Fixed Variable Dispersion Relations for the Pion–Nucleon System

D. H. LYTH*

Nordita, Blegdamsvej 17, Copenhagen, Denmark

It is shown that an earlier study by Hamilton and Woolcock of fixed momentum-transfer dispersion relations may be complemented by a study of fixed *energy* dispersion relations. Two main results are obtained. First, by demanding that the two types of relation give the same value for the amplitude, nontrivial restrictions are obtained on the amplitude (the f_0–N–N coupling constants and the values of certain integrals over the high-energy πN amplitude are obtained). Secondly, the fixed energy relation enables one to discuss quantitatively the validity of the "CGLN" approximate method which Hamilton and Woolcock had used to calculate the partial-wave amplitudes at low energies from the fixed momentum-transfer dispersion relation alone. The terms neglected by this approximation are evaluated, and found to be large except at low energies (the authors had themselves suggested that this might be the case). Even when these terms are included, undesirable cancellations occur, and the conclusion in fact is that fixed variable relations are not suitable for calculating the partial-wave amplitudes (except at low energies), but only for providing sum rules.

CONTENTS

INTRODUCTION

The starting point for this article is formed by some previous work of Hamilton and Woolcock.[1] In that work, the authors made a thorough numerical and theoretical investigation of the fixed momentum-transfer dispersion relations for the pion–nucleon scattering amplitudes. We show in this article that a similar study of fixed *energy* dispersion relations complements the study of fixed momentum-transfer dispersion relations in a very satisfactory way.

The idea of Hamilton and Woolcock was to obtain *accurate* values of the pion–nucleon coupling constant, the s- and p-wave scattering lengths, and even the s- and p-wave amplitudes at energies above threshold using as input data only a knowledge of the *resonant* partial waves and the total cross section, plus a *rough idea* of the other partial waves. We are willing to assume all this, and in addition to use a knowledge of the amplitude in the *crossed* channel ($\pi\pi \rightarrow NN$), so that the fixed energy dispersion relation may be used. Using this knowledge we shall on the one hand obtain new results (the f_0–N–N coupling constants and the values of certain integrals over the high-energy pion–nucleon scattering amplitudes), and on the other hand provide a critique of the H.W. calculation of the partial-wave amplitudes.

This calculation of the partial-wave amplitudes used the "CGLN" method,[2] which basically uses a truncated Taylor series in momentum transfer. We shall show that: (i) the evaluation by H. W. of this truncated series contained rather large errors, the correction of which tends to *worsen* their agreement with experiment; (ii) the fixed energy dispersion relation provides a means of calculating the remainder to this truncated series; (iii) when this remainder is included, the agreement with experiment is largely restored, but there is a rather large cancellation between the truncated series and its remainder and the conclusion is that the "GGLN" approach is not very good except at low energies.

It should be emphasized that the calculation of the partial-wave amplitudes was the least reliable part of the work of H. W., and that their values for the coupling constant and the scattering lengths (except possibly the $p^{(+)}$ scattering lengths) are unchanged.

* This work was done at University College, London, England.
[1] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 137 (1963) (to be referred to as "H.W."). See also W. S. Woolcock, Ph.D. thesis, University of Cambridge (unpublished).

[2] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

This is why we said that the study of fixed energy relations *complements* the study of fixed momentum-transfer relations. The complementarity is essentially a result of one's lack of knowledge of the distant singularities, which always arises in a dispersion relation. The fixed momentum-transfer relation will be of the form

$$G(st)=\frac{1}{\pi}\int_{\text{nearby cuts (at fixed } t)} ds'\frac{\operatorname{Im} G(s't)}{s'-s}+g_1(st),$$

where the integral can be evaluated, but the term $g_1(st)$, representing the distant cuts, cannot be evaluated; however, since g_1 has no nearby cuts it will be *slowly varying with respect to s.* On the other hand, the fixed energy relation will be of the form

$$G(st)=\frac{1}{\pi}\int_{\text{nearby cuts (at fixed } s)} dt'\frac{\operatorname{Im} G(st')}{t'-t}+g_2(st),$$

where now g_2 is *slowly varying with respect to t.* Thus one relation contains an unknown term rapidly varying in t only, whereas the other contains a similar term rapidly varying in s only, and it becomes eminently reasonable that where one fails the other may succeed.

In Sec. 1 the notation is introduced, and the basic dispersion relations are presented. The number of subtractions needed is investigated, and the rather troublesome question of the regions of convergence of the partial-wave expansions for the absorbtive parts is considered: a figure showing these regions in the s–t plane is given, which it is hoped will clarify the situation. In Sec. 2 the input data to be used (i.e., the first few partial waves in each channel) are presented.

In Sec. 3, the new results mentioned above are obtained, essentially by equating the fixed energy and fixed momentum-transfer relations, and in Sec. 4 the new results are used to correct the H. W. evaluation of the truncated Taylor series mentioned above. Finally in Sec. 5 it is shown how the fixed energy relations enable one to calculate the remainder to this series, and it is also shown that an alternative method of doing this due to Atkinson[3] does not give good results in practice.

SECTION 1

1.1. Notation and Kinematics

The units are such that pion mass $\mu=\hbar=c=1$. The nucleon mass is then 6.72 and the threshold s value $(M+\mu)^2=59.6\simeq 60$. The standard variables s, t, and u are used, which are the total four-momentum squared in channels 1 (πN scattering), 2 ($\pi\pi\rightarrow NN$), and 3 (crossed πN scattering), respectively; t is also related to the momentum-transfer in channel 1, so that it is natural to consider s and t as the basic variables, u being defined by the well-known relation

$$s+t+u=2M^2+2\mu^2\equiv\Sigma.$$

[3] D. Atkinson, Nuovo Cimento **30**, 551 (1963).

The scattering is described by the usual amplitudes[4]

$$A^{(\pm)}(st)=\pm A^{(\pm)}(ut), \tag{1a}$$

$$B^{(\pm)}(st=\mp B^{(\pm)}(ut), \tag{1b}$$

and in addition we define a third pair of amplitudes

$$F^{(\pm)}(st)=A^{(\pm)}(st)+[M(s-u)/(4M^2-t)]B^{(\pm)}(st)$$
$$=\pm F^{(\pm)}(ut). \tag{2}$$

The crossing relations (1) imply that at the point $s=u$, $B^{(+)}=A^{(-)}=0$. Hence it is useful to define

$$\tilde{B}^{(+)}(st)=B^{(+)}(st)/(s-u)=+\tilde{B}^{(+)}(u,t),$$
$$\tilde{A}^{(-)}(st)=A^{(-)}(st)/(s-u)=+\tilde{A}^{(-)}(u,t)$$

(remember that $u=\Sigma-s-t$).

Finally, $G(st)$ will denote any of the above amplitudes when general statements are made.

The partial waves for channel 1 are denoted as usual by[5]

$$f_{l\pm}^{(\pm)}(s)=\frac{\exp[2i\delta_{l\pm}^{(\pm)}(s)]-1}{2iq},$$

where q is the barycentric momentum,

$$q^2(s)=[s-(M+1)^2][s-(M-1)^2]/4s.$$

We also define $h_{l\pm}(s)=f_{l\pm}(s)/q^{2l}$ and the scattering lengths $a_{l\pm}=h_{l\pm}(s_0)$, where $s_0=(M+1)^2$, i.e., the threshold value of s. The connection between $f_{l\pm}$ and the full amplitudes is given by

$$f_{l\pm}(s)=\frac{1}{2}\int_{-1}^{1}\{F_1(st)P_l(x)+F_2(st)P_{l\pm1}(x)\}\,dx, \tag{3}$$

where

$$F_1(st)=\frac{(W+M)^2-1}{16\pi W^2}[A(st)+(W-M)B(st)],$$

$$F_2(st)=\frac{(W-M)^2-1}{16\pi W^2}[-A(st)+(W+M)B(st)],$$

$$W=+s^{\frac{1}{2}},\qquad x=1+t/2q^2$$

(= cosine of barycentric scattering angle). We shall also need the first few special cases of (3) at threshold $[A^1(s_00)\equiv\partial/\partial t|_{t=0}A(s_0t)$ and similarly for $B]$

$$[A(s_00)+B(s_00)]=4\pi(1+(1/M))a_{0+}, \tag{4a}$$

$$B(s_00)=4\pi[(1/2M)a_{0+}+2M(a_{1-}-a_{1+})], \tag{4b}$$

$$[A'(s_00)+B'(s_00)]=4\pi(1+(1/M))\tfrac{3}{2}a_{1+}. \tag{4c}$$

The partial waves for channel 2 are denoted as usual by $f_{\pm}^{J}(t)$, where $\pm$ refers to the nucleon helicity states,[6] *not* the isospin state; this is fixed by J as a consequence of the identity of the particles in the

[4] A, B correspond to the possibility of two spin states, $(\pm)$ to the two isospin states ($A^{\frac{3}{2}}=A^{(+)}-A^{(-)}$, $A^{\frac{1}{2}}=A^{(+)}+2A^{(-)}$, and similarly for B).

[5] The connection with the other commonly used notation is as follows: $f_{0+}^{\frac{1}{2}}, f_{1-}^{\frac{1}{2}}, f_{1+}^{\frac{1}{2}}, f_{2-}^{\frac{1}{2}}, \cdots$ correspond to s_{11}, p_{11}, p_{13}, $d_{13}, \cdots$, and $f_{0+}^{\frac{3}{2}}, f_{1-}^{\frac{3}{2}}, f_{1+}^{\frac{3}{2}}, f_{2-}^{\frac{3}{2}}\cdots$ correspond to s_{31}, p_{31}, p_{33}, $d_{33}\cdots$.

[6] $\pm$ means that the nucleons have the same and opposite helicities, respectively.

initial (and final) state, being (+) for even and (−) for odd J. The connection between $f_{\pm}{}^{J}$ and the full amplitudes is simpler if F and B are used rather then A and B (that is why we defined F). It is

$$F^{(\pm)}=-\sum_{J\left\{\substack{\text{even}\\ \text{odd}}\right\}}(J+\tfrac{1}{2})(8\pi/p^2)(pq)^J P_J(z)f_+{}^J(t), \quad (5a)$$

$$B^{(\pm)}=8\pi\sum_{J\left\{\substack{\text{even}\\ \text{odd}}\right\}}(J+\tfrac{1}{2})[J(J+1)]^{\frac{1}{2}}(pq)^{J-1} \times P'_J(z)f_-{}^J(t), \quad (5b)$$

where $p^2=(t/4-M^2)$, $q^2=(t/4-1)$, $z=(s-u)/4pq$ (= cosine of barycentric scattering angle in channel 2).

1.2. The Dispersion Relations

This subsection contains some essential facts about crossing symmetry in the dispersion relations, the number of subtractions needed and the possibility of evaluating the dispersion relations given the first few partial waves in each channel. It may be omitted if the reader is prepared to take the facts on trust when they are used later.

a. The Basic Relations, Ignoring Subtractions

All the amplitudes have cuts along $\infty>s>(M+1)^2$, $\infty>t>4$, and $\infty>u>(M+1)^2$, plus poles for $B^{(\pm)}$ at $s=M^2$ and $u=M^2$ with residue, for the s pole, $g^2=183.9\pm0.6$ (H.W. value). These poles will be formally included in the cuts when dispersion relations are written down.

Fixed t relations. For $A^{(+)}$, $\tilde{A}^{(-)}$, $\tilde{B}^{(+)}$, and $B^{(-)}$ one has [using the notation[7] $u\equiv\Sigma-s-t$, $u(s't)\equiv\Sigma-s'-t$]

$$G(st)=\pi^{-1}\int_{M^2}^{\infty}ds'\,\mathrm{Im}\,G(s't)\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]$$
$$=\pi^{-1}\int_{M^2}^{\infty}ds'\,\mathrm{Im}\,G(s't)\left[\frac{s'-u(s't)}{(s'-s)(s'-u)}\right]. \quad (6)$$

For $A^{(-)}$ and $B^{(+)}$, crossing symmetry requires the opposite sign

$$G(st)=\pi^{-1}\int_{M^2}^{\infty}ds'\,\mathrm{Im}\,G(s't)\left[\frac{1}{s'-s}-\frac{1}{s'-u}\right]$$
$$=\pi^{-1}\int_{M^2}^{\infty}ds'\,\mathrm{Im}\,G(s't)\left[\frac{s-u}{(s'-s)(s'-u)}\right]. \quad (7)$$

However, Eq. (7) gives us nothing new because it follows from Eq. (6) for $\tilde{A}^{(-)}$ and $\tilde{B}^{(+)}$ which gives (say for $\tilde{A}^{(-)}$)

$$\frac{A^{(-)}}{s-u}=\tilde{A}^{(-)}=\pi^{-1}\int ds'\,\frac{\mathrm{Im}\,A^{(-)}(s't)}{s'-u(s't)}\left[\frac{s'-u(s't)}{(s'-s)(s'-u)}\right]$$
$$=\pi^{-1}\int ds'\,\mathrm{Im}\,A^{(-)}(s't)[(s'-s)(s'-u)]^{-1}.$$

[7] Remember that u is always regarded as the subsidiary quantity, $u(st)=\Sigma-s-t$.

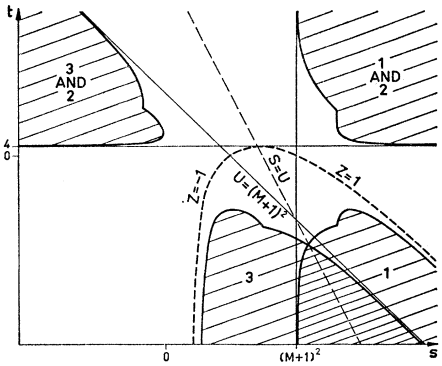

FIG. 1. Region of convergence of partial-wave expansions (explained in text). Shaded regions are those regions where the partial-wave expansions in channels 1, 2, and 3 diverge.

By using the variable $\omega(st)=\frac{1}{2}(s-u)=s-M^2-1+t/2$ instead of s (this merely shifts the origin of the s plane), all the above relations can be written in the form

$$G(st)=\int d(\omega'^2)[g(\omega't)/(\omega'^2-\omega^2)],$$

which is a standard dispersion relation in $y=\omega^2$ [8]; this fact is important when considering subtractions.

Fixed s relation. Here the relations for $B^{(+)}$, $A^{(-)}$ cannot be deducted from those for $\tilde{B}^{(+)}$, $\tilde{A}^{(-)}$; this is essentially because the fixed s relation does not automatically satisfy crossing symmetry as does the fixed t relation. The relations for all the amplitudes $A^{(\pm)}$, $B^{(\pm)}$, $\tilde{A}^{(-)}$, and $\tilde{B}^{(+)}$ are of the form

$$G(st)=\pi^{-1}\int_4^{\infty}dt'\,\frac{\mathrm{Im}\,G(st')}{t'-t}$$
$$\pm\pi^{-1}\int_{M^2}^{\infty}du'\frac{\mathrm{Im}\,G(u',\Sigma-u'-s)}{u'-u}, \quad (8)$$

where the + or − sign hold according to whether the amplitude is even or odd under crossing.

b. Regions of Convergence of the Partial-Wave Expansions

The absorptive parts $\mathrm{Im}\,G(st)$ are given in terms of partial waves essentially[9] by Legendre expansions

$$\mathrm{Im}\,G(st)=\Sigma a_l(s)P_l(x) \qquad [\text{For } s>(M+1)^2],$$
$$\mathrm{Im}\,G(st)=\Sigma b_l(t)P_l(z) \qquad [\text{for } t>4].$$

The regions of convergence of these expansions follow from the Mandelstam representation[10]; we show them in Fig. 1, which requires some explanation.

The figure is symmetrical under reflection at fixed t about the line $s=u$, as is required by crossing. For each of channels 1, 2, and 3 there is shown: (a) the region where the absorbtive part is nonzero for this channel, i.e., the regions $s>(M+1)^2$, $t>4$ and $u>(M+1)^2$

[8] Proof: note that $(\omega'-\omega)(\omega'+\omega)=(s'-u)(s'+u)$.

[9] The actual expressions, Eq. (5) and a similar one for channel 1 are more complicated, but the results stated are still true.

[10] See Ref. 1 for channel 1, and for channel 2 see W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960).

respectively; (b) the part of this region, indicated by two shaded areas, where the partial-wave expansion for the absorbtive part is divergent. Hence the region where the absorbtive part for a particular channel is both nonzero and calculable is given by the region [(a)−(b)].

From Fig. 1 we deduce the following. For the *fixed t relation*, the absorbtive part is given by a convergent partial-wave expansion, provided $-26<t<4$. For the *fixed s relation* the absorbtive part is given, on the *nearby* cuts, by convergent expansions provided that $0\lesssim s\lesssim(M+1)^2$, this upper limit may be extended to $s\sim(M+1)^2+20$ for the cut $4<t<\infty$ but *not* for the cut $(M+1)^2<u<\infty$ so that the fixed s relation is not useable as it stands in the physical region for πN scattering.

c. Subtraction

The relations given in Sec. a above may require subtractions. As is well-known, if, as $z\rightarrow\infty$, $F(z)/z^n\rightarrow 0$ for all $n\geq N$ (but not for $n<N$) then N subtractions are necessary in a dispersion relation for $F(z)$. For the fixed t relation, crossing allows the variable $y=(s-M^2-1+t/2)^2$ to be used, so that here if, as $s\rightarrow\infty$, $G(st)/s^n\rightarrow 0$ for all $n\geq N$ (but not for $n<N$) (at fixed t) N' subtractions are necessary, where N'=integral part of $N/2$.

The asymptotic behavior for $s<0$ ($t\rightarrow\infty$) and $t<0$ ($s\rightarrow\infty$) may be determined from unitarity because these regions are physical (see Ref. 1 for channel 1; a similar treatment for channel 2 is trivial, see Ref. 11). However, we require the behavior for $s>0$ and the only approach giving results here is that of Regge poles.[12] The details have been worked out for the πN system by Singh and Udgaonkar,[13] and when they are applied to determine the number of subtractions needed one finds that this number decreases as s, t decrease (for the fixed s, fixed t relations respectively). The results may be summarized as follows.

(*i*) *Fixed s relation.* For all amplitudes $A^{(\pm)}$ and $B^{(\pm)}$, Regge theory predicts the following:

$80\lesssim s\lesssim 150$, 2 subtractions are necessary;
$25\lesssim s\lesssim 80$, 1 subtraction is necessary;
$s\lesssim 25$, 0 subtractions are necessary.

For $s\lesssim 0$ the unitarity approach gives only the weaker statement: 1 subtraction is *sufficient*. In this article, only the following will need to be assumed:

$s\lesssim 90$, 3 subtractions sufficient (in Sec. 5);
$s\lesssim 60$, 1 subtraction sufficient (in Sec. 3);

so the Regge result for $s\lesssim 25$ is not, in fact, needed.

(ii) *Fixed t relation.* If the variable

$$y=(s-M^2-1+t/2)^2$$

is used:

$28\lesssim t\lesssim 80$, 1 subtraction necessary for $A^{(\pm)}$, $B^{(-)}$, none for $B^{(+)}$;
$t\lesssim 28$, 0 subtractions necessary for $A^{(-)}$, $B^{(\pm)}$, 1 for $A^{(+)}$.

The unitarity approach gives the same result for $t\lesssim 0$, and, in fact, we only need the result for $t\simeq 0$, the other results merely being given for completeness.

d. The Frazer Relation

For $A^{(+)}$ the fixed t relation is not valid without a subtraction even for $t<0$. An alternative relation is, however, *approximately* valid, that suggested by Frazer.[14] The relation is simpler for the amplitude $F^{(+)}$ (since $B^{(+)}$ does not require subtractions, $A^{(+)}$ can then be calculated without difficulty). Essentially the idea is to subtract off the contribution of $f_+{}^0(t)$ to give a convergent integral:

$$F^{(+)}(st)-\frac{16\pi}{4M^2-t}f_+{}^0(t)$$
$$=\pi^{-1}\int_{M^2}^{\infty}\text{Im }F^{(+)}(s't)\left\{\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]\right.$$
$$\left.-\frac{1}{2}\left[\int_{-1}^{1}\left(\frac{1}{s'-s}+\frac{1}{s'-u}\right)P_0(z)\,dz\right]\right\}ds'.$$

Then a dispersion relation is written for $f_0{}^+(t)/(4M^2-t)$ and the integration over z performed, giving

$$F^{(+)}(st)=\pi^{-1}\int_{M^2}^{\infty}\text{Im }F^{(+)}(s't)\left\{\frac{1}{s'-s}+\frac{1}{s'-u}-\frac{Q_0(z')}{pq}\right\}ds'$$
$$+\pi^{-1}\int_{-\infty}^{a}\int_{4}^{\infty}\frac{16\pi\text{ Im }F_+{}^0(t')}{(4M^2-t')(t'-t)}dt'$$
$$\simeq\pi^{-1}\int_{M^2}^{s_1}\text{Im }F^{(+)}(s't)\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]ds'$$
$$+\pi^{-1}\int_{s_1}^{\infty}\text{Im }F^{(+)}(s't)\left[\frac{1}{s'-s}+\frac{1}{s'-u}-\frac{Q_0(z')}{pq}\right]ds'$$
$$+\pi^{-1}\int_{4}^{\infty}\frac{16\pi\text{ Im }F_+{}^0(t')}{(4M^2-t')(t'-t)}dt'+\text{constant},\quad(9)$$

where the separation point s_1 is such that only s and p waves give appreciable contributions to Im $F(st)$ for $s<s_1$. The steps in the derivation of this equation, and their justification, are not given here[11] since the relation is only used at one point and is in any case checked numerically.

[11] D. H. Lyth, Ph.D. thesis, University of London, 1964 (unpublished).

[12] Although Regge poles give results conflicting with experiment in channel 1, this is only in the physical region, $t<0$. As s, t increase, the region of the J plane in which meromorphy need be assumed for our purposes decreases, and, in fact, none of the suggested modifications to the simple Regge theory (known to the author) would affect the results given here for s, $t>0$.

[13] V. Singh, Phys. Rev. **129**, 1889 (1962).

[14] W. R. Frazer, *Proceedings of the Rochester Conference* (University of Rochester, Rochester, New York, 1960), p. 282.

SECTION 2

In order to evaluate the relations, one needs as input the first few partial waves in each channel. The input which we shall use is presented in this section.

2.1. Channel 1 Input Data

The partial waves here are obtainable directly from experiment.

The dominant features are several well-established resonances (p_{33}, 180 MeV; d_{13}, 600 MeV: F_{15}, 900 MeV; F_{37}, 1320 MeV) for which a delta-function approximation is adequate:

$$\operatorname{Im} f_{l\pm}{}^{T}(s)=(\pi/q_r)\cdot R\cdot\delta(s-s_r) \qquad (10)$$

with the parameters listed in Table I.

The nonresonant partial waves have negligible imaginary parts below $s\sim100$. Hence only the p-wave resonance contributes here and one has the important result that the absorbtive parts are approximately *first-order polynomials* in t (p-wave dependence). For $100\lesssim s\lesssim200$ a recent analysis by Auvil and Lovelace[15] may be used to give a rough estimate of the (imaginary parts of the) nonresonant waves. Above $s\sim200$ little is known; we shall assume only that the cut $\infty>s>200$ gives a slowly varying contribution and with this assumption *predictions* about the amplitude in this region will be possible.

2.2. Channel 2 Input Data

Here the region of interest (i.e., low t values) is unphysical, but there have been several theoretical estimates of the partial waves. For $J=1$ and 2 there are resonances, and one may use

$$\operatorname{Im} f_{\pm}{}^{1}\simeq R_{\pm}{}^{1}\delta(t-28), \qquad (11a)$$

$$\operatorname{Im} f_{\pm}{}^{2}\simeq R_{\pm}{}^{2}\delta(t-80). \qquad (11b)$$

$R_{\pm}{}^{2}$ are not reliably known so they must be left free for the moment. $R_{\pm}{}^{1}$ are known from various sources; we take $R_{+}{}^{1}=25.0$, $R_{-}{}^{1}=15.5$ (corresponding[16] to $C_1=-1.0$).

For $J=0$, $f_{-}{}^{0}(t)=0$ since $J=0$ nucleons cannot have opposite helicities. $f_{+}{}^{0}(t)$ has been calculated in terms of the $J=0$, $T=0$ $\pi\pi$ phase shift (on the assumption that distant singularities in a dispersion relation give slowly varying terms).[16] By fitting πN partial-wave dispersion relations to experiment, two sets of $\operatorname{Im} f_{+}{}^{0}(t)$ giving good fits were obtained; they are given in Table II. (We are grateful to Dr. G. C. Oades for letting us have these values.)

TABLE I. Parameters R and s_r for the approximations to the resonant amplitudes $\operatorname{Im} f_{l\pm}{}^{T}(s)=(\pi/q_r)R\delta(s-s_r)$.

T	J	l	S_r	R
$\frac{3}{2}$	$\frac{3}{2}$	1	77	10
$\frac{1}{2}$	$\frac{3}{2}$	2	120	11
$\frac{1}{2}$	$\frac{5}{2}$	3	150	11
$\frac{3}{2}$	$\frac{7}{2}$	3	185	12

TABLE II. Various sets of $\operatorname{Im} f_{+}{}^{0}(t)$.

t	Set (1)	Set (2)	Set (3)
0	0	0	0
4.5	27.9	−8.3	8.8
5.5	23.6	−12.4	9.6
6.5	21.7	−14.6	9.8
7.5	20.8	−13.4	10.0
8.5	20.2	−7.8	10.1
9.5	19.8	3.0	10.3
10.5	19.5	18.1	10.5
20.0	18.8	99.0	8.1
30.0	18.6	104.5	7.2
40.0	18.5	107.6	6.2

SECTION 3

3.1. Equating the Fixed s and Fixed t Relations

The considerations of Sec. 1 show that for $0\lesssim s\lesssim(M+1)^2$ and small values of t both the fixed variable relations may be evaluated, except for distant cuts which give terms slowly varying in either s (for the fixed t relation) or t (for the fixed s relation). The basic assumption of this section is that this variation is *negligibly slow*, so that the dependence on s or t may be ignored entirely. Then, by requiring that the fixed s and fixed t relations agree we shall obtain an equation of the form

$$\pi^{-1}\int_{\text{nearby cuts}} ds'\frac{\operatorname{Im} G(s't)}{s'-s}+g_1(t)=\pi^{-1}\int_{\text{nearby cuts}} dt'\frac{\operatorname{Im} G(st')}{t'-t}+g_2(s) \qquad (12)$$

(the dependence of g_1 on s and g_2 on t being ignored), and we shall obtain as a consequence of this equation the results listed at the end of the section.

Basically the plan is as follows. First, g_2 is eliminated by differentiating with respect to t at $t=0$, giving (we drop the subscript on g_1 from now on)

$$\frac{n!}{\pi}\int\frac{\operatorname{Im} G(st')}{t'^{(n+1)}}dt'=\pi^{-1}\int\frac{\operatorname{Im} G^{(n)}(s'0)}{s'-s}ds'+g^{(n)}(0). \qquad (13)$$

Then, roughly speaking, we evaluate the integrals and obtain information about the $g^{(n)}(0)$, which gives the promised restrictions on the high-energy πN scattering amplitudes, since (if no subtraction is required)

$$g(t)=\pi^{-1}\int_{\text{distant cuts}} ds'\frac{\operatorname{Im} G(s't)}{s'-s}.$$

[15] P. Auvil and C. Lovelace, Nuovo Cimento **33**, 473 (1964).

[16] J. Hamilton, P. Menotti, G. C. Oades, and L. L. J. Vick, Phys. Rev. **128**, 1881 (1962).

TABLE III. $\tilde{B}^{(+)}$ amplitude.

	$s=0$	$s=60$		$s=0$	$s=60$
F_1	0.0009	0.0006	$\frac{1}{2}t_rF_2$	−0.0001	0.0004
Total	0.0009	0.0006	Total	0.0005	0.0010

We can also fix the unknown f_0-N-N coupling constants, $R_{\pm}^2$, which give a delta-function contribution to the t' integral in (13) with a known s dependence.

Three points must be added to the above outline:

(a) We use $\tilde{A}^{(-)}$, $\tilde{B}^{(+)}$ since they fall off faster than $A^{(-)}$, $B^{(+)}$ for large s and t.

(b) For $n\geq 3$ the t' integral in (13) is so weighted towards $t\sim 4$ that the resonance approximations for the $J=1$, 2 contributions to $\operatorname{Im} G(st')$ will hardly be good; hence the equation will only be used for $n=1$, 2.

(c) A straightforward numerical evaluation of each side, with subsequent comparison, would be very clumsy since the poles and part of the 33 resonance contribution are the same for both sides. Hence the relevant terms are first canceled.

The rest of this section is devoted to the detailed working out of the above ideas.

3.2. The Cases of $B^{(+)}$, $\tilde{A}^{(-)}$, and $B^{(-)}$

For these amplitudes the relations to be equated are:

(i) fixed t (the integrals can be taken up to $s=200$ using the information of Sec. 2)

$$G(st)=\pi^{-1}\int_{M^2}^{200}\frac{\operatorname{Im} G(s't)}{s'-s}\,ds'+\pi^{-1}\int_{M^2}^{200}\frac{\operatorname{Im} G(u't)}{u'-u}\,du'+g(t); \quad (14)$$

(ii) fixed s (letting the integrations go to infinity for the moment)

$$G(st)=\pi^{-1}\int_4^{\infty}\frac{\operatorname{Im} G(st')}{t'-t}\,dt'+\pi^{-1}\int_{M^2}^{\infty}\frac{\operatorname{Im} G(u',\Sigma-u'-s)}{u'-u}\,du'. \quad (15)$$

The difference between the last terms is

$$\Delta=\pi^{-1}\int_{M^2}\frac{\operatorname{Im} G(u',\Sigma-u'-s)-\operatorname{Im} G(u't)}{u'-u}\,du'.$$

Below $u'\sim 100$, $\operatorname{Im} G(u't)$ is, as we have noted, approximately a first-order polynomial in its second argument since the dominant terms (pole and p_{33} resonance) have a p-wave dependence. Hence

$$\operatorname{Im} G(u',\Sigma-u'-s)-\operatorname{Im} G(u',t)$$
$$\simeq(\text{function of } u')\ (\Sigma-u'-s-t)$$
$$=-(\text{function of } u')\ (u'-u)$$

and, therefore,

$$\Delta\simeq\pi^{-1}\int_{100}^{\infty}\frac{\operatorname{Im} G(u',\Sigma-u'-s)-\operatorname{Im} G(u't)}{u'-u}\,du'+\text{constant}.$$

Hence on equating (14) and (15), and differentiating with respect to t, the u' integrals cancel below $u'\simeq 100$. After slight rearrangement, one obtains, in fact,

$$\pi^{-1}\int_4^{\infty}\frac{\operatorname{Im} G(st')}{t'-t}\,dt'+\pi^{-1}\int_{100}^{\infty}\frac{\operatorname{Im} G(u',\Sigma-u'-s)}{u'-u}\,du'$$
$$\simeq\int_{M^2}^{100}\frac{\operatorname{Im} G(s't)}{s'-s}\,ds'+\int_{100}^{200}\operatorname{Im} G(s't)\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]ds'+g(t)+\text{constant}. \quad (16)$$

The two integrals of the right-hand side can easily be evaluated in terms of the data mentioned in Sec. 3. The second integral on the left-hand side involves $u'>100$ and Fig. 1 shows that $\operatorname{Im} G(u',\Sigma-u'-s)$ is unknown here except for $s\sim 0$. We have calculated the integral at $s=0$ and found it to give a negligable contribution to all the equations below; it is *assumed* that the integral can also be ignored for $s>0$.[17]

The first integral on the left-hand side is assumed for the moment to be dominated by the $J=1$ and $J=2$ resonant contributions, Eqs. (5) and (11). Thus we obtain, on differentiating (16) once and twice with respect to t at $t=0$

$$c/t_r^2=F_1(s)+g'(0), \quad (17a)$$
$$2c/t_r^3=F_2(s)+g''(0), \quad (17b)$$

where F_1 and F_2 come from the integrals on the right-hand side of (16) and are known. The constant c is given in terms of R_-^2 for $\tilde{B}^{(+)}$ or $R_{\pm}'$ for $\tilde{A}^{(-)}$, $B^{(-)}$ by Eqs. (5), and t_r is the position of the resonance ($t_r=$ 28, 80 for $J=1$, 2, respectively).

Equations (17) clearly require that, for all s in the range considered [i.e., $0\lesssim s\lesssim(M+1)^2$],

$$c/t_r^2=F_1(s)+g'(0)=t_r/2[F_2(s)+g''(0)]. \quad (18)$$

In the first row of Tables III, IV, and V we show, from left to right: $F_1(0)$, $F_1((M+1)^2)$, $(t_r/2)F_2(0)$, and $(t_r/2)F_2((M+1)^2)$. We cannot, however, assess how well (18) is satisfied without some idea of either the relative magnitudes of g', g'' and F_1, F_2 or of the value of c.

TABLE IV. $\tilde{A}^{(-)}$ amplitude.

	$s=0$	$s=60$		$s=0$	$s=60$
F_1	−0.002	−0.004	$\frac{1}{2}t_rF_2$	−0.003	−0.002
Total	−0.004	−0.006	Total	−0.006	−0.005

[17] The denominater increases and there seems no reason why Im G should increase rapidly.

Case of $\tilde{B}^{(+)}$

Here we do not know c, but a rough estimate of g', g'' is possible. A rough estimate of $\mathrm{Im}\,\tilde{B}^{(+)}(st)$ for $s>200$ has been made by H. W.,[1] so using

$$\tilde{b}^{(+)}(t)=\pi^{-1}\int_{200}^{\infty}\frac{\mathrm{Im}\,B^{(+)}(s't)}{(s'-s)(s'-u)}ds',$$

one obtains the order-of-magnitude predictions

$$\tilde{b}^{(+)\prime}(0)\sim 0.00005, \qquad \tilde{b}^{(+)\prime\prime}(0)\sim 0.00001. \quad (19)$$

Looking at Table III, we see that $\tilde{b}^{(+)\prime}$ is negligible compared with $F_1(s)$. F_1 is roughly constant, so the first equality of (18) is approximately satisfied with $c/t_r^2 \simeq 0.00075$. Using this value the second equality of (18) *requires* that $b''=0.000015$ [taking the average value for $F_2(s)$], which is in approximate agreement with the value (19).

In the second row of Table III are shown, from left to right; $F_1(0)+g'$, $F_1((M+1)^2)+g'$, $t_r/2[F_2(0)+g'']$, $t_r/2[F_2((M+1)^2)+g'']$. The equality (18) is quite well satisfied, and taking the average value of c gives finally

$$R_-^2=0.7\pm0.2$$

[with $\mathrm{Im}\,f_-^2(t)=R_-^2\delta(t-t_r)$].

Cases of $\tilde{A}^{(-)}$, $B^{(-)}$

Here c is known because $R_\pm^1$ are known (Sec. 3). Using Eqs. (5) and (11) we obtain $c/t_r^2=-0.0055$ for $\tilde{A}^{(-)}$ and $+0.19$ for $B^{(-)}$. The equality (18) together with the values for F_1, F_2 given in the first row of Tables IV and V now *requires* [taking average values for $F_1(s)$, $F_2(s)$]

$$\tilde{a}^{(-)\prime}\simeq -0.0025 \qquad \tilde{a}^{(-)\prime\prime}\simeq -0.00020 \quad (20)$$

$$b^{(-)\prime}\simeq 0.10 \qquad b^{(-)\prime\prime}\simeq 0.012. \quad (21)$$

Using these values, $F_1(s)+g'$ and $\frac{1}{2}t_r(F_2(s)+g'')$ are shown in the second row of Tables IV and V. The equality (18) is seen to be quite well satisfied.

Estimate of $\tilde{a}^{(-)}$ *and* $b^{(-)}$ *using* $\mathrm{Im}\,A^{(-)}$ *and* $\mathrm{Im}\,B^{(-)}$

H. W. assumed that $\mathrm{Im}\,A^{(-)}=\mathrm{Im}\,B^{(-)}=0$ for $s>200$ (except in one case which does not concern us). However, the Auvil–Lovelace analysis[15] gave quite large values for these quantities at $s=185$; after subtracting out the resonance contribution of f_{37} one

TABLE V. $B^{(-)}$ amplitude.

	$s=0$	$s=60$		$s=0$	$s=60$
F_1	0.08	0.11	$\frac{1}{2}t_rF_2$	0.036	0.003
Total	0.18	0.21	Total	0.20	0.17

obtains at $s=185$ and $t=0$

$$\mathrm{Im}\,A'=-0.44, \qquad \mathrm{Im}\,A''=-0.010,$$

$$\mathrm{Im}\,B'=0.065, \qquad \mathrm{Im}\,B''=0.0014.$$

Now $\tilde{a}^{(-)}$ and $b^{(-)}$ are given by

$$\tilde{a}^{(-)}(t)=\pi^{-1}\int_{200}^{\infty}\frac{\mathrm{Im}\,A^{(-)}(s't)}{(s'-s)(s'-u)}ds'$$

$$b^{(-)}(t)=\pi^{-1}\int_{200}^{\infty}\mathrm{Im}\,B^{(-)}(s't)\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]ds'.$$

The first integral is well convergent. Taking $\mathrm{Im}\,A\sim$ constant, and using the Auvil–Lovelace values one obtains

$$a'\sim -0.002,$$

$$a''\sim -0.0001,$$

which are in satisfactory agreement with the values (20), obtained by entirely different considerations. The second integral diverges logarithmically if $\mathrm{Im}\,B$ is assumed constant, so it depends sensitively on the manner in which $\mathrm{Im}\,B$ approaches zero. However, if the Auvil–Lovelace values were held constant up to a cutoff at $s'\sim 1000$, order of magnitude agreement with (21) would be obtained.

Fairly recently $h_{1,9}$ and $h_{3,11}$ resonances have been suggested around 2 BeV.[18] However, even if purely elastic these would only give contributions of -0.001, -0.00005, -0.015, and 0.0007 to $\tilde{a}'$, $\tilde{a}''$, b', and b'', respectively, which are not large enough to allow anything to be said concerning the existence of these resonances.

Summary of this Subsection

The requirements that the fixed s and fixed t relations should agree for $\tilde{B}^{(+)}$, $\tilde{A}^{(-)}$, and $B^{(-)}$, and that $\mathrm{Im}\,G$ in channel 2 should be dominated by the $J=1$ and 2 resonances leads to the equality (18). The equality is quite well satisfied with values of g' and g'' compatible with what is known of high-energy πN scattering, and with values of c compatible with the known amplitudes $\mathrm{Im}\,f_\pm^1(t)$. We also *predict* that $R_-^2=0.7\pm0.2$, where $\mathrm{Im}\,f_-^2\simeq R_-^2\delta(t-80)$.

In the next subsection, a study of $A^{(+)}$ will yield a value for R_+^2 also.

3.3. The Amplitude $A^{(+)}$

Here the Frazer relation (9) is used instead of the fixed t relation. The Frazer relation is for $F^{(+)}$, and $A^{(+)}$ may be calculated from this relation and the unsubtracted fixed t relation for $B^{(+)}$, using

$$A=F-[M(s-u)/(4M^2-t)]B.$$

[18] A. Diddens, E. Jenkins, T. Kycia, and K. Riley, Phys. Rev. Letters **10**, 262 (1963).

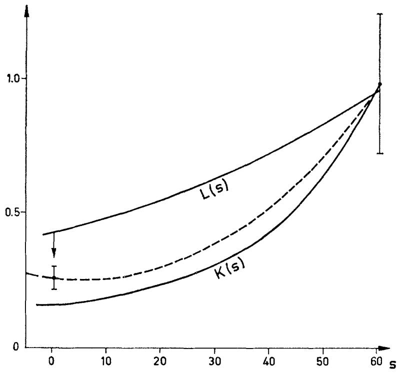

FIG. 2. $K(s)$ and $L(s)$. $K\equiv F_1$, $L\equiv t_rF_2/2$.

One obtains, after some cancellation,

$$A^{(+)}(st)=\pi^{-1}\int_{(M+1)^2}^{s_1}\operatorname{Im} A(s't)\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]ds'$$

$$+\pi^{-1}\int_{s_1}^{\infty}\operatorname{Im} F(s't)\left[\frac{1}{s'-s}+\frac{1}{s'-u}-\frac{Q_0(z')}{pq}\right]ds'$$

$$-\frac{M(s-u)}{4M^2-t}\pi^{-1}\int_{s_1}^{\infty}\operatorname{Im} B(s't)\left[\frac{1}{s'-s}-\frac{1}{s'-u}\right]ds'$$

$$+\pi^{-1}\int_4^{\infty}\frac{16\pi\operatorname{Im} F_+^0(t')}{(4M^2-t')(t'-t)}dt'+\text{constant}. \qquad (22)$$

Upon equating this relation with the fixed s relation, the low-energy u' integrals cancel as before, and so do the contributions from $\operatorname{Im} f_+^0(t')$. One thus ends up with relations similar to (17) above.[19]

$$\pi^{-1}\int\frac{\operatorname{Im}\bar{A}^{(+)}(st')}{t'^2}dt'=F_1(s),$$

$$\frac{2}{\pi}\int\frac{\operatorname{Im}\bar{A}^{(+)}(st')}{t'^3}dt'=F_2(s).$$

$\operatorname{Im}\bar{A}$ means $\operatorname{Im} A$ with the s-wave contribution $[16\pi\operatorname{Im} F_+^0/(4M^2-t)]$ subtracted out. As before F_1 and F_2 may be calculated except for the u' integration in channel 3, which is only calculable at $s\simeq 0$.

In Fig. 2 are shown $F_1(s)$ and $t_r/2F_2(s)$, with $t_r=80$ and $s_1=100$.[20] The channel 3 integral has been set equal to zero; the arrow indicates its contribution to F_1 at $s=0$, and its contribution to F_2 is negligable at $s=0$. It is seen that F_1 and $t_r/2F_2$ are approximately equal, hence we can *predict* that

$$\operatorname{Im} A(st')\simeq a(s)\delta(t'-80).$$

Assuming that the $J=2$ contribution dominates as before, Eqs. (2), (5a), (5b) and (11) give $a(s)$ in terms of R_+^2 and R_-^2. The dotted line in Fig. 2 shows $a(s)/t_r^2$ with R_-^2 fixed at the value of the last subsection ($R_-^2=0.7$) and R_+^2 fixed to give a best fit, $R_+^2=8.5$. The error bars show the change caused by a 20% change in R_+^2; a similar change in R_-^2 would have a much smaller effect. If a different partial wave had been assumed to dominate, or if t_r had been greatly altered, the fit would have been worse, so we have in fact predicted that

$$\operatorname{Im} F_\pm^2=R_\pm^2\delta(t-t_r),$$

with

$$R_+^2\simeq 8.5\pm 2,$$
$$R_-^2\simeq 0.7\pm 0.2,$$
$$t_r\simeq 80.$$

The only other estimates of $R_\pm^2$ are due to Kane and Spearman[21] ($R_+^2\simeq R_-^2\simeq 1.0$) and G. C. Oades[22] ($R_-^2\simeq 0.3$), but both these estimates involved the use of the partial-wave expansion for A, B in terms of $F_\pm^J$ at points far outside its region of convergence, and the first estimate did not clearly isolate the effect of the F_0 resonance from other possible effects.

A Check on the Frazer Relation

The accurate scattering lengths obtained in Ref. 1 give, via Eqs. (4), an accurate value for $F^{(+)\prime}((M+1)^2,0)$. On the other hand, the Frazer equation also gives a value, in terms of known channel 1 integrals and $\operatorname{Im} f_+^0(t)$. Equating these two values gives

$$\pi^{-1}\int\frac{16\pi\operatorname{Im} f_+^0(t')}{(4M^2-t')(t'-t)}dt'=0.5\pm 0.1.$$

The sets of values (1) and (2) of Table II give for this integral the values 0.58 and 0.45. The agreement gives added confidence both in the sets (1) and (2) and in the Frazer relation.

3.4. Summary of the Numerical Results of Sec. 3

Defining

$$g(t)=\pi^{-1}\int_{200}^{\infty}\frac{\operatorname{Im} G(s't)}{(s'-s)(s'-u)}ds',$$

$$g(t)=\pi^{-1}\int_{200}^{\infty}\operatorname{Im} G(s't)\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]ds'$$

[roughly independent of s for $0\lesssim s\lesssim (M+1)^2$], we

[19] There are no terms like g', g'' in (16) since order-of-magnitude estimates predict that they are negligible.

[20] The separation point s_1 may be taken anywhere between the first and second resonances (at $s=80$, 120) without altering the results very much.

[21] G. L. Kane and T. D. Spearman, Phys. Rev. Letters **11**, 45 (1963).

[22] G. C. Oades, Phys. Rev. **132**, 1277 (1963).

have obtained

$$\tilde{b}^{(+)\prime\prime}(0) \simeq 0.000015,$$
$$\tilde{a}^{(-)\prime}(0) \simeq -0.0025,$$
$$\tilde{a}^{(-)\prime\prime}(0) \simeq -0.00020,$$
$$b^{(-)\prime}(0) \simeq 0.10,$$
$$b^{(-)\prime\prime}(0) \simeq 0.012.$$

In addition we have obtained

$$16\int \frac{\operatorname{Im} f_+{}^0(t')}{(4M^2-t')(t'^2)}\,dt' = 0.5\pm 0.1,$$
$$R_+{}^2 = 8.5,$$
$$R_-{}^2 = 0.7,$$

with

$$\operatorname{Im} f_\pm{}^2 = R_\pm{}^2\delta(t-80).$$

Finally we shall need a value for $A^{(+)(2)}((M+1)^2, 0)$ in terms of Im $f_+{}^0$. Either the fixed s or the Frazer relation gives

$$A^{(+)\prime\prime}((M+1)^2, 0) = (0.009\pm 0.010) + \frac{2}{\pi}\int \frac{16\pi \operatorname{Im} f_+{}^0(t')}{(4M^2-t')t'^3}\,dt,$$

where the large error on the first term is due to a cancellation at large quantities; however, the second term is almost certainly dominant.

SECTION 4

4.1. The Hamilton–Woolcock Calculation of the Partial Waves

The rest of this article consists essentially of a discusion of the H.W. calculation mentioned in the Introduction. It is of interest because that calculation required no knowledge of the channel 2 ($\pi\pi\to NN$) amplitude (it essentially used the "CGLN" method, proposed by Chew *et al.*[2] in the first-ever application of dispersion relations to the pion–nucleon problem), and yet it appeared to give good results.

However, as H.W. pointed out, the results may be subject to large errors except at low energies. Our conclusion is that these errors are indeed present, and that, therefore, the "CGLN" method is not reliable except at low energies; furthermore, even when the method is modified by using a knowledge of the amplitude in channel 2 ($\pi\pi\to NN$), so as to give agreement with experiment up to higher energies, undesirable cancellations occur. It may fairly be said, therefore, that single-variable dispersion relations are not suitable for discussing the partial waves except at low energies, and that their principle role in a complete theory would be to provide *sum rules* of the type used by H.W. in Ref. 1 and by us in Sec. 3.

As was mentioned in the Introduction, H.W. used a Taylor series

$$G(st) \simeq G(s0) + tG'(s0) + \tfrac{1}{2}t^2G''(s0). \tag{23}$$

Error may occur, therefore, either from an incorrect evaluation of this series [i.e., incorrect values for the $G^{(n)}(s0)$], or from the neglected remainder term being large. In this section we investigate the errors from the first source, and show that the improvement of the H.W. values for $G^{(n)}(s0)$ makes the agreement of the H.W. results with experiment worse rather than better. Then in the next section we show how to calculate the remainder and restore to a large extent the agreement with experiment.

4.2. The H.W. Calculation of the $G^{(n)}(s0)$

H.W. used a fixed t dispersion relation, making subtractions for some amplitudes but not for others. We discuss the question of subtractions first, then we discuss the evaluation of the dispersion integrals.

a. The Question of Subtractions

In the case of $A^{(\pm)}$, $A^{(\pm)\prime}$, and $B^{(\pm)}$ the s- and p-wave scattering lengths which H.W. had already calculated were used to calculate the values of the amplitudes at $s=(M+1)^2$ using Eq. (4).[23] Using these values, the amplitudes were then calculated from the subtracted relation (for $A^{(+)}$, $\tilde{A}^{(-)}$, $\tilde{B}^{(+)}$, and $B^{(-)}$)

$$G(st) = G((M+1)^2, t) + \pi^{-1}\int_{M^2}^{\infty} \operatorname{Im} G(s't) \times\left[\frac{1}{s'-s}+\frac{1}{s'-u}-\frac{1}{s'-(M+1)^2}-\frac{1}{s'-[(M-1)^2-t]}\right]ds'.$$

However, there is a complication in the $(-)$ case. The s- and p-wave scattering lengths had been calculated by making a least-squares fit to several relations involving them; one of these relations was the *unsubtracted* fixed t relation

$$G((M+1)^2, t) = \pi^{-1}\int_{M^2}^{\infty} \operatorname{Im} G(s't) \times\left[\frac{1}{s'-(M+1)^2}+\frac{1}{s'-[(M-1)^2-t]}\right]ds'.$$

Now in the case of the $p^{(-)}$ scattering lengths this was in fact the most important piece of data. Hence it is true for our purposes to say that $A^{(-)}$, $A^{(-)\prime}$, and $B^{(-)}$ were calculated by H.W. from *unsubtracted* relations, except that the combination $[A^{(-)}(s0)+B^{(-)}(s0)]$ at $s=(M+1)^2$, which is proportioned to the *s-wave* scattering length, was known from other sources.

In the case of $A^{(-)\prime\prime}$, $B^{(\pm)\prime}$, and $B^{(\pm)\prime\prime}$, an unsubtracted relation was used since no accurate information was available with which to make a subtraction.

[23] In the case of $A^{(\pm)\prime}$ the value of $B^{(\pm)\prime}$ is also required; the value obtained from the unsubtracted relation is used.

In the case of $A^{(+)''}$ also there was no accurate information, but since the $A^{(+)}$ amplitude *requires* a subtraction H.W. were forced to make one. The subtraction constant was estimated roughly using the available data on d waves at 310-MeV pion lab energy; however, H.W. emphasized that this procedure was liable to serious error.

b. The Accuracy of the Dispersion Integral

The H.W. evaluation of the dispersion integrals was probably substantially correct[24] except that in the $(-)$ isospin case Im $G(st)$ was set equal to zero for $s>200$ Since H.W. used (actually or effectively) unsubtracted relations for the $(-)$ case, this means that they effectively used our relation

$$G^{(-)}(st)=\pi^{-1}\int_{M^2}^{200}\text{Im } G^{(-)}(s't)\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]+g(t)$$

with $g(t)=0$, except that the combination $[4Ma^{(-)}(0)+b^{(-)}(0)]$ was known from the s-wave scattering length. [Note that $s-u=4M$ at the point $(s=(M+1)^2, t=0)$.]

4.3. Corrections to the $G^{(n)}(s0)$

From the above discussion, two main errors seem likely in the H.W. evaluation of the quantities $G^{(n)}(s0)$: first, $A^{(+)''}(s, 0)$ may be in error by a constant term since the subtraction constant was unreliable; second, the $(-)$ amplitudes may have appreciable errors coming from setting $g^{(-)}(t)=0$. Our work of Sec. 3 enables these errors to be corrected.

a. Case of $A^{(+)''}$

Here we need an independent estimate of

$$A^{(+)''}((M+1)^2, 0)$$

to check the H. W. value. This is provided by the analysis of Sec. 3 which gave

$$A^{(+)''}((M+1)^2, 0)\simeq(0.009\pm0.010)+\frac{2}{\pi}\int_4^\infty\frac{16\pi \text{ Im} f_+^0(t')}{(4M^2-t')t'^3}dt',$$

where the large error was caused by a cancellation of large quantities. The integral is unfortunately strongly dependent on the set of values used for Im $f_+^0(t)$; we obtain $A^{(+)''}((M+1)^2, 0)=0.14$, 0.02 using sets (1) and (2) of Table II. However, the H.W. value was -0.056, and no set of Im $f_+^0(t)$ allowed by the analysis of Ref. 16 will produce this value; hence *the H.W. value of $A^{(+)''}(s, 0)$ must be increased by a constant amount, probably about* 0.1 *to* 0.2.

b. The $(-)$ Amplitudes

The corrections to be made to the $(-)$ amplitudes due to the neglect of high-energy integrals are

$$\Delta A^{(-)}(st)=(s-u)a^{(-)}(t),$$
$$\Delta B^{(-)}(st)=b^{(-)}(t),$$

[24] With the exception mentioned in footnote 26.

except that one must have $\Delta[A^{(-)}(s0)+B^{(-)}(s0)]=0$ for $s=(M+1)^2$, to preserve the s-wave scattering length. Using the values of $a^{(-)'}$, $b^{(-)'}$, $a^{(-)''}$, and $b^{(-)''}$ obtained in Sec. 3, one thus obtains

$$\Delta B^{(-)}(s0)=b^{(-)}(0),$$
$$\Delta B^{(-)'}(s0)=0.10,$$
$$\Delta B^{(-)''}(s0)=0.012,$$
$$\Delta A^{(-)}(s0)=-[(2s-\Sigma)/4M]b^{(-)}(0),$$
$$\Delta A^{(-)'}(s0)=a^{(-)}(0)-0.0025(2s-\Sigma),$$
$$\Delta A^{(-)''}(s0)=-0.0025-0.0004(2s-\Sigma).$$

$a^{(-)}(0)$ and $b^{(-)}(0)$ are unknown, but a connection can be established between them since the total cross section is quite well-known to high energies. This enables one to estimate[11] that

$$a^{(-)}(0)+b^{(-)}(0)/4M\simeq0.008.$$

Hence $b^{(-)}$ may be eliminated from the above equations.

4.4. The Effect on the Partial Waves of the Corrections to the $G^{(n)}(s0)$

Having calculated the $G^{(n)}$, H.W. calculated the partial waves using the Taylor series (23) and the projection formulas (3).[25] H.W. give results only for s and p waves up to $s=70$ (100 MeV) and $s=90$ (300 MeV) for the $(+)$ and $(-)$ cases, respectively. This was because they rightly considered the truncated Taylor series to be unreliable for the other cases (see 5.1 below). However we shall be calculating the re-

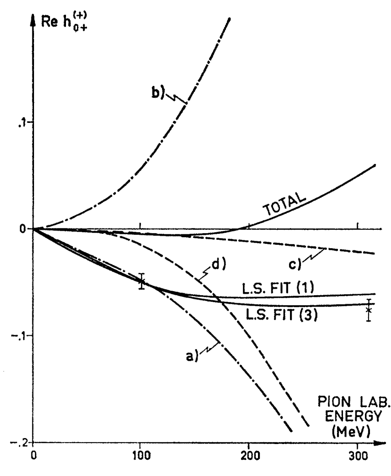

FIG. 3. $h_{0+}^{(+)}(s)$. (a), (b), (c), and (d) have the meanings assigned in Table VI. Experimental points are taken from Ref. 28 at 100 MeV, and Ref. 29 at 310 MeV.

[25] One obtains the series

$$f_{l\pm}(s)=\sum_{n=0}^{2}[c_{l\pm}^n(s)F_1^{(n)}(s0)+d_{l\pm}^n(s)F_2^{(n)}(s0)]$$

(with fairly simple coefficients $c_{l\pm}^n$, $d_{l\pm}^n$) already obtained using slightly different considerations by Chew *et al.* (Ref. 2). H.W. set $F_2^{(2)}=0$, but this makes a negligible difference.

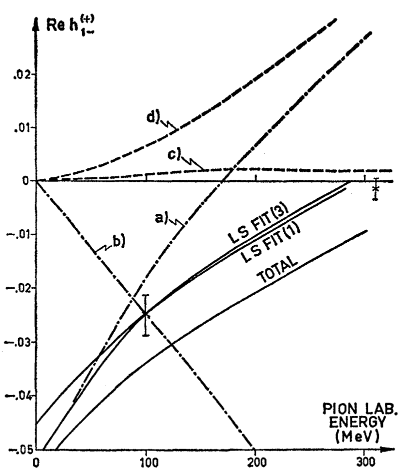

FIG. 4. $h_{1-}^{(+)}(s)$ (see Fig. 3 for explanation).

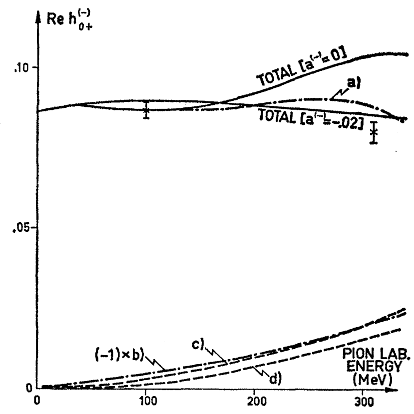

FIG. 6. $h_{0+}^{(-)}(s)$. (a), (b), (c), and (d) have the meanings assigned in Table VII. Experimental points are as in Fig. 3.

mainder term, so we give s, p, d, and f waves, up to $s=90$ for both $(+)$ and $(-)$ cases. In the cases where H.W. did not give results, we have calculated them using the values for $G^{(n)}(s0)$ given in Woolcock's thesis.[1]

The s- and p-wave[26] results are given in Figs. (3)–(8) [curve (a)] and those for d and f waves at $s=90$[27] in Tables VI and VII [row (a)].[28–30]

Next, we show in curves and rows (b) the corrections to the $h_{l\pm}$ coming from the corrections to the $G^{(n)}(s0)$ given in the last subsection, using the value 0.14 for $A^{(+)\prime\prime}(s_0, 0)$ (that obtained using column 2 of Table II) and setting $a^{(-)}=0$ for the moment. The corrections are seen to destroy the agreement with experiment for s and p waves; furthermore, the s waves especially are seen to be very sensitive to these comparatively small corrections [the situation for the $(-)$ case is worse than it looks because the various corrections tend to cancel], so the method already begins to look rather dangerous. Also in the $(+)$ case the results are much too large, indicating that the remainder term is going to tend to *cancel* the truncated series; this is to be expected according to the discussion of 5.1 below, but is a further objection to the method.

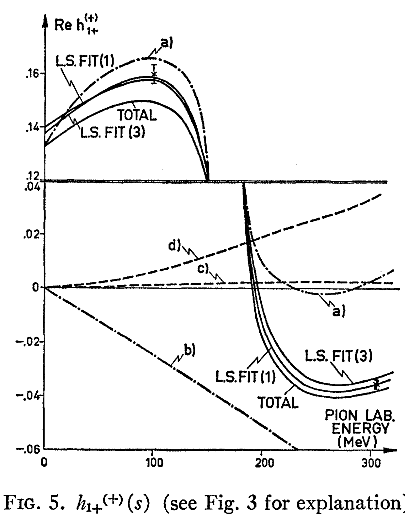

FIG. 5. $h_{1+}^{(+)}(s)$ (see Fig. 3 for explanation).

SECTION 5

5.1. Qualitative Discussion of the Remainder Term

The results for the partial waves given by the truncated Taylor series alone may now be obtained by adding curves or rows (a) and (b) of the figures and tables. The qualitative features are the same as those shown by (a) alone (H.W. result), namely that there is fair agreement with experiment at $s=70$,

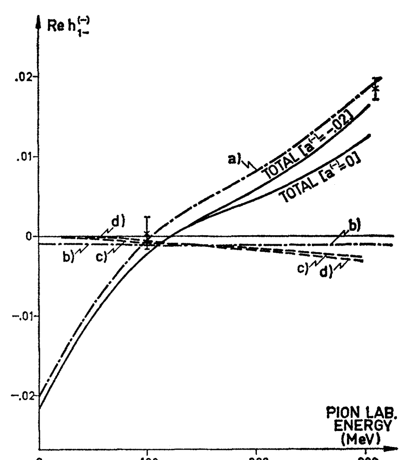

FIG. 7. $h_{1-}^{(-)}(s)$ (see Fig. 6 for explanation).

[26] In the case of $h_{1-}^{(\pm)}$ there is an additional source of error in the H.W. calculation due to the neglect of the rapidly varying $\mathrm{Im}\, f_{1-}^{\frac{3}{2}}(s)$ in calculating the $G^{(n)}(s0)$. This has been corrected for (see Ref. 11 for details and the reason why probably only $h_{1-}^{(\pm)}$ will be affected).

[27] No comparison with experiment is possible for $s<90$.

[28] D. Edmonds, S. Frank, and J. Holt, Proc. Phys. Soc. (London) **73**, 856 (1959); also D. Edwards and T. Massam (private communication to J. Hamilton, quoted in H.W.).

[29] O. Vik and H. Rugge, Phys. Rev. **129**, 2311 (1963).

[30] P. Auvil, A. Donnachie, C. Lovelace, and A. T. Lea (to be published).

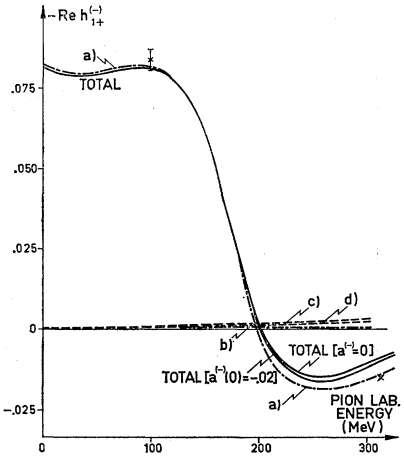

FIG. 8. $h_{1+}^{(-)}(s)$ (see Fig. 6 for explanation).

but that above this point the results become increasingly bad, especially for the (+) case. Hamilton and Woodcock had already explained this state of affairs in Ref. 1, as follows.

The full Taylor series converges for $|t|<|t_0|$, where t_0 is the position of the nearest singularity of $G(st)$ in the t plane, at fixed s. In fact, $t_0=4$ for all amplitudes and all s values, so the series converge for $|t|<4$.

In order to calculate the partial waves one needs $G(st)$ for all physical t values, which are (for a given s)

$$-4q^2 \leq t \leq 0.$$

Hence the Taylor series converges for all the t values required provided that $q^2<1$, i.e., $s<68$. If the full series does not converge, the truncated series will probably have a large remainder term, so this explains why the results for the partial waves become worse above $s \simeq 70$. H.W. explained the fact that the (−) results are better than the (+) ones by noticing that the (−) amplitudes will have a small discontinuity across the cut $t>4$ below about $t=28$ [the position of the resonance in $f_{\pm}^{1}(t)$, the lowest wave to contribute]. Hence t_0 is effectively ~ 28 and the series should give reasonable results for $q^2 \lesssim 7$, i.e., $s \lesssim 90$, as is indeed observed.

This discussion suggests that the way to calculate the remainder to the truncated series is to consider the analytic properties of the amplitudes at fixed s. We present two methods. The first (which is original) uses an explicit knowledge of the amplitude in channel 2 and gives good results; the second (due to Atkinson[3]) uses only a knowledge of the position of the nearest singularities, but it does not give good results in practice.

TABLE VI. d and f waves at 310 MeV, (+) case. (a) Truncated Taylor series using H.W.'s $G^{(n)}(s0)$. (b) Corrections to truncated series arising from our correction to $A^{(+)\prime\prime}(s0)$. (c) Born pole contribution to remainder to Taylor series. (d) Channel 2 contribution to remainder.

	$10^4\times h_{2-}$	$10^4\times h_{2+}$	$10^5\times h_{3-}$	$10^5\times h_{3+}$
(a)	−2.5	−7.0	0.6	0
(b)	31.1	31.3	−1.7	0
(c)	−1.6	−1.9	0.4	1.2
(d)	−16.9	−18.4	1.7	3.5
Total	10.1	2.0	1.0	4.7
Experiment (Ref. 29)	6.7	−6.5	1.0	2.8

5.2. First Method of Calculating the Remainder to the Taylor Series

This consists in noting that in the three-times subtracted fixed s dispersion relation,

$$G(st)=G(s0)+tG'(s0)+\tfrac{1}{2}t^2G''(s0)$$
$$+\pi^{-1}\int_4^\infty \operatorname{Im} G(st')\left[\frac{1}{t'-t}\frac{t^3}{t'^3}\right]dt'$$
$$+\pi^{-1}\int_{M^2}^\infty \operatorname{Im} G(u', \Sigma-u'-s)\left[\frac{1}{u'-u}\frac{(u-\Sigma+s)^3}{(u'-\Sigma+s)^3}\right]du', \tag{24}$$

the first three terms constitute the truncated Taylor series (23) and therefore the integrals must constitute the remainder. The projection of the partial waves may be carried out without difficulty, the denominators $(t'-t)$ and $(u'-u)$ giving rise to Q functions in the usual way.

There are three contributions to this remainder, each of which will be considered in turn.

The *pole at* $u'=M^2$ gives a small but not negligible contribution, which is shown in Figs. 3–8 [curve (c)] and Tables VI and VII [row (c)].

TABLE VII. d and f waves at 310 MeV, (−) case. (a)–(d) as for Table VI, except (b) corrections to truncated series arising from proposed corrections to A', A'', B', and B'' with $a^{(-)}(0)=0$.

	$10^4\times h_{2-}$	$10^4\times h_{2+}$	$10\times h_{3-}$	$10^5\times h_{3+}$
(a)	6.7	0.9	1.2	0
(b)	0.8	1.3	1.3	0
(c)	1.6	1.9	−0.4	−1.2
(d)	2.8	0.8	−1.6	−0.2
Total	11.9	3.8	0.5	−1.4
Experiment (Ref. 30)	4.7	2.6	2.0	−1.4

The *integral* $u'>(M+1)^2$ causes a difficulty, since Fig. 1 shows that the absorptive part is not given by a convergent partial-wave expansion except for $u'\simeq$

$(M+1)^2+3$ [we are in the region $s>(M+1)^2$]. However the three subtractions suppress this integral, and it will be assumed negligible. This assumption is supported by the fact that the contributions of the first two resonances to the (divergent) partial-wave expansion for the absorptive part give negligable contributions to the integral.

The *integral* $t'>4$ can be evaluated since Fig. 1 shows that the absorptive part on the nearby part of the cut is given by a convergent partial-wave expansion for $s \lesssim 80$ and hopefully for somewhat larger s values. The contributions from $\mathrm{Im}\, f_+{}^0$ and $\mathrm{Im}\, f_\pm{}'$ affect only the (+) (−) isospin cases, respectively; they are shown in Figs. 3–8 [curve (d)] and Tables VI and VII [row (d)], set (1) of $\mathrm{Im}\, F_0{}^+(t)$ (Table II) being taken for the moment. The contributions from $\mathrm{Im}\, F_+{}^0(t)$ are seen to be large, and to tend to cancel the divergent behavior of the Taylor series as is expected.

The final results for the partial waves are shown in Figs. 3–8 (curve marked "TOTAL") and Tables VI and VII. The addition of the remainder of the truncated Taylor series has clearly improved the results obtained by using the truncated Taylor series alone [i.e., the sum of curves (a) and (b)], and we conclude, therefore, that this method of calculating the remainder has been successful. It is shown in Sec. 5.4 that the results for s and p waves can be still further improved by making slight changes in the $G^{(n)}(s0)$, and it is also shown there that the results are not very different if set (2) of $\mathrm{Im}\, F_+{}^0(t)$ is used instead of set (1).

5.3. Second Method of Calculating the Remainder

This method was suggested by Atkinson,[3] and we refer to his paper for details. Atkinson only applied the method to one wave [$f_{0+}{}^{(-)}(s)$], and his result actually contains a numerical error (Atkinson, private communication).

The first step is to subtract out the Born term explicitly from the B amplitudes, defining a quantity

$$\bar{B}^{(\pm)}(st) = B^{(\pm)}(st) - g^2\left(\frac{1}{M^2-s} \mp \frac{1}{M^2-u}\right).$$

From now on $\bar{G}$ will represent any of $A^{(\pm)}$ or $\bar{B}^{(\pm)}$.

Next one transforms from the variable t to a variable

$$\omega(st) = \left[\left(\frac{x_2-x}{x_2-1}\right)^{\frac{1}{2}} - \left(\frac{x_1-x}{x_1-1}\right)^{\frac{1}{2}}\right] \Big/ \left[\left(\frac{x_2-x}{x_2-1}\right)^{\frac{1}{2}} + \left(\frac{x_1-x}{x_1-1}\right)^{\frac{1}{2}}\right],$$

where

$$x = 1 + t/2q^2,$$

$$x_1 = 1 + 4/2y^2,$$

$$x_2 = 1 + [\Sigma - (M+1)^2 - s]/2q^2.$$

It is then shown that the Taylor series in ω for $\bar{G}(st)$,

$$\bar{G}(st) = \sum_{n=0}^{\infty} a_n(s)\,\omega^n$$

is convergent for all values of s and t (on the physical sheet). This series may be rewritten

$$\bar{G}(st) = \lim_{N\to\infty} \sum_{n=0}^{N} C_n{}^N(s)\,\bar{G}^{(n)}(s0)\,t^n,$$

where the coefficients $C_n{}^N(s)$ are known functions of s. Atkinson now argues that since this expression converges for all s and t it will be more accurate even if N is kept finite than the corresponding truncated Taylor series. Thus he replaces the series

$$G(st) \simeq \sum_{n=0}^{2} \frac{1}{n!} G^{(n)}(s0)\,t^n \tag{25}$$

by the series

$$\bar{G}(st) \simeq \sum_{n=0}^{2} C_n{}^2(s)\,\bar{G}^{(n)}(s0)\,t^n$$

or

$$G(st) \simeq \sum_{n=0}^{2} C_n{}^2(s)\,\bar{G}^{(n)}(s0)\,t^n + g^2\left(\frac{1}{M^2-s} \mp \frac{1}{M^2-u}\right), \tag{26}$$

where the g^2 term only appears of course in the B amplitudes. The remainder to the Taylor series in this approach is clearly given by subtracting (25) from (26) giving

$$R(st) \simeq \sum_{n=0}^{2} \bar{G}^{(n)}(s0)\,[C_n{}^2(s) - (n!)^{-1}]\,t^n$$

$$+ g^2\left\{\left(\frac{1}{M^2-s} \mp \frac{1}{M^2-u}\right) - \sum_{n=0}^{2}\left[\frac{\partial^n}{\partial t^n}\bigg|_{t=0}\left(\frac{1}{M^2-s} \mp \frac{1}{M^2-u}\right)\right] t^n\right\}$$

The term in g^2 is equal to the corresponding term obtained by the first method (described in the last section). Hence the first term should correspond to the integral over t' in the first method (plus, strictly, the integral over u' but this is probably negligible as has been seen). The contributions to the partial waves of this first term and of the t' integral are compared in Table VIII. For $s=70$ there is complete disagreement, and even for $s=90$ there is only qualitative agreement. Looking back at Figs. 3–8 and Tables VI and VII it is clear that if curve (row) (d) were replaced by the corresponding term obtained from this method there would be violent disagreement with experiment.

Thus it appears that this second method of determining the remainder to the Taylor series is not satisfactory with this low order of truncation; it is necessary to know more than just the first three $G^{(n)}(s0)$ if the amplitude is to be determined from a knowledge of the analytic properties alone.

TABLE VIII. Taylor series remainders (non-Born). (a) From fixed s dispersion relation. (b) From the mapping method. The results for h_{1+}, h_{2+} and h_{3+} are very similar to those for h_{1-}, h_{2-}, and h_{3-}.

	(+) case, $s=70$		(+) case, $s=90$		(−) case, $s=70$		(−) case, $s=90$	
	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
h_{0+}	−0.015	0.000	−0.29	−0.445	0.001	−0.007	0.017	0.023
h_{1-}	0.006	−0.003	0.035	0.052	−0.0060	0.0012	−0.002	+0.008
$10^4\times h_{2-}$	−15.0	00.5	−20	−33	0.7	−6.3	2.9	2.4
$10^5\times h_{3-}$	9	6	2.0	0.3	−1.4	−0.4	−1.5	1.2

5.4. Further Improvement of the Results Obtained in Sec. 5.2

The results of Sec. 5.2 are in moderate agreement with experiment. In this subsection it is shown that if certain small changes are made in the $G^{(n)}(s0)$ good agreement can be obtained for s and p waves (leaving the d and f waves practically unchanged). However, since we cannot properly assess the error to which either the $G^{(n)}(s0)$ or the remainder term are liable, it is not clear whether these small changes are significant.

a. (+) Case

If the values used by H.W. for the s- and p-wave scattering lengths (a_{0+}, a_{1-}, and a_{1+}) were in error, this would affect their values for A, A', and B as follows:

$$\Delta A^{(+)}(s0)=4\pi\left[\frac{2M+1}{2M}\Delta a_{0+}-2M(\Delta a_{1-}-\Delta a_{1+})\right],$$

$$\Delta A^{(+)\prime}(s0)=4\pi\left[\frac{M+1}{M}\tfrac{3}{2}\Delta a_{1+}\right],$$

$$\Delta B^{(+)}(s0)=\left[\frac{2s-\Sigma}{4M}\right]4\pi\times\left[\frac{2M+1}{2M}\Delta a_{0+}-2M(\Delta a_{1-}-\Delta a_{1+})\right],$$

and the results for the s and p partial waves would be changed accordingly.

We have therefore carried out a least-squares fit as follows. Results to be fitted: the experimental s- and p-wave results at $s=70$ and $s=90$ and the values for the scattering lengths calculated by H.W.,[1] all weighted according to their quoted errors. Parameters to be varied: the scattering lengths a_{0+}, a_{1-}, and a_{1+}, giving rise to variations in A', A, and B and hence to variations in the theoretical values of the results to be fitted.

The partial waves obtained from this fit are shown in Figs. 3–8 [curve "L.S. Fit (1)"]. There is clearly a considerable improvement. The corresponding scattering lengths are shown in Table IX; a_{0+} is seen to agree well with the H.W. calculated value, $a_{1\pm}$ to agree only moderately well.

So far only one set of Im $f_+{}^0(t)$ has been used, set (1) of Table 2. We have repeated the whole calculation using several widely different sets of Im$f_+{}^0(t)$ (calculated by G. C. Oades[7] in terms of a two-parameter formula for the s-wave $\pi\pi$ phase shift; we are indebted to Dr. Oades for supplying the sets). The set giving best values[31] for the least-squares fit described above are shown in the last column of Table II and the results for the partial waves are shown in Figs. 3–8. The values obtained for a_{0+} and $a_{1\pm}$ are shown in Table IX. It is seen that the improvement in the fit to the partial waves and scattering lengths when these new values of Im$f_+{}^0(t)$ are used is not very great and in fact Im $f_+{}^0(t)$ is more likely to be given by one of the first two sets; however it is of interest that our fixed variable approach gives results for Im$f_+{}^0(t)$ in at least qualitative agreement with the partial-wave approach of Ref. 16.

TABLE IX. Alterations in scattering lengths, (+) case.

	Δa_{0+}	Δa_{1-}	Δa_{1+}
Set (1)	−0.003	0.009	0.007
Set (3)	−0.004	0.004	0.004
Changes proposed in Ref. 16	···	0.010	0.008
H.W. errors	0.004	0.005	0.005

b. (−) Case

In Sec. (4.2) $a^{(-)}(0)$ was set equal to zero. The results for s and p waves using $a^{(-)}(0)=-0.02$ are

TABLE X. Alterations in scattering lengths, (−) case.

	Δa_{1-}	Δa_{1+}
This calculation	−0.001	−0.0008
Changes proposed in Ref. 32	0.004	0.0007
H.W. errors	0.003	0.0020

[31] A set which changed sign also gave a good fit, in analogy with set (2), but the corresponding rate of change of the $\pi\pi$ phase shift is so large as to be hardly acceptable physically.

given in Figs. 3–8. The agreement with experiment is clearly improved, but again we cannot be sure that this is significant. The small changes[32] predicted in the p wave scattering lengths from the H.W. values are given in Table X.

SUMMARY OF CONCLUSIONS AND RESULTS

In Sec. 3 the fixed s and fixed t relations were equated; the assumption that distant singularities give slowly varying terms led to Eqs. (16) in the cases of $\tilde{A}^{(-)}$, $\tilde{B}^{(+)}$, and $B^{(-)}$, and to a similar equation in the case of $A^{(+)}$. The evaluation (where possible) of the integrals, using the input data described in Sec. 2, led to the results given at the end of the section for the unknown terms in the equations (i.e., the f_0–N–N coupling constants and the integrals over the high-energy πN amplitudes).

In Sec. 4, the results of Sec. 3 were used to improve the H.W. evaluation of the truncated Taylor series (23). The agreement of the result with experiment was made *worse* by this improvement, but in Sec. 5.2 the hitherto neglected remainder term was evaluated (by using a fixed energy dispersion relation) and fairly satisfactory agreement with experiment was obtained for s and p waves up to 300-MeV pion lab. energy.

In Sec. 5.3 an alternative suggestion due to Atkinson[3] for evaluating the remainder term *without* an explicit knowledge of the channel $2(\pi\pi\rightarrow NN)$ amplitude was shown not to give good results in practice. Finally in Sec. 5.4 it was shown that the results of Sec. 5.2 could be improved if small changes were made in the p-wave scattering lengths (used in evaluating the truncated Taylor series), and still further improved by using a set of values of $\mathrm{Im}\, F_+^0(t)$ somewhat smaller than those of Ref. 16; however, it was emphasized that these improvements may not be significant.

Note added in proof. The constants $R_\pm^2$, defined by

$$\mathrm{Im}\, f_\pm^2(t) \simeq R_\pm^2(t-80)$$

which we have loosely called 'the f^0–N–N coupling constants' are more precisely given by

$$R_\pm^2 = (\text{kinematic constant}) \times (f^0\text{–}\pi\text{–}\pi \text{ coupling constant}) \times (f^0\text{–}N\text{–}N \text{ coupling constant})$$

but of course the first two factors are known. The first is just a matter of definition and the second is known in terms of the width and elasticity of the observed f^0 resonance in $\pi\pi$ scattering.

[32] L. L. J. Vick, Nuovo Cimento **31**, 643 (1964).

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