exists a permutational symmetry or antisymmetry of the indices then this is respected by the unitary transformations, [H 1]. Thus one step toward getting irreducible representations is to find "tensors with permutational symmetry."

Moreover, the tracelessness condition itself can be expressed as a particular kind of permutational symmetry. Consider for example the tensor

$$t_\beta{}^\alpha,\ t_\alpha{}^\alpha=0. \tag{3.7}$$

This is a (N^2-1)-dimensional representation, the adjoint representation. The condition $t_\alpha{}^\alpha=0$ can also be expressed as follows.

$$\epsilon^{\beta\alpha_1\cdots\alpha_{N-1}}t_\beta{}^{\alpha_N}\pm\epsilon^{\beta\alpha_2\cdots\alpha_N}t_\beta{}^{\alpha_1}+\epsilon^{\beta\alpha_3\cdots\alpha_N\alpha_1}t_\beta{}^{\alpha_2}$$
$$\pm\cdots\pm\epsilon^{\beta\alpha_N\alpha_1\cdots\alpha_{N-2}}t_\beta{}^{\alpha_{N-1}}=0, \tag{3.8}$$

where all signs are $+$ for N odd, while signs alternate for N even. It is easily checked that (3.8) is tantamount to the one condition $t_\alpha{}^\alpha=0$. We can therefore associate to $t_\beta{}^\alpha$ a new tensor:

$$T^{[\alpha_1\alpha_2\cdots\alpha_{N-1}]\alpha_N}=\epsilon^{\beta\alpha_1\cdots\alpha_{N-1}}t_\beta{}^{\alpha_N}, \tag{3.9}$$

where T is totally antisymmetric in its first $N-1$ indices and where the symmetrization condition

$$T^{[\alpha_1\cdots\alpha_{N-1}]\alpha_N}\pm T^{[\alpha_2\cdots\alpha_N]\alpha_1}$$
$$+\cdots\pm T^{[\alpha_N\alpha_1\cdots\alpha_{N-2}]\alpha_{N-1}}=0 \tag{3.10}$$

is imposed in addition. It is therefore clear that to any irreducible tensor $T_{\beta_1\cdots\beta_n}{}^{\alpha_1\cdots\alpha_m}$ can be associated a tensor with upper indices *only*. For this new tensor there must then exist mixed symmetry–antisymmetry conditions between these upper indices. Unless specified otherwise, we mean in the following by tensor always this latter kind with upper indices only.

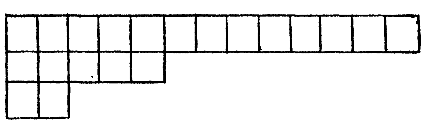
According to a general theorem [H 1] we can find *all* irreducible representations of $SU(N)$ by classifying all tensors with permutational symmetry and with one kind of index (say upper) only.

(2) There are other standard forms for the tensors which can be arrived at by the use of the ϵ's. For example, in the case of $SU(3)$ one can make all tensors totally symmetric in upper and likewise in lower indices and furthermore require tracelessness [B 1]. The form chosen here is perhaps most easily suited for the application of Young tableau rules, which give a quick and exhaustive way of treating all possible symmetry/antisymmetry situations. The rules are the following [H 1, L 1].

To an irreducible tensor of rank n of the group $SU(N)$ there corresponds an "allowed" partition

$$(n_1, n_2, \cdots, n_N), \sum_i n_i = n, \qquad n_1 \geq n_2 \geq \cdots \geq n_N. \tag{3.11}$$

Some of the n_i at the back end may be zeros in which case we do not write them explicitly. In evident shorthand then, for example for $N=6$, the partition (31^2) stands for (311000). If all $n_i=0$ we write (0). This is the identity representation. The partition can be pictured by a "tableau" like



with N rows of successive length $n_1, \cdots, n_N$. The structure of this tableau is meant to prescribe symmetrizations and antisymmetrizations as follows. Fill each square of the tableau with one of the configuration labels, as they occur for example in (3.1). For a given filling, antisymmetrize in the labels occurring in the columns; then symmetrize in the labels occurring in the rows. Thus a Young tableau stands for a permutation operator. When these operators act on $T^{\alpha_1\cdots\alpha_r} \sim x^{\alpha_1}(1)\cdots x^{\alpha_r}(r)$, many of the N^r ($r=$ rank of the tensor) combinations of tensor indices get annihilated. The remaining combinations are $D_N(n_1, \cdots, n_N)$ in number. D_N is the dimension of the representation.

Remarks. (1) Alternatively, we can first symmetrize, then antisymmetrize, or follow more complex procedures of ordering symmetrizations and antisymmetrizations. Each order gives a set of D_N vectors spanning the representation space. Different orders give different sets. Nor are these sets in general orthogonal sets. For procedures to get orthogonal sets see [Y 1, J 1, T 1].

(2) Except for purely symmetric or purely antisymmetric tensors there are more (independent) ways than one to distribute the configuration labels over the squares. The various independent ways yield *equivalent* representations of $SU(N)$, for an example see [B 1, p. 22]. [The number of equivalent representations is equal to the number of times the tensor $(n_1, \cdots, n_N)$ occurs in the direct product of $1\otimes 1\otimes\cdots\otimes 1$; n factors).

Proceeding as in (3.11) for all n and all allowed partitions of n one obtains all irreducible representations of $SU(N)$. The association: tensor$\leftrightarrow$Young tableau is unique.

(3) Formula for D_N.

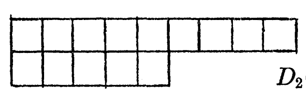
$$\begin{aligned} D_N(n_1, \cdots, n_N) = [(n_1-n_2+1)/1!]\cdot(n_1-n_3+2)\cdots(n_1-n_N+N-1)\,. \\ \cdot(n_2-n_3+1)/2!\cdots(n_2-n_N+N-2)\,. \\ \vdots \qquad\qquad \vdots \\ (n_{N-1}-n_N+1)/(N-1)!. \end{aligned} \tag{3.12}$$

This formula gives the dimension for any N. For $N=2$ take only the first column on the right-hand side of (3.12); for $N=3$, take both first and second columns, etc. Note that

$$D_N(n_1, \cdots, n_N) = D_N(n_1-n_N, n_2-n_N, \cdots, n_{N-1}-n_N, 0) \tag{3.13}$$

so we always may strike off the columns of length N. This corresponds to the fact that $\epsilon_{\lambda_1\ldots\lambda_N} x^{\lambda_1}(1)\cdots x^{\lambda_N}(N)$ is a scalar under $SU(N)$.

The case $N=2$:



$$D_2(n_1, n_2) = n_1 - n_2 + 1 \tag{3.14}$$

is quite familiar. The n_2 pairs of "spins" are each paired off to zero resultant. The residual n_1-n_2 spins are in a totally symmetric state, and therefore correspond to a total spin $(n_1-n_2)/2=S$. Hence $D_2=2S+1$. Thus given the total spin of an n spin $\frac{1}{2}$ system its symmetry is determined. That is why for $SU(2)$ problems like normal atomic coupling there is not such a great advantage to use tableaux.

Examples.

(a)

$$D_N(1)=D_N(1^{N-1})=N. \tag{3.15}$$

(1) is the defining representation, (1^{N-1}) its conjugate. (1^{N-1}) corresponds to the y tensor in (3.5). The representation (1) is self-conjugate if and only if $N=2$, cf. (3.6).

(b)

$$D_N(21^{N-2})=N^2-1, \tag{3.16}$$

this is the adjoint representation.

(c)

$$D_N(n_1)=\binom{N+n_1-1}{n_1}, \tag{3.17}$$

these are the totally symmetric representations. Thus $D_3(3)=10$, $D_6(3)=56$, etc.

(d)

$$D_N(1^k)=\binom{N}{k}, \tag{3.18}$$

the totally antisymmetric representations; $D_6(1^3)=20$, etc.

(4) Conjugate representations. We have the identity

$$D_N(n_1,\cdots,n_N)$$
$$=D_N(n_1-n_N, n_1-n_{N-1},\cdots,n_1-n_2,0). \tag{3.19}$$

The partition on the right-hand side (which is an allowed one if the one on the left-hand side is) corresponds to a representation called the conjugate representation of $(n_1,\cdots,n_N)$. The special case (1) versus (1^{N-1}) has been met already. Other examples: (3^2) is the *10** of $SU(3)$, (3^5) is the *56** of $SU(6)$.

These are examples of inequivalent representations with the same dimensions. However, conjugation does not exhaust the possible equality of dimensions. Example: $D_4(3)=D_4(21)=D_4(22)=20$. Where possible, representations will often be referred to by their dimensions (in italics), with the understanding that this is not adequate in general. [For $SU(2)$ it is always adequate.]

Using the language of upper and lower indices as in (3.1), the connection between a representation and its conjugate amounts to an interchange of roles of upper and lower indices. Thus for $SU(3)$, the *10* corresponds to $T^{\alpha\beta\gamma}$, the *10** to $T_{\alpha\beta\gamma}$ with total symmetry in the 3 indices in each case. In $SU(6)$, $D_6(31^3)=D_6(3^22^3)=280$. To each representation belongs a traceless tensor $T_{\gamma\delta}{}^{\alpha\beta}$. For (31^3), the *280*, we have symmetry in (α,β), antisymmetry in (γ,δ). For $(3^2,2^3)$, the *280**, we have antisymmetry in (α,β), symmetry in (γ,δ).

If the conjugate representation is identical with the original one, we have a self-conjugate case. In tensor language, we have equivalent properties of upper and lower indices. Thus (21^{N-2}), the adjoint representation, is the self conjugate tensor $T_\beta{}^\alpha$. Other examples: $D_6(42^4)=405$, $D_6(2^21^2)=189$ are each self-conjugate and correspond to a tensor $T_{\gamma\delta}{}^{\alpha\beta}$, symmetric (*405*), antisymmetric (*189*), both in the lower and in the upper indices.

Whether a tensor is self-conjugate depends both on the partition and on the dimension. Thus $D_3(21)=8$ is self-conjugate, but $D_6(21)=70$ is not. In either case we can represent the tensor by $T^{[\alpha\beta]\gamma}$, antisymmetric in $[\alpha,\beta]$ which satisfies in addition $T^{[\alpha\beta]\gamma}+T^{[\gamma\alpha]\beta}+T^{[\beta\gamma]\alpha}=0$. For $N=3$ we can revert from $T^{[\alpha\beta]\gamma}$ to $t_\beta{}^\alpha$ see (3.9); for $N=6$ we cannot do so.

Finally, we recall that a familiar way to construct explicit representations for the rotation group is to find the spherical harmonics. One can in principle construct analogous "harmonic functions" for all unitary groups [M 1, B 4, K 2, I 2].

B. Representations of $U(N)$

Let us return to the tensor T, Eq. (3.1), defined as a tensor under $U(N)$. By definition a transformation of T involves a product of m factors A, n factors $A^\dagger$ where A is an element of $U(N)$. Now generally

$$A=(e^{i\phi}\cdot 1)\cdot a, \tag{3.20}$$

where a is an element of $SU(N)$, 1 is the $N\times N$ unit matrix. The factorization (3.20) says that $U(N)$ contains an invariant subgroup $U(1)$. [For a more refined discussion of the connection between $U(N)$ and $SU(N)$ see, e.g., [E 1].] The transformed T is therefore equal to its transformed under $SU(N)$ times a "gauge" factor $\exp[i(m-n)\phi]$. As $\det A=\exp(iN\phi)$ we may therefore put

$$D^{-1}=\exp(-iN\phi) \tag{3.21}$$

in Eq. (3.2).

Even though, as noted above, the Levi–Civita symbol is not a tensor under $U(N)$ we may nevertheless apply the entire apparatus of Sec. III A also to the case of $U(N)$, with one additional stipulation. All we have to do is to assign to the general tensor T of Eq. (3.1) a "baryon number" $B=\lambda(m-n)$, where λ is a number independent of m and n. For the defining representation $B=\lambda$. Note that B is independent of any possible symmetry between upper or between lower tensor indices. After B has been assigned the problem is reduced to $SU(N)$ and we may again consider tensors with only upper (or only lower) indices.

Example. Consider $SU(3)$. Let x^α be a "quark" [G 2, Z 1] with baryon number $B=\frac{1}{3}$. Then $x_\alpha{}^*$ is an "anti-quark", $B=-\frac{1}{3}$. $M_\beta{}^\alpha$ has $B=0$ and may represent a meson octet. Its Young tableau is (21). Consider the tensor $T^{[\alpha\beta]\gamma}$ with

$$T^{[\alpha\beta]\gamma}+T^{[\gamma\alpha]\beta}+T^{[\beta\gamma]\alpha}=0. \tag{3.22}$$

T has $B=1$. After having established the value of B, we now put

$$T^{[\alpha\beta]\gamma}=\epsilon^{\alpha\beta\delta}B_\delta{}^\gamma$$

as in Eq. (3.9). Then $B_\delta{}^\gamma$ is traceless. It may represent the baryon octet. Its Young tableau is also (21). Thus the same (21) describes a quark–antiquark set and a three-quark set. There is no source of confusion as long as the assignment of a B value is separately established.

Once this point is out of the way we can restrict the rest of Sec. III to $SU(N)$. The generalizations to $U(N)$ will be obvious.

C. Reduction of Product Representation

Tableaux are also useful for the decomposition of $(n_1, \cdots, n_N)\otimes(n_1', \cdots, n_N')$ into irreducible representations. The general procedure is best arrived at as follows [L 1].

(I). $(n_1, \cdots, n_N)\times(1)$, product of the general tableau with □. Add □ to the original tableau in all possible ways such that one still has an allowed partition (3.11) (of $n+1$) as a result. Each of these ways corresponds to an irreducible representation in the product.

Examples.

$$\begin{aligned}
(1)\times(1)&=(2)+(1^2)\\
(21)\times(1)&=(31)+(22)+(21^2), && N\geq 3,\\
&=(31)+(22)+(1), && N=3,\\
&=(2)+(0), && N=2,\\
(2^2)\times(1)&=(32)+(2^21), && N\geq 3,\\
&=(32)+(1^2), && N=3.
\end{aligned}$$

(II). $(n_1, \cdots, n_2)\times(m)$. We now multiply with a totally symmetric representation □□□□□□ of m squares. Proceed successively square by square of this set of m as under (I). But never add two or more squares of the set of m to the same column.

Examples.

(1) $SU(2)$. Let $m\geq n$, then

$$\begin{aligned}
(m)\times(n)&=(m+n)+(m+n-1,\ 1)+(m+n-2,\ 2)\\
&\qquad+\cdots+(m,\ n),\\
&=(m+n)+(m+n-2)+(m+n-4)\\
&\qquad+\cdots+(m-n)
\end{aligned}\tag{3.23}$$

where columns of length 2 have been struck away, see (3.13). Put $m=2S$, $n=2S'$, then the dimension check for Eq. (3.23) reads

$$(2S+1)\times(2S'+1)=[2(S+S')+1]+\cdots+[2(S-S')+1].$$

Thus with the tableau rules we have the reduction for angular momentum as a special case.

Equation (3.23) provides an example of simple reducibility. We say that a product representation is simply reducible if any representation which occurs in the product appears only once. Equation (3.23) is a special case of a slightly more general situation.

Lemma 1. For any $SU(N)$ the product $(m, m, m, \cdots, m)\times(n)$ (where the number of m's is $\leq N$) is simply reducible.

The following special case of Lemma 1 will be important for later purposes.

Lemma 2. The conjugate representation of (n) is $(n, n, \cdots, n)$ where the number of n's is equal to $N-1$. The product of (n) and its conjugate is simply reducible; in particular it contains the adjoint representation (21^{N-2}) once and only once.

(2) $SU(3)$.

$$\begin{aligned}
(3^2)\times(3)&=(63)+(531)+(432)+(333)\\
&=(63)+(42)+(21)+(0).
\end{aligned}$$

Dimension check [use (3.12)],

$$10^*\times 10=64+27+8+1.$$

Also

$$\begin{aligned}
(21)\times(3)&=(51)+(42)+(41^2)+(321),\\
&=(51)+(42)+(3)+(21),\\
8\times 10&=35+27+10+8.
\end{aligned}$$

(3) $SU(6)$.

$$\begin{aligned}
(3^5)\times(3)&=(3^6)+(43^42)+(53^41)+(63^4)\\
&=(0)+(21^4)+(42^4)+(63^4).
\end{aligned}$$

(Strike columns of length 6). Check:

$$56^*\times 56=1+35+405+2695.\tag{3.24}$$

Also

$$\begin{aligned}
(21^4)\times(3)&=(41^5)+(321^4)+(421^3)+(51^4),\\
&=(3)+(21)+(421^3)+(51^4),\\
35\times 56&=56+70+1134+700.
\end{aligned}\tag{3.25}$$

(III). $(n_1, \cdots, n_N)\times(n_1', \cdots, n_N')$. First proceed with the top row of length n_1' as under (II); then add on the next row of length n_2' as under (II); $\cdots$, etc. But now mark one last additional rule.

Put a 1 in each of the n_1' squares of the top row in $(n_1', \cdots, n_N')$, put a 2 in each of the n_2' squares of the second row, $\cdots$, an N in each of the n_N' squares of the last row. Now look at one of the product tableaux obtained as per previous instructions *with* the markings 1, or 2, $\cdots$ or N in the various squares inserted. Read from right to left the numbers found in the first row; continue to read from right to left what is marked in the second row, $\cdots$, from right to left in the last row. This gives one total sequence of numbers which should obey the rule: up to any point within the sequence the number of 1's $\geq$ number of 2's $\cdots \geq$ number of n's. (This is known as a lattice permutation [L 1]).

Examples.

(1) $SU(3)$

$$(2^2)\times(2^2)=(4^2)+(431)+(42^2),$$
$$=(4^2)+(32)+(2),$$
$$6^*\times 6^*=15+15'+6.$$

(Note the two distinct representations of dimension *15*).

$$(21)\times(21)=(42)+(41^2)+(3^2)+(321)+(321)+(2^3)$$
$$=(42)+(3)+(3^2)+(21)+(21)+(0)$$
$$8\otimes 8=27+10+10^*+8+8+1.$$

(2) $SU(6)$.

$$(21^4)\times(21^4)=(2^6)+(32^41)+(32^41)+(3^22^21^2)+(3^22^3)+(42^31^2)+(42^4)$$
$$=(0)+(21^4)+(21^4)+(2^21^2)+(3^22^3)+(31^3)+(42^4).$$

Check:

$$35\times 35=1+35+35+189+280^*+280+405. \quad (3.26)$$

Exercise. (21) is the *70*, (2^41) the *70**. Show that

$$70^*\times 70=1+2\times 35+189+280+280^*+405+3675. \quad (3.27)$$

It is important not only to know which representations occur in the product of two representations. Also one wants to know what is the structure of the transformation which expresses the product of any tensor component of one representation times any component of the other as a sum of components contained in the irreducible parts of the product. This is the problem of the Clebsch–Gordan coefficients. For a number of cases this has been discussed in the literature. For $SU(6)$ see [C 2, C 3, S 8].

D. Reduction of $SU(MN)$ with Respect to $SU(M)\times SU(N)$

Consider the invariant quadratic form $x_\alpha{}^*x^\alpha$ of $SU(MN)$, where x^α is the defining representation, $\alpha=1, \cdots, MN$. We can replace each index value α by an index pair Ai, $A=1, \cdots, M$; $i=1, \cdots, N$, so

$$x_\alpha{}^*x^\alpha=x_{Ai}{}^*x^{Ai}. \quad (3.28)$$

For fixed i (or A) we recognize within (3.28) an invariant form of $SU(M)$ [or $SU(N)$]. In other words, for fixed i (or A) but running A (or i), x^{Ai} is a fundamental representation of $SU(M)$ [or $SU(N)$]. We can represent this content of Eq. (3.28) in tableau language as follows.

$$\square=(\square;\square). \quad (3.29)$$

where the left-hand side is the (1) of $SU(MN)$ while on the right-hand side the first $\square$ is the (1) of $SU(M)$, the second $\square$ is the (1) of $SU(N)$. Thus, equivalently to (3.29):

$$1=(1;1). \quad (3.30)$$

Still a third way of representing the contents of (3.28) is by dimensions

$$MN=(M;N). \quad (3.31)$$

For the conjugate to the defining representation,

$$(1^{MN-1})=(1^{M-1};1^{N-1}), \quad (3.32)$$
$$(MN)^*=(M^*,N^*). \quad (3.33)$$

Another obvious example for this kind of notation is

$$(0)=(0;0), \quad \text{(tableaux)} \quad (3.34)$$
$$1=(1;1), \quad \text{(dimensions)} \quad (3.35)$$

linking the respective identity representations.

We have now three examples where to an irreducible representation of $SU(MN)$ there belongs *one* irreducible representation pair of $SU(M)$; $SU(N)$. For all other representations of $SU(MN)$ we will have more than one pair. We can find all such pairs by operating separately with the tableau rules for each of the three groups as will be clear from the following examples.

$$(1)\quad (1)\times(1)=(2)+(1^2)=(1,1)\times(1,1)=(2+1^2;2+1^2),$$

or

$$(2)+(1^2)=(2;2)+(1^2;2)+(2;1^2)+(1^2;1^2). \quad (3.36)$$

From (3.17, 18) this has the dimension check

$$\tfrac{1}{2}MN(MN+1)+\tfrac{1}{2}MN(MN-1)$$
$$=(\tfrac{1}{2}M(M+1);\tfrac{1}{2}N(N+1))+(\tfrac{1}{2}M(M-1);\tfrac{1}{2}(N(N+1))+(\tfrac{1}{2}M(M+1);\tfrac{1}{2}N(N-1))+(\tfrac{1}{2}M(M-1);\tfrac{1}{2}N(N-1)), \quad (3.37)$$

with the observation that a dimension pair $(a;b)$ counts for ab. We want of course to know the separate content of the (2) and the (1^2) of $SU(MN)$. This also follows uniquely from the dimension check (3.12):

$$(2)=(2;2)+(1^2;1^2),$$
$$(1^2)=(1^2;2)+(2;1^2). \quad (3.38)$$

For $SU(6)$, $M=3$, $N=2$ (3.38) corresponds to (in dimensions)

$$21=(6;3)+(3^*;1),$$
$$15=(3^*;3)+(6;1). \quad (3.39)$$

Of course (3.38) could also have been found from the observation that the symmetric tensor (2) of $SU(MN)$ corresponds either to symmetry in both the tensor indices of $SU(M)$ and $SU(N)$ or to anti-symmetry in both these cases.

$$(2)\quad (1^{MN-1})\times(1)=(21^{MN-2})+(0)$$
$$=(1^{M-1},1^{N-1})\times(1,1)$$
$$=(21^{M-2}+0,\,21^{N-2}+0).$$

Using (3.35) we therefore have at once for the adjoint representation (21^{MN-2}):

$$(21^{MN-2})=(21^{M-2},21^{N-2})+(21^{M-2},0)+(0,21^{N-2}),\quad (3.40)$$

corresponding to

$$(MN)^2-1=(M^2-1,N^2-1)+(M^2-1,1)+(1,N^2-1).\quad (3.41)$$

For $M=3$, $N=2$:

$$35=(8;3)+(8,1)+(1,3).\quad (3.42)$$

The general procedure will now be clear and we give some further results for $SU(6)\supset SU(3)\otimes SU(2)$. Multiply both (2) and (1^2) with (1). This gives, in dimensions:

$$56=(8;2)+(10;4),\quad (3.43)$$
$$70=(1;2)+(8;2)+(10;2)+(8;4),\quad (3.44)$$
$$20=(1;4)+(8;2),\quad (3.45)$$

for the three (three-sextet) tableaux (3), (21), (1^3), respectively.

Conjugate tableaux have conjugate content. Thus from (3.39):

$$21^*=(6^*;3)+(3;1),\quad (3.46)$$
$$15^*=(3;3)+(6^*,1),\quad (3.47)$$

for the $SU(6)$ tableaux (2^5) and (1^4), respectively. Now

$$21^*\times 21=405+35+1,\quad (3.48)$$
$$15^*\times 15=189+35+1.\quad (3.49)$$

Thus with the help of (3.35) and (3.42)

$$189=(8+1;5)+(8+8+10+10^*;3)+(27+8+1;1),\quad (3.50)$$
$$405=(27+8+1;5)+(27+8+8+10+10^*;3)+(27+8+1;1).\quad (3.51)$$

Likewise, $15^*\times 21$ gives

$$280=(10+8;5)+(27+8+8+10+1;3)+(10+10^*+8,1)\quad (3.52)$$

while 280^* has the conjugate content.

More details and examples are found in [H 3].

In equations such as (3.39–47) we find examples where the contents of an $SU(6)$ representation is completely specified in terms of labels provided by the subgroup $SU(3)\otimes SU(2)$. On the other hand, Eqs. (3.50–52) show that this specification is not adequate in general, as $(8;3)$ occurs twice in each instance. This raises the general problem of a full labeling of the D_N states which span a given representation of $SU(N)$. This question is discussed in the next section.

E. Generators; Labeling Problems; Currents

A unitary transformation acting on a tensor with D_N linearly independent components is a linear transformation in the D_N-dimensional space spanned by these components. The transformation can be represented by a $D_N\times D_N$ matrix U which itself is unitary and which can be represented by [H1, B1]

$$U=\exp(i\epsilon_B F_B),\quad (3.53)$$

where a summation over B is implied. For $U(N)$, $B=1,\cdots,N^2$; for $SU(N)$, $B=1,\cdots,N^2-1$. The ϵ_B are parameters which may be chosen to be real. Thus $SU(N)$ is an (N^2-1)-parameter group. Like U, the *generators* F_B can be represented by $D_N\times D_N$ matrices which are hermitian for real ϵ_B. Moreover, for $SU(N)$ we have the trace condition

$$(F_B)_i{}^i=0.\quad (3.54)$$

The F_B satisfy commutation relations

$$[F_B,F_C]=if_{BCD}F_D.\quad (3.55)$$

Here the totally antisymmetric real f_{BCD} are the so called structure constants of the Lie algebra of the F's. The f_{BCD} have the all-important property to be independent of the particular representation on which the F's act. The f's are therefore fully specified by what will be called the defining generators (DG). The DG are defined as the representation of the generators by $N\times N$ sized matrices in as far as they act on the defining representation.

Example. For $SU(2)$ the three generators are the angular momentum operators; $f_{BCD}=\epsilon_{BCD}$, the 3-dimensional Levi-Civita symbol. The DG are $\sigma_B/2$ where the σ's are the three Pauli matrices.

It is easy to write down the DG for $SU(N)$. Put $F_B=(C_\beta{}^\alpha, C'_\beta{}^\alpha, H_k)$. Here $\alpha>\beta=1,\cdots,N$ so there are $N(N-1)/2$ C and the same number of C' matrices; $k=1,\cdots,N-1$. We may put

$$(C_\beta{}^\alpha)_j{}^i=\tfrac{1}{2}(\delta^{\alpha i}\delta_{\beta j}+\delta^{\alpha j}\delta_{\beta i}),$$
$$(C'_\beta{}^\alpha)_j{}^i=(i/2)(\delta^{\alpha i}\delta_{\beta j}-\delta^{\alpha j}\delta_{\beta i}),\quad (3.56)$$

while the H_k are diagonal matrices whose elements

along the diagonal may be chosen as follows:
For H_k:

$$[2k(k+1)]^{-\frac{1}{2}}(\underbrace{1, 1, \cdots, 1}_{k}, -k, 0, \cdots, 0),$$

$$k=1, \cdots, N-1. \tag{3.57}$$

It is very convenient to normalize such that Tr F_B^2 is independent of B, as has been done here. From this realization of the F_B, the f_{BCD} can be computed. For $SU(3)$ this is done in [G 1].

The H_k are a set of commuting matrices. From the main property of the f's it follows then that for any representation there is such a commuting subset of $(N-1)$ generators. It is known that this is the maximal commuting subset. Thus for $SU(N)$ a partial labeling is provided by $N-1$ additive quantum numbers. (The number $N-1$ of H_k is the "rank" of the group.)

The important question arises to find the matrix respresentation of the F's acting on any representation of $SU(N)$. This problem has been solved explicitly [B 2].

We now turn to the question of a complete labeling of the D_N "vectors" spanning the representation space for a tensor in $SU(N)$. For $SU(2)$ the answer is familiar. The representation is characterized by the angular momentum squared, which we write as

$$\sum_{B=1}^{3} F_B^2$$

in the present language and for which we introduce the symbol $C_2^{(2)}$:

$$C_2^{(2)} = \sum_{B=1}^{3} F_B^2. \tag{3.58}$$

Furthermore the individual vectors in the representation space are labeled by the magnetic quantum number, which we now call H_1. The labeling by $C_2^{(2)}$ and H_1 is complete. How is this for $SU(N)$?

(1) The number of commuting operators whose eigenvalues are sufficient for the specification of a state is $(N-1)(N+2)/2$, see [R 1, B 2].

(2) $N-1$ such operators are given by the H_k.

(3) A further set of $N-1$ operators is given by the "Casimir operators" $C_i^{(N)}$, $i=2, \cdots, N$. $C_i^{(N)}$ is a polynomial of degree i in the F_B. It is known [R 2, B 3] that there are just $N-1$ independent nonlinear expressions in the F_B which commute with all F_B.

The $C_2^{(N)}$ are easy:

$$C_2^{(N)} = \sum_{1}^{N} F_B^2. \tag{3.59}$$

The other $C_i^{(N)}$ have been explicitly constructed, see [B 2] (also for the earlier literature on the subject) and [K 1]. The answer is as follows (sum over repeated labels from 1 to N).

$$C_3^{(N)} = d_{ABC}F_AF_BF_C,$$

$$C_4^{(N)} = d_{ABE}d_{ECD}F_AF_BF_CF_D,$$

$$C_5^{(N)} = d_{ABP}d_{PQC}d_{QDE}F_AF_BF_CF_DF_E, \quad \text{etc.}$$

where the d's are defined by the anticommutator of the DG which can generally be written as

$$F_AF_B + F_BF_A = 2C\delta_{AB}F_0 + d_{ABC}F_C, \tag{3.60}$$

where the real d_{ABC} are totally symmetric, C is a number depending on the choice of normalization for the F_B and F_0 is the unit matrix.

A perhaps simpler way of writing the C operators is as follows [O 1]. Go to $U(N)$ and write the N^2 generators as A_β^α, $\alpha, \beta=1, \cdots, N$. Then a (non-Hermitian) representation of the DG is

$$(A_\beta^\alpha)_j^i = \delta^{\alpha i}\delta_{\beta j}, \tag{3.61}$$

so that

$$[A_\beta^\alpha, A_\delta^\gamma] = f_{\beta\delta\sigma}{}^{\alpha\gamma\rho}A_\rho^\sigma, \tag{3.62}$$

$$f_{\beta\delta\sigma}{}^{\alpha\gamma\rho} = \delta_\sigma^\alpha\delta_\beta^\gamma\delta_\delta^\rho - \delta_\delta^\alpha\delta_\sigma^\gamma\delta_\beta^\rho. \tag{3.63}$$

These f's are again the structure constants (in a slightly different guise). The C operators are simply traces of powers of A:

$$C_i^{(N)} = \text{Tr}\,(A^i) \equiv A_{\alpha_2}{}^{\alpha_1}A_{\alpha_3}{}^{\alpha_2}\cdots A_{\alpha_1}{}^{\alpha_i}, \tag{3.64}$$

where now i runs from 1 to N (rather than from 2). $C_1^{(N)} = A_\alpha^\alpha$ (summed over α) commutes with all A. Hence $(A_\alpha^\alpha)_j^i = c\delta_j^i$. The restriction $U(N)\rightarrow SU(N)$ amounts to the condition that on any representation $C_1^{(N)}=0$.

The $C_i^{(N)}$ label fully a representation of $SU(N)$. We found a previous labeling, namely the partition numbers $(n_1, \cdots, n_N)$ of which only $N-1$ are independent for $SU(N)$ see Eq. (3.13). These two ways of labeling are equivalent; $C_i^{(N)}$ is a polynomial of degree i in the n_j.

Examples. For $SU(3)$, [B 3], partition (pq):

$$C_2^{(3)} = \tfrac{1}{3}(p^2+q^2-pq+3p), \tag{3.65}$$

$$C_3^{(3)} = \tfrac{1}{18}(p-2q)(2p+3-q)(p+q+3). \tag{3.66}$$

For $SU(4)$, partition (pqr), [B 5],

$$C_2^{(4)} = \tfrac{3}{8}(p^2+q^2+r^2) - \tfrac{1}{4}(pq+qr+rp) + \tfrac{1}{2}(3p+q-r). \tag{3.67}$$

{There are always constant cofactors to be fixed by a normalization convention. The present ones correspond to the normalizations in (3.56–57). See further [W 6].}

(4) We have now found $2(N-1)$ of the desired commuting operators which is not yet enough (except for $N=2$). One way to get a complete set is to consider

the factorization [W 3, B 3]

$$SU(N)\supset U(1)\otimes SU(N-1),$$

where $U(1)$ is generated by a linear combination of the $N-1$ H_i and where the generators of $SU(N-1)$ commute with the generator of $U(1)$. For example, the DG of $U(1)$ can be given by H_{N-1} of Eq. (3.57) and the DG of $SU(N-1)$ by Eqs. (3.56–57) with $N\rightarrow N-1$. We may then further label by the $C_i^{(N-1)}$. Continue next likewise: $SU(N-1)\supset U(1)\otimes SU(N-2)$ which yields the $C_i^{(N-2)}$, etc. A full labeling is then obtained by: $N-1$ H_i (or linear combinations thereof) and the $C_i^{(k)}$, $k=2, 3, \cdots, N$; $i=2, 3, \cdots, k$ which yields the desired total number.

Example. $SU(3)\supset U(1)\otimes SU(2)$. The 5 labels are: $C_2^{(3)}$, $C_3^{(3)}$ or equivalently the partition (pq); $C_2^{(2)}$: isospin; the hypercharge [generator of $U(1)$]; and the z component of isospin.

While this procedure is sufficient for any N, it is not the only one and in fact it is not the one commonly used. To conclude this section, we discuss the method followed [B 5] in the discussion of $SU(6)$.

For a full labeling in $SU(6)$ one needs 20 operators. The following have been used:

(I) The partition numbers $(n_1, \cdots, n_5)$ of the $SU(6)$-Young tableau, equivalent to the $C_i^{(6)}$: five in number.

(II) The contents classification by the chain

$$SU(6)\supset SU(3)\otimes SU(2) \qquad (3.68)$$

where the two subgroups are commuting ones. This gives: 5 labels from $SU(3)$, 2 from $SU(2)$, thus seven in total.

From (I) and (II) we get 12 labels. These are sufficient to label fully the representations *35*, *56*, *70*, *20* as was already seen in Eqs. (3.42–45).

(III) The labeling by a second chain [B 5]

$$SU(6)\supset U(1)\otimes SU(2)\otimes SU(4) \qquad (3.69)$$

$$\hookrightarrow SU(4)\supset SU(2)\otimes SU(2). \qquad (3.70)$$

The subgroup in the first link (3.69) was introduced in [G 4]. In (3.69) we pick up: three labels from the $SU(4)$-tableau plus two labels from $SU(2)$ (the "strange quark spin"):the $U(1)$ chosen in this particular case (hypercharge) is already contained in the $SU(3)$ of the chain (3.68). Finally the subgroup labeling (3.69) yields two more labels from one of the $SU(2)$ ("nonstrange quark spin") while the other $SU(2)$ (isospin) is identical with the $SU(2)$ used in the first chain.

Thus under (III) we find seven new labels and we have now 19 in all. This is ample to classify the contents of *189*, *405*, *280*, Eqs. (3.50–52). A 20th label has not so far been necessary. It would suffice to take one of those H_k of $SU(6)$ that have not been explicitly used till now.

It should be stressed [B 5] that the operators of the first chain (3.68) do not all commute with those of the second one (3.69–70). This, however, is not an objection of principle, and such a situation has been met before [M 1]. The use of noncommuting chains is dictated by physical arguments to be discussed below [Sec. IV]. It necessitates additional recoupling transformations to which correspond such phenomena as $\omega-\phi$ mixing.

We conclude this section by recalling the well-known properties of the operators F_A under the transformation

$$F_A'=U^\dagger F_A U, \qquad (3.71)$$

where U is given by Eq. (3.53). This transformation can also be written as

$$F_A'=C_{AB}(\epsilon)F_B, \qquad (3.72)$$

where C is an $N\times N$-sized matrix which is unitary. In fact, with a hermitian choice for the generators, C is orthogonal. C depends on the values of the parameters ϵ_A. These properties of C are easily verified when the ϵ_A are infinitesimal. From Eqs. (3.55), (3.71) we have in this case

$$C_{AB}(\epsilon)=\delta_{AB}+\epsilon_C f_{CAB} \qquad (3.73)$$

so that C is orthogonal as the f's are antisymmetric.

Thus the set of operators F_A themselves form an (N^2-1)-dimensional representation of $SU(N)$ ("angular momentum is a vector"); and if we add the identity operator we have a representation of $U(N)$.

Consider two unitary vectors x, y and construct the quantity

$$J_A^{(N)}(x,y)\equiv x^\dagger F_A y=x_\alpha^*(F_A)_\beta{}^\alpha y^\beta, \qquad (3.74)$$

where $(F_A)_\beta{}^\alpha$ are the DG matrices. From Eqs. (2.1–2) and (3.71–72) the $J_A^{(N)}$ behave as follows under a transformation of $U(N)$:

$$J_A'^{(N)}(x,y)=C_{AB}(\epsilon)J_B^{(N)}(x,y). \qquad (3.75)$$

Thus they transform as a representation of $U(N)$. We shall call the $J_A^{(N)}(x,y)$ the currents of $U(N)$. Note in particular that

$$J_A^{(N)\dagger}(x,x)=J_A^{(N)}(x,x) \qquad (3.76)$$

if we choose a Hermitian set F_A. Finally

$$\sum_A J_A^{(N)}(x,y)J_A^{(N)}(u,v)=\text{invariant under } U(N). \qquad (3.77)$$

F. Contents of $SU(M+N)$ with Respect to $SU(M)\times SU(N)$

In Eq. (3.69) we met the chain $SU(6)\supset SU(4)\otimes SU(2)$. This raises the question stated in the title of this section. Once again we start with the quadratic invariant in terms of the defining representation, now for $SU(M+N)$ and write

$$\sum_1^{M+N} x_\alpha{}^* x^\alpha = \sum_1^M x_\alpha{}^* x^\alpha + \sum_{M+1}^{M+N} x_\alpha{}^* x^\alpha. \qquad (3.78)$$

The first term on the right-hand side is the corresponding quadratic invariant for an $SU(M)$, the second one for an $SU(N)$. In terms of Young tableaux the following identification is evident.

$$\square = (\square, 0) + (0, \square), \qquad (3.79)$$

where 0 stands for the identity tableau. Equivalently

$$1 = (1, 0) + (0, 1), \qquad (3.80)$$

or in dimensions

$$M+N = (M;\, 1) + (1;\, N).$$

In a similar (but of course not identical) way to $SU(MN) \supset SU(M) \otimes SU(N)$ one now starts building up higher representations of $SU(M+N)$ in terms of sums of representation pairs. Full details are found in [H 4]. Some examples are given at the end of Sec. IV B.

G. Pseudo-Unitary Groups

Following the notations of Sec. IIA, let x denote a complex vector in an $(M+N)$-dimensional space and $x^\dagger$ its Hermitian conjugate. We define an adjoint $\bar{x}$ as

$$\bar{x} = x^\dagger \Gamma, \quad \text{or} \quad \bar{x}_\alpha = x_\beta{}^* \Gamma_\alpha{}^\beta, \qquad (3.81)$$

where

$$\begin{aligned} \Gamma_\beta{}^\alpha &= \delta_\beta{}^\alpha, && 1 \le \alpha, \beta \le M \\ &= -\delta_\beta{}^\alpha, && M+1 \le \alpha, \beta \le M+N \qquad (3.82) \\ &= 0, && \text{otherwise.} \end{aligned}$$

Under the transformation

$$x' = Ax, \qquad (3.83)$$

$\bar{x}$ becomes

$$\bar{x}' = \bar{x}\bar{A}, \qquad (3.84)$$

$$\bar{A} = \Gamma A^\dagger \Gamma, \qquad (3.85)$$

as $\Gamma^2 = 1$. The transformations (3.83–84) which leave invariant the "pseudo-unitary length"

$$\bar{x}x = \sum_1^M x_\alpha{}^* x^\alpha - \sum_{M+1}^{M+N} x_\alpha{}^* x^\alpha \qquad (3.86)$$

form a "pseudo-unitary" group denoted [H 5] by $U(M, N)$ and defined by the matrices A for which

$$\bar{A}A = 1. \qquad (3.87)$$

[One may restrict oneself to $M \ge N$; $U(M, 0) \equiv U(M)$.]

The further restriction

$$\det A = 1$$

corresponds to the subgroup $SU(M, N)$.

[The restriction: A real, would lead to the pseudo-orthogonal group $O(M, N)$ of which the Lorentz group $O(3, 1)$ is an example.]

Also for $SU(M, N)$ we can start to build tensors T, again defined as in Eq. (3.1), but where now "transforms like" means that an upper index transforms with A as in Eq. (3.83) and a lower index with $\bar{A}$ as in (3.84). It is therefore clear that there exists a one-to-one correspondence between the representations of $SU(M+N)$ and the finite dimensional representations of $SU(M, N)$. Therefore we have at once a full classification of these latter tensors in terms of $SU(M+N)$. Moreover the product reduction $(n_1, \cdots, n_{M+N}) \otimes (n'_1, \cdots, n'_{M+N})$ in $SU(M, N)$ is one-to-one to the same reduction in $SU(M+N)$. Also the considerations of Sec. IIB, D, F adapt at once to the pseudo-unitary case.

In regard to the question of generators of $U(M, N)$ one has to be careful with questions of Hermiticity. The action of a transformation in $U(M, N)$ on a D_N-dimensional representation may again be represented as in Eq. (3.53). (We consider only finite dimensional representations.) However, for real ϵ_B, not all the generators are Hermitian. In fact the Hermiticity condition

$$F_A{}^\dagger = F_A \quad \text{(unitary case)} \qquad (3.88)$$

is replaced by

$$\bar{G}_A \equiv \Gamma G_A{}^\dagger \Gamma = G_A \quad (ps\text{-unitary case}), \qquad (3.89)$$

where we denote the generators of $U(M, N)$ by G_A. Note that Γ itself may be taken as one of the G_A.

Using the DG it is easy to establish a one-to-one correspondence between the G_A and the F_A of $U(M+N)$. As Γ is Hermitian it may also be taken as one of the DG of $U(M+N)$. We have for the DG

$$\Gamma F_A + \eta_A F_A \Gamma = 0 \qquad (3.90)$$

where

$$\eta_A = -1 \text{ for } M^2+N^2 \text{ of the } F_A, \qquad (3.91)$$

$$= +1 \text{ for the other } 2MN\ F_A, \qquad (3.92)$$

and we have [by verifying that (3.87) is satisfied]

$$\begin{aligned} G_A &= F_A && \text{if } \eta_A = -1, \\ &= iF_A && \text{if } \eta_A = +1. \qquad (3.93) \end{aligned}$$

[The subset of G_A which satisfy (3.93) generate a subgroup $U(M) \otimes U(N)$ of $U(M, N)$. This is the "maximal compact subgroup" of $U(M, N)$.]

Example: $U(2)$ for which $x_1{}^* x^1 + x_2{}^* x^2 =$ invariant, has DG: σ_1, σ_2, σ_3, and 1. For $U(1, 1)$: $x_1{}^* x^1 - x_2{}^* x^2 =$ invariant. $\Gamma = \sigma_3$ and the DG are $i\sigma_1$, $i\sigma_2$, σ_3, and 1.

Finally we define a set of currents for $U(M, N)$ by

$$J_A^{(M,N)}(x, y) = \exp\{(i\pi/4)(1+\eta_A)\}\cdot\bar{x}G_A y. \quad (3.94)$$

The phase factor is introduced so that [see (3.90)]

$$J_A^{(M,N)}(x, x)^\dagger = J_A^{(M,N)}(x, x). \quad (3.95)$$

As we shall see in Sec. VI [see the remark after Eq. (6.14)] for SU(2, 2) these phase factors are quite familiar from relativistic theory. Now $\bar{x}G_A y = x^\dagger\Gamma G_A y$ and $\Gamma G_A = \exp\{(i\pi/4)(1+\eta_A)\}\cdot\Gamma F_A$. As we have remarked before, Γ may be considered as one of the DG of $SU(M+N)$, thus for some fixed X we may put $\Gamma = F_X$. Now from Eqs. (3.55) and (3.60)

$$F_X F_A = \sum_C \rho_{XAC} F_C,$$

$$\rho_{XAC} = C\delta_{XA}\delta_{CO} + \tfrac{1}{2}(d_{XAC} + if_{XAC}), \quad (3.96)$$

and where F_O is the identity. If we insert (3.96) into (3.94) and compare with (3.74) we get (sum over C, but keep X final!)

$$J_A^{(M,N)}(x, y) = \exp\{(i\pi/2)(1+\eta_A)\}\cdot\sum_C \rho_{XAC} J_C^{(M+N)}(x, y) \quad (3.97)$$

so that the currents $J_A^{(M,N)}$ are linear combinations of the currents $J_A^{(M+N)}$ belonging to the compact group $U(M+N)$. It is this connection which plays a role in the algebra of currents, see Sec. VI, Appendix.

IV. SU(6)

A. Introduction

Consider the fundamental triplet u^A of $U(3)$, $A=1$, 2, 3. Name them: $u^1=p$, $u^2=n$, $u^3=\lambda$. (p, n) is the isodoublet, λ the isosinglet. (To avoid confusion, we denote proton and neutron by P, N, respectively, in this paper.) I_3, the charge Q, the hypercharge Y, and the baryon number B are as follows.

	p	n	λ
I_3	$\frac{1}{2}$	$-\frac{1}{2}$	0
Q	q_0	q_0-1	q_0-1
Y	y_0	y_0	y_0-1
B	b	b	b.

(4.1)

There are well-known degrees of freedom of choice for q_0, y_0, b [G 1, B 7, G 5, N 1]. Until further notice [in Sec. IVJ] we shall take

$$q_0 = \tfrac{2}{3}, \quad y_0 = \tfrac{1}{3}, \quad b = \tfrac{1}{3}. \quad (4.2)$$

Correspondingly,

$$Q = I_3 + Y/2. \quad (4.3)$$

More specifically, let the triplet have spin $\frac{1}{2}$. We write the one particle wave function for zero three-momentum as

$$u^\alpha = u^{Ai}, \quad \alpha = 1, \cdots, 6; \quad i = 1, 2, \quad (4.4)$$

with the following correspondence between α and (Ai):

$$\begin{array}{rcccccc} \alpha = & 1, & 2, & 3, & 4, & 5, & 6 \\ A = & 1, & 2, & 3, & 1, & 2, & 3 \\ i = & 1, & 1, & 1, & 2, & 2, & 2 \end{array} \quad (4.5)$$

where the spin index $i=1, 2$, corresponds to spin "up," "down." For any dynamical situation where $SU(3)$ invariance holds and where, in the *static* limit, no spin dependence appears, the physical wave functions u^α satisfy the property that

$$u_\alpha^* u^\alpha = \text{invariant under } SU(6). \quad (4.6)$$

Thus the u^α may be considered as the defining representation of that group. Call X_n the DG of $SU(6)$. The X_n can be expressed in terms of the DG of $SU(2)$: S_a and the DG of $SU(3)$: F_P as follows.

$$X_n:\ S_a\otimes 1,\ 1\otimes F_P,\ S_a\otimes F_P; \quad a=1, 2, 3,\ P=1, \cdots, 8, \quad (4.7)$$

so $n=1, \cdots, 35$. The $\otimes$ means the following. The X_n are 6×6 matrices and

$$(X_n)_\alpha{}^\beta:\ (S_a)_i{}^j\delta_A{}^B,\ \delta_i{}^j(F_P)_A{}^B,\ (S_a)_i{}^j(F_P)_A{}^B;$$
$$\alpha = (Ai),\ \beta = (Bj). \quad (4.8)$$

Having explained the $\otimes$ notation, we now go to the customary shorthand:

$$X_n:\ X_a = S_a, \quad X_P = F_P, \quad X_{aP} = S_a F_P, \quad (4.9)$$

with the commutation relations

$$[X_a, X_b] = i\epsilon_{abc}X_c,$$
$$[X_a, X_P] = 0,$$
$$[X_P, X_Q] = if_{PQR}X_R,$$
$$[X_a, X_{bP}] = i\epsilon_{abc}X_{cP},$$
$$[X_P, X_{bQ}] = if_{PQR}X_{bR},$$
$$[X_{aP}, X_{bQ}] = (i/2)\delta_{ab}f_{PQR}X_R + i\epsilon_{abc}(\tfrac{1}{3}\delta_{PQ}X_c + d_{PQR}X_{cR}). \quad (4.10)$$

Here the f_{PQR}, d_{PQR} refer to $SU(3)$, cf. Eqs. (3.55,60). [We have used the normalization $c=\frac{1}{6}$ in (3.60).] They are tabulated in [G 1]. Equation (4.10) specifies the structure constants of $SU(6)$. (Note that X_{aP} is a simple direct product only in the defining representation of the generators.)

We are now going to discuss the consequences of the assumption that static phenomena involving hadrons are approximately described by the dynamical group $SU(6)$. We are not yet going to raise questions about the meaning of such an assumption as regards the underlying dynamics. To this question we return in Sec. IV I. The order of business is here reversed as compared with for example the discussion of Russell–Saunders coupling where we could start with the secure underpinning of the Hamiltonian (2.7). For the present it cannot be otherwise. The exciting thing about $SU(6)$ is that, if there is anything to it, it must give stronger clues to the almost totally unsolved problem of strong dynamics than can be gotten from kinematical symmetries.

The following items will be discussed. In Sec. IVB we treat the question of $SU(6)$ representations for hadron states. It may at once be noted that, whereas all generators of $SU(6)$ commute with parity, all states within a $SU(6)$ supermultiplet should have the same parity. In Sec. IVC, mass formulae are discussed. Then we turn to the static electromagnetic and semi-leptonic vertices, Sec. IVD, E, F, non-leptonic decays, IVH, and the S-wave nucleon–nucleon scattering Sec. IVG. Questions about the structure of the dynamics lead us into the discussion of the reality of triplets Sec. IVJ. Up till there we will often use triplets u^α as a good mathematical tool, but without prejudice as to their actual existence.

Concerning the interpretation of $SU(6)$ symmetry, the early papers contain the following comments. The nonrelativistic nature of $SU(6)$ is stated in [S 1]. A possible kinematic origin of the symmetry is contemplated in [G 3]. Limitations on $SU(6)$ due to recoupling of spin to orbital angular momentum are stated in [P 1]. The study of baryon–meson interaction then begins with the analysis of effective vertices [G 4]. Here one is already one step beyond the static limit, however, as will be further discussed in Secs. VA and C. The survey in this section is in part an elaboration of an earlier review of $SU(6)$ as a dynamical group [B 8].

B. Some Specific Supermultiplets

The first kind of clue to $SU(6)$ stems from a simple counting problem. Can one fill representations of this group with known hadrons?

(I) The *56*.+ This is the dimension of the representation (3), corresponding to the tensor structure $B^{\alpha\beta\gamma}$, with total symmetry in (α, β, γ). Its $SU(3)\otimes SU(2)$ content is given in Eq. (3.43). Thus it can be filled with the usual baryon octet and decuplet which have indeed the same parity [G 3]. There are other baryon resonances inside the *56*-mass range. Thus closeness in mass alone is not a sufficient ground for this symmetry.

Next we construct $B^{\alpha\beta\gamma}$, at first using quarks (q) as a tool. Thus we put

$$B^{\alpha\beta\gamma}\sim\tfrac{1}{6}\sum_P u^\alpha(1)u^\beta(2)u^\gamma(3). \qquad (4.11)$$

The summation is over all permutations of the configuration labels 1, 2, 3. Otherwise the notations are as in Eq. (3.1). Let $\|u^\alpha\|$, the norm of u^α for fixed α be unity. Then

$$\begin{aligned}\|B^{\alpha\beta\gamma}\| &= 1, &&\alpha=\beta=\gamma,\\ &=\tfrac{1}{3}, &&\alpha=\beta\neq\gamma,\\ &=\tfrac{1}{6}, &&\alpha, \beta, \gamma \text{ distinct.}\end{aligned} \qquad (4.12)$$

Consider now as an example the state $|N^{*+}\tfrac{1}{2}\rangle$, where $\tfrac{1}{2}$ is the S_z value. From (4.1, 4, 5, 11) we see that it is a mixture of B^{115} and B^{124}. The mixture is determined from the fact that this state is totally symmetric in spin and in unitary spin *separately*. Hence

$$\begin{aligned}|N^{*+}\tfrac{1}{2}\rangle=\tfrac{1}{3}[&u^1(1)u^1(2)u^5(3)+u^1(1)u^5(2)u^1(3)\\ &+u^5(1)u^1(2)u^1(3)+u^1(1)u^4(2)u^2(3)\\ &+u^1(1)u^2(2)u^4(3)+u^2(1)u^1(2)u^4(3)\\ &+u^4(1)u^1(2)u^2(3)+u^4(1)u^2(2)u^1(3)\\ &+u^2(1)u^4(2)u^1(3)]\\ =B^{115}&+2B^{124},\end{aligned} \qquad (4.13)$$

where the over-all norm is such that $\|N^{*+}\tfrac{1}{2}\|=1$, see (4.12). Also $|P\tfrac{1}{2}\rangle$ is a normed mixture of B^{115} and B^{124} and it must be orthogonal to $|N^{*+}\tfrac{1}{2}\rangle$, hence we may put $|P\tfrac{1}{2}\rangle=2^{\frac{1}{2}}(B^{115}-B^{124})$.

Note. One does not need to keep track of the I-spin in such derivations. The reason for this is that for $SU(2)$ the tensor symmetry itself determines I, see (3.14).

One can construct likewise by hand the other members of the *56*. However there is a quicker way. From Eq. (3.43) we know the content of *56* with respect to spin and unitary spin. One knows the tensor structure of the individual $SU(3)$ and $SU(2)$ representations involved. This makes it possible to write down the answer almost at once [B9].

$$\begin{aligned}B^{\alpha\beta\gamma}=\chi^{ijk}d^{ABC}+(1/3\sqrt{2})[&\epsilon^{ij}\chi^k\epsilon^{ABD}b_D{}^C\\ &+\epsilon^{jk}\chi^i\epsilon^{BCD}b_D{}^A+\epsilon^{ki}\chi^j\epsilon^{CAD}b_D{}^B],\end{aligned} \qquad (4.14)$$

where the following definitions have been used:

$$(1)\quad \chi^{111}=\psi^{\frac{3}{2}},\qquad \chi^{112}=(1/\sqrt{3})\psi^{\frac{1}{2}},$$
$$\chi^{122}=(1/\sqrt{3})\psi^{-\frac{1}{2}},\qquad \chi^{222}=\psi^{-\frac{3}{2}}, \qquad (4.15)$$

where ψ^a is a normed wave function for $S=\tfrac{3}{2}$, $S_z=a$. The χ^i are normed spin-$\tfrac{1}{2}$ wave functions.

(2) d^{ABC} is the decuplet tensor:

$$d^{111}=N^{*++},\quad d^{112}=(1/\sqrt{3})N^{*+},\quad d^{122}=(1/\sqrt{3})N^{*0},\quad d^{222}=N^{*-},$$
$$d^{113}=(1/\sqrt{3})Y^{*+},\quad d^{123}=(1/6^{\frac{1}{2}})Y^{*0},\quad d^{223}=(1/\sqrt{3})Y^{*-},$$
$$d^{133}=(1/\sqrt{3})\Xi^{*0},\quad d^{233}=(1/\sqrt{3})\Xi^{*-},\quad d^{333}=\Omega^{-}. \tag{4.16}$$

[The numerical coefficients are of course derived in a way similar to the derivation of Eq. (4.13).]

(3) ϵ^{ij} and ϵ^{ABC} are the Levi-Civita symbols for $SU(2)$ and $SU(3)$, respectively.

(4) $b_B{}^A$ is the baryon octet

$$b_1{}^1=(\Sigma^0/\sqrt{2})+(\Lambda/6^{\frac{1}{2}}),\qquad b_2{}^2=-(\Sigma^0/\sqrt{2})+(\Lambda/6^{\frac{1}{2}}),\qquad b_3{}^3=-(2\Lambda/6^{\frac{1}{2}}),$$
$$b_2{}^1=\Sigma^+,\quad b_3{}^1=P,\quad b_1{}^2=\Sigma^-,\quad b_3{}^2=N,$$
$$b_1{}^3=\Xi^-,\quad b_2{}^3=-\Xi^0. \tag{4.17}$$

The coefficient $(3\sqrt{2})^{-1}$ in Eq. (4.14) follows from the condition:

$$\sum_{\alpha\beta\gamma} \| B^{\alpha\beta\gamma} \| = \text{sum of norms of the individual particle states, each with norm 1.} \tag{4.18}$$

The following two identities are often useful.

$$\epsilon^{ij}\chi^k+\epsilon^{jk}\chi^i+\epsilon^{ki}\chi^j=0, \tag{4.19}$$

$$\epsilon^{ABD}b_D{}^C+\epsilon^{BCD}b_D{}^A+\epsilon^{CAD}b_D{}^B=0. \tag{4.20}$$

(4.20) is a special case of (3.8).

(II) The *35*$^-$. This is the dimension of (21^4), the tensor is $M_\beta{}^\alpha$ with $M_\alpha{}^\alpha=0$. Its content is given by Eq. (3.42). Choosing odd parity, it is filled by a nonet of vector mesons and octet of pseudoscalar mesons [S 1, G 3].

However, it is now well established that there exists a ninth pseudoscalar meson as well with a mass $\simeq$960 MeV which is inside the *35*-mass range [G 6, G 7, K 3, K 4, L 2]. It is called X^0 (also η' or η^*). The mixing with the η appears to be rather small, $\simeq 12^0$ [L 2]. We shall see later on in Sec. V that a set of $9V$ and $9P$ mesons comes in naturally from relativistic considerations.

The structure of $M_\beta{}^\alpha$ can be determined as for the *56*, either by building up from a $\bar{q}q$ system or directly from the tensor structure read off from (3.42), [G 4].

$$M_\beta{}^\alpha=-i\delta_j{}^iP_B{}^A-(\boldsymbol{\delta}\boldsymbol{\varepsilon})_j{}^iV_B{}^A, \tag{4.21}$$

$$P_1{}^1=(\pi^0/\sqrt{2})+(\eta/6^{\frac{1}{2}}),\qquad P_2{}^2=-(\pi^0/\sqrt{2})+(\eta/6^{\frac{1}{2}}),\qquad P_3{}^3=-2\eta/6^{\frac{1}{2}},$$
$$P_1{}^2=\pi^-,\quad P_1{}^3=K^-,\quad P_2{}^1=\pi^+,\quad P_3{}^1=K^+,\quad P_3{}^2=K^0,\quad P_2{}^3=\bar{K}^0; \tag{4.22}$$

$$V_1{}^1=(\rho^0/\sqrt{2})+(\omega^0/6^{\frac{1}{2}})+(\phi^0/\sqrt{3}),\qquad V_2{}^2=-(\rho^0/\sqrt{2})+(\omega^0/6^{\frac{1}{2}})+(\phi^0/\sqrt{3}),\qquad V_3{}^3=-(2\omega^0/6^{\frac{1}{2}})+(\phi^0/\sqrt{3}),$$
$$V_1{}^2=\rho^-,\quad V_1{}^3=K^{-*},\quad V_2{}^1=\rho^+,\quad V_3{}^1=K^{+*},\quad V_3{}^2=K^{0*},\quad V_2{}^3=\bar{K}^{0*}. \tag{4.23}$$

$\| M_\beta{}^\alpha \| = M_\alpha{}^{*\beta}M_\beta{}^\alpha$ satisfies again a relation like (4.18). In (4.23) ω^0 is the isosinglet member of the vector *8*. ϕ^0 is the unitary singlet vector meson.

(III) The *70*$^-$. The $SU(6)$ multiplets contained in the product of *35*$^-\times$*56*$^+$ are given in Eq. (3.25). With the parity assignments chosen for *35* and *56*, the specific product considered reduces to a sum of representations with odd parity.

It has been suggested [P 1] that the *70*$^-$ might perhaps be useful for the classification of some higher baryon resonances which decay strongly into baryon (in the *8*)+meson (in the *35*). From the content formula (3.44) the *70*$^-$ could accomodate a $(\frac{3}{2})^-$ octet. This "γ octet" has often been suggested, but its status is presently quite unclear. Another candidate for the *70*$^-$ would be the $Y_0^*(1405)$. A recent analysis [K 5] indicates that this is a $(\frac{1}{2})^-$ state, as was often supposed. For a detailed discussion of the *70*$^-$ see [G 8, G 9].

The tensor structure of the *70* is $T^{[\alpha\beta]\gamma}$, antisymmetric in (α, β) and subject to the constraint equation (3.22). $T^{[\alpha\beta]\gamma}$ has been constructed explicitly, [B 10, Eq. (4.2)].

The *70* with odd parity provides a good starting point for a very preliminary discussion of strong vertices. As *35*$\times$*56* contains *70* only once, Eq. (3.25), the "decay vertex" of the *70* is unique, namely,

$$T^*{}_{[\alpha\beta]\gamma}B^{\alpha\gamma\delta}M_\delta{}^\gamma. \tag{4.24}$$

Equation (4.24) is $SU(6)$ invariant and parity-conserving. One may now discuss relative decay rates in a way often done in a discussion of higher symmetries:

compute matrix elements from (4.24) and correct for mass differences within a multiplet by adjusting the phase space. Regardless of whether this is a reliable procedure, it should be stressed that for $SU(6)$ there arise new questions as compared to a purely internal symmetry group like $SU(3)$, for the following connected reasons. (1) In an actual decay we cannot have all particles with zero momentum; but we have not yet given meaning to "an $SU(6)$ multiplet with finite momentum." (2) The *70* contains $S=\frac{3}{2}$ states. These cannot decay into baryon ($S=\frac{1}{2}$) +pseudoscalar meson unless orbital angular momentum appears. Correspondingly, (4.24) gives zero for this particular kind of decay, as one can readily verify from Eqs. (4.14, 20) and [B 10, Eq. (4.2)]. Again the question of including (angular) momentum appears.

We shall discuss in Sec. V the attempts which have been made to include momentum in a systematic way in vertex structures. Other examples of the urgent need do so are given by decays like $V\rightarrow 2P$, $N^*\rightarrow N\pi$, etc.

(IV) A discussion has been given recently of static baryon resonances in a 700^+ [G 10].

(V) "Recoupled" multiplets. By way of model, let us consider the *35* as given by Eq. (4.21) to be a $\bar{q}q$ system in an orbital S state. We can imagine that this system has "higher excited states" with $L>0$. Take for example P states. The contents of these states is given by 35×3, where the *3* denotes the dimension of the $L=1$ representation of $O(3)$. Coupling L to S one gets from Eq. (3.42)

$$35\otimes 3=(8+1;\,5)+(8+8+1;\,3)+(8+1;\,1), \qquad (4.25)$$

where the number following the semicolon is $2J+1$, J being the *total* angular momentum of the system. For a *35* with odd parity, the multiplets on the right-hand side of Eq. (4.25) all have even parity.

One can phrase this procedure without necessarily talking of quarks. The formal question then is to discuss supermultiplets which correspond to the group $SU(6)\times O(3)$, [M 2].

Possible assignments of meson resonances to the states given in Eq. (4.25) have been discussed [B 11]. These "P-state mesons" may indeed be of interest in view of the recent evidence for a 2^+ nonet of vector mesons [B 12]; for a theoretical discussion of their decay rates see [G 11]. *Added note:* see also [A 19].

As the coupling with orbital angular momentum does not of course affect $SU(3)$, one can generate in this way only unitary octets and singlets from the *35*. In this respect the situation is different if one asks, on the other hand, whether spin 2 mesons should be accomodated in higher $SU(6)$ multiplets without recoupling to L. To get spin 2 one needs at least systems which transform like $\bar{q}\bar{q}qq$. These are found in Eqs. (3.50, 51, 52). In particular the spin 2 content of the *189* is a nonet only, while *280, 280** do not contain a nonet of this kind. As the f^*-f^0 mixing [B 12] appears to be comparable to the $\omega-\phi$ mixing, the 189 is perhaps a more plausible subject for further study than the *405*. At any rate, the main qualitative distinction between these $SU(6)$ multiplets and the recoupled *35* is the appearance of the *27* of $SU(3)$. For this reason it would be of great interest to know if further experiments will confirm the indication of a $Y=2$ meson resonance [F 3], in the same mass range as the 2^+ states. For the $Y=2$ state one needs at least a *27*. Either picture, the recoupled *35*, or a new $SU(6)$ multiplet, leads to a quite complex set of resonant states. The region of resonances above the *35* may well give further dynamical clues.

We conclude this section with a few examples of the decomposition of $SU(6)$ multiplets under the subgroup given in Eq. (3.69). Specifically, $U(1)$ corresponds to hypercharge, $SU(2)$ to λ-spin and $SU(4)$ to the union of $(p,\,n)$-spin and isospin. With the help of Eqs. (4.4, 5, 14, 21) one finds

$$56=(20,\,1)_1+(10,\,2)_0+(4,\,3)_{-1}+(1,\,4)_{-2}$$
$$=(N,\,N^*)+(\Lambda,\,\Sigma,\,Y_1^*)+(\Xi,\,\Xi^*)+(\Omega^-).$$

The first (second) number in brackets is the dimension of the $SU(4)$ [$SU(2)$] representation, the subscript is the Y value. [The *20* of $SU(4)$ corresponds to the tableau (3).] Likewise

$$35=(15,\,1)_0+(1,\,1)_0+(1,\,3)_0+(4,\,2)_1+(4^*,\,2)_{-1}$$
$$=(\pi\rho\omega)+(\eta)+(\phi)+(K,\,K^*)+(\bar{K},\,\bar{K}^*),$$

where the ω and ϕ are the physical states given in Eq. (4.31) below.

C. Mass Formulae

As an introduction to the discussion of $SU(6)$-mass formulae, the following two remarks on the $SU(3)$ case may be helpful.

(1) As is well known [G 1], in $SU(3)$ one assumes that the mass split operator transforms like the hypercharge operator F_8, which is a member of the octet of $SU(3)$ generators. There are two operators of this kind which may be written as

$$\partial C_2^{(3)}/\partial F_8\sim F_8, \qquad \partial C_3^{(3)}/\partial F_8\sim d_{8AB}F_AF_B, \qquad (4.26)$$

see Eqs. (3.59, 60). Thus from two independent "scalar operators," the Casimir operators, one gets two "vector operators" by formal differentiation. A procedure of this kind holds for all $SU(N)$ (and more generally) [G 12]. The effective mass operator is therefore generally of the form

$$m+aF_8+bd_{8AB}F_AF_B, \qquad (4.27)$$

which is equivalent to the usual $SU(3)$ mass formula.

The number of vector operators determines the *maximum* number of parameters in the mass formula. For a specific representation R, the actual number is equal to the number of times that *8* appears in the reduction $R^*\times R$, where R^* is conjugate to R. For $R=8$, *8* occurs twice and we have a three-parameter formula. For $R=10$, *8* occurs *once* and a two-parameter formula results (equal spacing). Correspondingly, for the *10*, the expectation values of the second operator in (4.26) is itself of the form $\langle m+aF_8\rangle$.

(2) Call dynamical mass split operator the part of a Hamiltonian which induces the effective mass split. If we say that the dynamical mass operator transforms like *8*, then this is equivalent to the statement under (1) only in first-order perturbation theory. Because the $SU(3)$ breaking effects are not obviously small in any known scale it is a much discussed puzzle how to understand the simple structure of the effective mass operator from a simple structure of a dynamical mechanism.

Now let us turn to $SU(6)$. At least two mechanisms for effective mass splits are involved. (1) A spin-dependent effect to separate different spins within the supermultiplet. (2) $SU(3)$ breaking effects. It is not obvious *a priori* that these effects should be linearly independent. For example the $SU(3)$ theory does not preclude that the two mass parameters in (4.27) should be spin-dependent. Nor is there a known ground to exclude that the spin dependent terms should have $SU(3)$-dependent coefficients.

Let us nevertheless consider an additivity assumption [P 1]: (1) $SU(3)$ breaking transforms like F_8 but with spin-independent coefficients [at least within an $SU(6)$ multiplet], (2) a spin-dependent effect independent of $SU(3)$ [at least within an $SU(6)$ multiplet]. This leads to the following results. For the *56*, the mass operator is

$$M+aY+b[I(I+1)-Y^2]+c(S), \qquad (4.28)$$

where a, b are constants and c depends in some way on $S(S+1)$. M is the "central mass," the value to which all masses collapse in the absence of $SU(6)$ breaking. Equation (4.28) enables one to calculate from the *8* an equidistance $\simeq$*130* MeV for the *10* which compares rather well with the experimental value $\simeq$*145* MeV [P 1]. It may be noted that this result is independent of the way c depends on S.

For the *35* we use Eq. (4.21) and write the mass operator as

$$\alpha M_\beta{}^{*\alpha}M_\alpha{}^\beta+\beta[\boldsymbol{\sigma}M^*]_B{}^A[\boldsymbol{\sigma}M]_A{}^B+\gamma M_\beta{}^{*i3}M_{i3}{}^\beta, \qquad (4.29)$$

where the α term corresponds to the central mass, β gives the $SU(3)$-independent and spin-dependent effect. [ ] denotes spin trace. In the internal symmetry breaking γ term, the summation over $i=1, 2$ guarantees spin independence; the singling out of the unitary index value "3" corresponds to the proper hypercharge dependence. Equation (4.29) has the following consequences.

(1) For P:

$$4K^2-\pi^2=3\eta^2. \qquad (4.30)$$

(2) Mixing between ω^0 and ϕ^0 such that the physical states are

$$\omega=(1/\sqrt{3})(\omega^0+\phi^0\sqrt{2}),$$
$$\phi=(1/\sqrt{3})(\omega^0\sqrt{2}-\phi^0). \qquad (4.31)$$

These are the mixtures proposed before [O 2, S 2]. They appear here in a rather natural way because of the (8+1)-vector meson degeneracy in the strict $SU(6)$ limit.

Remark. $\omega-\phi$ mixing was introduced to explain deviations from an octet mass formula for vector mesons. It also serves to explain the suppression of such modes as $\phi\rightarrow\rho\pi$ compared to $\phi\rightarrow K\bar{K}$. From phase space, barrier factors, and spin–isospin weights one estimates a rate ratio [L 2]

$$(\phi\rightarrow\rho\pi)/(\phi\rightarrow K\bar{K})\simeq 4,$$

in the absence of any other inhibitions. Recent experimental results for this ratio are: 0.22 ± 0.09 [L 3]; 0.3 ± 0.15 [L 2] which indicate a $\rho\pi$ supression by one order of magnitude. This fits well with the mixing given by Eq. (4.31) which forbids $\phi\rightarrow\rho\pi$. On the other hand, another recent measurement [B 13] gives a quite high relative rate for $\phi\rightarrow\pi$ mesons which appears to be in confiict with the other two measurements. For other discussions of $\omega-\phi$ mixing see [K 6, C 7]. For the connection between $\omega-\phi$ mixing and a subgroup of $SU(6)$ see [G 4].

(3) For V:

$$\rho^2=\omega^2,$$
$$2K^{*2}-\phi^2=\rho^2. \qquad (4.32)$$

(4) The $P-V$ relation

$$K^{*2}-\rho^2=K^2-\pi^2 \qquad (4.33)$$

which is well satisfied [P 1].

Thus for the *56* and the *35* the simple further assumption of additivity seems to work rather well. This may indicate that spin-dependent effects appear dynamically in a different mass scale than $SU(3)$ breaking effects.

A different starting point is to assign definite tensor properties under $SU(6)$ to the $SU(6)$-breaking effective mass operator. A first attempt was to assign this operator to a representation *35*. By an argument similar to the derivation of Eq. (4.26) there are 5 such operators [K 7]. But by Lemma 2 (Sec. IIIC), *56**$\times$*56* contains *35* only once. Hence one gets a one parameter mass formula for the *56*. In particular no spin dependence is introduced: *35* does not contain (*1*; *1*). Also Σ and Λ remain degenerate.

A more systematic approach in this direction then developed [B 6]. For a representation R of $SU(6)$, project out from all representations in $R^*\otimes R$ the parts which transform like $(8;1)$. These give each a candidate for $SU(6)$ breaking with the desired $SU(3)$ properties. In addition project out all parts which transform like (1; 1). These do not break $SU(3)$ but may give $S(S+1)$ terms.

The detailed application of this program [B 5] necessitates a further labeling of the representations *189*, *405* as briefly discussed in Sec. IIIE. More details are given in [B 6] and we state the results.

(1) For the *56* one obtains Eq. (4.28). The central mass M is found to be

$$M \simeq 1065 \text{ MeV}. \tag{4.34}$$

(2) For the *35* one finds Eq. (4.30), but no other relations, unless more restrictive assumptions are made. The central mass of the *35* is found to be

$$\mu \simeq 615 \text{ MeV}. \tag{4.35}$$

(3) A detailed discussion of the *70*-mass formula has also been given [B 5]. Here several mixing problems have to be resolved. These mass formulae are analyzed further in [G 8, G 9].

In conclusion, there are several quite suggestive indications from mass parameters and $\omega-\phi$ mixing which fit in with $SU(6)$ in a natural way. Relations like (4.32, 33) are compatible with the tensor analysis of mass splits under special conditions [B 8]. It appears that the tensor method is too general.

The foregoing discussion refers to the effective mass split operator in $SU(6)$. As for $SU(3)$, we must also ask the deeper question of the dynamical mechanism for this effective split. It is at this point that the distinction between kinematical and dynamical symmetry plays an important role. As was discussed in Sec. IIB, the $SU(3)$ breaking could well be characterized by nondynamical parameters while for $SU(6)$ this is impossible. As will be discussed further in Sec. V, the kinetic energy terms in the free part of a Hamiltonian must give rise to spin-dependent mass terms in the presence of interaction. Thus there is a natural mechanism for the breakdown $SU(6)\rightarrow SU(3)\otimes SU(2)$.

Further remarks. (1) It has been noted [H 6] that the coefficients of tensors which appear in the general tensor method must be baryon number-dependent. (2) A more specific model for mass breaking is found in [D 1]. (3) It has been attempted to unify the *56*- and the *70*-mass formulae in terms of a higher symmetry [B 17].

D. Magnetic Moments

The electric charge and the magnetic moment of particles are static limits of effective vertices. Their discussion within the static $SU(6)$ picture is therefore appropriate. We recall first some $SU(3)$ results.

(1) The electric charge operator Q is supposed to transform like the component of an octet operator, in accordance with equation (4.3) which can also be written as $Q=F_3+F_8/\sqrt{3}$. In the defining representation Q is the 3×3 matrix

$$Q_A{}^B=\tfrac{1}{3}\begin{pmatrix}2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}. \tag{4.36}$$

(In Sec. IVJ a more general form of the charge operator is discussed.) Charge conservation implies the F-coupling

$$e\bar{b}_A{}^B(Qb-bQ)_B{}^A \tag{4.37}$$

for the baryon *8*, where e is the proton charge. For the *10*, the coupling is $3e\bar{d}_{ABD}d^{ABC}Q_C{}^D$, etc.

(2) The magnetic moment operator is also supposed to transform like an *8*. The most general form for the baryon *8* magnetic moments,

$$\mu_8=\mu_D\bar{b}_A{}^B(Qb+bQ)_B{}^A+\mu_F\bar{b}_A{}^B(Qb-bQ)_B{}^A, \tag{4.38}$$

contains an arbitrary mixture of D and F coupling. From (4.38):

$$\mu(P)=\mu_F+\tfrac{1}{3}\mu_D, \tag{4.39}$$

$$\mu(N)=-\tfrac{2}{3}\mu_D, \text{ etc.} \tag{4.40}$$

For the *10*, Lemma 2 of Sec. IIIC implies that there can be only one type of magnetic coupling. Hence for the members of the *10* their respective magnetic moments are proportional to the charge:

$$\mu_{10}=\text{const. } Q_{10}. \tag{4.41}$$

(3) If one assumes in $SU(3)$ that the electromagnetic form factors transform like an *8* for all momentum transfers q, then one has generally a D and an F contribution for all q which may be different for different q.

We now turn to the $SU(6)$ theory.

(1) The charge operator is assumed to transform like an (*8*; *1*) member of a *35*. The $SU(6)$ analog of (4.36) is

$$q_\alpha{}^\beta=\delta_i{}^jQ_A{}^B. \tag{4.42}$$

The static electric charge of the *56* is

$$3eB^*{}_{\alpha\beta\gamma}q_\delta{}^\gamma B^{\alpha\beta\delta} \tag{4.43}$$

which gives the respective charges of the *56* members, as is verified by inserting Eqs. (4.14–17). The electrostatic interaction is equal to (4.43) times an external Coulomb potential.

(2) The magnetic moment operator $\mathbf{m}$ is assumed to transform like an (*8*; *3*) member of a *35*:

$$\mathbf{m}_\alpha{}^\beta=(\boldsymbol{\sigma})_i{}^jQ_A{}^B. \tag{4.44}$$

The crucial consequence of this straightforward assumption is that the D/F ratio in (4.38) becomes unique because, by Lemma 2, the *35* can be coupled in only one way to the *56*. The magnetic moments for the *56* are given by

$$3\mu(P)B^*_{\alpha\beta\gamma}(\mathbf{m})_\delta{}^\gamma B^{\alpha\beta\delta}, \tag{4.45}$$

where $3\mu(P)$ ensures proper normalization. The magnetic interaction is equal to the scalar product of the 3-vector (4.45) and the magnetic field [which is a 3-vector external to $SU(6)$]. The consequences of (4.45) which go beyond $SU(3)$ are the following.

(a) Baryon-*8* magnetic moments [B 9; S 3]. The ratio of μ_D and μ_F in Eq. (4.38) is now fixed and is given by

$$\mu_D/\mu_F=\tfrac{3}{2}. \tag{4.46}$$

Hence from Eqs. (4.39, 40)

$$\mu(N)/\mu(P)=-\tfrac{2}{3}, \tag{4.47}$$

in good agreement with the experimental ratio -0.68. The *8*-magnetic moments are discussed further in Sec. IVJ.

(b) Baryon-*10* magnetic moments [B 9]. In Eq. (4.45) the *56*-wave function in the $SU(6)$ limit is used. Hence the constant in (4.41) is expressible in terms of $\mu(P)$:

$$\mu_{10}=\mu(P)Q_{10}. \tag{4.48}$$

In particular

$$\mu(\Omega^-)=-\mu(P). \tag{4.49}$$

See further Sec. IVJ for remarks on mass corrected magnetic moments.

(c) Transition moments between *10* and *8*. By Lemma 1, 8×10 is simply reducible. Hence $SU(3)$ says that all transition moments ($8\rightarrow 10$) are expressible in one of them. We denote the M_1-transition moment between states with $S_z=\frac{1}{2}$ by $\langle P\mid\mu\mid N^{*+}\rangle$, etc. Then [B 9]

$$\begin{aligned}\langle P\mid\mu\mid N^{*+}\rangle&=-\langle\Sigma^+\mid\mu\mid Y^{*+}\rangle=\langle N\mid\mu\mid N^{*0}\rangle\\&=2\langle\Sigma^0\mid\mu\mid Y_0^*\rangle=(2/\sqrt{3})\langle\Lambda\mid\mu\mid Y_0^*\rangle\\&=\langle\Xi^0\mid\mu\mid\Xi^{*0}\rangle.\end{aligned} \tag{4.50}$$

According to (4.45) these moments can be expressed in terms of $\mu(P)$. One finds [B 9]

$$\langle P\mid\mu\mid N^{*+}\rangle=(2\sqrt{2}/3)\mu(P). \tag{4.51}$$

Comparisons of (4.51) have been made with photo- and electroproduction data at the *33*-resonance. This involves the assumption that transition form factors do not vary much from their $SU(6)$-limit value to the region of physical interest. While such an assumption is open to question, it is nevertheless interesting that the $E2$ transition at resonance is much smaller than $M1$ [G 13]; in the static limit there is of course no room for $E2$ [B 18]. Regarding $M1$, an earlier analysis of photoproduction [G 13] gives a value 1.6 times the right-hand side of (4.51) [B 9]; while a much closer agreement is claimed with electro production [G 14, G 15]. I am indebted to R. H. Dalitz for the communication of a re-analysis of photoproduction and electroproduction by him and D. Sutherland, which continues to indicate an effective discrepancy by a factor $\sim$1.6.

Further remarks. (1) The same assumptions applied to the *35* yield one additional $SU(6)$ relation: $\mu(\rho^+)=3(\pi^+\mid\mu\mid\rho^+)$ [B 9].

(2) Magnetic moments have been calculated for the baryon *20* [S 3].

(3) The magnetic moments of the baryon *70* can be related to those of the *56* by a symmetry stronger than $SU(6)$ [R 3].

(4) It has been observed [A 2] that Eq. (4.47) is more stable against symmetry breaking than are $SU(3)$ relations like $\mu(\Lambda)=\mu(N)/2$. This is perhaps not surprising as the neutron and proton belong to the same isomultiplet.

(5) Several attempts have been made to relate Eq. (4.47) to the weaker assumption of invariance under the subgroup $SU(4)$ discussed in [G 4]. This author finds himself in agreement with the conclusions on this subject in [B 14].

(6) Several authors have discussed radiative decays ρ^0, ω, $\phi\rightarrow\pi^0\gamma$ [B 15, T 2, A 3, R 4, S 4, B 20].

E. Electromagnetic Mass Differences

There have been several investigations of these mass differences. The starting point is to consider to second (and higher) order the effects of the charge operator and of the magnetic moment operator which have been discussed to first order in the previous section. We shall discuss the various attempts in increasing order of generality.

(1) Charge operator to second order [S 3]. The charge operator is assumed to be in the (*8*; *1*), part of a *35*, as in Eq. (4.42). Consider the electromagnetic mass splits as second-order effects in $q_\alpha{}^\beta$. For the *56* this amounts first to finding the independent $SU(6)$ invariants contained in $56^*\times 56\times 35\times 35$ and secondly to project out those terms in the *35*'s which transform like $q_\alpha{}^\beta$. With the help of Eqs. (3.24) and (3.26) it is easily seen [S 3] that there are only two such invariants, corresponding to one tensor *35* and a *405* contained in the product of $56^*\times 56$ as well as in 35×35. A two-parameter mass split operator results which leads to the following relations.

$$N^{*+}-N^{*0}=Y_1^{*+}-Y_1^{*0}=P-N=\Sigma^+-\Sigma^0, \tag{4.52}$$

$$\Xi^{*-}-\Xi^{*0}=N^{*-}-N^{*0}=Y_1^{*-}-Y_1^{*0}=\Xi^--\Xi^0=\Sigma^--\Sigma^0, \tag{4.53}$$

$$N^{*++}-N^{*-}=3(N^{*+}-N^{*0}). \tag{4.54}$$

(2) Charge operator to all orders [C 4]. This leads to only one additional parameter in the mass split operator. This is so because if Q_β^α is a *35*, then for arbitrary powers of Q we can at most have the following three distinct baryon contractions:

$$B^*_{\alpha\beta\gamma}B^{\alpha\beta\delta}Q_\delta^\gamma;\quad B^*_{\alpha\beta\gamma}B^{\alpha\delta\epsilon}Q_\delta^\beta Q_\epsilon^\gamma;\quad B^*_{\alpha\beta\gamma}B^{\delta\epsilon\zeta}Q_\delta^\alpha Q_\epsilon^\beta Q_\zeta^\gamma. \tag{4.55}$$

The first two terms correspond to the *35* and *405* mentioned above; the third one is a contraction of Q's with the *2695* in $56^*\times 56$. The inclusion of higher powers of Q can no longer give any new baryon density structure.

The consequence of the inclusion of the *2695* density is [C 4] that the relation (4.54) no longer follows. On the other hand, the relations (4.52, 53) both survive.

(3) Charge operator and magnetic moment operator to second order [K 8, D 2, V 1]. One includes here to second order not only the effects of q_β^α but also of $\mathbf{m}_\beta^\alpha$, Eq. (4.44). The resulting mass relations are as follows.

$$\Xi^- - \Xi^0 = \Sigma^- - \Sigma^+ - N + P, \tag{4.56}$$

$$N^{*0} - N^{*+} = Y_1^{*0} - Y_1^{*+} = N - P, \tag{4.57}$$

$$N^{*-} - N^{*0} = Y_1^{*-} - Y_1^{*0} = \Xi^{*-} - \Xi^0 = N - P + (\Sigma^- + \Sigma^+ - 2\Sigma^0), \tag{4.58}$$

$$N^{*-} - N^{*++} = 3(N-P). \tag{4.59}$$

These relations are weaker than (but of course compatible with) Eqs. (4.52–54).

The agreement of the strong relations (4.52, 53) with experiment is not too impressive [S 3, C 4]. The weaker relations (4.56–59) involve first of all the $SU(3)$ relation (4.56) [C 5] which is well satisfied [K 8]:

$$\Sigma^- - \Sigma^+ - N + P = 6.38 \text{ MeV}$$

$$\Xi^- - \Xi^0 = 6.5 \pm 1.0 \text{ MeV}.$$

Furthermore one derives [K 4] a predicted value $Y_1^{*-} - Y_1^{*+} = 4.47$ MeV. This is to be compared with two experimental results for this difference: 17 ± 7 MeV [C 6] and 4.3 ± 2.2 MeV [H 7]. One can therefore not yet judge the agreement with experiment, but the third approach outlined above seems so far the more reasonable one.

The last method has also been applied to the *35* mesons. Three relations are obtained which for the present are not very useful [K 4]. For applications to the baryon *70* see [D 11].

F. The Semi-Leptonic Vertex

Some of the most interesting results obtained with $SU(3)$ follow from the assumption that the effective hadron currents associated with semi-leptonic processes transform like the adjoint representation of $SU(3)$ [C 8]. The simplest extension of this idea to $SU(6)$ is to assume that these currents transform like members of a *35* [B 19]. This incorporates the $SU(3)$ results and in addition gives rise to new predictions. As has been stressed several times now, we may only use $SU(6)$, as so far defined, provided we neglect hadron recoil. On the other hand, the lepton current which is an external probe may carry any momentum. This makes it possible to include such nonstatic effects as weak magnetism.

To begin with, we write the vertex for the *56* in the following form

$$3B^*_{\alpha\beta\gamma}C_\delta^\gamma(q)B^{\alpha\beta\delta} \tag{4.60}$$

and put

$$C_\delta^\gamma(q) = (G_V/\sqrt{2})[\delta_i^j(L_0)_A{}^B + i\mu_W(\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{L}_A{}^B)_i{}^j] + (3G_A/\sqrt{2})(\boldsymbol{\sigma}\mathbf{L}_A{}^B)_i{}^j, \tag{4.61}$$

$$\mu_W = \tfrac{3}{5}[\mu(N) - \mu(P)]/e. \tag{4.62}$$

The lepton currents are defined as follows.

$$(L_\mu)_A{}^B = \sqrt{2}\begin{pmatrix} 0 & l_\mu\cos\theta & l_\mu\sin\theta \\ l_\mu^+\cos\theta & 0 & 0 \\ l_\mu^+\sin\theta & 0 & 0 \end{pmatrix},$$

$$-il_\mu = \bar{\mu}(p)\gamma_\mu(1+\gamma_5)\nu_\mu(p+q) + \bar{e}(p)\gamma_\mu(1+\gamma_5)\nu_e(p+q). \tag{4.63}$$

θ is the Cabibbo angle. Normalizations are such that

$$[N\tfrac{1}{2}\rightarrow P\tfrac{1}{2}]_V = G_V/\sqrt{2};\qquad [N\tfrac{1}{2}\rightarrow P\tfrac{1}{2}]_A = G_A/\sqrt{2}. \tag{4.64}$$

Individual transition elements are computed with the help of (4.14–17). From $SU(3)$ only one finds the amplitude relations

$$(\Omega^-\rightarrow\Xi)\cot\theta = (N^{*-}\rightarrow N) = -(N^{*+}\rightarrow N)\sqrt{3} \tag{4.65}$$

while the specific $SU(6)$ predictions are as follows.

(1) The semi-leptonic D/F ratio is

$$D/F = \tfrac{3}{2} \tag{4.66}$$

which agrees within the error with the best experimental value 1.7 ± 0.35 [W 4]. See also [R 5].

Remark. The D/F value given in (4.66) was first obtained in [G 4] for a somewhat different, but closely related, situation namely the pseudo-vector baryon–meson vertex. If in the latter instance one neglects the baryon recoil and treats the pseudo-vector coupling P as an "external field" the value (4.66) follows also in that case. We return in Sec. V to a more systematic exposé of the strong vertex.

(2)

$$(N^{*+}\tfrac{1}{2}\rightarrow N\tfrac{1}{2})_V = -(2\sqrt{2}/5)(N\tfrac{1}{2}\rightarrow P\tfrac{1}{2})_{V,\text{magn}}, \tag{4.67}$$

$$(N^{*+}\tfrac{1}{2}\rightarrow N\tfrac{1}{2})_A = -\tfrac{2}{5}G_A. \tag{4.68}$$

In Eq. (4.67) the subscript "V, magn" refers to the contribution from the μ_W term in Eq. (4.61) only.

With a similar warning as was made for the electromagnetic transition elements, one may attempt to compare (4.67, 68) with experiments on N^* production by neutrinos. Given the present rough data, the comparison is not discouraging [A 4, P 2, A 5]. Added note. See further [A20].

(3) Ω^- decays should be strongly dominated by the axial vector contribution, as is discussed in more detail in [M 3].

The foregoing discussion is independent of the ratio G_A/G_V in (4.61). It is a natural further assumption that the A and V currents belong to the *same 35*- representation of $SU(6)$. From this condition it follows that in Eq. (4.61):

$$G_A/G_V=1. \tag{4.69}$$

In physical terms, Eq. (4.69) means that the (effective) *coupling of the fundamental sextet, but not of the nucleon, is of the V–A type in the SU(6) limit.* In fact from (4.60) and (4.69) one can compute the corresponding ratio for nucleons:

$$(G_A/G_V)_{\text{nucleon}}=\tfrac{5}{3}. \tag{4.70}$$

Remarks. (1) With the same comment as was made after Eq. (4.66), it should be noted that Eq. (4.70) also appears in [G 4].

(2) For a connection between generalized Goldberger–Treiman relations and the ratio G_A/G_V see [B 19], and also [S 5, L 4].

(3) It has recently been shown [V 2] that many of the consequences of the static $U(6)$ actually follow from the weaker requirement of invariance under the subgroup $U(3)\times U(3)$ generated by 1, S_3, F_P, S_3F_P [notations as in Eq. (4.7)]. For a different (chiral) $U(3)\times U(3)$ see [H 11].

(4) For the question of the renormalization of V- and A-coupling constants see [A 12, G 25].

G. S-Wave Nucleon–Nucleon Scattering

Within the static theory, only $L=0$ scattering channels are amenable to treatment. For the nucleon–nucleon scattering $1+2\rightarrow 3+4$ this has been discussed in detail in [B 21]. We summarize their argument.

As [D 3]

$$56\times 56=462+1050+1134'+490, \tag{4.71}$$

there can be only four independent amplitudes. Moreover, the Pauli principle allows the *490* and *1050* only [D 3], so that only two parameters remain. The scattering amplitude can be written as follows.

$$\begin{aligned}&aB^*_{\alpha\beta\gamma}(3)B^{\alpha\beta\gamma}(1)\cdot B^*_{\lambda\mu\nu}(4)B^{\lambda\mu\nu}(2)\\&+bB^*_{\alpha\beta\gamma}(3)B^{\alpha\beta\nu}(1)\cdot B^*_{\lambda\mu\nu}(4)B^{\lambda\mu\gamma}(2)\\&+cB^*_{\alpha\beta\gamma}(3)B^{\alpha\mu\nu}(1)\cdot B^*_{\lambda\mu\nu}(4)B^{\lambda\beta\gamma}(2)\\&+dB^*_{\alpha\beta\gamma}(3)B^{\lambda\mu\nu}(1)\cdot B^*_{\lambda\mu\nu}(4)B^{\alpha\beta\gamma}(2),\end{aligned} \tag{4.72}$$

with the additional constraint

$$c=-b,\qquad d=-a. \tag{4.73}$$

Equations (4.72, 73) are then applied only to nucleons. In terms of the usual exchange operators, the nucleon–nucleon scattering matrix becomes

$$[a-(b/27)](1-P_\sigma P_\tau) \tag{4.74}$$

which is bad in the sense that equal scattering lengths are predicted for ^1S and ^3S [K 9, A 6]. The reasonable suggestion is made [B 21] to apply the theory only at energies large compared to the $^3S-{}^1S$ mass split.

It is then seen that the curves for phase shifts $\delta(^1S)$ and $\delta(^3S)$ are close in the region of 100–400 MeV [B 21]. Of course this is necessarily a qualitative statement. Perhaps the main thing we learn is that there appears nothing untoward from the comparison of an $SU(6)$ four-point function with experiment.

Further remarks. (1) An attempt has been made to classify two-baryon resonances by means of the *490* representation. It appears that such an assignment to a single $SU(6)$ multiplet does not work well [D 3].

(2) Λ- and Σ-nucleon scattering is discussed in [C 9].

Addendum. S-wave scattering is also discussed in [B 48].

H. Non-Leptonic Decays

In $SU(3)$ one commonly assumes that the nonleptonic decay interaction transforms like an *8*-component. Under more stringent conditions the following amplitude equality was derived [L 4, S 6]

$$\sqrt{3}\langle\Sigma^+\mid p\pi^0\rangle+\langle\Lambda\mid p\pi^-\rangle-2\langle\Xi^-\mid\Lambda\pi^-\rangle=0. \tag{4.75}$$

for both the S and the P wave. For the variety of the additional conditions, see, e.g. [L 5, P 3]. Experimentally these relations appear to hold well [S 7].

S-wave hyperon decay is a natural case for treatment with $SU(6)$. For this channel the simplest extension of the $SU(3)$ discussion to $SU(6)$ is to assume that the interaction transforms like a 35 component. The S-wave problem can then be treated as follows.

Let $P_\alpha{}^\beta=\delta_i{}^jP_A{}^B$ be the pseudoscalar part of the meson *35*, see (4.22). Introduce a corresponding spurion $S_\alpha{}^\beta=\delta_i{}^js_A{}^B$ with $K^0(s)\equiv s_3{}^2=1$; $\bar{K}^0(s)\equiv s_2{}^3=1$, all other $s_A{}^B=0$. The $K^0(s)$ are the spurions for $|\Delta I|=\frac{1}{2}$ transitions. For example write $\Lambda\rightarrow P+\pi^-$ as $K^0(s)+\Lambda\rightarrow P+\pi^-$. The amplitude for this and similar processes can then be obtained by taking a linear combination of all fully $SU(6)$-contracted Hermitian scalars made from B^*, B, P, and S. These are

$$B^*_{\alpha\beta\gamma}B^{\alpha\beta\gamma}P_\sigma{}^\rho S_\rho{}^\sigma;\qquad B^*_{\alpha\beta\gamma}B^{\alpha\beta\delta}(PS+SP)_\delta{}^\gamma;$$
$$B^*_{\alpha\beta\gamma}B^{\alpha\delta\epsilon}P_\delta{}^\beta S_\epsilon{}^\gamma,$$

of which the first one does not contribute to the processes of interest. Thus we get a two parameter expression for the set of all S-wave decays. As a result, Eq. (4.75) follows at once for S waves, without any additional assumptions. See [R 5, S 9, A 7] and [K 10, B 22, I 1, M 4]. These papers deal also with other amplitude relations. For example, $\langle\Sigma^+ | N\pi^+\rangle_S = 0$ from $SU(6)$, a current×current interaction and $|\nabla I| = \frac{1}{2}$ [A 8].

This concludes the survey of consequences of the static $SU(6)$. Non-leptonic P wave relations belong properly to the subject of Sec. V, but they will be reviewed briefly at this point. There have been many discussions of the P-wave situation. The common feature is a much greater theoretical complexity and less transparent predictions. The methods used are to introduce a further spurion (*1*; *3*) to describe one unit of orbital angular momentum; or alternatively to use dynamical groups of the kind mentioned in Sec. V. There appears to be a consensus that the P-wave relation (4.75) does not appear in any natural way. However, a rather straightforward analysis gives the P-wave relation (4.75) but with the rhs equal to $2\langle\Omega^- | \Lambda K^-\rangle_P/\sqrt{3}$, so that the validity of the P-wave triangle gets linked with a presumed smallness of this Ω^--decay amplitude [A 9, I 1]. For further literature see [A 10, G 19, H 8, I 3, K 11, K 12, O 4, R 6, R 7, S 19]. Finally, it may be recalled [P 3] that non-leptonic asymmetry parameters depend very sensitively on the way mass split effects are introduced.

I. The Master Problem

We have now learned that $SU(6)$ gives interesting results for counting states, for mass regularities and particle mixing, for magnetic moments and semi-leptonic vertices, for S-wave non-leptonic decays, and that the S-wave nucleon scattering results, though not compelling, are not unreasonable or paradoxical either. Faced with this situation, we must consider a much more difficult question.

The master problem: to find the structure of *dynamical equations* such that $SU(6)$ appears as a symmetry *of some of their approximate solutions*; and to state the dynamical nature of what is meant by "approximate."

If one assumes that the answer to this question lies within the framework of present-day theory, the following comments and further questions are in order.

(1) The dynamics should be relativistically invariant.

(2) The question before us is

$$\text{dynamics?} \rightarrow SU(6). \qquad (4.76)$$

It may be good to emphasize once more [B 10] that this question is distinct from the one to be discussed in Sec. V:

$$SU(6) \rightarrow \text{larger or other dynamical groups?} \qquad (4.77)$$

In Sec. VA the motivation is discussed for this question, to ask for larger dynamical groups which contain $SU(6)$. These groups appear in connection with the so-called relativistic completion procedures, to be discussed below. The question whether there is such a useful group is evidently not identical with the question of the underlying dynamics. In fact, a group which contains $SU(6)$ is a group which contains spin. Hence by the argument sketched in Sec. II, such a larger group must itself be of approximate dynamical character.

(3) Consider as an example the magnetic moment result (4.47). We have all learned in school that the "anomalous" values of the nucleon moments are due to the virtual meson cloud. In the cloud virtual scatterings take place and many-particle virtual states appear, involving arbitrarily high virtual momenta. Is this picture correct? If so by what mechanism do spin orbit interactions in the cloud average out to give the $SU(6)$ answer?

(4) For dynamical approaches involving bootstrap methods see for example [B 25, C 10]. A short discussion of the algebra of currents is deferred till Sec. VI.

(5) An interesting dynamical question is the connection between the nuclear physics $SU(4)$ [W 2] in which the nucleon is the representation *4* with that subgroup $SU(4)$ of $SU(6)$ in which the nucleon is in the *20* [G 4]. For a discussion see [C 11].

(6) In the nuclear $SU(4)$ case, the dynamics is divided in two stages. First, it should be shown that the (mean) potential between nucleons (in a many nucleon system) is spin- and isospin-independent. Secondly, one discusses energy levels in such a potential.

Is there a similar adiabatic approximation in $SU(6)$? Should one understand the $SU(6)$ dynamics in terms of spin-unitary spin-independent properties of real triplets? The concluding part of Sec. IV is devoted to some aspects of this question.

J. Triplets: Formal Tool or Reality?

In Sec. IVB we started to construct baryon states "as if" they were three triplet configurations. This was a mathematical illustration and we switched soon to a tensorial method. Likewise, certain selection rules can be expressed vividly in terms of triplet properties [L 6, L 7], but equally well as a property of tensor products. Added note: see also [L 14].

The question is whether one can make any predictions from properties of the *35*, *56* or higher representations of $SU(6)$ which are predicated on the real existence of triplets. Thus we must ask for the dynamics of multi-triplet systems.

It has been suggested that the internal dynamics of these systems might be nonrelativistic [N 6, M 5]. Consider a two-triplet (or triplet–antitriplet) system bound by a square well. If the range is a, the velocities are $\sim h/M_t a$, where M_t is the triplet mass. Thus for $a \sim h/M_{\text{nucleon}}c$, $v/c \sim M_{\text{nucleon}}/M_t$ which is ~ 0.1 for $M_t \sim 10$ BeV. However, such estimates depend very importantly on the potential shape. They may not be applied to a

Yukawa potential for example [M 5]. This point will be discussed in a forthcoming paper by O. W. Greenberg, to whom I am indebted for an illuminating discussion. At any rate, a potential in which nonrelativistic dynamics can be justified is just what lends itself to an $SU(6)$ picture. With its help an analysis of higher meson resonances has recently been attempted [D 4].

The baryon *56*, viewed as a genuine three-body problem raises interesting questions. As the *56* is symmetric in the internal variables, the generalized Pauli principle requires that the space part be totally antisymmetric, if the triplets satisfy Fermi–Dirac statistics. In the nonrelativistic picture under consideration, this is a puzzlement, at least if we suppose that simple (non-exchange) two-body forces are responsible for the binding. Several suggestions have been made for a way out of this dilemma.

(1) The baryon is not just a three-triplet structure but has a "core" in addition; attractive core–triplet forces overcome repulsive triplet–triplet forces [G 3].

(2) Attractive three-body forces overcome repulsive two-body forces [K 13].

(3) There exists more than one kind of $SU(3)$ triplet, see for example [H 9], and the *56* is made up out of triplet members belonging to different kinds. The internal part of the wave function is then not necessarily totally symmetric in *all* internal variables—which now further include a variable specifying which triplet we use. We come back to this point below.

(4) The triplets do not obey Fermi–Dirac statistics but follow para-statistics [G 16]. Because of the distinction between triplet and antitriplet this assumption does not affect the interpretation of the *35*. Higher para-states have been constructed which may possibly accomodate higher resonances [G 16]. For a survey of parafield theory see [G 17].

Let us see next what realistic triplet models can say about electromagnetic properties.

(α) Assume one set of triplets only, embedded in an internal symmetry not larger than $SU(3)$. For definiteness one may imagine a baryon model of the kind (1) or (2). Instead of using the tensor method described in Sec. IVD, one can (equivalently) use vector addition of triplet magnetic moments [B 9, S 10], very much as in nuclear physics calculations [B 26]. We do this next, but use the general charge assignments of Eq. (4.1) instead of Eq. (4.2). The corresponding charge operator is given by $Q=I_3+Y/2+(q_0-\frac{2}{3})t$. t is the triality, see, e.g. [B 7] and q_0 is defined in Eq. (4.1). The corresponding static magnetic moment operator $\mathbf{M}$ is assumed to be proportional to Q, as in (4.44). Hence for a set of triplets and anti-triplets

$$\mathbf{M}=\mu\{[q_0\boldsymbol{\sigma}^{(p)}+(q_0-1)(\boldsymbol{\sigma}^{(n)}+\boldsymbol{\sigma}^{(\lambda)})] -[q_0\boldsymbol{\sigma}^{(\bar{p})}+(q_0-1)(\boldsymbol{\sigma}^{(\bar{n})}+\boldsymbol{\sigma}^{(\bar{\lambda})})]\}. \quad (4.78)$$

Take the expectation value of M in the composite three-body states of the *56*. Then [B 8]

$$\mu(P):\mu(N):\mu(\Lambda)=(3q_0-4):(3q_0+1):(3q_0-3). \quad (4.79)$$

For $q_0=\frac{2}{3}$ we recover the results of Sec. IVD. The "nuclear physics' calculation described here rests on the additional assumption that "internal exchange currents" between the triplets, which may give additional terms in (4.78) do not play an appreciable role. (Such additional assumptions cannot appear in the formal method of Sec. IVD.)

Equation (4.79) shows that $q_0\neq\frac{2}{3}$ will spoil the good relation (4.47). Thus if only one set of triplets exists [and the internal symmetry is not larger than $SU(3)$], then it is indicated that they should be fractionally charged. Note that $SU(6)$ considerations play an essential role at this point. For example in an $SU(3)$ model with a core, the core should be an $SU(3)$ singlet but may carry spin. According to $SU(6)$ a possible core should be an $SU(6)$ singlet, hence spinless. Thus in $SU(3)$ but not in $SU(6)$ could the core give an additional magnetic moment itself.

It has been noted [N 1] that for general q_0 the hyperon magnetic moments need not take on the values predicted [C 5] on the basis of Eq. (4.3). This would give two independent sources for deviations from a relation like $\mu(\Lambda)=\mu(N)/2$: (a) because of the q_0 effect, (b) because of breakdown of $SU(3)$. Let us now exclude cause (a) on the grounds just mentioned. Then cause (b) only remains. As an extension of the successful description of mass splits as first-order effects, the assumption is plausible that the $SU(6)$ magnetic moment ratios for the *56* should be corrected by the inverse of the true mass ratios [B 8]. The resulting magnetic moment values are found in the "Mass-corrected" column of [Table I, B 8]. In particular

$$\mu(\Lambda)=-0.78\mu(P)$$

which agrees within the error with the weighted average of available experiments [H 10]

$$\mu(\Lambda)=(-0.73\pm0.17)\mu(P).$$

Remarks. (1) If one takes Eq. (4.78) seriously, then one can express the magnetic moment of $\mu(p)$ in terms of $\mu(P)$. For $q_0=\frac{2}{3}$: $\mu(p)=2\mu(P)/3$. At first sight it seems curious that the triplet moment should be of the order of the nucleon moment, if the triplet mass is quite high ($\gtrsim 10$ BeV). It has been noted, however [B 27], that triplet magnetic moments can be enhanced by strong binding in an external field of the scalar type. The Dirac equation in a scalar potential is

$$[M+V+\beta\boldsymbol{\alpha}(\mathbf{p}-e\mathbf{A})]u=\beta Eu.$$

In a region where V is constant, the effective mass is $M+V$ [L 8]. Applied to triplets, a large M can be compensated by a large V. The situation is distinct for external fields other than the scalar kind [L 8, G 18].

(2) For attempts to deduce radiative vector meson decay rates from a realistic triplet model see [T 2, B 20]. The importance of these calculations lies in the fact that the $M1$-transition element for $\omega\rightarrow\pi^0+\gamma$ is expressed in terms of the *nucleon* magnetic moment via Eq. (4.78). The method appears to depend more decisively on the real existence of quarks than in any of the problems mentioned earlier, as is also evident from the fact that the results obtained *cannot* be deduced from $SU(6)$-algebra alone.

(β) The case of three triplets [H 10, B 27]. The introduction of more triplets allows one to retain Eq. (4.47) in a theory with integrally charged triplets. I am indebted to Professor A. Tavkhelidze for an illuminating discussion of this point. The argument is as follows.

Consider a second group $SU(3)'$ with corresponding tensor indices A', B', $\cdots$. Let $u^{A,A'}$ denote a three triplet set, where A is the usual $SU(3)$ index. Define a charge operator as follows

$$Q_{B,B'}{}^{A,A'}=Q_B{}^A\delta_{B'}{}^{A'}+\delta_B{}^A\tilde{Q}_{B'}{}^{A'}, \tag{4.80}$$

where $Q_B{}^A$ is given by (4.36) and

$$\tilde{Q}_{B'}{}^{A'}=\tfrac{1}{3}\begin{pmatrix}1 & & \\ & 1 & \\ & & -2\end{pmatrix}. \tag{4.81}$$

This corresponds to three $SU(3)$ triplets with respective charges: (1, 0, 0); (1, 0, 0); (0, −1, −1). The 56 is supposed to be made up of one triplet member of each kind and instead of $B^{\alpha\beta\gamma}$, Eq. (4.14), the baryon tensor is taken to be

$$B^{\alpha\beta\gamma}\rightarrow\epsilon^{A'B'C'}B^{\alpha\beta\gamma} \tag{4.82}$$

which is now totally *anti*symmetric in the internal variables.

Next define a magnetic moment operator as in (4.44) but with $Q_A{}^B$ replaced by (4.80). The baryon moments are computed by replacing $B^{\alpha\beta\gamma}$ in (4.45) by (4.82).

Observe the identity

$$\epsilon_{A'B'D'}\tilde{Q}_{C'}{}^{D'}\epsilon^{A'B'C'}=0 \tag{4.83}$$

from which it follows that the second term in (4.80) does not contribute to the magnetic moments. Hence Eq. (4.47) has recovered.

Much more experimental information is needed before one can know which if any of these preliminary models of multi-triplet systems will survive. Nor are more complex models excluded [F 5]. The possibility may finally be noted that $SU(3)$ is perhaps not the full group of internal symmetry and that there are larger groups (see, e.g. [N 2]) in which there is room for the *8*- and *10*-representations of $SU(3)$; but not for triplets.

In summary, $SU(6)$ results obtained from quark models are often suggestive, yet they do not appear to be compelling for the reality of quarks. On the other hand, such stronger than $SU(6)$ results as, for example, the $\omega\rightarrow\pi^0+\gamma$ decay rate seem more difficult to understand without a real structure of the nucleon with three quarks in its "outer shell."

V. RELATIVISTIC EXPLORATIONS

A. Introduction

In parallel with the sorting out of the consequences of the static $SU(6)$, discussed in the previous section, several distinct attempts began to develop to embed the static $SU(6)$ in a relativistic description. Up till now there does not exist a satisfactory solution to this question. The main approaches to the question stated in Eq. (4.77) will be outlined in this Introduction.

(A) It was realized from the start [G 3] that it is possible to give a classification of supermultiplet states also for $\mathbf{p}\neq0$. To see this, recall that for spin $\frac{1}{2}$ a relativistic spin $S(\mathbf{p})$ can be defined [F 6] which commutes with the free Hamiltonian $\boldsymbol{\alpha}\mathbf{p}+\beta m$. For each $\mathbf{p}$ it is possible to define a group by DG as in Eq. (4.7) but with S replaced by $S(\mathbf{p})$. This is the group called $SU(6)_p$, see [G 20].

It is possible to generalize the Foldy–Wouthuysen spin description [and hence $SU(6)_p$] to a relativistic N-body system with interaction. More precisely, one can construct for such a system a corresponding representation of the Poincaré group [F 7]. One must next ask whether a system so described is necessarily a physically reasonable system, in particular whether it is separable [F 7]. In a concrete instance, one means the following by separability. Consider a three-body system with its assigned representation. Move one of the particles to infinity. Do we approach in this way a system of two interacting particles plus a free particle in all frames? This is a far from trivial question. In particular separability in the center-of-mass frame does not guarantee separability in all other frames [F 7]. The question is clearly of interest if one wishes to find connections with local dynamical theories, because of creation and annihilation processes.

In the context of such theories, $SU(6)_p$ runs into difficulties. For specific examples it has been shown that local interaction violates the symmetry [M 6, R 8] and it has been proved, more generally, that this symmetry implies a unit S-matrix [J 2]. This approach will not be discussed further in this paper.

(B) The starting point for a second approach is to seek for an approximate dynamical symmetry which, in the language of a Lagrangian field theory, is broken by free kinetic energy terms [B 8, B 16]. This new symmetry is supposed to contain the static $SU(6)$ and to be of use for $\mathbf{p}\neq0$.

It may be helpful to illustrate the idea by an example

[B 8]. Consider a 12×12 matrix

$$^{(1)}\mathfrak{M}_b{}^a(q)=if_V(q^2)\,(\gamma\epsilon(q))_\mu{}^\lambda V_B{}^A$$
$$-f_A(q^2)\{\gamma_5[(\gamma q)/\mu]\}_\mu{}^\lambda P_B{}^A, \quad (5.1)$$

where V and P are given by Eqs. (4.22, 23). $q=(\mathbf{q}, iq_0)$. $\gamma_\mu=(\boldsymbol{\gamma}, \gamma_4)$ are Hermitian Dirac matrices, $(\gamma q)=(\gamma_\mu q_\mu)$, etc. The polarization four-vector $\epsilon_\mu(q)=(\boldsymbol{\varepsilon}(q), i\epsilon_0(q))$ is related to the $\boldsymbol{\varepsilon}$ in Eq. (4.21) as follows

$$\boldsymbol{\varepsilon}(q)=\boldsymbol{\varepsilon}+[\mathbf{q}(\mathbf{q}\boldsymbol{\varepsilon})/\mu(q_0+\mu)],$$
$$\epsilon_0(q)=\mathbf{q}\boldsymbol{\varepsilon}/\mu, \quad (5.2)$$

so that

$$q\epsilon(q)=0. \quad (5.3)$$

In regard to indices, the ranges for α, i, A will be maintained as in Sec. IVA. In addition we shall use from here on the following conventions. The early letters a, b, $\cdots$ of the Latin alphabet run from 1–12. The middle letters λ, μ, $\cdots$ of the Greek alphabet run from 1–4.

Unless otherwise specified, we use the representation

$$\boldsymbol{\gamma}=\rho_2\boldsymbol{\sigma}, \qquad \gamma_4=\rho_3, \qquad \gamma_5=\rho_1. \quad (5.4)$$

Then $^{(1)}\mathfrak{M}_b{}^a$ can be written as

$$^{(1)}\mathfrak{M}_b{}^a(q)=\begin{pmatrix} N_\beta{}^\alpha(q), & -M_\beta{}^\alpha(q) \\ M_\beta{}^\alpha(q), & -N_\beta{}^\alpha(q)\end{pmatrix}, \quad (5.5)$$

$$M_\beta{}^\alpha(q)=-f_V(q^2)\,(\boldsymbol{\sigma}\boldsymbol{\varepsilon}(q))_j{}^i V_B{}^A-if_A(q^2)\,(q_0/\mu)\,P_B{}^A\delta_i{}^j, \quad (5.6)$$

$$N_\beta{}^\alpha(q)=-f_V(q^2)\epsilon_0(q)\,\delta_j{}^iV_B{}^A-if_A(q^2)[(\boldsymbol{\sigma}\mathbf{q})_j{}^i/\mu]P_B{}^A. \quad (5.7)$$

Define the static limit q^0 of q by $q^0=(0, i\mu)$. Then

$$N_\beta{}^\alpha(q^0)=0, \quad (5.8)$$

while

$$M_\beta{}^\alpha(q^0)=M_\beta{}^\alpha, \quad (5.9)$$

provided that

$$f_V(-\mu^2)=f_A(-\mu^2). \quad (5.10)$$

The right-hand side of Eq. (5.9) is just the static $SU(6)$ matrix of Eq. (4.21) and Eq. (5.10) may therefore be regarded as the *static* $SU(6)$ form factor condition.

Note that

$$^{(1)}\mathfrak{M}_b{}^a(q^0)=\begin{pmatrix} 0, & -M_\beta{}^\alpha \\ M_\beta{}^\alpha, & 0\end{pmatrix}. \quad (5.11)$$

This doubling of the matrix is well known from the treatment of the $\mathfrak{D}^{[\frac{1}{2},\frac{1}{2}]}$ representation (four-vector) of the orthochronous Lorentz group in the $SL(2, C)$ language [S 11]. The doubling is necessary in order to assign a definite parity to the representation.

$M_\alpha{}^\beta$ is odd, $N_\alpha{}^\beta$ is even under space reflection. We now define the baryon–meson vertex in the no-baryon recoil limit to be [B 8]

$$B^*{}_{\alpha\beta\gamma}N_\delta{}^\gamma(q)\,B^{\alpha\beta\delta}.$$

The vertex depends on the single three-momentum $\mathbf{q}$. To order v/c (P-wave vertex) one should retain only terms of the first power in $|\mathbf{q}|$. This means that to this order Eq. (5.10) is valid.

Using the methods of Sec. IVD, one obtains from Eq. (5.12) all the nonrelativistic vertex results first given in [G 4] and reviewed in [B 8]. The use of Eq. (5.10) is essential. It follows that such strong interaction relations as

$$D/F=\tfrac{3}{2}, \qquad g_A/g_V=\tfrac{5}{3} \text{ (strong int.)}$$

do not involve any further relation between form factors than those implied by the static $SU(6)$. It has been shown in [B 10] and will be discussed further in Sec. VC that under the same conditions the inclusion of baryon recoil to order v/c is also possible. (It may be noted incidentally that up to this order the Foldy–Wouthuysen spin is identical with the nonrelativistic spin.) See further the derivation of Eqs. (5.63) and (5.76) below.

The prescription just described amounts to associating to the static meson tensor its "relativistic completion" Eq. (5.1), the even parity part of which is coupled to the baryon current. The general theme to be discussed in several variations may now be stated as follows.

(a) To associate to any static $SU(6)$ representation their relativistic completion or completions. This is done by "boosting," a procedure described in Sec. VB. It will be shown there that there are in general more than one completions to a given static $SU(6)$ tensor.

(b) To give rules for the coupling of the completed $SU(6)$ tensors to each other in the form of effective vertices which are covariant. This can be done in more than one way and the variety of these ways can be classified by means of a "booster group" which is $U(6, 6)$ [S 13, B 28].

$U(6, 6)$ is a special case of the pseudo-unitary groups discussed in Sec. IIIG. It was seen there that the transformations of such groups leave invariant pseudo-unitary quadratic forms, Eq. (3.86). What is this form for $U(6, 6)$? Let $u^\lambda(p)$ be a four-component Dirac spinor, $\lambda=1$–4, and let $u^{\lambda A}(p)$ be a triple of such spinors describing the relativistic $SU(3)$ triplet. $U(6, 6)$ is the group of transformations for which the "relativistic mass term"

$$\bar{u}_{\lambda A}(p)\,u^{\lambda A}(p)=\text{invariant}, \quad (5.12)$$

where $\bar{u}=u^\dagger\gamma_4$. Using the representation (5.4) one sees that this quadratic form has six $+$, and six $-$ signs, whence $U(6, 6)$. The structure of this group is explained in more detail in Sec. VB.

Equation (5.12) may be looked upon as a covariant generalization of the invariance requirement (4.6) for $SU(6)$. As $SU(6)$ is an approximate dynamical symmetry, the same is true therefore for $U(6, 6)$, as the latter contains $SU(6)$. Moreover, one knows *explicitly* at least one mechanism by which $U(6, 6)$ is broken: As will be seen in Sec. VB, $U(6,6)$ does not leave invariant the "kinetic energy term" $\bar{u}_{\lambda A}(p)(\gamma p)_{\mu}{}^{\lambda}u^{\mu A}(p)$. Thus *in the presence of interactions* [where the identity $\bar{u}(\gamma p)u=im\bar{u}u$ is not valid], $U(6, 6)$ is intrinsically broken by the spin-orbit couplings inherent in (γp) terms. While the intrinsically broken character of $U(6, 6)$ was often well appreciated, the applications of $U(6, 6)$ and some of its subgroups was nevertheless pursued further, motivated by the idea (see Sec. IVI) that, where the static $SU(6)$ is successful, an *effective* damping out of these symmetry breaking spin-orbit couplings must be at work. However, it was soon realized that no systematic application of such invariance ideas is possible for all n-point functions. This is because of the unitarity problem, as is explained in Sec. VE.

The approach just sketched is only one of several which were followed in the fall of 1964 in order to find approximate dynamical symmetries which contain $SU(6)$. For example, others took the structure of four quark interactions as a starting point. Independently, the following groups were suggested: the chiral group $U(6)\otimes U(6)$ [F 4, D 5, B 16]; the nonchiral group $U(6)\otimes U(6)$ [O 3, B 16]: $SL(6, C)$ [F 8, R 9, S 12]; $U(6, 6)$ [S 13, B 28, B 16, R 10]. [The latter group went for a while under a variety of names: $\tilde{U}(12)$, $U(12)_{\mathcal{L}}$, $M(12)$, $V(12)$.] These groups will be discussed in Sec. VD. As they are all related to $U(6, 6)$ it is convenient to take the latter as a starting point.

It may directly be noted that the matrix ${}^{(1)}\mathfrak{M}_b{}^a(q)$ has a vanishing 12-trace:

$$ {}^{(1)}\mathfrak{M}_a{}^a(q)=0. \qquad (5.13) $$

This is true for all q and therefore in particular in the static limit $q=q_0$ see Eq. (5.11). Observe an important difference between the $SU(6)$ condition $M_\alpha{}^\alpha=0$ and Eq. (5.13). The former implies that $P_A{}^A=0$, corresponding to 8 pesudoscalar mesons. On the other hand, Eq. (5.13) is also true if $P_A{}^A\neq 0$. It will be shown more systematically below that in fact all the larger groups mentioned above accomodate a 36th meson along with the 35 of $SU(6)$. It is natural to identify this extra meson with the X^0 mentioned in Sec. IVB. Accordingly we shall from now on redefine $P_B{}^A$ as follows.

$$ P_B{}^A=[P_B{}^A \text{ of Eq. (4.22)}]+(X^0/\sqrt{3})\,\delta_B{}^A. \qquad (5.14) $$

(c) Closely related to the methods of (B) are attempts to embed $SU(6)$ in a kinematical symmetry by means of the formal introduction of more than the four usual translation operators p_μ. The idea, first mention dine [S 1], is the following. Consider the kinetic energy $\gamma_\mu p_\mu$ for a free relativistic quark. In the notation of Eq. (5.1) this term can be written as a 12×12 matrix $(\gamma_\mu p_\mu)_b{}^a\delta_B{}^A$. Thus the kinetic energy can be brought in formal correspondence with the ϕ^0-meson part of the meson matrix and the fact that the kinetic energy breaks all of the symmetries mentioned above can be expressed as the formal statement that $(\gamma_\mu p_\mu)_b{}^a\delta_B{}^A$ is not a full representation of any of these groups. In essence, the idea is now to enlarge the set of 4 translations to a set of 36 (or a multiple thereof). In this way one can obtain descriptions which are invariant inclusive of these generalized translations, which contain the four usual ones. See [F 8, B 29, B 30, R 11, N 3, K 14, N 4, R 12, Z 2]; also [N 5]. *Added note.* See further [B 49].

For physical applications it is necessary, however, to introduce the boundary condition that the operators of the theory act on such states which depend on the four translational variables p_μ only. Thus till now the extra translations have not led to physical predictions that cannot be obtained as well without their introduction. In practice, we are back to some of the methods included in (B). Extra translations will not be discussed further in what follows.

(D) Another line of development finds its origin in the observation [B 31] that for a vertex in the brick wall system one can define a conserved "spin" W:

$$ \mathbf{W}=\tfrac{1}{2}\gamma_4\sigma_1,\ \tfrac{1}{2}\gamma_4\sigma_2,\ \tfrac{1}{2}\sigma_3, \qquad (5.15) $$

(where all vertex momenta are in the 3-direction) which has the same commutation relations as $\frac{1}{2}\boldsymbol{\sigma}$. Thus one can define a group isomorphic to $SU(6)$ by replacing $\mathbf{S}$ by $\mathbf{W}$ in Eq. (4.7). This group is called $SU(6)_W$. It can be applied to any collinear configuration [L 9]. It is a subgroup of $U(6, 6)$ see Sec. VD.

The W-spin will be discussed in more detail in Sec. VF, but we state some of its main consequences right here. (A detailed survey of W-spin predictions is given in [H 12].)

(1) When applied to baryon vertices it leads to the relations [B 31]

$$ G_m{}^N(q^2)/G_m{}^P(q^2)=-\tfrac{2}{3}, \qquad G_e{}^N(q^2)=0 \quad (5.16) $$

for all q^2 where G_m, G_e are the magnetic and electric Sachs form factors, respectively. Slightly more generally it gives [B 31, P 4]

$$ G_m(q^2)/G_m{}^P(q^2) \quad \text{and} \quad G_e(q^2)/G_e{}^P(q^2) $$
$$ \text{are independent of } q^2, \qquad (5.17) $$

where the respective numerators refer to any member of the baryon octet or to the $(\Sigma^0\mid\Lambda)$-transition form factors.

$G_e{}^N(q^2)$ is known to have a nonzero slope near $q^2=0$ but $G_e{}^P(q^2)/G_e{}^N(q^2)<0.2$ in the known region. Within such $\sim 20\%$ margins the relations (5.16) appear to be reasonable up to $q^2\sim(1\text{ BeV}/c)$,^{2} see, e.g. [D 6].

(2) Applied to the forward meson–baryon scattering amplitudes one obtains the Johnson–Treiman relations [J 3]

$$\tfrac{1}{2}[f(K^+)-f(K^-)]=[f(K^0)-f(\bar{K}^0)]=[f(\pi^+)-f(\pi^-)] \quad (5.18)$$

for scattering off protons. Of the many ways in which this relation has been derived, the one using W-spin seems to be the most transparent. The J–T relations appear to be well satisfied for not too low energies [G 21, L 10] (especially for the K-particle equalities). There are many puzzles why these relations work, in the face of $SU(3)$ breaking effects [H 2], but these relations are currently regarded as one of the more promising consequences of W spin. It has been noted, however, [S 14], that the J–T relations may perhaps be understood by dynamical approximations which do not invoke W-spin or other symmetries.

$SU(6)_W$ is necessarily an approximate symmetry of the dynamical kind, as collinear and noncollinear configurations are dynamically coupled. It is interesting that at least some noncollinear diagrams do not violate the J–T relations [H 13].

(E) Finally we mention the kinetic spurion approach [A 11, G 22, F 10, B 32, O 5, F 9]. One introduces in all possible ways spurions $(\gamma p)_\mu{}^\lambda \delta_B{}^A$ into $U(6, 6)$ vertices or other effective matrix elements. Here p can be any 4-momentum characteristic for the problem at hand. This implies the breakdown of $U(6, 6)$ to a lower symmetry (see Sec. VD). One finds the following symmetry survives: (a) $U(6)\otimes U(6)$ (nonchiral) for one-particle states. (b) $SU(6)_W$ for collinear configurations. (c) $U(3)\otimes U(3)$ for coplanar configurations (d) $SU(3)$ only for more complicated situations. This hierarchy has been noticed by several authors [O 6, D 7, H 14].

The remainder of Sec. V is devoted to the discussion of techniques, consequences, and difficulties for the approximate dynamical symmetries mentioned above. In VB, boost matrices are defined, boosts are performed on the *6*, *36*, *56* and the group $U(6, 6)$ is introduced. In VC the meson–baryon coupling is discussed under the weakest conditions which give the "good" results for the phenomenological vertex. In VD some subgroups of $U(6, 6)$ are treated. Section VE deals with the implications of the higher symmetries in the nonstatic domain; in particular the unitarity difficulties are discussed. Finally in VF more details about W spin are given.

B. The Boosted 6, 36, 56; the Group $U(6, 6)$

1. Boost Matrices

We return briefly to the labeling explained in Eq. (2.6) by means of components and base vectors. For $SU(2)$ we consider the two-base vectors $u_{(1)}{}^i$, $u_{(2)}{}^i$:

$$u_{(1)}{}^i=\begin{pmatrix}1\\0\end{pmatrix}, \qquad u_{(2)}{}^i=\begin{pmatrix}0\\1\end{pmatrix}. \quad (5.19)$$

These u's, enlarged as follows,

$$u_{(1)}{}^\lambda=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \qquad u_{(2)}{}^\lambda=\begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad (5.20)$$

represent Dirac spinors for $p=0$, $p_0>0$. The number of *components* for given state has doubled. In addition we introduce the $\mathbf{p}=0$ negative energy spinors

$$v_{(1)}{}^\lambda=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \qquad v_{(2)}{}^\lambda=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}. \quad (5.21)$$

Note. In the nonrelativistic two-component language, u^i (upper two u^λ) and v^i (lower two v^λ) may both be considered as *6*-representations of one and the same $U(6)$. Even at $\mathbf{p}=0$, this is no longer true in the four component language, as is shown by the distinct structure of (5.20) and (5.21). The four-component quantities u^λ and v^λ are actually representations of a group $U(6)\otimes U(6)$, as will be seen in more detail after Eq. (5.88) below.

We have now an array $D_{(\mu)}{}^\lambda(0)$ defined as

$$D_{(\mu)}{}^\lambda(0)=(u_{(1)}{}^\lambda, u_{(2)}{}^\lambda, v_{(1)}{}^\lambda, v_{(2)}{}^\lambda)=\delta_{(\mu)}{}^\lambda, \quad (5.22)$$

where the index λ numbers components, and the bracketed index (μ) labels states. In the representation (5.4) we have correspondingly for $\mathbf{p}\neq 0$ (and mass m)

$$D_{(\mu)}{}^\lambda(p)=\left[\frac{m-i(\gamma p)\gamma_4}{\{2m(p_0+m)\}^{\frac{1}{2}}}\right]_{(\mu)}^{\lambda}, \quad (5.23)$$

where to $(\mu)=1$, 2 correspond solutions of the Dirac equation for $\mathbf{p}$, p_0 and to $(\mu)=3$, 4 the solutions for $-\mathbf{p}$, $-p_0$ $(p_0>0)$. From (5.22):

$$D_{(\mu)}{}^\lambda(p)=D_\nu{}^\lambda(p)D_{(\mu)}{}^\nu(0). \quad (5.24)$$

$D_{(\mu)}{}^\lambda$ is a matrix with one index in component space, and one in base vector (state) space. $D_\nu{}^\lambda(p)$ is the *same* matrix but with both indices in component space. $D_\nu{}^\lambda(p)$ is called the boost matrix [W 5].

Define as usual

$$\bar{D}_\lambda{}^{(\mu)}(p)=D_\nu{}^{*(\mu)}(p)(\gamma_4)_\lambda{}^\nu \quad (5.25)$$

for all p. Hence (5.24) gives

$$\bar{D}_\lambda^{(\mu)}(p)=\bar{D}_\nu^{(\mu)}(0)\bar{D}_\lambda^{\nu}(p), \qquad (5.26)$$

where

$$\bar{D}_\lambda^{\nu}(p)=(\gamma_4 D^\dagger(p)\gamma_4)_\lambda^{\nu}. \qquad (5.27)$$

From now on the matrices D, $\bar{D}$ defined by Eqs. (5.23, 24, 27) will always be meant to be the boost matrices. We list some of their properties.

$$D\bar{D}=1, \qquad (5.29)$$

$$D\gamma_5\bar{D}=\gamma_5, \qquad (5.30)$$

$$D\gamma_4\bar{D}=-i(\gamma p)/m, \qquad (5.31)$$

$$D\boldsymbol{\gamma}\boldsymbol{\varepsilon}\bar{D}=\gamma\epsilon(p), \qquad (5.32)$$

$$D\gamma_4\boldsymbol{\gamma}\boldsymbol{\varepsilon}\bar{D}=\sigma_{\mu\nu}p_\mu\epsilon_\nu(p)/m,$$

$$\sigma_{\mu\nu}=-i[\gamma_\mu,\gamma_\nu]/2,$$

where $\epsilon_\mu(p)$ is defined in Eq. (5.2). Let C be the charge conjugation matrix, $C^{-1}\gamma_\mu C=-\gamma_\mu^t$, $C^t=-C$; t=transpose. In the representation (5.4) C may be taken as

$$C=i\gamma_5\sigma_2. \qquad (5.33)$$

We have

$$CD^t=\bar{D}C, \qquad \bar{D}^tC^{-1}=C^{-1}D. \qquad (5.34)$$

2. The Boosted 6

We go from the static sextet u^{iA} defined in Eq. (4.4) to the corresponding $u^{\lambda A}=u^a$ by using the transition (5.19)→(5.20). The boosted sextet $u^a(p)$ is

$$u^a(p)=u^{\lambda A}(p)=D_\mu^{\lambda}(p)u^{\mu A}, \qquad (5.35)$$

and likewise

$$v^a(p)=v^{\lambda A}(p)=D_\mu^{\lambda}(p)v^{\mu A}. \qquad (5.36)$$

u and v are associated to the sextet and antisextet, respectively, in the sense of the canonical expansion

$$\psi^a(x)=\sum_{(i)=1,2}(m/E)^{\frac{1}{2}}$$
$$\times[a_{(i)}u_{(i)}{}^a(p)\exp(ipx)+b_{(i)}{}^\dagger v_{(i)}{}^a(p)\exp(-ipx)].$$

The particle–antiparticle conjugation operation is not part of $SU(6)$, but it can be implemented as an additional rule. In Eqs. (5.35, 36) we have reverted to the dropping of state labels and will continue to do so from now on.

3. The Boosted 36

Start from Eqs. (4.21) and (5.14). The static $SU(6)$ matrix $M_{jB}{}^{iA}$ has first to be "enlarged" in a similar way as was done for the *6* in going from (5.19) to (5.20). This can be done as follows.

$$M_{jB}{}^{iA}\to\mathfrak{M}_b{}^a(0)=\mathfrak{M}_{\mu B}{}^{\lambda A}(0)=[\tfrac{1}{2}\gamma_5(1+\gamma_4)M]_{\mu B}{}^{\lambda A},$$

that is,

$$\mathfrak{M}_b{}^a(0)=\begin{pmatrix}0 & 0\\ M_\beta{}^\alpha & 0\end{pmatrix}.$$

Next we boost the static matrix to momentum q with the help of $D(q)$ and $\bar{D}(q)$ (for mass μ):

$$\mathfrak{M}_b{}^a(q)=\mathfrak{M}_{\mu B}{}^{\lambda A}(q)=D_\nu{}^\lambda(q)\mathfrak{M}_{\rho B}{}^{\nu A}(0)\bar{D}_\mu{}^\rho(q), \qquad (5.37)$$

and find with the help of Eqs. (5.29–32):

$$\mathfrak{M}_b{}^a(q)={}^{(1)}\mathfrak{M}_b{}^a(q)+{}^{(2)}\mathfrak{M}_b{}^a(q), \qquad (5.38)$$

$${}^{(1)}\mathfrak{M}_b{}^a(q)=i(\gamma\epsilon(q))_\mu{}^\lambda V_B{}^A-\{\gamma_5[(\gamma q)/\mu]\}_\mu{}^\lambda P_B{}^A, \qquad (5.39)$$

$${}^{(2)}\mathfrak{M}_b{}^a(q)=-(i/\mu)\{\sigma_{\mu\nu}q_\mu\epsilon_\nu(q)\}_\mu{}^\lambda V_B{}^A-i(\gamma_5)_\mu{}^\lambda P_B{}^A. \qquad (5.40)$$

Dropping indices, one notes that

$$\gamma_5\mathfrak{M}(q)\gamma_5=-{}^{(1)}\mathfrak{M}(q)+{}^{(2)}\mathfrak{M}(q). \qquad (5.41)$$

Remarks. (1) Equation (5.39) is proportional to the expression (5.1) *provided that* $f_V(q^2)=f_A(q^2)$ *for all* q^2. (2) ${}^{(2)}\mathfrak{M}$ is a second matrix which has static $SU(6)$ limit properties in the sense described in Sec. VA. (3) We could have started from $\gamma_5\mathfrak{M}(0)\gamma_5$ instead of from Eq. (5.39) and would have obtained (5.41) directly by boosting because of Eq. (5.30). (4) $\mathfrak{M}$ and $\gamma_5\mathfrak{M}\gamma_5$ obviously have the same parity.

4. The Boosted 56

Once again we enlarge the zero momentum $SU(6)$ tensor $B^{\alpha\beta\gamma}$, Eq. (4.14).

$$B^{iA,jB,kC}\to B^{\lambda A,\mu B,\nu C}(0)=B^{abc}(0),$$

$$B^{abc}(0)=u^{(\lambda\mu\nu)}(0)d^{ABC}+(1/3\sqrt{2})[\epsilon^{\lambda\mu}u^\nu(0)X^{ABC}$$
$$+\epsilon^{\mu\nu}u^\lambda(0)X^{BCA}+\epsilon^{\nu\lambda}u^\mu(0)X^{CAB}], \qquad (5.42)$$

where $u^\mu(0)$ is given by Eq. (5.20) and $X^{ABC}=\epsilon^{ABD}b_D{}^C$. $u^{\lambda\mu\nu}(0)$ is the correspondingly enlarged spin-$\frac{3}{2}$ wave function [We may construct $u^{(\lambda\mu\nu)}(0)$ as a totally symmetric direct product of three functions $u^\lambda(0)$.] We have to enlarge ϵ^{ij} of Eq. (4.14) as follows.

$$\epsilon^{ij}\to\epsilon^{\lambda\mu}=\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}=\left[\frac{1+\gamma_4}{2}\gamma_5C\right]^{\lambda\mu} \qquad (5.43)$$

see Eq. (5.33). [Note that $\epsilon^{\lambda\mu}$ represents a spin-zero state constructed out of two *four*-component spin-$\frac{1}{2}$ states given by Eq. (5.20).] We now define the *totally symmetric boost* to momentum p (for mass M):

$$B^{abc}(p)=D_\rho{}^\lambda(p)D_\sigma{}^\mu(p)D_\tau{}^\nu(p)B^{\rho A,\sigma B,\tau C}(0). \qquad (5.44)$$

Note in particular that

$$\epsilon^{\lambda\mu}u^{\nu}(0)\rightarrow\tfrac{1}{2}\{[1-(i\gamma p/M)]\gamma_5 C\}^{\lambda\mu}u^{\nu}(p). \quad (5.45)$$

By means of Eq. (5.44) we have obtained a boosted *56* which is totally symmetric in (a, b, c). By means of γ_5-insertion we can obtain alternative boosts, in a similar way as was described for the *36*. For example we can start from $(\gamma_5)_\rho{}^\lambda(\gamma_5)_\sigma{}^\mu B^{\rho A,\sigma B,\nu C}(0)$ instead of from Eq. (5.43), and we can insert γ_5 pairs in a variety of ways, always ending up with a boosted *56* which has the same parity as does $B^{abc}(p)$. This is the variety of boosts discussed in [B 28] (where a γ-representation was chosen in which γ_5 is diagonal). These asymmetric boosts will not be used explicitly in what follows, but we shall comment further on them in Sec. VC and D.

The totally symmetric boosted anti-*56* is given by

$$C^{abc}(p)=D_\rho{}^\lambda(p)D_\sigma{}^\mu(p)D_\tau{}^\nu(p)C^{\rho A,\sigma B,\tau C}(0), \quad (5.46)$$

where $C(0)$ is obtained from $B(0)$, Eq. (5.42) by the replacement

$$u^{(\lambda\mu\nu)}(0)\rightarrow v^{(\lambda\mu\nu)}(0), \qquad u^{\lambda}(0)\rightarrow v^{\lambda}(0), \quad (5.47)$$

where $v^\lambda(0)$ is given in (5.21) and $v^{(\lambda\mu\nu)}(0)$ is the negative energy function which corresponds similarly to $u^{(\lambda\mu\nu)}(0)$.

Remarks. (a) The enlargment of the number of components which we have met several times is always necessary in order to have boosted super multiplets with a definite parity [S 11]. (b) The wave equations satisfied by the boosted functions are the Bargmann–Wigner equations [B 32, S 13, S 15].

5. *The Group* $U(6, 6)$

Consider the set Γ^X of 16 Hermitian matrices defined by

$$\Gamma^X: \rho^\lambda\otimes\sigma^\mu \qquad X=1,\cdots,16.$$

$$\rho^\lambda=(\boldsymbol{\rho}, 1), \qquad \sigma^\mu=(\boldsymbol{\sigma}, 1). \quad (5.48)$$

The ρ and the σ sets are each made up of the 2×2 Pauli matrices and a 2×2 unit matrix. The direct product notation in (5.48) is explained after Eq. (4.7), we write briefly Γ^X: $(\rho^\lambda\sigma^\mu)$. The Γ^X may be considered as the DG of a $U(4)$. Equivalent to (5.48) is the set

$$\Gamma^X: (1, \gamma_\mu, \sigma_{\mu\nu}, i\gamma_\mu\gamma_5, \gamma_5) \quad (5.49)$$

of 4×4 Hermitian Dirac matrices. In analogy with Eq. (4.7) we can define a 144 set of 12×12 matrices

$$T^K: \Gamma^X F^P; \qquad (T^K)_b{}^a=(\Gamma^X)_\mu{}^\lambda(F^P)_B{}^A, \quad (5.50)$$

$$P=0, 1, \cdots, 8, \qquad K=1, \cdots, 144.$$

The F^P are the DG of $U(3)$, and the T^K are those of a group $U(12)$.

It has been explained in Eq. (3.89) how one can associate the DG of pseudo-unitary groups to the DG of a unitary group. Following the prescriptions given there we define a set $\tilde{T}^K$ by

$$\tilde{T}^K=\tfrac{1}{2}(1+i)T^K+\tfrac{1}{2}(1-i)\Gamma T^K\Gamma, \quad (5.51)$$

where the 12×12 matrix Γ is given by

$$\Gamma_b{}^a=(\gamma_4)_\mu{}^\lambda\delta_B{}^A, \qquad a=(\lambda A), \qquad b=(\mu B). \quad (5.52)$$

Thus Γ is the direct product of γ_4 and the unit DG of $U(3)$. Γ is of the form (3.82) with $M=N=6$. Hence the $\tilde{T}^K$ are the DG of $U(6, 6)$.

For any p the $u^a(p)$ defined in (5.24) may be regarded as a fundamental representation of $U(6, 6)$. Indeed, the infinitesimal transformation (with real ϵ^K)

$$u^a(p)\rightarrow(1+i\epsilon^K\tilde{T}^K)_b{}^a u^b(p) \quad (5.53)$$

satisfies

$$\bar{u}_a(p)u^a(p)=\text{invariant}; \qquad \bar{u}_a=u_a{}^*\Gamma_b{}^a \quad (5.54)$$

where we have a special case of Eqs. (3.81–86). Here we have used Eq. (3.89), namely $\tilde{T}^K=\Gamma\tilde{T}^{K\dagger}\Gamma$. $\bar{u}_a(p)u^a(p)$ is "the mass term" and $U(6, 6)$ is the group which leaves the mass term invariant. But it does not leave invariant "the kinetic energy term" $\bar{u}_a(p)(\gamma p)_b{}^a u^b(p)$ as is evident from Eq. (5.53). (At this point we recognize the formal device of introducing more translations as a means to get an "invariant generalized kinetic energy term," as discussed in Sec. VA.)

The quadratic form (5.54) is also a scalar density with respect to the Poincaré group, as is seen by the application [S 11] of an inhomogeneous Lorentz transformation (a, Λ):

$$u^{\lambda A}(p)\rightarrow e^{ipa}S_\mu{}^\lambda u^{\mu A}(\Lambda^{-1}p). \quad (5.55)$$

It is important to note that the matrix

$$(S_\mu{}^\lambda\delta_B{}^A) \text{ is contained in the } (\tilde{T}^K)_b{}^a. \quad (5.56)$$

One can build a tensor calculus for $U(6, 6)$. For example $\bar{u}_a(p_1)u^b(p_2)$ is a (reducible) $U(6, 6)$ tensor *for any* p_1, p_2. In particular we meet the following finite dimensional, and therefore nonunitary, representations.

(a) $\mathfrak{M}_b{}^a(q)$ given in Eq. (5.37) is the *143* of $U(6, 6)$.

(b) $B^{abc}(p)$ given in Eq. (5.44) is also a $U(6, 6)$ tensor. [It belongs to the representation with partition (3) and dimension $D_{12}(3)=364$ of $U(6, 6)$ see Eq. (3.12). For more details on $U(6, 6)$ tensor calculus see [D 8].]

Once one has a tensor calculus one can build interactions. Consider as a simple example

$$g(q^2)\bar{u}_a(p_1)\mathfrak{M}_b{}^a(q)u^b(p_2), \qquad q=p_2-p_1 \quad (5.57)$$

where $g(q^2)$ is a form factor. For definiteness we consider the sextet to be on the mass shell. (5.57) is an $U(6, 6)$ scalar. It also has the right covariance structure because of (5.56). Equation (5.57) involves an assumption of analytic continuation. By boosting, $\mathfrak{M}(q)$ was obtained for q's for which $q^2=-\mu^2$. It is assumed that this expression may be continued off the mass shell, as in (5.57).

Crossing symmetry is not part of $U(6,6)$ but is not at variance with the latter symmetry. For example [B 28] the substitution rules $u^\mu(p_2)\rightarrow v^\mu(-p_2)$; $q=(p_2-p_1)\rightarrow q=-p_2-p_1$ applied to Eq. (5.57) bring us to a crossed channel.

The baryon–meson vertex proposed for $U(6,6)$ is $f(q^2)\bar{B}_{abc}(p_1)\mathfrak{M}_d{}^c(q)B^{abd}(p_2)$, [S 13, B 28, S 15]. This form implies a common form factor for all values of q^2.

This is a far stronger assumption than the one given in Eq. (5.10) in connection with a preliminary discussion of the P-wave vertex. It has become clear in the intervening period that the strong $f(q^2)$-assumption is by no means necessary to derive any of the vertex predictions that are considered "good." In Sec. VC we turn to a discussion of the vertex under what appear to be minimal conditions for obtaining the good results.

Remarks. (1) Concerning mass formulae for $U(6,6)$ and related groups see [R 13, C 12]. (2) Trilinear meson couplings are discussed in [B 32, G 23, H 15, G 24]. (3) Other boosted supermultiplets and their vertices are discussed in [S 16, D 9, D 10, D 12, H 18]. Note in particular that in the U(6, 6) limit many decay modes of higher resonances are forbidden [H 16]. (4) For parity assignments to $U(6,6)$ representations see [C 13]. (5) Instead of considering the vertices mentioned above to be of the effective kind, one can also try to think of them as a Lagrangian interaction. Then the free kinetic energy terms are explicit symmetry breaking terms. In this way one formulates at least in principle a dynamical theory. See the survey [D 8] and also [O 6]. In this approach the boosted *36*, *56*, etc., are considered as primitive fields. The frustrating thing is that one does not know how to calculate reliably with such a scheme. (6) *Added note.* For a relativistic version of the "recoupled" multiplets discussed in Sec. IVB, see [G 28].

C. The Baryon–Meson Vertex

We consider the covariant coupling of the baryon octet to pseudoscalar and to vector mesons in the $SU(3)$-symmetry limit and make only one additional assumption [P 4].

(A) The baryon octet is part of the totally symmetrical boosted *56*.

The vertex then has the following form

$$\bar{B}_{abc}(p_1)\mathfrak{M}_d{}^c(q)B^{abd}(p_2), \qquad q=p_2-p_1, \quad (5.58)$$

where we mean by B^{abd} the octet part of Eq. (5.44) and where

$$\mathfrak{M}_d{}^c(q)=\{f_V(q^2)i\gamma\epsilon(q)V-f_A(q^2)\gamma_5[(\gamma q)/\mu]P$$
$$-f_T(q^2)[i\sigma_{\mu\nu}q_\mu\epsilon_\nu(q)/\mu]V-if_P(q^2)\gamma_5P\}_d{}^c. \quad (5.59)$$

The form factors f_V, f_A, f_T, f_P are considered to be four independent functions of q^2. We take the baryons to be on the mass shell, so no questions arise of analytic continuation of boosted functions.

$SU(3)$ contractions reduce (5.58) to

$$4(D+F)\bar{u}\mathfrak{M}u[1+(q^2/4M^2)]$$
$$-\tfrac{1}{4}(D-F-2T)\bar{u}\{Z_2Z_1\mathfrak{M}\}u$$
$$-\tfrac{1}{2}(D-T)\bar{u}Z_2\mathfrak{M}'Z_1u, \quad (5.60)$$

where

$$Z_i=1-(i\gamma p_i/M), \qquad \mathfrak{M}'=\gamma_5C\mathfrak{M}^tC^{-1}\gamma_5.$$

{ } denotes a trace with respect to the Dirac matrices. The following definitions have also been used:

$$D\bar{u}\mathfrak{M}u=\bar{u}_B{}^A(\mathfrak{M}u+u\mathfrak{M})_A{}^B,$$
$$F\bar{u}\mathfrak{M}u=\bar{u}_B{}^A(\mathfrak{M}u-u\mathfrak{M})_A{}^B,$$
$$T\bar{u}\mathfrak{M}u=\bar{u}_B{}^A\mathfrak{M}_C{}^Cu_A{}^B. \quad (5.61)$$

Equation (5.60) yields the following results (divide by 6 and drop the $\bar{u}$, u symbols).

1. *Pseudoscalar Vertex*

$$-i[D+(2F/3)-T/3][1+(q^2/4M^2)]\gamma_5$$
$$\times[(2M/\mu)f_A(q^2)+f_P(q^2)], \quad (5.62)$$

so that

$$D/F=\tfrac{3}{2}\ \textit{for all}\ q^2. \quad (5.63)$$

Remarks. (1) From $SU(3)$ alone it follows in general that the D/F ratio may be a function of q^2. (2) The alternative boosts of the *56* with γ_5-insertions as described in Sec. VB give different values for D/F [B 28, R 9]. I learned from N. Cabibbo and M. Veltman the following elegant way to obtain these various alternative vertices from the $\bar{B}\mathfrak{M}B$ vertex, where $\mathfrak{M}$ is given by Eq. (5.38): insert in this vertex in all possible ways pairs of "γ_5 spurions" $(\gamma_5)_\mu{}^\lambda\delta_B{}^A$. This gives the various D/F ratios including the "regular" as well as the "irregular" couplings [R 9]. In Sec. VD the connection between these γ_5 insertions and the group $SL(6,C)$ will be explained. The preceding remarks make clear the necessity of assumption (A) to obtain (5.63).

2. *Vector Vertex*

$$i[\gamma_\mu F_1(q^2)+\sigma_{\mu\nu}q_\nu F_2(q^2)]\epsilon_\mu(q), \quad (5.64)$$

$$F_1(q^2)=\left\{F+T+\frac{q^2}{4M^2}\left(D+\frac{2F}{3}-\frac{T}{3}\right)\right\}f_V(q^2)$$
$$+\frac{q^2}{2M\mu}\left(D-\frac{F}{3}-\frac{4T}{3}\right)f_T(q^2), \quad (5.65)$$

$$2MF_2(q^2)=\left(D-\frac{F}{3}-\frac{4T}{3}\right)f_V(q^2)$$
$$+\frac{2M}{\mu}\left\{D+\frac{2F}{3}-\frac{T}{3}+\frac{q^2}{4M^2}(F+T)\right\}f_T(q^2). \quad (5.66)$$

The Sachs-type form factors are defined by

$$G_e = F_1 - (q^2/2M) F_2, \qquad G_m = (F_1/2M) + F_2, \quad (5.67)$$

so that

$$G_e(q^2) = (F+T)[1+(q^2/4M^2)] \times [f_V(q^2) - (q^2/2M\mu) f_T(q^2)], \quad (5.68)$$

$$2MG_m(q^2) = \left(D + \frac{2F}{3} - \frac{T}{3}\right)\left(1+\frac{q^2}{4M^2}\right) \times [f_V(q^2) + (2M/\mu) f_T(q^2).] \quad (5.69)$$

Note that the $D/F/T$ ratios factor out for G_e, G_m but not for the Dirac–Pauli form factors F_1, F_2.

(1) Equation (5.17) follows from (5.68, 69) and the $SU(3)$ assumption that the electromagnetic couplings are proportional to the strong $(\rho^0 + \omega^0/\sqrt{3})$ coupling.

(2) The relation [B 34]

$$G_e^P(q^2) = \text{const.}\ G_m^P(q^2) \quad (5.70)$$

which appears to be in good agreement with experiment [D 6] does not follow from the present considerations. It demands an additional constraint between f_V and f_T [B 35, P 4]. In this connection it is worth while to ask "how relativistic" the Eqs. (5.16, 70) really are and the following remark is perhaps instructive [F 11]. Consider the proton as a rigid sphere which bounces off a brick wall with a momentum transfer of say 500 MeV/c. We have neglected the Lorentz contraction, but $(1-v^2/c^2)^{\frac{1}{2}} \simeq 0.97$ so we make only a 3% error. Thus the "experimental" relations (5.16, 70) may be approximately described as a property of the *static* charge and magnetic moment distributions. In this connection it is of interest that these relations have been derived from an essentially static quark model for the region $q^2/M^2 \lesssim 1$ [B 27]. From this point of view it is of considerable interest to know how good the relations (5.17, 70) are in the truly relativistic region.

(3) The $U(6, 6)$ limit corresponds to

$$f_V(q^2) = f_A(q^2) = f_P(q^2) = f_T(q^2). \quad (5.71)$$

This does give a connection between G_e and G_m [see also the preceding remark (1)], namely [S 13, S 15] $G_e^P/G_m^P = 1 + q^2/2M\mu$. This relation can be modified, however, by arguments about how to continue analytically to $q^2 = 0$ [F 12].

(4) Returning to the strong vertex, we make one additional assumption.

(B) The form factors in (5.59) satisfy the *static* $SU(6)$ limit conditions [B 10]

$$f_V(-\mu^2) = f_A(-\mu^2); \qquad f_P(-\mu^2) = f_T(-\mu^2), \quad (5.72)$$

one of which was already encountered, Eq. (5.10). Put [B 10]

$$f_V(-\mu^2)/f_T(-\mu^2) = \xi, \quad (5.73)$$

where ξ is a free parameter. From Eqs. (5.62–66) one finds the following results for g_A, g_V defined by the P-wave vertex part $g_V P^\dagger P \rho_0^0 + g_A P^\dagger \boldsymbol{\sigma} P \nabla \pi^0/\mu$ (ρ_0^0 is the longitudinal component of ρ^0):

$$g_A = \tfrac{5}{3}[f_A(-\mu^2) + (\mu/2M) f_P(-\mu^2)], \quad (5.74)$$

$$g_V = f_V(-\mu^2) + (\mu/2M) f_T(-\mu^2). \quad (5.75)$$

Thus

$$g_A/g_V = \tfrac{5}{3} \text{ independent of } \xi. \quad (5.76)$$

This concludes the discussion of the octet vertex under minimal conditions. For more details see [S 13, S 15]. The closely related vertex for semi-leptonic processes is discussed in [R 15, A 13, K 15].

D. Groups Related to $U(6, 6)$

Equations (5.48–52) imply that the group $U(2, 2)$ has the DG

$$\tilde{\Gamma}^X = \tfrac{1}{2}(1+i)\Gamma^X + \tfrac{1}{2}(1-i)\gamma_4 \Gamma^X \gamma_4. \quad (5.77)$$

A subgroup of $U(4)$ has as DG a subset of the Γ^X, call it the set $\Gamma^{(X)}$. The corresponding subgroup of $U(2, 2)$ has the DG

$$\tilde{\Gamma}^{(X)} = \tfrac{1}{2}(1+i)\Gamma^{(X)} + \tfrac{1}{2}(1-i)\gamma_4 \Gamma^{(X)} \gamma_4. \quad (5.78)$$

We are interested in such sub groups of $U(6, 6)$ which have as DG those matrices obtained by substituting $\Gamma^{(X)}$ for Γ^X in Eqs. (5.50, 51):

$$\tilde{T}^{(K)}: \tilde{\Gamma}^{(X)} F^P. \quad (5.79)$$

(a)

$$U(6, 6) \supset GL(6, C). \quad (5.80)$$

The corresponding $\Gamma^{(X)}$ and $\tilde{\Gamma}^{(X)}$ are

$$\Gamma^{(X)}: 1, \gamma_5, \sigma_{\mu\nu}; \ \tilde{\Gamma}^{(X)}: 1, i\gamma_5, \boldsymbol{\sigma}, i\gamma_5\boldsymbol{\sigma}. \quad (5.81)$$

The corresponding group (5.79) is homomorphic with $GL(6, C)$.

This group has two bilinear invariant forms, namely,

$$\bar{u}_a u^a; \ i\bar{u}_{\lambda A}(\gamma_5)_\mu{}^\lambda u^{\mu A}. \quad (5.82)$$

Corresponding to this doubling there are more higher rank tensors in this group than in $U(6, 6)$ (for given number of indices). Thus of the set

$$\mathfrak{M}, \ \gamma_5 \mathfrak{M} \gamma_5, \quad (5.83)$$

with $\mathfrak{M}$ defined by Eqs. (5.37–40) only one is a $U(6, 6)$ tensor while both are $GL(6, C)$ tensors. The "γ_5 insertions" discussed in Eq. (5.41) and after Eq. (5.47) just correspond to the breakdown Eq. (5.80). For more details about the tensors of $SL(6, C)$ see [H 17].

(a′)

$$GL(6, C) \to GL(6, C) \otimes GL(6, C) \quad (5.84)$$

by means of the following procedure. Use Eq. (5.81) and define

$$\tilde{\Gamma}_{\pm}^{(X)}=\tfrac{1}{2}(1\pm\gamma_5)\tilde{\Gamma}^{(X)}. \quad (5.85)$$

(This is a "complexification" because the γ_5 appears without an i.) $\tilde{\Gamma}_{\pm}^{(X)}F^P$ generate $GL(6, C)\otimes GL(6, C)$. (The subgroup $SL(2, C)\otimes SL(2, C)$ is associated with the proper complex Lorentz group [S 11].)

(b)

$$GL(6, C)\otimes GL(6, C)\supset U(6)\otimes U(6), \text{ chiral} \quad (5.86)$$

is achieved by contracting $\tilde{\Gamma}_{\pm}^{(X)}$ such that $\tilde{\Gamma}^{(X)}$ is restricted to $(1, \boldsymbol{\sigma})$. For details on this group see [F 4, D 5, B 16].

(c)

$$U(6, 6)\supset U(6)\otimes U(6); \text{ nonchiral.} \quad (5.87)$$

In this case

$$\Gamma^{(X)}: 1, \gamma_4, \boldsymbol{\sigma}=\tilde{\Gamma}^{(X)}. \quad (5.88)$$

Consider a one quark state with its $U(6, 6)$ breaking kinetic energy operator. In its rest frame $\gamma p=i\gamma_4 p_0$. In this situation only that subgroup of $U(6, 6)$ survives generated by those DG of $U(6, 6)$ which commute with γ_4. This leads to the set (5.88) [O 6, D 7, H 14].

The determination of the particle content of $U(6, 6)$ multiplets by means of this compact subgroup $U(6)\otimes U(6)$ proceeds as follows. The factors of the $SU(2)\otimes SU(2)$ subgroup of $U(6)\otimes U(6)$ with respective DG $(1\pm\gamma_4)\boldsymbol{\sigma}/2$ correspond to quark spin ($\gamma_4=1$) and antiquark spin ($\gamma_4=-1$), respectively (in the rest frame). Correspondingly, the $U(6)$ factors refer to particles and anti particles, respectively. The sextet is *(6, 1)*, the anti sextet is *(1, 6*)*. See also the note after Eq. (5.21). The meson *36* corresponds to *(6, 6*)* and the *56* to *(56, 1)*, the anti-*56* to *(1, 56*)*, etc.

(d) $SU(6)_W$. Consider two Lorentz-orthogonal unit vectors ϵ_μ^1, ϵ_μ^2: $(\epsilon_\mu^i\epsilon_\mu^j)=(\epsilon^i\epsilon^j)=\delta^{ij}$. Define

$$W_1=\tfrac{1}{2}i(\gamma\epsilon^1)\gamma_5, \qquad W_2=\tfrac{1}{2}i(\gamma\epsilon^2)\gamma_5, \qquad W_3=\tfrac{1}{2}i(\gamma\epsilon^1)(\gamma\epsilon^2). \quad (5.89)$$

W satisfies the same algebra as does $\mathbf{S}=\boldsymbol{\sigma}/2$. Choose the ϵ_μ^i to be orthogonal to a four-vector p_μ: $(\epsilon^i p)=0$. Then

$$[W, (\gamma p)]=0 \quad (5.90)$$

so W commutes with "the kinetic energy." Choose $p_\mu=(0, 0, p, ip_0)$; $\epsilon_\mu^1=(1, 0, 0, 0)$: $\epsilon_\mu^2=(0, 1, 0, 0)$, then W reduces to the expression (5.15).

One also arrives at (5.15) as follows [O 6, D 7, H 14]. Consider the kinetic energy operator (γp). For p in the 3-direction this becomes $\gamma_3 p_3+i\gamma_4 p_0$. Let p represent any of the four-vectors which may occur in a collinear configuration. The breakdown of $U(6, 6)$ to a lower symmetry then amounts to finding the subset of the Γ^X, Eq. (5.49) which commute with γ_3, γ_4. This is the W-spin set (5.15). The consequences of $SU(6)_W$-symmetry are discussed further in Sec. VF.

(e) $U(3)\otimes U(3)$. Consider a coplanar configuration with "3" the direction normal to the plane. The corresponding surviving set $\Gamma^{(X)}$ is

$$\Gamma^{(X)}=\tilde{\Gamma}^{(X)}: 1, \gamma_4\sigma_3, \quad (5.91)$$

and the group in question has DG $(1\pm\gamma_4\sigma_3)F^P/2$.

$U(6)$ representations are reducible under this group, for example [D 7]: *56*=*(3, 6)*+*(6, 3)*+*(1, 10)*+*(10, 1)*. The decuplets are the $S_3=\pm\frac{3}{2}$ components of the *56*. The remainder corresponds to the $S_3=\pm\frac{1}{2}$ parts.

Equation (5.91) implies spin conservation normal to the plane. From this symmetry alone nucleon–nucleon scattering predictions have been derived [D 7] which were found and are further discussed in [K 9, A 6].

E. Unitarity; Other Implications

For any of the groups discussed in Sec. VD one can by tensor contraction construct scalars which (whenever applicable) can represent effective matrix elements. This is true not only for vertices but also for scattering, annihilation processes etc. $SU(3)$-violating effects must play a quite important role for many of these processes [A 1, H 2] and a systematic method for dealing with such violations is not available at present. This is to be compounded with violations of the higher symmetries under discussion.

1. The Unitarity Problem

Soon after the examination of the consequences of $U(6, 6)$ and related groups had begun, it was realized that these symmetries are not compatible in general with the unitarity conditions [B 36, B 37, G 26]. An example will clarify the point.

Consider an $SU(6, 6)$ invariant scattering amplitude

$$f(s, t)\sum_K \bar{u}_a(p_4)(\tilde{T}^K)_b{}^a u^b(p_1)\cdot\bar{v}_c(p_2)(\tilde{T}^K)_d{}^c v^d(p_3) \quad (5.92)$$

for the scattering $S(p_1)+\bar{S}(p_2)\rightarrow\bar{S}(p_3)+S(p_4)$, ($S$=sextet, $\bar{S}$=antisextet). The summation over K ranges over the 143 DG of $SU(6, 6)$, see Eq. (5.51). Equation (5.92) is $SU(6, 6)$ invariant for any set of values p_1-p_4. Strict $SU(6, 6)$ implies a common form factor $f(s, t)$ for all the 143 terms in the sum. But this is in conflict with unitarity in the elastic region, as a simple calculation shows [B 36]. Briefly stated, the closure in the unitarity sum is effected by the introduction of projection operators which behave like the kinetic energy terms often referred to, and which almost always break the invariance. The unitarity conflict exists independently of any details regarding a possible underlying local field theory. A few technical comments follow.

(a) Stripped of its inessential $SU(3)$ details, the mentioned example amounts to saying that a four-fermion interaction cannot be an equal mixture of P, V, A, and T for all s, t without violating unitarity.

(b) It was noted in [B 36] that the unitarity conflict is not confined to $U(6, 6)$ but holds true for more general relativistic completions. This aspect is discussed in more detail in [A 14, B 39], where it is shown that the same problem arises also for $SL(6, C)$ and also if one allows for the introduction of all possible (γp) spurions.

(c) The unitarity condition is satisfied "asymptotically" as we go to zero velocities [B 36]. The unitarity violations are of the characteristic order $(v/c)^2$. In the nonrelativistic limit one can implement one-particle unitarity for meson–baryon scattering. This leads to $SU(6)$-coupling constant sum rules [B 36] which are closely related to the J–T relations. Sum rules of this kind for broken $SU(6)$ are considered in [C 14].

(d) For some cases unitarity can be saved at the price of crossing symmetry [B 38]. *Added note.* See also [B 49].

(e) As far as I understand it, it is a matter of language whether or not the collinear group $SU(6)_W$ is compatible with unitarity. The group makes no claims concerning nonforward directions which are coupled to the forward direction in the unitarity sum. One may say that this coupling is not comprehended by $SU(6)_W$.

(f) A formal unitarity condition can be satisfied if extra translations are introduced [H 19].

The unitarity conflict would be a violation of our physical principles if an approximate kinematical symmetry were involved. From the present phenomenological point of view it means a most severe restriction: the two sides of a unitarity relation cannot both be right. This circumstance together with the $SU(3)$-breaking problem mentioned earlier makes it no great surprise that not many encouraging phenomenological clues have emerged in the subsequent investigation of specific processes.

2. *Meson–Baryon Scattering*

Static $SU(6)$-type studies have been made along the lines of Sec. IVG [C 2, B 40, B 41]. However, no S-wave projection has been studied separately as was done for the nucleon–nucleon case. It is therefore not easy to interpret the results. Relativistic calculations [R 16, C 15, B 37, M 7] give several bad predictions in the symmetry limit such as no Ξ polarization in $K^-p\rightarrow\Xi^-K^+$.

3. *Baryon–Baryon Scattering* [K 9, A 6, S 17, F 13]

No suggestive regularities are found. Some disagreeable features remain also in the presence of spurions. The most detailed numerical comparisons are in [K 9]. For $B\bar{B}$-scattering see [A 6, B 42].

4. *Nucleon–Antinucleon Annihilation at Rest*

The following discussion is for annihilation purely from the S state. First consider the case of $U(6, 6)$ symmetry. Then the nucleon is in $B^{abc}(0)$, Eq. (5.44) and the antinucleon in $C^{abc}(0)$, Eq. (5.46).

(a) The following property of the once contracted product is important

$$\bar{C}_{abc}(0)B^{ade}(0)=0. \qquad (5.93)$$

This is a direct consequence of $\bar{v}_a(0)u^a(0)=0$, Eqs. (5.20, 21). This type of orthogonality can be expressed in terms of a selection rule [H 23]. Equation (5.93) applied to the $\bar{C}B$-current gives [H 21]

$$\bar{p}+p\rightarrow e^+e^-\text{: forbidden.} \qquad (5.94)$$

(b) *2-meson annihilation.* Equation (5.93) gives

$$\text{nucleon}+\text{antinucleon}\rightarrow 2\text{ mesons: forbidden,} \qquad (5.95)$$

not only for PP channels (P=pseudoscalar meson) [H 20, H 21] but also for PV combinations [D 13, C 16]. There are some models which involve some PP suppression, but in any case (5.95) is not good for the $\rho\pi$ mode which is appreciable. Its rate is $\sim$10 times the 2π rate [B 43].

Several investigations have been made about 2-meson annihilation using lower symmetry. First, a single spurion of the type $(\boldsymbol{\sigma}\mathbf{q})$ was introduced but this leads to a new difficulty. It predicts (R=rate)

$$\frac{R(K^+K^-)}{R(K_1{}^0K_2{}^0)}=\frac{16}{1} \qquad (5.96)$$

see [H 20, D 14, A 15]. (The result of [K 16] does not agree with this answer.) This is to be compared with an experimental ratio $(1.1\pm0.1)/(0.61\pm0.09)$, [B 43]. Other attempts to treat 2-meson annihilation proceed by the introduction of a pair of (γq) or of γ_5 spurions [H 22, L 11]. This leads to better results for the ratio in Eq. (5.96). Also the result $R(\rho\pi)/R(\pi\pi)\cong6$ was obtained [M 8]. It appears that so far no compelling conclusion can be drawn from 2-meson annihilation.

(c) *3-meson annihilation.* Returning to $U(6, 6)$, the identity Eq. (5.93) implies a unique coupling in this case. A main consequence is that annihilations involving a ϕ or strange mesons are forbidden [D 13, C 16]. In $\bar{p}p$ annihilation, ϕ production is rare and the $KK\pi$ to 3π ratio is small [B 44, B 45]. (Note that $KK\pi$ means a true three-body channel, not K^*K.) The predictions also hold for $\bar{n}p$ annihilation for which no detailed data are yet available.

More detailed predictions in the 3-meson case include the following [C 16]. [Note that these allow for the inclusion of the parameter ξ defined in Eq. (5.73).]

(α) The $3P$ annihilations should be in the ^1S state. This is in agreement with an experiment [B 45] on nonresonant 3π annihilation.

(β) The ratio $R(\pi^+\pi^-\pi^0)/R(\pi^+\pi^-\eta^0)$ should be $\simeq$3. The experimental ratio is $(3.3\pm0.5)/(1.2\pm0.3)$ [F 14].

(γ) The PPV annihilation should be in the ^3S state. This is in agreement with an experiment on $\omega\pi\pi$ annihilation [B 46].

(δ) The ratio $R(\pi^+\pi^-\rho^0)/R(\pi^+\pi^-\omega)$ should be $\simeq 1$. This is to be compared with an experimental ratio $(5.8\pm1.0)/(3.9\pm0.5)$ [F 14].

The $U(6, 6)$ predictions therefore reproduce some of the striking experimental regularities about preferred angular momentum states. Moreover the predicted $(3\pi/2\pi\eta)$ and $(2\pi\rho/2\pi\omega)$ ratios are not in disagreement with the present experimental information. All these results do *not* depend on a prescribed relative strength of pseudoscalar meson/vector meson coupling. They do depend on the totally symmetric boosted structure for baryons and antibaryons. It will be very interesting to see how the $\bar{p}n$ predictions [C 16] will fare.

(d) *4-meson annihilation.* $U(6, 6)$ forbids $\bar{p}p\rightarrow K^+K^-\pi^+\pi^-$ [H 23]; for more details on these channels see [L 12].

Added note. The reactions $\bar{P}P\rightarrow\bar{B}B$ are discussed in [B 50].

F. W-Spin

We now continue the discussion of $SU(6)_W$ started in Sec. VC. It was noted there that **W**, Eq. (5.15), acts in the same way on a sextet (but not on an antisextet) state at rest as does **S**. The same is therefore true for a multi-sextet state. It follows that the baryon *56*, looked upon as a representation of static $SU(6)$ is likewise a representation *56* of $SU(6)_W$. In addition, W commutes with the DG $\gamma_5\sigma_3$ for Lorentz transformations in the 3-direction of motion. Thus also for $p_3\neq0$ we may assign the baryon to the same $SU(6)_W$ representation *56* as for $p_3=0$; and likewise for other baryon representations. This is sometimes called the boost-free property (in the direction of motion) of $SU(6)_W$ representations [L 9].

For mesons we have a *35* (not *36*) of $SU(6)_W$ which is $\neq$*35* of static $SU(6)$ (even for $p_3=0$), because mesons are like a sextet–antisextet system. To see what happens, let $V_{1,0,-1}$ and P be a degenerate quartet of vector mesons (with helicities 1, 0, -1) and a pseudoscalar meson which are (for illustration purposes only) written as fermion pairs as follows.

$$V_1=\bar{f}_\uparrow f_\uparrow,\qquad V_0=(1/\sqrt{2})(\bar{f}_\uparrow f_\downarrow-\bar{f}_\downarrow f_\uparrow),\qquad V_{-1}=-\bar{f}_\downarrow f_\downarrow,$$
$$P=(1/\sqrt{2})(\bar{f}_\uparrow f_\downarrow+\bar{f}_\downarrow f_\uparrow).$$

f is the fermion, $\bar{f}$ its antiparticle, the indices denote S_3 values. Let S_- be the spin lowering operator: $S_-V_1=V_0$, $S_-V_0=V_{-1}$, $S_-V_{-1}=0$. There is a corresponding W-spin operator W_- which acts the same on f but with different phases on $\bar{f}$ [for which $\gamma_4=-1$ in Eq. (5.15)]. We now have $W_-V_1=P$, $W_-P=V_{-1}$, $W_-V_{-1}=0$. Going from a single f to the $SU(3)$ triplets we therefore get the following multiplets for $SU(6)_W$ [C 17].

$$35=(8, 3)+(8, 1)+(1, 3);\ (W),$$

where (a) (8, 3) is an $SU(3)$ octet with W-spin 1. It contains the vector octet with $S_3=\pm1$ and the pseudoscalar octet. (b) (8, 1): W-spin 0. This is the vector octet with $S_3=0$. (c) (1, 3): unitary singlet, W-spin 1. Its members are ϕ_1^0, X^0, ϕ_{-1}^0. Note how the inclusion of X^0 in a $U(6, 6)$ multiplet remains necessary in the $SU(6)_W$ subgroup.

In addition one has the representation $1=(1, 1)$ to which the ϕ_0^0 is assigned.

Furthermore a photon $\gamma_{\pm1}$ with helicity ±1 may be considered as a W-spin 1 (and U-spin zero) member of an $(8, 3)_W$, while a virtual photon γ_0 is like a member of an $(8, 1)_W$ [C 17]. Such assignments enable the analysis of photo- and electroproduction by means of W-spin methods.

The following are some applications of $SU(6)_W$.

(a) Recalling that $W=S$, $W_3=S_3$ for baryons, one easily verifies that $\rho\rightarrow2\pi$; $N^*\rightarrow N\pi$ are W-spin allowed decays.

(b) MB→MB, elastic and inelastic. From Eq. (3.25) applied to 35_W and 56_W, it follows that there are four independent amplitudes. This leads to many relations [C 17] *always for forward or backward configurations.*

(α) The J–T relation. It should be noted that this relation had been shown to be independent of any particular form of relativistic completion [B 36, C 18].

(β) A number of reactions are forbidden, like $K^-P\rightarrow K^-N^{*+}$, $K^+P\rightarrow K^0N^{*++}$ etc.

(γ) Relations for $Y=2$ systems:

$$\begin{aligned}d\sigma(K^+P\rightarrow K^{*+}P)&=2/3d\sigma(K^+P\rightarrow K^0N^{*++})\\&=16d\sigma(K^0P\rightarrow K^{0*}P)\\&=16/3d\sigma(K^0P\rightarrow K^+N).\end{aligned}$$

(δ) Many relations for $Y=1$ and 0 amplitudes. Note in particular:

$$d\sigma(\pi^-P\rightarrow\pi^-N^{*+}):d\sigma(\pi^-P=\pi^0N^{*0}):d\sigma(\pi^-P\rightarrow\pi^+N^{*-})=2:9:24.$$

See [O 7]. Several MB-scattering predictions are in disagreement with experiment [C 19, J 4].

(ϵ) For photoproduction see [C 17] and also [D 15]. *Added note.* See further [J 5, K 18].

(c) For the application of non-leptonic decays see [H 8].

(d) For the vertex one obtains the results stated in [B 31].

G. Conclusion

From the phenomenological study of approximate dynamical symmetries which contain static $SU(6)$ we have so far learned the following. Regarding the main motivation, the inclusion of the baryon–meson vertex, the treatment of the vertex to order v/c is possible with the only additional nonstatic assumption that the boosted *56* remains totally symmetric, Sec. VD. Phenomenologically I regard this vertex to be neither better nor worse understood to v/c than the static vertices discussed in Sec. IV. From the order $(v/c)^2$ on, the unitarity troubles set in which make illusory as a matter of principle any systematic phenomenological comparisons.

Those that have been made give many bad results along with a few interesting ones, such as the J–T relations which emerge as the result of whatever relativistic completion one chooses. Annihilation processes into two mesons have not led to very constructive conclusions so far. There appear to be a number of regularities for $\bar{P}P\rightarrow 3$ mesons (in which the number of triplets and of anti-triplets are separately conserved!). Whether these will be of further interest will perhaps be clearer after the corresponding $\bar{P}N$ experiments will be available. Regarding the form factor relations like Eq. (5.17), one may question the extent to which they are truly relativistic for $q^2<(1\ \text{BeV}/c)^2$, Sec. VC. A number of new and interesting methods have emerged, notably the W spin. No compelling directions have emerged so far from asking the question stated in Eq. (4.77).

VI. CURRENT ALGEBRAS

This section was written in collaboration with M. A. B. Bég.

A. Introduction

This section deals with some of the applications of current commutator algebras [G 27] to dynamical problems involving hadrons. These applications have so far gone in two rather distinct directions.

(1) Current commutation relations supplemented by external information (such as the experimental values of scattering cross sections) have been used for the derivation of strict sum rules. This direction is very promising. In particular an approximately quantitative evaluation of the absolute value of $|C_A/C_V|$ has been achieved in this way [A 16, W 7].

(II) It has been attempted to derive $SU(6)$ and related results as internal consistency properties of some of the approximate solutions of sets of current commutation relations. This approach is aimed at the master problem (B), Sec. I (see also Sec. IVJ). Its status is presently unclear. In particular it is not yet understood in what sense the solutions are approximate.

This survey would not be complete without a discussion of current algebras, but for two reasons this section must be very brief. First, because it is restricted to such applications only which bear on the interpretation of the symmetries discussed in this paper. Secondly, because of the many still open questions which are met in this connection.

B. The Current Algebra $U(12)$

In Eq. (5.50) quantities T^K were defined ($K=1,\cdots,144$) which are the DG of $U(12)$. In the language of Eq. (3.55) the generators T^K satisfy

$$[T^K, T^L]_-=if^{KLM}T^M, \tag{6.1}$$

where f^{KLM} are the structure constants of $U(12)$. Acting on the defining representation, Eq. (6.1) is a matrix relation between the $(T^K)_b{}^a$, a, $b=1,\cdots,12$. The $(T^K)_b{}^a$ are Hermitian matrices.

Let $\psi^a(\mathbf{x},t)=\psi^{\lambda A}(\mathbf{x},t)$, ($\lambda=1$–$4$, $A=1$–3) denote the bare field operator for an $SU(3)$ triplet of spin-$\frac{1}{2}$ particles. $\psi_a{}^\dagger(\mathbf{x},t)$ is its Hermitian conjugate. They satisfy the equal time anticommutation relations

$$[\psi_a{}^\dagger(\mathbf{x},t),\psi^b(\mathbf{y},t)]_+=\delta_a{}^b\delta(\mathbf{x}-\mathbf{y}). \tag{6.2}$$

Define 144 Hermitian current densities

$$J^K(\mathbf{x},t)=\psi_a{}^\dagger(\mathbf{x},t)(T^K)_b{}^a\psi^b(\mathbf{x},t). \tag{6.3}$$

From an uncritical application of Eq. (6.2) and with the help of Eq. (6.1) one obtains

$$[J^K(\mathbf{x},t), J^L(\mathbf{y},t)]_-=if^{KLM}J^M(\mathbf{x},t)\delta(\mathbf{x}-\mathbf{y}). \tag{6.4}$$

Define

$$Q^K=\int d^3x J^K(\mathbf{x},t). \tag{6.5}$$

Then

$$[J^K(\mathbf{x},t), Q^L]=if^{KLM}J^M(\mathbf{x},t). \tag{6.6}$$

$$[Q^K, Q^L]=if^{KLM}Q^M. \tag{6.7}$$

However, it is well known [S 18] that the above "derivation" of Eq. (6.4) is open to serious objection. More strongly singular terms may appear in the right-hand side of Eq. (6.4) because of the singular nature of bilinear products of $\psi^\dagger$ and ψ taken at the same space–time point. Concrete counter examples to special cases of Eq. (6.4) have been given [S 18]. At this stage the study of current commutators branches out in two directions.

(a) For such cases where the converse has not been proved, one introduces relations of the kind (6.4) as an explicit new postulate [G 27]. Furthermore one abstracts from the fact that the relations (6.4) are initially suggested by a representative quark model and makes the additional assumption [G 27] that the relations (6.4) hold (qualified as above) for the set of full (rather than quark) current densities which we shall continue to call $J^K(\mathbf{x},t)$. Which particular current density one has in mind is specified by the nature of the corresponding representative T^K.

Example. T^K: $\tau^3/2$, ($\tau^i=$isospin). J^K is the isocharge density. Call I^3 the corresponding Q^K, Eq. (6.5). From T^K: $\gamma_5\tau^i/2$ one obtains the isocharge raising and lowering axial densities $\gamma_5\tau^\pm=\gamma_5(\tau^1\pm i\tau^2)/2$. Denote the corresponding Q^K by $Q_a{}^\pm$. Then

$$[Q_a{}^+, Q_a{}^-]=2I^3. \tag{6.8}$$

This is the relation which has been applied successfully in the calculation of $|C_A/C_V|$, [A 16, W 7].

(b) One asks for specific dynamical conditions under which at least some of the Eq. (6.4) can be guaranteed to have no extra singular terms on the right-hand side. See for example [A 17] where in particular Eq. (6.8) is justified for some models. Also, nonforward neutrino reactions have been studied with the inclusion of such extra terms in those current commutators for which their existence has not been disproved [A 18].

Remarks. (α) Because integrations as in Eqs. (6.6, 7) may average out some extra singularities [such as for example a constant times a $\delta'(\mathbf{x}-\mathbf{y})$-function], Eq. (6.6) may be safer than Eq. (6.4) and Eq. (6.7) may be still safer.

(β) The circumstance that Eq. (6.2) does not depend on mass (because of its *equal time* character) has led to the view [G 27] that the validity of the generalized Eq. (6.4) may persist in the presence of symmetry breaking.

(γ) The density corresponding to I^3 is the 4th component of the isovector 4-vector. Likewise $Q_a^{\pm}$ corresponds to the 4th component of the isovector axial vector. As the Γ^X run through the set defined in Eq. (5.49), we meet *all* the invariant S, V, T, A, P densities with factors i automatically included to guarantee Hermiticity. For example, $\psi^\dagger \boldsymbol{\sigma} F^P \psi = -i\bar{\psi}\gamma_5\boldsymbol{\gamma} F^P\psi$, the space part of the axial vector unitary nonet density. The group $U(12)$ of Eq. (6.4) [D 7] is therefore the group of the *Hermitian* current densities. See also subsection D) below.

(δ) Subgroups of $U(12)$ can be recognized by the same method as was used in Sec. VD for $U(6, 6)$, see for example [F 4]. In fact, $U(6, 6)$ and $U(12)$ have such subgroups in common which are less than or equal to the maximal compact subgroups $U(6)\otimes U(6)$ of $U(6, 6)$. Thus the recognition of the role of the chiral or nonchiral $U(6)\otimes U(6)$, of $SU(6)_W$ and of $U(3)\otimes U(3)$ is no unique property of the current algebras per se.

(ϵ) The group $U(12)$ has also been recognized [O 8] in a different physical context as the maximal compact unitary symmetry of quark–antiquark systems. This symmetry group $U(12)$ has quite distinct implications such as the inclusion within a given supermultiplet of states with different baryon numbers.

C. Applications

Take Eq. (6.8) as an example. From this equation:

$$\sum\{\langle P_q \mid Q_a^+ \mid \nu\rangle\cdot\langle \nu \mid Q_a^- \mid P_{q'}\rangle - \text{“}Q_a^+ \leftrightarrow Q_a^-\text{”}\} = \langle P_q \mid I^3 \mid P_{q'}\rangle, \quad (6.9)$$

P is a proton state with 4-momentum q_μ (or q_μ'), ν is a complete set of intermediate states. The $\mid C_A/C_V \mid$ calculation proceeds by separating out the neutron from ν and relating the full remaining ν-set to off mass-shell nucleon cross sections via *PCAC*. For the present purpose the following is relevant.

(a) Sum rules are obtained for each value of $q_0(=q_0')$. In the $\mid C_A/C_V \mid$-calculation the limit $q_0 \to \infty$ is sometimes taken, but this is not essential [W 7].

(b) Treating the ν-set in full gives $\mid C_A/C_V \mid \simeq 1.2$ [A 16, W 7].

(c) Truncating the scattering cross section integral so as to take only the 33-region into account gives $\mid C_A/C_V \mid \simeq 1.44$, [A 16]. The mentioned results are for a calculation of the type (I) mentioned above.

In order to go to type (II), consider the set of Eqs. (6.7), or a subset thereof. As in (6.9), put these relations between baryon octet states (for example) and consider a further approximation of the ν set, in which one takes out not only the possible baryon octet states, but also decuplet states of zero width, degenerate (or approximately degenerate) with the octet, in the spirit of approximate $SU(6)$ symmetry.

The input of scattering information is now lost. Instead, one asks [L 13, D 7] if there are self-consistent solutions of such sets of equations (not identities) obtained from (6.7) by truncating the ν-set to *56* states only (“one-particle saturation” of intermediate states).

The subset chosen for this purpose is the group of currents corresponding to the chiral group $U(6)\otimes U(6)$, [L 13, D 7] but one may also consider smaller sets, see further [B 47, O 9, R 18]. In this way self-consistent solutions $\mid C_A/C_V \mid = \frac{5}{3}$, $\mid D/F \mid = \frac{3}{2}$ were obtained. These include the $SU(6)$ answers $g_A/g_V = \frac{5}{3}$, $D/F = \frac{3}{2}$. No explicit N^*-N mass degeneracy need be used. (This was also true for the initial derivation, Sec. IVF).

This result raises the general question whether all $SU(6)$ results can be obtained from sum rules like (6.4) and related ones by truncating the intermediate ν set. This seems possible. For example, the characteristic *56* results for the magnetic moments can be obtained by taking matrix elements of magnetic moment operators between *56* states only. In fact, such matrix elements are just the Clebsch–Gordan coefficients [F 15] of the group $U(12)$, or of a specific subgroup such as $SU(6)_W$ which is of interest for vertex calculations.

The question arises if we have learned anything really new from this one particle saturation. In this connection the following theorem [C 20] is important. If one assumes [D 7] that matrix elements are strictly saturated by states which belong to the same $SU(6)_W$ representation then $SU(6)_W$ would be a group of the Hamiltonian. Thus the saturation assumption is equivalent to earlier results and difficulties, but in another language, and the master problem remains open. See also [S 21]. Also, the notion of “approximate saturation” leads to complications [C 20]. Moreover, the uniform application of one-particle saturation for all q_μ appears to lead to difficulties [K 17].

Alternatively one may proceed without the explicit assumption of saturation, but hope that higher states in the ν set cancel in the *commutator* so as to produce the $SU(6)$ results wherever desired [L 13]. Now the burden of proof changes to the demonstration that this cancelation indeed takes place.

There are many other interesting predictions from commutator algebras, combined with the restriction to one-particle ν states only, such as for the charge radius of the proton [L 13, D 7]. But also new questions are raised. Under what dynamical conditions is the algebra consistent (question of extra singularities)? What is the extent to which the algebra exhausts dynamical information? Can the approximate sense of one particle state calculations be justified by a degree of cancelation of higher states in the commutator?

D. Appendix. Symmetry Group $U(6, 6)$, Current Group $U(12)$ and $GL(12, C)$

The following mathematical remark may perhaps help to make clear how such distinct 12-dimensional unitary groups as the noncompact $U(6, 6)$ and the compact $U(12)$ have both put in an appearance in the discussion of strongly interacting particles.

Consider the four 2×2 matrices $\sigma^\mu = (\boldsymbol{\sigma}, 1)$ with matrix elements $\sigma_{mn}{}^\mu$. One sees that

$$\sum_\mu \sigma_{mn}{}^\mu \sigma_{rs}{}^\mu = 2\delta_{ms}\delta_{nr}. \tag{6.10}$$

With the help of this relation and of Eq. (5.48) one obtains a similar relation for the 16 Dirac matrices. It is in fact easy to show the following [for example with the help of Eqs. (3.56, 57].

Let the Γ^X be the Hermitian DG of a unitary group $U(N)$, represented by matrices $(\Gamma^X)_{mn}$; $X=1, \cdots, N^2$; $m, n=1, \cdots, N$. Norm the matrices so that $\mathrm{Tr}(\Gamma^X)^2$ is independent of X. Then

$$\sum_X \Gamma_{mn}{}^X \Gamma_{rs}{}^X = \text{const.}\, \delta_{ms}\delta_{rn}, \tag{6.11}$$

where the constant depends on the scale set by $\mathrm{Tr}(\Gamma^X)^2$.

Equation (6.11) is invariant under the similarity transformations

$$\Gamma^X \to S^{-1}\Gamma^X S, \tag{6.12}$$

where S is any nonsingular complex valued matrix. Thus the group of Eq. (6.11) is $GL(N, C)$. Of course the transformed Γ^X remain Hermitian only for those S which are in $U(N)$.

For $N=12$, consider the following special choices for S.

(α) S satisfies the $U(6, 6)$ condition

$$\bar{S}S=1, \qquad \bar{S}=\gamma_4 S^\dagger \gamma_4. \tag{6.13}$$

Multiply Eq. (6.11) by $\bar{\psi}_m(x)\psi_n(y)\bar{\psi}_r(u)\psi_s(v)$, where ψ_n is a relativistic quark field operator and $\bar{\psi}_n = \psi_n{}^\dagger \gamma_4$. One gets

$$\sum_X \bar{\psi}(x)\Gamma^X\psi(y)\cdot\bar{\psi}(u)\Gamma^X\psi(v) = \bar{\psi}(x)[\psi^t(y)\bar{\psi}^t(u)]\psi(v) \tag{6.14}$$

which is a $U(6, 6)$ current identity in the restricted (quark) sense. The corresponding bilinear invariant $\bar{\psi}(x)\psi(x)$ is "the mass term." Note that $\bar{\psi}(y)\Gamma^X\psi(y)$ is not generally a Hermitian density. Shrinking $SU(3)$ to the identity one obtains $U(2, 2)$ relations of a kind that have long been known [P 5].

(β) S satisfies the $U(12)$ condition

$$S^\dagger S = 1. \tag{6.15}$$

Multiply Eq. (6.11) by $\psi_m{}^\dagger(x)\psi_n(y)\psi_r{}^\dagger(u)\psi_s(v)$:

$$\sum_X \psi^\dagger(x)\Gamma^X\psi(y)\cdot\psi^\dagger(u)\Gamma^X\psi(v) = \psi^\dagger(x)[\psi^t(y)\psi^{\dagger t}(u)]\psi(v) \tag{6.16}$$

which is a $U(12)$ identity for non local current densities. The corresponding bilinear invariant $\psi^\dagger(x)\psi(x)$ plays no special role. The connection between unitary and pseudo-unitary currents was referred to before in Eqs. (3.93, 94).

Note that $U(12)$ is a group acting on tensor indices only [like $U(6, 6)$] and not on the arguments like x of the fields (or field operators).

VII. NO-GO THEOREMS FOR CERTAIN APPROXIMATE KINEMATIC SYMMETRIES

At the time that the $SU(6)$ theory began to develop, several theorems were known already which showed that, under a number of rather general conditions, a union of the Poincaré group and an internal symmetry group can only be achieved in the trivial sense of a direct product, unless violence is done to some generally accepted physical principles.

In the language of Sec. IIB, these theorems are applicable to approximate kinematic symmetries. As was noted there, it is a relativistically invariant operation to take the limit "coupling constant→zero." Hence the no go theorems may be applied to this limit world. If inconsistencies appear in the limit world then, in the kinematic case, it is no excuse that the symmetry is broken in the real world. The inconsistency would apply to the real world as well.

The no-go theorems have no bearing on approximate symmetries of the dynamical kind. In this case the symmetry appears in an approximation which itself is not defined in a covariant way. Then it has no meaning to apply to the limit world a symmetry which contains the full Poincaré group. In this class belong static $SU(6)$ and the dynamical groups discussed in Sec. V which contain the static $SU(6)$.

Neither the scope of this paper nor the competence of this author permit a detailed discussion of what has been achieved in the study of no go theorems. However, as they have added so much to the hilarity and confusion in recent discussions of the symmetries at hand, and as they are quite important for the future use of symmetry arguments, it may not be out of place to mention just a few examples. For detailed references, see, e.g. [J 2, S 20, O 10]. For these rigorous theorems, the rigorous definition of a symmetry operation applies: a one-to-one correspondence which associates to any physically realizable state another such state in such a way that all transition probabilities are preserved [S 11].

A number of no-go theorems were derived in response to the following question. In the conventional way of dealing with internal symmetries, one assumes that the over-all symmetry is the direct product $T\otimes P$ (T=internal symmetry group, P=Poincaré group) in the kinematic limit in which T is exact. As a result, all particles in a T multiplet have the same mass. Is it possible to have a symmetry group G which (in the kinematical limit) contains T and P, but not as a direct product, so that the imposition of G would allow for the possibility of a unified description of particles with different mass?

Consider first the case: T is any semi-simple Lie group, with generators T_A; the set of generators of G is neither less nor more than the combined set T_A and the ten generators L_i of P. One can next proceed in either of two ways.

(α) Assume in addition that $[T_A, J_{\mu\nu}]=0$ where the six $J_{\mu\nu}$ are the subset of L_i corresponding to the homogeneous Lorentz group. It follows from these assumptions [M 9] that also $[T_A, P_\mu]=0$, where the four P_μ are the translation operators, so that $G=T\otimes P$ again.

(β) Assume in addition that $[H_a, L_i]=0$. Here the H_a are the maximal commuting subset of the T_A in terms of which the (additive) quantum numbers of T are defined, see Eq. (3.57). It follows also from these assumptions that $G=T\otimes P$ [C 22]. See also [T 3].

Thus, in physical terms, the Lorentz invariance of the internal quantum numbers is sufficient to prove that all members of multiplets must have the same mass and spin. The two theorems just mentioned can also be applied to the case that T is any compact group. Their proofs do not exclude [M 9, C 22] the possibility that something new might happen if the number of generators of G is larger than was specified above.

The next important step in weakening the conditions on G is contained in the following theorem [O 11]. Let G be a Lie group of finite order which contains P as a subgroup. The irreducible representations of G define a Hilbert space. The mass operator P_μ^2 and any power of P_μ^2 are assumed to be self-adjoint over this Hilbert space. Then the spectrum of P_μ^2 is either a single point or else it is continuous. (It was not shown that the continuous case is actually realizable.)

Subsequently, this theorem was still further strengthened as follows [R 17]. Let G again be a Lie group of finite order, the Lie algebra of which contains the Lie algebra of P as a subalgebra. Then in any irreducible representation of G, P_μ^2 cannot have more than one eigenvalue.

Thus for finite order Lie groups there is no way known to escape from the $T\otimes P$ structure. There are no general theorems known for infinite parameter groups, but such groups do not seem too attractive for other reasons. One special no go theorem in this category is the result [J 2] mentioned in Sec. IVA.

Finally, a conjecture should be noted [C 23] which has been stated for the case that the kinematic group G is a connected Lie group which is "particle finite." This means that (a) $G\supset P$, (b) G has at least one locally faithful unitary representation which under P decomposes into the direct sum of a finite number of positive-mass representations of P with mass $<M$ (where M is any finite positive mass). The conjecture says that any connected particle finite Lie group is locally isomorphic to $T\otimes P'$, where T is a compact Lie group, and P' is a trivial extension of P. The conjecture has been proved for the case that G is locally isomorphic to the semi direct product of a semi-simple Lie group and an Abelian group (the latter containing the P_μ); no counter examples to the conjecture are known.

VIII. QUESTIONS

Many of the questions collected in this concluding section were met in the previous parts of this survey.

(a) In explaining the distinction between kinematical and dynamical symmetry, atomic analogies may have been helpful. It is clear, however, that this distinction can be fully abstracted from the question of a possible substructure of baryons or mesons. Nevertheless, the question remains: is there such a substructure in terms of prime matter? Could such matter have unusual properties (such as parastatistics) which "average out" for the usual hadrons? As to whether or not triplets are the answer, it has been seen that one must further ask: is $SU(3)$ the biggest internal symmetry group?

(b) Is it at all true that $SU(3)$ is an approximate symmetry of the kinematic type? There is a perhaps related technical question: why have all higher symmetries that have had some measure of success been more manifest for masses and vertices while scatterings etc. are much less transparent? Higher symmetry means: higher than the direct product of isospin and hypercharge. Are even the latter two of the kinematical kind?

(c) Are there other ways to think of symmetry in particle physics than the approximate ways discussed before? (Lorentz invariance is globally an approximate kinematic symmetry in the presence of gravitational coupling. Yet it is a strict symmetry in local inertial frames!)

(d) The master problem (B), Sec. I is so far still open. In spite of its successes for the derivation of sum rules, it has not been shown that the algebra of currents provides the answer. Is it possible to show, where desired, that the contributions of higher intermediate states inserted in the commutators cancel out, sometimes to a very high degree?

(e) The intrinsic breaking of some of the symmetries discussed here by kinetic energy terms is reminiscent of strong coupling approximations [P 1, B 9] where the interaction terms are diagonalized first. In the old-fashioned strong coupling calculations some masses are put equal to infinity from the start. Can one obtain some dynamical symmetries in strong coupling approximations with masses kept finite? [C 21].

(f) To what extent are dynamical symmetries other than static $SU(6)$ useful for a phenomenological description? Is the independence of q^2 of a number of form factor ratios an indication for the approximate validity of W-spin, or should this approximate q^2-independence [for $q^2\lesssim(1\ \mathrm{BeV}/c)^2$] be explained as a property of static charge and magnetic moment distributions?

Are noncompact spectrum generating algebras a useful tool in particle physics [D 16, C 21]?

(g) Are there any alternatives left to the internal symmetry⊗Poincaré group picture in the face of the no-go theorems?

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