Dispersion Relations and Direct Nuclear Reactions*

HOWARD J. SCHNITZER†

Department of Mathematics, University of California, Berkeley, California

Department of Physics,‡ Brandeis University, Waltham, Massachusetts

This review is concerned with the application of dispersion relation methods to direct nuclear reactions. We begin with a discussion of the assumptions required to relate analyticity properties to the conventional descriptions of direct-reaction amplitudes. This leads naturally to a description of phenomenological methods such as extrapolation to poles and similar topics. The use of the unitarity condition provides a framework for the discussion of the distorted wave Born approximation from the S-matrix point of view Finally we call attention to some dynamical calculations of relevance to our discussion.

I. INTRODUCTION

Some five years ago Amado[1] considered the plane-wave Born approximation for (d, p) reactions from the point of view of its analytic properties. He noted that the Born amplitude exhibits a pole because of single nucleon exchange in the complex $\cos\theta$ plane very near the physical region. The characteristic forward peaking of stripping angular distributions could then be thought of as resulting from a dominant contribution of this pole near $\cos\theta=1$. Furthermore, the location of the pole can be calculated from a Feynman diagram involving single-nucleon exchange. Since this situation has some analogy with many examples from elementary particle physics, it has prompted attempts to apply the methods of analyticity, unitarity, and dispersion relations to nuclear reactions. These efforts have been in various directions, but as yet there has been no review of the status of these "new methods." It is the purpose of this paper to give an account of this research which will be accessible to experimentalists as as well as theorists.

A few words about the over-all objectives of the program might be in order. One should recognize that the application of dispersion methods to nuclear reactions is still exploratory, and as such should not be considered as an immediate replacement for detailed numerical studies of distorted-wave Born calculations and the like. What then can be expected at this stage of development? Much of the work that has been done to date is somewhat pedagogical in that it has reformulated what was already known in terms of the language of analyticity and unitarity. However, new derivations of old results are useful when they give additional insight into the limitations of approximation schemes and suggest new experiments. Furthermore, they may serve as a conceptual guide to new calculational schemes or approximations which, depending on the type of theory being attempted, may or may not include the familiar approximations as special cases. Since dynamical dispersion theories involve nonlinear integral equations, but only on-the-energy-shell quantities, these extensions of the theory may be different than those obtained from Schrödinger's equation or the equivalent Lipmann–Schwinger linear integral equation which involve off-the-energy-shell quantities and the construction of potentials. If exact solutions to both calculational methods were available, one would expect the results to be identical, but where approximations must be made, one or the other may have considerable calculational or conceptual advantage. In addition, one may prefer alternate formulations when nuclear reaction theory is extended to higher energies, as the construction of potentials for these cases may not be well defined. These are some of the more ambitious aims of this program; however definite conclusions must await further work.

Here we limit ourselves to explicit applications of dispersion methods, but omit discussing such related topics as applications of the Bethe–Salpeter equation and Feddeev equations. In Sec. II we discuss the analytic properties of reaction amplitudes, while the methods of pology are emphasized in Sec. III. Section IV is devoted to a discussion of unitarity; in particular a derivation by Omnes[2] of the distorted-wave Born approximation (DWBA) from unitarity and some reasonable statistical assumptions based on empirical information. Attempts at dynamical calculations are briefly surveyed in Sec. V and some concluding remarks are given in Sec. VI. Here we emphasize the status of the theory without giving a great deal of detail, except for a few selected topics of interest. The topics emphasized necessarily reflect the tastes of the author. We apologize to those whose work we have inadvertetly omitted or, misrepresented.

II. LOCATION OF SINGULARITIES

We are interested in using dispersion relations and analytic properties as tools for extending our under-

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† Alfred P. Sloan Foundation Fellow.

‡ Permanent address.

[1] R. D. Amado, Phys. Rev. Letters **2**, 399 (1959).

[2] Roland Omnes, Phys. Rev. **137**, B649 (1965). We wish to thank Professor Omnes for making a copy of his manuscript available to us in advance of publication.

standing of nuclear transfer reactions, and in particular the so-called direct interactions. It is customary to consider the analytic properties of the transition matrix as functions of complex variables as a prelude to phenomenological applications or dynamical calculations. For two-body reactions $(a+A\to b+B)$ there are two independent kinematic variables, energy and momentum transfer, which can be analytically continued from physical to unphysical values. In general, the n-point amplitude has $3n-10$ independent kinematic variables, all of which may be continued. However, in this review we will emphasize the discussion of the 4-point amplitude, although in many places some generalizations of the results are possible.

The first step in this program was made by Amado[1] who pointed out that the Butler mechanism for deuteron stripping led to a pole in the complex $\cos\theta$ plane on the real axis at value of $\cos\theta>1$, but near the physical region. This result can be reinterpreted as a pole of a Feynman graph due to particle exchange, shown in Fig. 1, for the reaction $A(a, b)B$. In this language, the pole dominates the exchange reaction because it is the nearest singularity to $\cos\theta=1$, giving the qualitative features of the forward-peaked angular distribution characteristic of these reactions. (Later we give a more systematic discussion of the assumptions made in locating singularities.) One can explicitly write down the plane-wave Born approximation for a particle exchange of the type $A(a, b)B$, where particle x is exchanged. Aside from inessential kinematic factors (and neglecting spin complications), the matrix element is[3]

$$T(\text{PWBA})\sim I_l(q)\,(K^2+\alpha^2)\,J(K), \qquad \text{(II.1)}$$

where

$$\mathbf{K}=(m_b/m_a)\mathbf{k}_a-\mathbf{k}_b, \qquad \mathbf{q}=-[\mathbf{k}_A-(m_A/m_B)\mathbf{k}_B],$$

are the Galilean invariant momentum transfer associated with the vertices of the diagram in Fig. 1. The overlap integrals are defined by

$$J(K)=\int_0^\infty dr r^2 G(r) j_0(Kr) \qquad \text{(II.2)}$$

and

$$I(q)=\int_0^\infty dr r^2 F_l(r) j_l(qr), \qquad \text{(II.3)}$$

where A, a, b, and x are assumed to have spin zero, l is the captured orbital angular momentum, $G(r)$ is the bound-state radial wave function of particle $a\to b+x$, $F(r)$ is the bound-state wave function of particle $B\to A+x$, j_l are the spherical Bessel functions, and $i\alpha$ is the bound-state wave number of particle x associated with the vertex $a\to b+x$. We assume that the potentials that bind x, $V(r_{bx})$, and $V(r_{Ax})$, can be described by superpositions of Yukawa potentials. We hope that this is not unreasonable, since the nucleon–nucleon interactions from which these potentials are derived are of this type.[4] This then enables us to use the results of Bertocchi *et al.*[5] to discuss the analyticity of the form factors occurring in Eqs. (II.2) and (II.3). The bound-state wave function $F_l(r)$ is given by a superposition of spherical Hankel functions:

$$F_l(r)=-\int_0^\infty \rho_l(\sigma)\sigma h_l^{(1)}(i\sigma r)\,d\sigma,$$

where

$$\rho_l(\sigma)=N_l\delta(\sigma-k_x)+\rho_l'(\sigma)\theta(\sigma-\beta^2),$$

$$\beta^2=(k_x+\mu),$$

$$\int_0^\infty \rho_l(\sigma)\sigma^m\,d\sigma=0 \qquad (m=-l, \cdots, +l), \qquad \text{(II.4)}$$

ik_x is the wave number for x in the vertex $B\to A+x$, and μ is the longest range Yujawa potential in $V(r_{Ax})$. Throughout this paper we take μ to be the pion mass since we wish to identify the longest range interaction (and hence the nearest singularities) with the lightest strongly interacting particle that can be exchanged between A and x. On the other hand, if one wanted $V(r_{Ax})$ only to be an equivalent interaction (as is customary in DWBA calculations), then we would use μ_{eff} instead of μ, where μ_{eff}^{-1} represents the phenomenological size of this interaction. In general, $\mu_{\text{eff}}\ll\mu$, hence we are faced with the problem of an interaction which is large over a distance much larger than the range characterized by the lightest particle exchange. We postpone a discussion of the problem, emphasized by Eden and Goldstone,[4] to Sec. V and concentrate on the formal question of locating singularities. Using the wave function of (II.4), we find

$$I_l(q)=N_l\left(\frac{q}{k_x}\right)^l\frac{1}{q^2+k_x^2}+\int_{\beta^2}^\infty d\sigma\rho_l(\sigma)\left(\frac{q}{\sigma}\right)^l\frac{1}{q^2+\sigma^2} \qquad \text{(II.5)}$$

and a similar expression for $J(K)$. Therefore, aside from trivial factors,[6]

$$T(\text{PWBA})\sim(q/k_x)^l N_l N_0[\Gamma_B(q)\,\Gamma_a(K)/(q^2+k_x^2)],$$

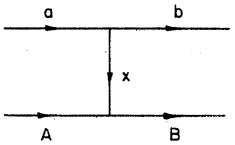

FIG. 1. Pole diagram or single-particle exchange in the reaction $A(a, b)B$.

[3] For example, see N. Austern in *Selected Topics in Nuclear Theory* (International Atomic Energy Agency, Vienna, 1963), p. 17 and in particular pp. 42–43.

[4] J. Charap and S. Fubini, Nuovo Cimento **14**, 540 (1959); R. J. Eden and J. Goldstone, Nucl. Phys. **49**, 33 (1963).

[5] L. Bertocchi, C. Ceolin, and M. Tonin, Nuovo Cimento **18**, 770 (1960).

[6] C. Dullemond and H. J. Schnitzer, Phys. Rev. **129**, 821 (1963); and *Proceedings of the Padua Conference on Direct Interactions and Nuclear Reaction Mechanisms* (Gordon and Breach, Science Publishers, New York, 1963), p. 420.

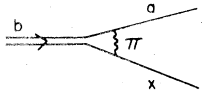

FIG. 2. Diagram for one-pion contribution to the vertex $b \to a+x$.

with

$$(q^2+k_x^2)I_l(q)=N_l(q/k_x)^l\Gamma_B(q), \qquad \text{(II.6)}$$

and

$$(K^2+\alpha^2)J(K)=N_0\Gamma_a(K).$$

The existence of the pole at $q=ik_x$ is independent of our assumption of Yukawa potentials; it just comes from the asymptotic part of the wave functions. However, the vertex functions $\Gamma_B(q)$ and $\Gamma_a(K)$ have branch cuts, coming from the continuum of integral (II.5), which do depend on this assumption. It is easily seen from (II.6) that the normalization of the asymptotic amplitudes, i.e., the reduced widths, play a role analogous to that of coupling constants in elementary particle theories. One can also discuss the analytic properties of the DWBA if the Yukawa assumption is again made for the potentials. This has been studied by Yazaki,[7] using methods similar to those of Bosco.[8] He assumes that both optical potentials have the spectral representation

$$U(E,r)=v(r)+\pi^{-1}\int_{E_t}^{\infty}\frac{dE'\mathrm{Im}U(E',r)}{E'-E-i\epsilon}, \qquad \text{(II.7)}$$

where $v(r)$ is real and E_t is the threshold for inelastic processes. As usual, it is further assumed that

$$U(E,r)=\int_{\mu}^{\infty}d\sigma\tilde{U}(E,\sigma)\frac{e^{-\sigma r}}{r},$$

where μ has a meaning similar to that discussed in connection with the pole term. The initial and final distorted waves are solutions of the Schrödinger equation with optical potentials, Eq. (II.7), and the bound-state wave functions taken as in (II.4). To discuss the wave functions, the work of Bertocchi *et al.*[5] and de Alfaro and Rossetti[9] is relevant. One then can do enough integrations to exhibit the singularities for a single partial wave explicitly. One can verify that this calculation gives the same analyticity properties as those computed from Feynman diagrams (taken in the nonrelativistic limit, of course). One again obtains a pole and the contributions of the vertex functions of

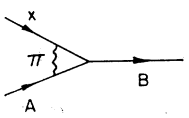

FIG. 3. One-pion contribution to the vertex $B \to A+x$.

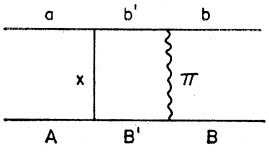

FIG. 4. A one-pion contribution to the final-state rescattering in the reaction $A(a,b)B$.

(II.6). These can also be obtained from Feynman graphs, with leading cuts coming from one-pion exchange.[10] The location of the singularities are found to be identical in the two approaches with the graphs contributing to these nearest singularities being those of Figs. 1, 2, and 3. Similarly, diagrams in Figs. 4, 5, and 6 give the same singularities as the rescattering terms of the DWBA.[7,11] We find it quite natural to define the class of singularities we have been discussing as potential singularities. This terminology will be used throughout.

To extend our discussion of the analyticity of nuclear-reaction matrix elements beyond the DWBA, still further assumptions are necessary. Our previous discussions make the following hypotheses plausible[12]:

(a) The singularities of a nuclear-reaction matrix element can be determined from the set of all possible *nonrelativistic* Feynman graphs for this process. Therefore, all nuclei (including excited states stable under strong interactions) are treated on the same basis as the nucleon. Virtual meson exchanges must not be omitted from these diagrams if potential singularities are to be included in the discussion of analytic properties. With these assumptions nuclei can emit either virtual nucleons or mesons.

(b) To add dynamics to the discussion, one can further assume that the discontinuity across a branch cut of a matrix element is given by the discontinuity of the corresponding reduced graph.[13]

Although these assumptions are very reasonable, they certainly cannot be considered to have been derived from the A-nucleon Schrödinger equation. Only the two-nucleon problem and single-particle potential scattering can be considered to be solved in this sense.[5,10,14] In fact, the study of the analyticity of three-nucleon Schrödinger amplitudes is still not complete. On the other hand, the analytic properties of Feynman

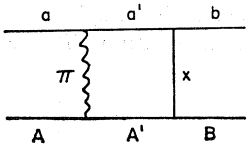

FIG. 5. A one-pion contribution to the initial-state rescattering in the reaction $A(a,b)B$.

[7] K. Yazaki, Progr. Theoret. Phys. (Kyoto) **31**, 772 (1964). However, the location of left-hand singularities by Yazaki at $k_i=i_i/2$ and $k_f=i_f/2$ for this DWBA matrix element is in error. These singularities are absent.

[8] B. Bosco, Phys. Rev. **123**, 1072 (1964).

[9] V. DeAlfaro and M. Rossetti, Nuovo Cimento **18**, 783 (1960).

[10] R. Blankenbecler and L. Cook, Phys. Rev. **119**, 1745 (1960).

[11] H. J. Schnitzer, Nucl. Phys. **36**, 505 (1962).

[12] These assumptions have been emphasized by P. Stechel, Nuovo Cimento **31**, 250 (1964).

[13] L. B. Okun and A. P. Rudik, Nucl. Phys. **15**, 261 (1960).

[14] R. Blankenbecler, M. L. Goldberger, N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) **10**, 62 (1960).

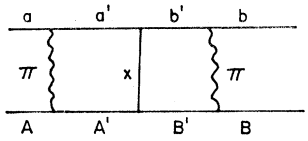

FIG. 6. A diagram for the reaction $A(a, b)B$ with both final- and initial-state rescattering.

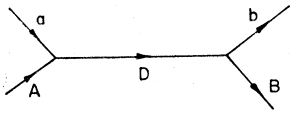

FIG. 8. Quasi compound-nucleus contribution to $A(a, b)B$.

graphs can be determined by means of Landau rules.[15,16] Furthermore, if one is considering *nonrelativistic* dispersion theory, one can assume that the kinetic energy of a nucleon or nucleus is small compared with its rest mass. Then the relativistic Feynman denominators $p^2-m^2+i\epsilon$ are replaced with the nonrelativistic denominator $2mE-\mathbf{p}^2+i\epsilon$, and the integration variables for independent loop momenta are $d^3k dE_k$. A detailed study of nonrelativistic Feynman graphs is possible and has been made for a subset of possible graphs by Blokhintsev *et al.*,[17] where all graphs containing *no meson lines* are considered. This is equivalent to the analyticity obtained using only zero-range potentials. We will call such graphs cluster graphs, and the resulting singularities cluster singularities. They exhibit some of the many-body structure of the nuclei involved, but with zero-range potentials. The remaining diagrams contain many-body features, but the inclusion of meson lines means that the finite range of nucleus–nucleus interactions are included.

The preceding discussion summarizes the assumptions needed to compute the singularities of reaction amplitudes from distorted-wave theories or from Feynman graphs. We now give a brief sketch of some results, but detailed derivations, which can be found in published papers, will be omitted. There are problems involving analyticity which cannot be answered by the general formalism, but require reference to specific reactions; among these is the location of nearest singularities. It is rather simple to exhibit those graphs giving the nearest singularities. Recall that graphs such as shown in Fig. 1 give rise to a pole in the cos θ plane at a value of cos $\theta>1$. Similarly there is an exchange stripping pole (Fig. 7) which occurs for cos $\theta<-1$. In addition, there are graphs which are of a quasi compound-nucleus character (Fig. 8), where nucleus D represents any stable (to strong interactions) nuclear state which can be reached by $a+A$. This diagram gives a pole on the real axis of the energy plane *below* the energy threshold for any physical two-particle reaction connected to $a+A$. Multiparticle exchanges give rise to branch cuts in the appropriate energy or momentum transfer complex planes. For example, Figs. 4 and 5 will contribute cuts to both planes. In general, real two-particle intermediate states, which can be reached from $a+A$, give branch cuts in the energy planes associated with the channel thresholds.[18] When momentum transfer is held fixed, nonrelativistic two-body reaction graphs only have "right-hand" cuts (cuts running from E_0 to $E^2=+\infty$) in the energy plane. These just reflect the consequences of unitarity[18] and do not give much insight into the reaction mechanism. One also can transcribe compound-nucleus reactions into this language, where single-particle resonances give rise to a pair of complex conjugate poles in the second energy sheet which can be reached by analytic continuation through the unitarity cuts. Much more interesting from the point of view of understanding the nature of the reaction mechanism is the singularities in the invariant momentum transfer or (linearly related) cos θ plane. The potential singularities nearest to the particle exchange pole are given by Figs. 2–5. It can also be shown that Fig. 2(3) has the same cos θ branch point as does Fig. 4(5).[16] To illustrate the above points for a specific example, we plot the leading singularities for a typical reaction, $Si^{28}(d, p)\,Si^{29}$, in Fig. 9 when the deuteron laboratory energy is 6.2 MeV. The direct pole is at cos $\theta=1.4$, and the exchange pole is at cos $\theta=-15$. The singularity at cos $\theta=2.7$ is from the nuclear vertex and that at cos $\theta=3.0$ is from the deuteron vertex. The nearest cluster contribution to the nucleus form factor is given in Fig. 10. There is a similar contribution to the reaction amplitude, a triangle singularity, shown in Fig. 11. In general, one finds that the cluster singularities are more distant than the nearest potential singularities. Again appealing to the $Si^{28}(d, p)\,Si^{29}$ example, one finds the nearest cluster singularities of the triangle diagrams of Figs. 12 and 13 to be at cos $\theta=5.4$ ($E_d=6.2$ MeV). This

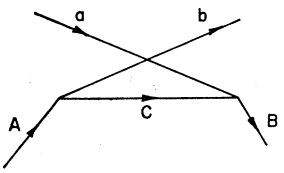

FIG. 7. Exchange (stripping) pole contribution to $A(a, b)B$.

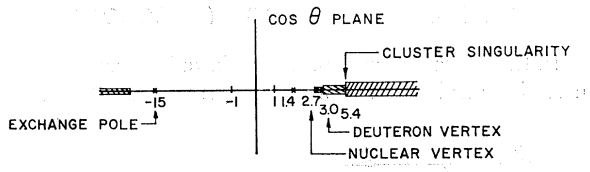

FIG. 9. The positions of the leading singularities in the complex cos θ plane for Si^{28} (d, p) Si^{29} when the deuteron laboratory kinetic energy is 6.2 MeV.

[15] L. D. Landau, Nucl. Phys. **13**, 181 (1959).
[16] See R. Eden in *Brandeis Lectures in Theoretical Physics, 1961* (W. A. Benjamin, Inc., New York, 1962), Vol. I, for detailed applications of this method.
[17] L. D. Blokuntsev, Nucl. Phys. **40**, 117 (1963).
[18] I. S. Shapiro, Nucl. Phys. **28**, 244 (1961); and *Selected Topics in Nuclear Theory* (International Atomic Energy Agency, Vienna, 1963), p. 85, contain further details.

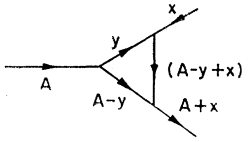

FIG. 10. A simple cluster contribution to the nuclear vertex $(A+x)\rightarrow(A)+x$.

illustrates a typical case in which the singularity nearest the pole is a potential cut.

Although this situation is generally true, there are some exceptional cases[19] where the cluster singularities are the closest to the Butler pole. This occurs for reactions with light nuclei, usually when $(A-y)$ [Fig. 10] has a closed-shell structure and y is a single proton. If x is a neutron and y is a proton, Stechel finds that the cluster diagram gives the leading singularity if either

(1) the nucleus $(A-y)$ has a doubly closed shell or subshell,

or

(2) the nucleus $(A-y)$ has a closed proton shell (or subshell) and one hole in the same shell (or subshell) for the neutrons.

The examples he found of this type are shown in Table I.[19] The entry (β/μ) indicates the range of the equivalent Yukawa potential $e^{-\beta r}/r$, normalized to a one-pion exchange potential $e^{-\mu r}/r$. Additional examples for $x=1$ are known if nucleus $(A-y)$ has an α-particle structure. They are tabulated in Table II.[19] There are probably more examples, but an exhaustive search for them has not been made. In the region of heavier nuclei, the cluster singularities are in general not the nearest singularities.

In the preceding we have summarized the assumptions used in finding the singularities of reaction amplitudes and given some of the simpler results. However, we remind the reader that this should not be considered a closed problem, since the analyticity of even the three-body Schrödinger amplitudes is not yet completely known. It is certain that there is much to be learned from the study of the analyticity of the N-body Schrödinger problem. Even though our knowledge of analyticity is still limited, the properties discussed here still can be a useful tool in the study of various reaction problems.

III. PHENOMENOLOGY (POLOLOGY)

In the preceding section we reviewed the assumptions used in obtaining the analytic properties of reaction

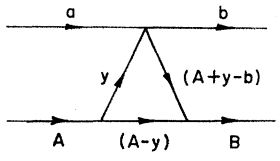

FIG. 11. A simple cluster contribution to the reaction $A(a,b)B$.

[19] P. Stechel, Nuovo Cimento **31**, 250 (1964).

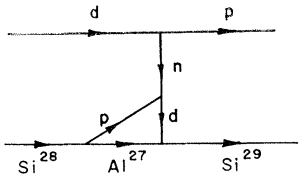

FIG. 12. A cluster contribution to Si^{28} (d, p) Si^{29}.

matrix elements. One of the results obtained was the analogy between renormalized coupling constants of elementary particle reactions and reduced widths of nuclear reactions of the form $A\rightarrow B+x$, which came from the connection between the residue of the particle exchange pole and the asymptotic normalization of the bound-state wave function. This defines the interaction strength (reduced width) for the vertex $A\rightarrow B+x$ since at the pole all three particles are on the energy shell. Since this definition is kinematically unique, it provides an invariant description of the reduced width which does not depend on which of the three particles, A, B, or x, is "exchanged" in a nuclear reaction; at the pole all three are physical.

One can press the analogy with particle physics further and attempt to obtain reduced widths by an extrapolation to the pole[20] in a way similar to the successful extrapolation to the pion pole in nucleon–nucleon scattering.[21,22] However, this procedure encounters many complications when applied to experimental direct-interaction angular distributions. The pole is generally further from the physical region than in the nucleon–nucleon applications and is not far enough removed from branch cuts to make things simple.[23] The Coulomb interactions further complicate matters. In diagram language, the lowest-order final-state Coulomb scattering contributes to diagrams of the form shown in Fig. 14. These give singularities in the $\cos\theta$ plane which begin at the Butler pole. Finally, to extrapolate experimental angular distributions with any degree of reliability, very accurate measurements would be required near the forward direction. Even if such an extrapolation were made, it would not be clear that the reduced widths so obtained would be meaningful.

One can attempt to gain some insight into the severity of these difficulties by performing computer

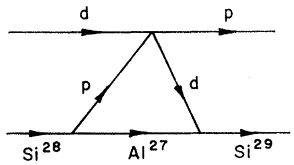

FIG. 13. A cluster contribution to Si^{28} (d, p) Si^{29}.

[20] To our knowledge, this was first suggested as applicable to nuclear physics by R. D. Amado, Ref. 1.

[21] G. Chew and F. Low, Phys. Rev. **113**, 1640 (1959).

[22] P. Cziffra, M. H. MacGregor, M. J. Moravscik, and H. P. Stapp, Phys. Rev. **114**, 880 (1959).

[23] This is also the conclusion of N. R. Gibbs and W. Tobocman, Phys. Rev. **124**, 1496 (1961), based on DWBA calculations.

experiments in which the input parameters can be controlled and the expected reduced width is known. An exploratory calculation of this type was carried out by Dullemond and Schnitzer[6] on $Si^{28}(d, p)$ Si^{29}, where the data treated was generated by the DWBA calculation of Bassel *et al.*[24,25] Since computer experiments may be of some importance, it might be useful to consider the extrapolation procedure in some detail. Suppose the reaction amplitude can be written, for fixed energy, as

$$T(E, \cos\theta) = T_{(\text{pole})}(E, \cos\theta) + \int_{(\cos\theta)_0}^{\infty} dx' \frac{\rho(E, x')}{(x' - \cos\theta)}, \quad \text{(III.1)}$$

and suppose that

$$T_{(\text{pole})}(E, \cos\theta) = gG/(A - B\cos\theta), \quad \text{(III.2)}$$

where g and G are constants, A and B are functions of energy, $(\cos\theta)_0 > A/B$, and $d\sigma/d\Omega = |T|^2$. It is clear that

$$\lim_{\cos\theta \to A/B} (A - B\cos\theta)^2 (d\sigma/d\Omega) = g^2G^2. \quad \text{(III.3)}$$

Define an extrapolation function

$$F(E, \cos\theta) = (g^2G^2/|T_{(\text{pole})}|^2)(d\sigma/d\Omega)_{\text{exp}}, \quad \text{(III.4)}$$

where $(d\sigma/d\Omega)_{\text{exp}}$ is the experimental angular distribution. For our particular example,

$$F(E, \cos\theta) = (A - B\cos\theta)^2 (d\sigma/d\Omega)_{\text{exp}}. \quad \text{(III.5)}$$

For values of $\cos\theta < \cos\theta_0$ it is possible to expand (III.1) in the form

$$F(E, \cos\theta) = \sum_{n=0}^{\infty} b_n (A - B\cos\theta)^n. \quad \text{(III.6)}$$

To perform an extrapolation one generally assumes that

$$F(E, \cos\theta) \cong \sum_{n=0}^{N} a_n (A - B\cos\theta)^n \quad \text{(III.7)}$$

is valid, where one chooses the best value of N by statistical methods.[6,26] In doing the extrapolation one fits $F(E, \cos\theta)$, as given by (III.5), by a least-squares polynomial for a fixed value of N, and then computes the weighted least-squares error $\chi_N^2 = \sum_i w_i \epsilon_i^2$ for this value of N, choosing the best value of N as the minimum of χ_N^2 vs. N. One should find that χ_N^2 decreases rapidly with increasing N, reaches a minimum N_0, and then increases slowly for larger values of N.

For the best value of N,

$$g^2G^2 = a_0 = F(E, A/B), \quad \text{(III.8)}$$

which gives the value of the residue at the pole.

Extrapolations were made for the $Si^{28}(d, p)$ Si^{29} reaction for deuteron laboratory energies between 5 and 15 MeV, and for the Si^{29} in its ground state and first excited state.[6] Three cases were examined: (a) Coulomb interactions of nuclei due to a uniformly charged sphere and optical potentials chosen to fit angular distributions to known experimental results; (b) same as case (a) but with no Coulomb interaction; and (c) same as case (a) but with the optical potentials considerably reduced in strength. The theoretical value of the reduced width was given *a priori* by the known asymptotic neutron wave functions used in the computer-generated cross sections. The results of the extrapolation were somewhat disappointing, with typical values of G^2 being too small by roughly a factor of 4. In addition, the extrapolated values of G^2 were found to have considerable energy dependence for the cases (b) and (c). This result, together with the fact that high degree polynomials (typical best value of $N \simeq 6$) were required, indicated that nearby singularities were not sufficiently accounted for by the extrapolation procedure.

TABLE I.[a] Examples where the cluster singularity is the nearest singularity of the nucleus vertex function $(A+n) \to (A-p) + (np) \to (A) + (n)$. The equivalent Yukawa potential for a vertex with a potential singularity at the same position as the cluster singularity is $e^{-\beta r}/r$. The range (β/μ) is given in units of the pion mass. Case 1 indicates that nucleus $(A-p)$ has a doubly closed shell or subshell. Case 2 indicates that nucleus $(A-p)$ has a closed proton shell (or subshell) and one hole in the same shell (or subshell) for the neutrons.

$(A+n)$ nucleus	$(A-p)$ nucleus	β/μ	Case
^{13}N	^{11}C	0.78	2
^{14}N	^{12}C	0.89	1
^{18}F	^{16}O	0.54	1

[a] P. Stechel, Nuovo Cimento **31**, 250 (1964).

TABLE II.[a] Examples where the cluster singularity is the nearest singularity of the nucleus vertex function $(A+n) \to (A-y) + (y+1) \to (A) + 1$. The range of the equivalent Yukawa potential for a potential singularity at the same position is (β/μ) in units of the pion mass.

$A+1$	A	$A-y$	β/μ
^7Li	^6Li	α	0.86
^7Li	^6He	α	0.58
^7Be	^6Li	α	0.73
^{10}B	^9Be	^8Be	0.75

[a] P. Stechel, Nuovo Cimento **31**, 250 (1964).

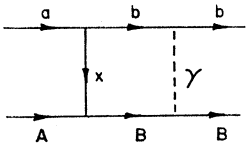

FIG. 14. A one-photon contribution to the final-state Coulomb interaction in the reaction $A(a, b)B$.

[24] R. H. Bassel, R. M. Drisko, and G. R. Satchler, Oak Ridge National Laboratory report ORNL-3240 (unpublished).
[25] G. R. Satchler (private communication).
[26] P. Cziffra and M. Moravscik, Lawrence Radiation Laboratory report UCRL-8523 (rev.) (unpublished).

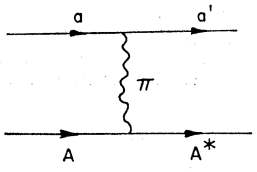

FIG. 15. A one-pion exchange contribution to inelastic scattering $A(a, a')A'$.

Although the computer experiment of Dullemond and Schnitzer was not as successful as one would have hoped, more work is indicated before a final statement is made about the usefulness of extrapolating angular distributions to find absolute reduced widths. More effort along these lines appears to be justified, particularly since a workable extrapolation procedure would give a model-independent value for the width. Some modifications of the outlined extrapolation procedure should be attempted before this question can be answered. In particular, one may replace the plane waves by Coulomb waves[3,27] to account for these additional nearby singularities. Moringo[27] claims that the Coulomb-wave Born approximation predicts smaller absolute cross sections than the PWBA (using the same widths). This correction would be in a direction to improve the extrapolation. On the other hand, Dullemond and Schnitzer extrapolated theoretical cross sections without Coulomb interactions which did not differ significantly from the other cases examined. In order to include the possibility of other forms for the modified pole term, we should now write

$$F(E, \cos\theta) = (N_l^2 N_0^2/|\, T_{(\text{theory})}\,|^2)\,(d\sigma/d\Omega_{\text{exp}}), \quad \text{(III.9)}$$

where we have divided out the theoretical reduced widths N_l and N_0 to ensure the correct normalization at the pole. Presumably the extrapolations are more successful if the experimental data used is confined to the first peak. Even if a procedure is satisfactory for extrapolating computer-generated data, the problem still remains of obtaining sufficiently accurate experimental data in the forward direction before these methods can be meaningfully applied.

In a similar context, Robson[28] has suggested that one may be able to extrapolate angular correlation coefficients to the pole in direct interactions of the form $A(a, b)B^*(\gamma)B$. He has a computer experiment in progress to test whether this is feasible, but no results are known. He also suggests that it may be possible to extrapolate to a pole in the amplitude for inelastic scattering, $A(a, a')A^*$. Such a pole would correspond to the exchange of a pion, as shown in Fig. 15. We think that in practice this would be much more difficult to carry out than for particle-transfer reactions, since this represents the exponential fall-off of a potential whose equivalent range is much larger than $1/\mu$ (see Sec. V).

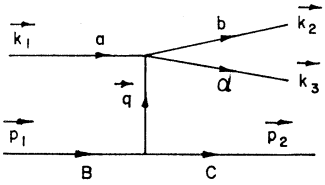

FIG. 16. A single-particle exchange diagram for $A+a\rightarrow b+c+d$.

Another simple, but useful, result which one finds on examining the location of exchange stripping poles is that in general they are quite far from the physical region.[11,29] This implies that for these cases backward peaks in angular distributions cannot be attributed to a mechanism analogous to that which gives the forward peaks.

A rather interesting suggestion of Bleiden[30] involves the application of the Treiman–Yang test[31] to nuclear-reaction physics. The Treiman–Yang test,[31] which is a necessary condition for the validity of the one-pion exchange model, could be used to test for the dominance of pole graphs involving the exchange of spin-zero

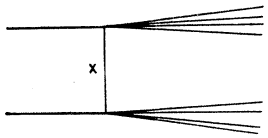

FIG. 17. A single-particle exchange diagram for $A+a\rightarrow b+c+d+e+\cdots$.

clusters (He^4, Be^8, C^{12}, etc.) in nuclear reactions. To treat an example, consider a nuclear reaction of the type

$$A+a\rightarrow b+c+d. \quad \text{(III.10)}$$

Suppose we want to test the hypothesis that the pole diagram (Fig. 16) dominates the reaction, where x has spin zero. In the coordinate system where a is at rest (with momenta as in Fig. 16),

$$\mathbf{q}=\mathbf{p}_1-\mathbf{p}_2=\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_1=\mathbf{k}_2+\mathbf{k}_3. \quad \text{(III.11)}$$

Then if particle a is unpolarized and x has spin zero, the differential cross section in this coordinate system is invariant under *simultaneous* rotation of the momentum vectors $\mathbf{k}_2$ and $\mathbf{k}_3$ about the axis defined by $\mathbf{q}$. An equivalent statement is that in the a rest frame there is no correlation between the planes defined by $\mathbf{n}_d=\mathbf{k}_2\times\mathbf{k}_3/|\,\mathbf{k}_2\times\mathbf{k}_3\,|$ and $\mathbf{n}_p=\mathbf{p}_1\times\mathbf{p}_2/|\,\mathbf{p}_1\times\mathbf{p}_2\,|$. Therefore, for the above-stated conditions, the distribution of events as a function of $\cos\phi$, where

$$\cos\phi=\mathbf{n}_p\cdot\mathbf{n}_d,$$

must be isotropic. Blieden lists several suitable re-

[27] F. B. Moringo, Phys. Rev. **134**, B1243 (1964). A detailed discussion of these matters is to be found in C. F. Clement, Nucl. Phys. **66**, 241 (1965).
[28] D. Robson, Bull. Am. Phys. Soc. **9**, 467 (1964).
[29] For additional remarks on this subject see J. B. French, *Proceedings of the Rutherford Conference* (Heywood and Company, Ltd., London, 1962), p. 429; and N. Austern, Ref. 3, p. 75.
[30] H. R. Blieden, Phys. Letters **9**, 176 (1964).
[31] S. B. Trieman and C. N. Yang, Phys. Rev. Letters **8**, 140 (1962).

actions for this test. One may extend the discussion to the general case of single-particle, spin-zero exchange for graphs of the type of Fig. 17. The reader is referred to the original papers[30–32] for the appropriate details in this case. For some situations the single-particle, spin-zero exchange might only be expected to dominate near the forward direction, so that one might restrict the measurement of the cos ϕ distribution to that region. It is interesting to note that these results are valid for arbitrary modifications of the vertex functions connected to the ends of the propagator in Fig. 17.

A subtle and (we think) important suggestion comes to us by way of analogy with elementary particle physics.[33] Kacser and Aitchison discuss reactions typified by $He^3+Li^6\to\alpha_1+Li^{5*}\to\alpha_1+p+\alpha_2$. They notice that for certain energies α_2 can move in the same direction as α_1 and fast enough to interact with it. This will happen even though the relative 2α energy does not correspond to an excited state of Be^8. Such an interaction results in an apparent Be^8 resonance, as

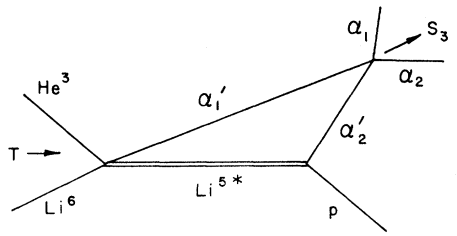

FIG. 18. A triangle graph for $He^3+Li^6\to\alpha_1'+Li^{5*}\to\alpha_1+\alpha_2+p$.

detected by the 2α spectrum, whose energy depends on the beam energy, in distinction from true resonances. Although such rescattering can be expected from all analogous processes, the magnitude of the effect depends crucially on the intermediate resonance width. This pseudoresonance is predicted from a detailed study of the triangle graph[16] (Fig. 18). In order to exhibit the features of this process most clearly, Kacser and Aitchison discuss the analysis of the relevant experiment in terms of the invariant quantities

$$S_1=(p+\alpha_2)^2,\qquad S_2=(p+\alpha_1)^2,\qquad S_3=(\alpha_1+\alpha_2)^2, \tag{III.12}$$

where p, α_1, and α_2 are the four-momenta of the particles. It is clear that

$$S_1=(M_p+M_{\alpha_2})^2+M_pM_{\alpha_2}(\mathbf{v}_p-\mathbf{v}_{\alpha_2})^2 \tag{III.13}$$

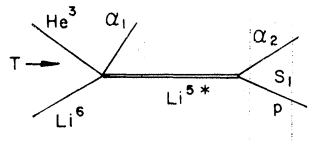

FIG. 19. A graph for a Breit-Wigner contribution to $He^3+Li^6\to\alpha_1+Li^{5*}\to\alpha_1+\alpha_2+p$.

is the square of the invariant mass of the $(p+\alpha_2)$ pair. One also easily finds that

$$S_1+S_2+S_3=[M(He^3)+M(Li^6)]^2+2TM(Li^6)+2M_\alpha^2+2M_p^2, \tag{III.14}$$

where T is the kinetic energy of the He^3 in the laboratory system. For fixed T, one can construct a phase-space (Dalitz) plot for each event when (S_1, S_2, S_3) is known for the event in question. A large number of events are expected to collect in the bands $S_1=[M(Li^{5*})]^2$ corresponding to the mechanism

$$He^3+Li^6\to\alpha_1+Li^{5*}\to\alpha_1+p+\alpha_2, \tag{III.15}$$

and in the bands $S_3=[M(Be^{8*})]^2$ corresponding to

$$He^3+Li^6\to p+Be^{8*}\to p+\alpha_1+\alpha_2 \tag{III.16}$$

(Figs. 19 and 20, respectively). However, these enhancements are independent of the beam energy T, while the effect we wish to emphasize depends on T. Of course there can be other variations in the density of events in the Dalitz plot due to nonresonant interactions, e.g., an attractive zero-energy scattering length for a pair of particles.

Let us consider the process of Eq. (III.15), with intermediate state Li^{5*} but including rescattering corrections. After a time related to the Li^{5*} level width, there will be a breakup into $p+\alpha_2'$. Under certain kinematical conditions α_2' can catch α_1' and interact with it, namely if α_2' moves in the same directions as α_1' and with a speed greater than that of α_1'. One can give a prescription for these conditions in terms of the allowed region in the S_1, S_3 plane (for fixed T). Following Kacser and Aitchison,[33] we plot the band corresponding to the Li^{5*} intermediate state $S_1=[M(Li^{5*})]$ in Fig. 21(a–c) corresponding to different He^3 energies T. These intersect the Dalitz boundary at S_{3+} and S_{3-}, with $S_{3+}>S_{3-}$. It can be shown that if and only if S_{3-}

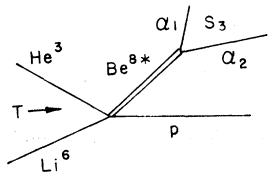

FIG. 20. A graph for a Breit-Wigner contribution to $He^3+Li^6\to Be^{8*}+p\to\alpha_1+\alpha_2+p$.

[32] Some generalizations have been stated by A. S. Goldhaber, Phys. Rev. **135**, B508 (1964). Additional tests of the one-particle exchange model involving the angular correlations of the decays of unstable states are discussed by K. Gottfried and J. D. Jackson, Nuovo Cimento **33**, 309 (1964). The connection of these decay distributions with other angular correlations and generalizations to exchanges of spins other than zero are discussed by K. Gottfried and J. D. Jackson (to be published).

[33] C. Kacser and I. J. R. Aitchison, Rev. Mod. Phys. **37**, 350 (1965); Phys. Rev. (to be published); and Nucl. Phys. (to be published).

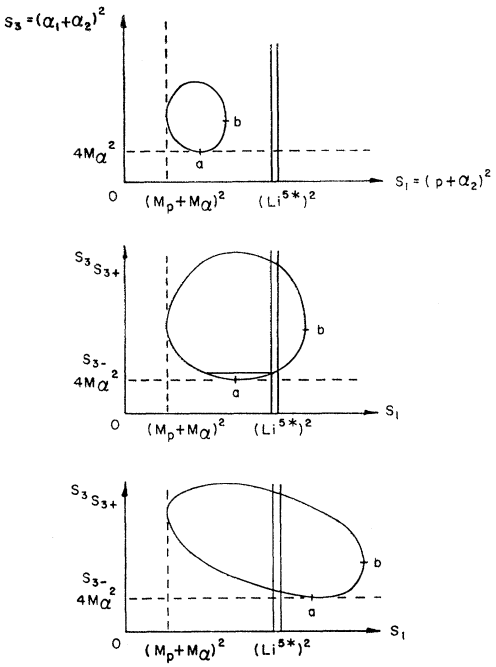

FIG. 21. (a) (After Kacser and Aitchison, Ref. 33.) The Dalitz region in the S_1–S_3 plane for the reaction $He^3+Li^6\rightarrow \alpha_1+\alpha_2+p$ at a deuteron energy T below the energy for pseudoresonances. (b) Same as (a), but at an energy T where pseudoresonances are possible. (c) Same as (a), but at an energy T above the energy for pseudoresonances.

is on the lower right-hand arc *ab* (Fig. 21b), these kinematic conditions can be satisfied. If these conditions are met, there will be an enhancement at $S_3=S_{3-}$ (shown as a shaded band in Fig. 21b) independent of whether these is a Be^{8*} level at this value of S_3 and dependent on only *some* attractive α–α interaction at this energy. The pseudoresonance is then an enhancement whose position S_{3-} is a rapidly varying function of T.

Kacser and Aitchison[33] have performed numerical calculations for various reactions, and find that the largest effects are of the order of 25%. In particular, they have studied

$$\text{I:}\qquad He^3+Li^6\rightarrow\alpha_1+Li^{5*}(16.81)\rightarrow p+\alpha_1+\alpha_2,$$

with rescattering between α_1 and α_2. Since S_{3-} is fairly near the Be^{8*} level at 2.90 MeV, they have used this to estimate the $\alpha_1\alpha_2$ interaction in this region. The contribution (in the Breit–Wigner form) of

$$\text{II:}\qquad He^3+Li^6\rightarrow p+Be^{8*}(2.90)\rightarrow p+\alpha_1+\alpha_2$$

(Fig. 20) is added coherently as a background to the amplitude computed from the triangle graph.[34] The square of the matrix element $(\text{I}+\text{II})^2$ shows about a 25% enhancement in the region of $T=5$–6 MeV as compared to the Breit–Wigner term, $|\text{ II }|^2$, taken alone. It is to be emphasized that this enhancement in the $(\alpha_1\alpha_2)$ spectrum occurs at different relative $(\alpha_1\alpha_2)$ energies E, depending on the beam energy. For this reason, the excitation function for producing $(\alpha_1\alpha_2)$ pairs at fixed E versus T shows a peak which varies with T. The effects should be striking experimentally. In addition, it is of considerable theoretical importance to find experimental examples of these (as yet unobserved) effects since they are very necessary consequences of our present understanding of final-state interactions.

[34] See also C. Kacser, Phys. Letters **12**, 269 (1964); and I. J. R. Aitchison, "Final State Interactions Among Three Particles," Nuovo Cimento (to be published).

IV. PHENOMENOLOGY (UNITARITY)

Our discussion of direct nuclear reactions has centered on questions of analyticity, with an emphasis on the phenomenological considerations of nearest singularities. Although the location of singularities alone is not sufficient to provide an understanding of the reaction mechanism, since detailed questions of dynamics ultimately cannot be avoided, these methods can give useful insights into the gross features of the reaction. Another question frequently discussed in the S-matrix approach is the effects of unitarity. These are of great importance in particle physics, and they are expected to be of no less importance for our problem.

The requirement of unitarity can be stated in terms of the S matrix as

$$\sum_b \langle a\,|\,S\,|\,b\rangle\langle b\,|\,S^+\,|\,c\rangle=\sum_b \langle a\,|\,S^+\,|\,b\rangle\langle b\,|\,S\,|\,c\rangle=\delta_{ac}, \tag{IV.1}$$

which in terms of operators means that

$$S^+S=SS^+=1. \tag{IV.2}$$

If we write

$$S=1+2iT, \tag{IV.3}$$

then one can write for the transition operator T,

$$T^+T=\tfrac{1}{2}i(T^+-T). \tag{IV.4}$$

Time-reversal invariance implies that T is symmetric. Therefore, one obtains the result that

$$T^+T=\mathrm{Im}T. \tag{IV.5}$$

Although the PWBA reproduces some of the gross features of direct interactions, since the amplitude is real it is clear that it cannot satisfy unitarity. On the other hand, the DWBA obtained from formal scattering theory, takes into account the rescattering of the incident and outgoing waves, and thus includes some effects of unitarity. Since in many cases the DWBA gives excellent agreement with experiment, it is of considerable interest to be able to derive the results of DWBA from the S matrix (and unitarity) without reference to potentials. Such a derivation has been

attempted by several authors.[2,35] This connection between dispersion theory and conventional methods would allow us to avoid the ambiguous procedure of constructing optical potentials to produce distorted wave functions. Instead one deals directly with measureable quantities such as the phase shifts. This change in emphasis is useful since there is no guarantee that these wave functions are reliable even though it is always possible to fit elastic scattering experiments with an energy-dependent optical potential.

As one example of this approach let us sketch Omnes' derivation of the DWBA from dispersion theory.[2] Consider a complete set of states, specified by energy, total angular momentum, quantum numbers of nuclei, and incident particles, and denote them by $|\,a\rangle$, $|\,b\rangle$, $\cdots$. These are the states usually used in specifying an experiment. The unitarity of S permits us to consider the set of eigenstates of S, $|\,\alpha\rangle$, $|\,\beta\rangle$, $\cdots$ which satisfy

$$S\,|\,\alpha\rangle=\exp\,(2i\delta_\alpha)\,|\,\alpha\rangle, \tag{IV.6}$$

where δ_α is a real number. The two bases are related by an (energy-dependent) unitary transformation

$$|\,a\rangle=\sum_\alpha U_{a\alpha}\,|\,\alpha\rangle. \tag{IV.7}$$

If one chooses for $|\,a\rangle$ time-reversal-invariant states, and requires S to be time-reversal invariant, then U will be real, i.e.,

$$\sum_\alpha U_{a\alpha}{}^2=1. \tag{IV.8}$$

Then from Eqs. (IV.6) and (IV.7) the elastic scattering amplitude can be written

$$\langle a\,|\,S\,|\,a\rangle=\sum_\alpha U_{a\alpha}{}^2\exp\,(2i\delta_\alpha). \tag{IV.9}$$

The empirical evidence is that the scattering amplitude is almost pure imaginary for nuclear reactions at moderate energies, which means the matrix elements (IV.9) are almost real. As a function of the orbital angular momentum l, it is very small for small l, and tends to 1 for large l. From (IV.8) and (IV.9) we see that $\langle a\,|\,S\,|\,a\rangle$ is obtained by adding complex numbers of phase exp $(2i\delta_\alpha)$, whose moduli add up to 1. Omnes makes the assumption, which fits these experimental facts, that most of the eigenphases δ_α are near either 0 or $\pi/2$ (modulo π). This is a crucial step in the argument, but unfortunately it still has the status of an arbitrary assumption. In some sense this divides the eigenchannels into two groups at a given energy, those near resonance and those far from resonance. The ansatz then states that in some statistical way one can take the eigenphases to be grouped about 0 and $\pi/2$. To go on with the argument we will accept this guess about the behavior of the eigenphases and hope that it can be eventually put on a firmer footing.

[35] J. S. Ball and W. R. Frazer, Phys. Rev. Letters **14**, 746 (1965); E. J. Squires, Nuovo Cimento **34**, 1328 (1964).

To proceed consider the reaction process $a\to b$ and split the reaction matrix into two parts:

$$T=T_1{}^0+T_2, \tag{IV.10}$$

where $T_1{}^0$ gives the Born approximation for $a\to b$ and T_2 is everything else. When l is large, the cross section due to $T_1{}^0$ is smaller than that due to T_2, so one may approximately write the unitary operator

$$S_2=1+2iT_2$$

and

$$S\cong S_2. \tag{IV.11}$$

This separation is similar to the usual distorted wave approach, but it is not the range of $T_1{}^0$ that is smaller that that of T_2, but for large l, $T_1{}^0<T_2$.

We are now interested in the modifications of the final and initial state interactions on the real amplitude $\langle b\,|\,T_1{}^0\,|\,a\rangle$, when there is absorption in $|\,a\rangle$ and $|\,b\rangle$. It is convenient to first change to the eigenphase basis,

$$\langle\beta\,|\,T_1{}^0\,|\,\alpha\rangle=U_{a\alpha}U_{b\beta}{}^{-1}\langle b\,|\,T_1{}^0\,|\,a\rangle. \tag{IV.12}$$

The relation of $\langle\beta\,|\,T_1\,|\,\alpha\rangle$ (which includes final state interactions) to $\langle\beta\,|\,T_1{}^0\,|\,\alpha\rangle$ is a standard dispersion relations problem.[2,36] When $\delta_\alpha(E)$ and $\delta_\beta(E)$ vary slowly with energy in the range 0 or $\pi/2$, and $T_1{}^0(E)$ varies slowly with energy, the solution simplifies to

$$\langle\beta\,|\,T_1\,|\,\alpha\rangle=\exp\,[i(\delta_\beta-\delta_\alpha)]\langle\beta\,|\,T_1{}^0\,|\,\alpha\rangle. \tag{IV.13}$$

This result is probably more general than the stated deviation.[2] Since $T_1{}^0$ only has matrix elements between $|\,a\rangle$ and $|\,b\rangle$, then from (IV.12) and (IV.13),

$$\langle b\,|\,T_1\,|\,a\rangle=[\sum_\beta U_{b\beta}{}^2\exp\,(i\delta_\beta)]\times[\sum_\alpha U_{a\alpha}{}^2\exp\,(-i\delta_\alpha)]\langle b\,|\,T_1{}^0\,|\,a\rangle. \tag{IV.14}$$

We can evaluate the quantities in the brackets, which contain the effects of final state interactions, for large l. We write

$$\langle a\,|\,S\,|\,a\rangle=1-\epsilon_a$$

and

$$\langle b\,|\,S\,|\,b\rangle=1-\epsilon_b. \tag{IV.15}$$

For large l, ϵ_a and ϵ_b are *small* and real. With the above simplifying assumptions about the eigenphases, we find

$$\langle a\,|\,S\,|\,a\rangle=[1-(\epsilon_a/2)]+\exp\,(2\pi i/2)\,(\epsilon_a/2), \tag{IV.16}$$

$$\sum_\alpha U_{a\alpha}{}^2\exp\,(i\delta_\alpha)=[1-(\epsilon_a/2)]+\exp\,(i\pi/2)\,(\epsilon_a/2), \tag{IV.17}$$

and

$$|\sum_\alpha U_{a\alpha}{}^2\exp\,(i\delta_\alpha)\,|=|\,1-(\epsilon_a/2)\,|=|\,\langle a\,|\,S\,|\,a\rangle\,|^{\frac{1}{2}}. \tag{IV.18}$$

[36] R. Omnes, Nuovo Cimento **8**, 316 (1958); M. Jacob, G. Mahoux, and R. Omnes, Nuovo Cimento **23**, 838 (1962); J. D. Jackson and G. L. Kane, Nuovo Cimento **23**, 444 (1962).

Hence, in this approximation,

$$\langle b \mid T_1 \mid a \rangle = [\langle b \mid S \mid b \rangle]^{\frac{1}{2}} \times \langle b \mid T_1^0 \mid a \rangle [\langle a \mid S \mid a \rangle^*]^{\frac{1}{2}}. \quad \text{(IV.19)}$$

This gives the results of the DWBA.

Omnes' analysis cannot be applied to cases where the absorption is large. In that case ϵ_a and ϵ_b are no longer small compared to 1, and $\langle b \mid T_1^0 \mid a \rangle$ is of the order or larger than $\langle a \mid T_2 \mid a \rangle$. It is clear from most analyses that unitarity[3] considerably reduces the contribution of the PWBA to small partial waves. However, the preceding discussion shows that the detailed predictions of a single-channel DWBA calculation are not expected to be accurate for low partial waves. Thus many-channel calculations appear to be called for if there is to be improved understanding of the reaction mechanism for low l.[2,3]

In an effort to avoid some of the difficulties with Omnes' approach Ball and Frazer[35] have given an alternative discussion of the DWBA. An essential part of their treatment is to truncate the unitarity series [Eq. (IV.1)] to include only initial or final states in the intermediate sum. By using a set of plausible assumptions and the requirements of unitarity they are again led to Eq. (IV.19), but for applications to particle physics this seems to work only for the exchange of spin zero particles. However, it appears likely that in application to nuclear physics the difficulties found for the exchange of particles with spin do not occur, since these Born terms are damped at large momentum transfers by the nuclear form factors at the vertices. Once again the arguments are easier to justify for the high partial waves, where the rescatterings are weak. In still another approach to the same problem Squires[35] gives a derivation based on a random-phase multichannel N/D model, where again (IV.19) is obtained.

One can also discuss the strong absorption model of Blair[37] in terms of the unitarity relation. The assumptions of the strong-absorption model is

$$\langle a \mid S_l \mid a \rangle = 1, \quad l \geq kR$$
$$= 0, \quad l \leq kR, \quad \text{(IV.20)}$$

where k is the three-momentum in the center of mass, and R is a radius parameter. This means

$$\langle a \mid T_l \mid a \rangle = \langle b \mid T_l \mid b \rangle = \tfrac{1}{2}i, \quad l \leq kR$$
$$= 0, \quad l \geq kR. \quad \text{(IV.21)}$$

If one substitutes (IV.20) into (IV.19) one predicts that the reaction cross section is identically zero for $l \leq kR$. But we have just stated that the derivation of (IV.19) is not valid when $\epsilon_a, \epsilon_b \sim 1$. In fact, it is trivial to see that if one sets $\epsilon_a = \epsilon_b = 1$, and evaluates Eq. (IV.11) one finds (IV.19)],

$$\langle b \mid T_1 \mid a \rangle = \tfrac{1}{2} \langle b \mid T_1^0 \mid a \rangle \neq 0 \text{ and real.} \quad \text{(IV.22)}$$

But even the derivation of (IV.14) and hence (IV.22) fails for small l, so that nothing reliable can be stated about $\langle b \mid T_1 \mid a \rangle$ for low l, except that the PWBA is considerably reduced.

V. DYNAMICAL CALCULATIONS

The objectives of applying dispersion theory to nuclear reactions are at least twofold: first, to present a calculational scheme, and secondly, to help clarify the derivation of known results and approximations. Sections II–IV of this review were devoted to some of the latter aspects of dispersion theory, while in this section we wish to survey some dynamical calculations. One class of dynamical calculations attempts to formulate an approximation which will include the effects of unitarity and yet go beyond the limitations of the DWBA. Such a stripping calculation was attempted by Amado.[38] The methods applied were analogous to those used by Amado[39] in his dispersion relation solution of V–θ scattering in the Lee model. However, V–θ scattering occurs only in the S wave, while a stripping reaction has contributions from all partial waves; hence it is difficult to assess the validity of the procedure. In work of this type, the reduced widths must be specified from outside the calculation, since they must be obtained either from experiment or a nuclear structural calculation. The advantage of a dispersion formulation of dynamics would be that only on-the-energy-shell quantities enter, but one would have to deal with nonlinear integral equations.

To our knowledge, Amado's attempt is the only dispersion relation calculation of a direct interaction amplitude. However, calculations for somewhat simpler reactions have been performed, principally for the photodisintegration of the deuteron and deuteron electromagnetic form factor. Many of the same features as in direct interactions appear, but comparison with potential theory is simpler and there are many fewer complications. We summarize some results for the case $\gamma = D \rightarrow N + P$, since it bears some resemblance to direct-reaction theory. It has been shown by Martin and Vinh Mau[40] that the nonrelativistic matrix element for deuteron photodisintegration satisfies the Mandelstam representation. One can also make a relativistic calculation, and hence include the effects of such things as exchange currents, meson production, and relativistic corrections. This is a rather complicated theory; hence it is necessary to show that the relativistic theory is similar to the nonrelativistic theory in the low- and medium-energy region. Several rela-

[37] A. Dar and W. Tobocman, Phys. Rev. Letters **12**, 511 (1964); J. S. Blair, Phys. Rev. **108**, 827 (1957).

[38] R. D. Amado, Phys. Rev. **127**, 261 (1962).

[39] R. D. Amado, Phys. Rev. **122**, 696 (1961).

[40] A. Martin and R. Vinh Mau, Nuovo Cimento **20**, 246 (1961); see also F. Troyon, University of Rochester, thesis, 1962 (unpublished).

tivistic calculations have been made [41–44] which attempt to get agreement with experiment below the meson production threshold. The earliest papers[41,42] retain only the asymptotic part of the deuteron wave function, while subsequent calculations[43,44] attempt to take into account the deuteron structure. The results are encouraging, with improved agreement as additional effects are included in the theory.[44] This work indicates that dispersion methods might have some hope of successfully describing nuclear-reaction processes. There have also been attempts to study the three-body problem by various authors. These have dealt with formal questions such as integral equations for the connected part of three-body potential scattering[45] and the Bethe–Salpeter amplitude for the three-body problem.[46] Exactly soluble models for N–D potential scattering have also been studied.[47] These papers are somewhat outside the scope of our paper so we will not go into further details.

Another interesting dynamical formulation which should be mentioned is due to Villars.[48] He adapts a definition of asymptotic states similar to the one used in field theory. One can then express the reaction amplitudes as matrix elements of a "current operator," J, similar to that occurring in dispersion theory, which can be shown to satisfy a Low equation. These nonlinear coupled integral equations, typical of dispersion relations, can be used as the starting point for an approximation scheme for nuclear reactions, although Villars makes no attempt in this direction. Rather, he derives a set of linear integral equations for the transition amplitudes, whose structure is similar to equations obtained by Feshbach.[49]

A different and important sort of dynamical question concerns the distinction between nuclear size and range of nuclear interactions. For the purpose of discussion, consider the nuclear size (or radius) associated with in interaction to be the distance for which the interaction between two nuclear clusters is substantially different from zero. On the other hand, the range of an interaction will mean the distance associated with the exponential fall-off of the interaction in the asymptotic region, i.e., in the region outside the nuclear radius.[50] It is clear from our definition of the nuclear vertex function and its relation to the wave function that the longest range part of the interaction comes from the nearest singularities in the cos θ plane. Only for the deuteron does the range agree closely with the size. Even for relatively low-A nuclei, the range, determined from the potential singularities, is considerably smaller than the size. This apparent paradox can be resolved if one notices that a superposition of shorter range terms with an oscillating weight function can give a size which is larger than the range.[50,51] Loosely speaking, one has the picture of a nucleus, with the conventional idea of nuclear size, and with the range due to the exchange of a pion between the *surfaces* of the two nuclear clusters involved. (For the few exceptional cases[19] where cluster singularities are nearer, this heuristic picture is easily modified.) Since the understanding of nuclear size is not simply related to the location of singularities, one must appeal to dynamics to obtain an estimate of the nuclear size. It would be surprising if such dynamics did not involve some features of the nuclear many-body problem. The nuclear radius enters in direct interactions in a variety of ways, but in conventional calculations it is described phenomenologically. For example, in the Butler model a cut-off is arbitrarily introduced in a radial overlap integral. In more elaborate calculations such as the DWBA one assumes a size empirically for each potential used in the calculation. Even in dispersion calculations such as Amado's the bound-state wave functions are specified. However, an optimist would hope to be able to calculate reaction-matrix elements from two-body, nucleon–nucleon interactions, where a size would occur naturally. A more modest aim would be to find a subset of graphs, which might exhibit the size features.

The simplest element which enters reaction theory in which the size occurs is in the vertex part for nucleon absorption by a nucleus. Eden and Goltstone[50] discussed the application of dispersion theory to the nuclear many-body problem; in particular, the vertex part. Dispersion theory can be used for the many-body problem because there is detailed knowledge of the analytic properties of the nucleon–nucleon scattering amplitude. They formulate a self-consistency requirement for the nucleon–nucleon vertex part, using dispersion techniques, which can be thought of as an analogue to Hartree or Brueckner equations. There are additional developments in a sequel by Froggatt,[52] but we will not enter into the technical details here. However, we feel that this is one of the more promising areas for the application of dispersion relations to nuclear physics.

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[41] B. Sakita and C. J. Goebel, Phys. Rev. **127,** 1787 (1962).
[42] A. Donnachie, Nucl. Phys. **37,** 595 (1962).
[43] M. H. Skolnick, Phys. Rev. **136,** B1493 (1964).
[44] M. LeBellac, F. M. Renard, and J. Tran Than Van, Nuovo Cimento **33,** 594 (1964); **34,** 450 (1964).
[45] C. Lovelace, Phys. Rev. **135,** B1225 (1964); and C. Lovelace in *Strong Interactions and High Energy Physics,* edited by R. G. Moorhouse (Oliver and Boyd, London, 1964), and references cited therein.
[46] P. Stechel, Nucl. Phys. **52,** 189 (1964).
[47] R. D. Amado, Phys. Rev. **132,** 485 (1963); R. Aaron, R. Amado, and Y. Yam, *ibid.* **136,** B650 (1964); Phys. Rev. Letters **13,** 574 (1964); **13,** 701(E) (1964), and references cited therein.
[48] F. Villars, MIT preprint (to be published).
[49] H. Feshbach, Ann. Phys. (N. Y.) **19,** 287 (1964).
[50] This distinction has been emphasized by R. J. Eden and J. Goldstone, Nucl. Phys. **49,** 33 (1963).
[51] I. S. Shapiro, Ref. 18, suggests the alternate possibility that the nuclear radius is characterized by an essential singularity at infinity.
[52] C. D. Froggatt, Nucl. Phys. **61,** 303 (1965).

VI. CONCLUDING REMARKS

In the preceding we have discussed a selection of topics involving the use of dispersion methods for problems in direct nuclear reactions. It should be clear to the reader that the concrete successes of this approach are not overwhelming; however, it must be further admitted that the work in these areas has been limited. We do not wish to be too pessimistic about the usefulness of these techniques, so we give emphasis to those areas which clearly deserve further effort.

The derivation of the DWBA from unitarity and some reasonable statistical assumptions based on empirical information is quite interesting. It remains to be seen whether this approach can be extended to low partial waves. There remain many formal questions concerning the analytic structure of the many-body amplitudes. Even the simplest example, the three-body problem, has not been completely understood from this point of view. On the whole, the practical implications of pology techniques are disappointing. An exception may be the application of the Treiman–Yang test to multiparticle exchange reactions. This sort of correlation measurement may give some insight into the reaction mechanism for cases where spin-zero exchange is possible. The extrapolation to particle exchange poles has proved more difficult than originally anticipated. In addition, the nearest (anomalous) singularities considerably underestimate nuclear size, so that one cannot relate the nearby branch points in the cos θ plane directly to the phenomenological radii associated with nuclear reactions. This feature of reactions, which is not exhibited by potential scattering, reflects the many-body character of a nuclear reaction. In dynamical calculations of reaction amplitudes, this problem is usually avoided by introducing average potentials or interactions from empirical data and bound-state wave functions from nuclear structure theory. A formulation which may give a more fundamental description of the nucleon–nucleon vertex has been given by Eden and Goldstone[50] in their work on the nuclear many-body problem. They concentrate on a self-consistent dispersion theoretical description of nuclear vertex functions, an essential component of reaction amplitudes. We must wait to see how far this program can be pursued.

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