Distorted-Wave Born Approximation for Stripping to Virtual Levels*

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INTRODUCTION

Proton groups are often observed from deuteron-induced reactions leading to three-body final states by the two-stage process $A+d\rightarrow p+B^*\rightarrow p+n+A$ and these closely resemble the proton groups seen in the usual stripping reaction which produces B in a bound state. Our problem is to adapt the familiar distorted-wave Born approximation (DWBA) to this case of stripping to virtual levels. A procedure for evaluating the stripping amplitude is found and a numerical application is mentioned.

THE SCATTERING AMPLITUDE

The distorted-wave theory for the ordinary stripping reaction[1] $A+d\rightarrow p+B$ leads to the following exact expressions for the scattering amplitude, or more strictly, the T-matrix element (Spin coordinates are omitted for simplicity):

$$\langle\Psi_f^{(-)} \mid V_{nA}+V_{pA}-\bar{V}_{dA} \mid \psi_d(\mid \mathbf{r}_{nA}-\mathbf{r}_{pA} \mid) \times\chi_i^{(+)}(\mathbf{k}_d, \mathbf{r}_{dA})\psi_A(\xi)\rangle \quad (1)$$

and

$$\langle\chi_f^{(-)}(\mathbf{k}_p, \mathbf{r}_{pB})\psi_B(\xi, \mathbf{r}_{nA}) \mid V_{np}+V_{pA}-\bar{V}_{pB} \mid \Psi_i^{(+)}\rangle. \quad (2)$$

These are the so-called prior and post expressions. Here $\Psi_i^{(+)}$ and $\Psi_f^{(-)}$ are exact solutions of the full Hamiltonian

$$\begin{aligned} H&=H_A+T_{np}+T_{dA}+V_{np}+V_{pA}+V_{nA} \\ &=H_A+T_{nA}+T_{pB}+V_{nA}+V_{np}+V_{pA} \end{aligned} \quad (3)$$

satisfying the appropriate boundary conditions.

$$\chi_i^{(+)}(\mathbf{k}_d, \mathbf{r}_{dA}) \qquad \text{and} \qquad \chi_f^{(-)}(\mathbf{k}_p, \mathbf{r}_{pB})$$

are distorted deuteron and proton wave functions generated using the optical model potentials $\bar{V}_{dA}$ and $\bar{V}_{pB}$. $\psi_A(\xi)$ and $\psi_B(\xi, \mathbf{r}_{nA})$ are, respectively, the wave functions of the target and residual nuclei and $\psi_d(\mid \mathbf{r}_{nA}-\mathbf{r}_{pA} \mid)$ is the deuteron internal wave function. The DWBA consists of replacing the exact wave functions Ψ in (1) and (2) by the corresponding distorted waves yielding the approximate formulas

$$\langle\chi_f^{(-)}(\mathbf{k}_p, \mathbf{r}_{pB})\psi_B(\xi, \mathbf{r}_{nA}) \mid V_{nA}+V_{pA}-\bar{V}_{dA} \mid \psi_d \times(\mid \mathbf{r}_{nA}-\mathbf{r}_{pA})\chi_i^{(+)}(\mathbf{k}_d, \mathbf{r}_{dA})\psi_A(\xi)\rangle \quad (4)$$

and

$$\langle\chi_f^{(-)}(\mathbf{k}_p, \mathbf{r}_{pB})\psi_B(\xi, \mathbf{r}_{nA}) \mid V_{np}+V_{pA}-\bar{V}_{pB} \mid \times\psi_d(\mid \mathbf{r}_{nA}-\mathbf{r}_{pA} \mid)\chi_i^{(+)}(\mathbf{k}_d, \mathbf{r}_{dA})\psi_A(\xi)\rangle. \quad (5)$$

These approximate prior and post formulas are identically equal. The well-known zero-range DWBA calculations are performed by taking the post expression, neglecting $(V_{pA}-\bar{V}_{pB})$, and replacing $V_{np}\psi_d$ by a δ function. There does not appear to be any practical method of evaluating the prior expression.

Gerjuoy[2] has studied the scattering problem where there are three bodies in the final state and has shown how an exact prior expression may be obtained, analogous to (1). He actually used plane waves in place of the χ's, but the result is simply extended to distorted waves. When we make the DWBA, by analogy of going from (1) to (4), we obtain

$$\langle\chi_f^{(-)}(\mathbf{k}_p, \mathbf{r}_{pB})\psi_B^{(-)}(\mathbf{k}_n, \mathbf{r}_{nA}, \xi) \mid V_{nA}+V_{pA}-\bar{V}_{dA} \mid \times\psi_d(\mid \mathbf{r}_{nA}-\mathbf{r}_{pA} \mid)\chi_i^{(+)}(\mathbf{k}_d, \mathbf{r}_{dA})\psi_A(\xi)\rangle. \quad (6)$$

The only modification from the two-body case, Eq. (4), is that the bound state wave function ψ_B has been replaced by a positive energy wave function $\psi_B^{(-)}(\mathbf{k}_n, \mathbf{r}_{nA}, \xi)$, an exact eigenfunction for the system $(A+n)$ in a scattering state. Unfortunately, as in the two-body case, there appears to be no practical way of evaluating this matrix element. One might suppose that the post matrix element for the three-body case

$$\langle\chi_f^{(-)}(\mathbf{k}_p, \mathbf{r}_{pB})\psi_B^{(-)}(\mathbf{k}_n, \mathbf{r}_{nA}, \xi) \mid V_{np}+V_{pA}-\bar{V}_{pB} \mid \times\psi_d(\mid \mathbf{r}_{nA}-\mathbf{r}_{pA} \mid)\chi_i^{(+)}(\mathbf{k}_d, \mathbf{r}_{dA})\psi_A(\xi)\rangle \quad (7)$$

obtained by similarly replacing the bound-state wave function in (5) by $\psi_B^{(-)}$ should also be valid, and identically equal to (6). However, (7) is not even a convergent integral, as we can see by making the zero-range approximation. Then after integrating over $\mathbf{r}_{pB}$

* A brief account of this work was published in the *Proceedings of the Padua Conference* (Gordon and Breach, Science Publishers, New York, 1963), p. 530.

[1] W. Tobocman, *Theory of Direct Nuclear Reactions* (Oxford University Press, London, 1961).

[2] E. Gerjuoy, Ann. Phys. (N. Y.) **5**, 58 (1958).

and ξ, (7) becomes

$$\text{const}\int d\mathbf{r}_{nA}[\exp(-i\mathbf{k}_p\cdot\mathbf{r}_{nA})+\text{scattered waves}]$$
$$\times[\exp(-i\mathbf{k}_n\cdot\mathbf{r}_{nA})+\text{scattered waves}]$$
$$\times[\exp(i\mathbf{k}_d\cdot\mathbf{r}_{nA})+\text{scattered waves}]$$

and gives a δ function together with terms which oscillate as a function of the upper limit. Hence the matrix element (7) cannot represent the required scattering amplitude. The source of this difficulty is the fact that the V_{np} interaction is not localized in the vicinity of the target nucleus. On the other hand, the prior expression (6) certainly does converge as the interaction V_{nA} vanishes outside the nucleus. In order to obtain a satisfactory post expression for the three-body problem it is necessary either to retreat to the basic formalism or more simply to transform the valid prior expression (6), using Green's theorem and the equations defining the distorted waves. The latter course yields, if we take the r_{nA} integral out to a radius R_n:

$$\int_0^{R_n} r_{nA}^2\,dr_{nA}\int_\infty d\mathbf{r}_{pB}\int d\omega_{nA}\int d\xi\chi_F^{(-)*}(V_{nA}+V_{pA}-\bar{V}_{dA})\chi_I^{(+)}$$
$$=\int_0^{R_n} r_{nA}^2\,dr_{nA}\int_\infty d\mathbf{r}_{pB}\int d\omega_{nA}\int d\xi\chi_F^{(-)*}(V_{np}+V_{pA}-\bar{V}_{pB})\chi_I^{(+)}$$
$$-\frac{\hbar^2R_n^2}{2m_n^*}\int_\infty d\mathbf{r}_{pB}\int d\omega_{nA}\int d\xi\left(\chi_F^{(-)*}\frac{\partial}{\partial r_{nA}}\chi_I^{(+)}-\chi_I^{(+)}\frac{\partial}{\partial r_{nA}}\chi_F^{(-)*}\right)_{r_{nA}=R_n},\quad(8)$$

where we have written $\chi_I^{(+)}$ for

$$\psi_d(|\mathbf{r}_{nA}-\mathbf{r}_{pA}|)\chi_i^{(+)}(\mathbf{k}_d,\mathbf{r}_{dA})\psi_A(\xi)$$

and similarly for $\chi_F^{(-)*}$. The above relation is valid for both the two-body and three-body final state problems. Indeed if we take R_n to be the radius of A then the second term on the right-hand side is Butler's Wronskian expression for the scattering amplitude for bound or unbound neutrons with distorted waves. If R_n is taken to be large then in the bound state case this surface integral vanishes and we have derived the usual equality of prior and post distorted wave expressions. In the three-body case the left-hand side converges to the prior expression (6) but while the sum of the two terms on the right-hand side also converges to the required scattering amplitude, the separate volume and surface integrals have equal and opposite fluctuations as R_n increases.

EVALUATION OF THE SCATTERING AMPLITUDE

We have found that for sufficiently large R_n both sides of Eq. (8) give the required scattering amplitude but that the left-hand side is not suitable for evaluation. On the right-hand side the volume integral may be found for a particular R_n using the well-known computing methods of the zero-range DWBA for the bound-state problem but the surface term is plainly far too difficult to evaluate. Hence we are forced to consider various mathematical devices in order to extract the scattering amplitude. In the following paragraphs we present two such methods.

Our approach to such devices is that the volume integral in the post matrix element (7) which, though it "formally" represents the scattering amplitude, fails to converge to the correct result, may be compared to a formal series (e.g., a perturbation expansion) which fails to converge. Such series are often summable to the proper result by such devices as the rearrangement of terms, and we shall see that the same applies here. We present two applicable devices.

Firstly, we expand all the scattering wave functions in partial waves. The valid prior matrix element (6) then becomes,

$$\sum_{l_p l_n l_d}\langle\chi_{f,l_p}^{(-)}\psi_{B,l_n}^{(-)}\,|\,V_{nA}+V_{pA}-\bar{V}_{dA}\,|\,\psi_d\chi_{i,l_d}^{(+)}\psi_A\rangle,\quad(9)$$

where, for instance, $\psi_{B,l_n}^{(-)}$ is an exact eigenfunction for $(A+n)$, the component of $\psi_B^{(-)}(\mathbf{k}_n,\mathbf{r}_{nA},\xi)$ which survives if the incident neutron plane wave in the latter is replaced by its lth component only. Now the transformation (8) from prior to post matrix elements applies equally well if each directed wave function, e.g., $\psi_B^{(-)}(\mathbf{k}_n,\mathbf{r}_{nA},\xi)$ is replaced by a single partial-wave component, e.g., $\psi_{B,l_n}^{(-)}$. There is the vital difference, however, that when R_n is taken to infinity, the volume matrix element on the right-hand side of (8) actually converges in the case of partial waves, and the surface integral vanishes: the integrand of the r_{nA} integral is of the form (oscillatory function of r_{nA})$/r_{nA}$. This means that we have validly, in place of (8), the post–prior equality for partial waves:

$$\langle\chi_{f,l_p}^{(-)}\psi_{B,l_n}^{(-)}\,|\,V_{nA}+V_{pA}-\bar{V}_{dA}\,|\,\psi_d\chi_{i,l_d}^{(+)}\psi_A\rangle$$
$$=\langle\chi_{f,l_p}^{(-)}\psi_{B,l_n}^{(-)}\,|\,V_{np}+V_{pA}-\bar{V}_{pB}\,|\,\psi_d\chi_{i,l_d}^{(+)}\psi_A\rangle.\quad(10)$$

One can easily see that the sum over l values in (9) is strongly convergent and the same must apply if we use (10) to replace each prior matrix element in (9) by its post counterpart. Therefore we have proved that a valid expression for the scattering amplitude is the partial-wave, post sum:

$$\sum_{l_p l_n l_d} \langle \chi_{f,l_p}{}^{(-)} \psi_{B,l_n}{}^{(-)} \mid V_{np}+V_{pA}-\bar{V}_{pB} \mid \psi_d \chi_{i,l_d}{}^{(+)} \psi_A \rangle. \tag{11}$$

The only difference between this valid, convergent expression for the scattering amplitude, and the invalid, divergent one (7) resides in a reversal of the order of spatial integration and summation over l values.

A straightforward evaluation of (11) would mean following precisely the conventional procedure used for bound final states, except for replacing a bound-state wave function ψ_B by a positive-energy one $\psi_{B,l_n}{}^{(-)}$. However, the radial partial-wave integrals would converge excessively slowly.

Our second method consists in expanding only the wave function $\psi_B{}^{(-)}$ of the prior matrix element in partial waves, i.e., we take

$$\sum_{l_n} \langle \chi_f{}^{(-)}(\mathbf{k}_p, \mathbf{r}_{pB}) \psi_{B,l_n}{}^{(-)} \mid V_{nA}+V_{pA}-\bar{V}_{dA} \mid \times \psi_d \chi_i{}^{(+)}(\mathbf{k}_d, \mathbf{r}_{dA}) \psi_A \rangle. \tag{12}$$

On transforming these matrix elements to the post form by Green's theorem as in (8), we find that again the post volume integrals do not converge when $R_n \to \infty$, nor do the surface integrals vanish. However, one can prove that a correct post–prior correspondence is obtained by introducing a weak convergence factor into the post matrix element, i.e.,

$$\begin{aligned} &\langle \chi_f{}^{(-)}(\mathbf{k}_p, \mathbf{r}_{pB}) \psi_{B,l_n}{}^{(-)} \mid V_{nA}+V_{pA}-\bar{V}_{dA} \mid \times \psi_d \chi_i{}^{(+)}(\mathbf{k}_d, \mathbf{r}_{dA}) \psi_A \rangle \\ &= \lim_{\alpha \to 0} \langle \chi_f{}^{(-)}(\mathbf{k}_p, \mathbf{r}_{pB}) \psi_{B,l_n}{}^{(-)} \mid \\ &\times (V_{np}+V_{pA}-\bar{V}_{pB}) \Phi(r_{nA}, \alpha) \mid \psi_d \chi_i{}^{(+)}(\mathbf{k}_d, \mathbf{r}_{dA}) \psi_A \rangle, \end{aligned} \tag{13}$$

where, for instance,

$$\Phi(r_{nA}, \alpha) = \exp(-\alpha r_{nA}). \tag{14}$$

When a narrow virtual state B^* is present, one needs take account of only one value of l_n, or very few values. Our final prescription, then, is to evaluate the V_{np} matrix element on the right of (13) for some finite values of α, extrapolate numerically to $\alpha=0$, and lastly sum over l_n, as in (12), if necessary. Since one will, in practice, expand $\chi_f{}^{(-)}$ and $\chi_i{}^{(+)}$ in (13) in partial waves, this is actually only equivalent to evaluating (11) with convergence factors $\Phi(r_{nA}, \alpha)$ instead in the partial-wave integrals, where they are not required in principle because the integrals converge anyhow; but the factors Φ will help numerically to expedite the convergence of the procedure.

The square modulus of the resulting T-matrix element yields the differential cross section, or correlation function, for production of neutrons and protons of definite energies and directions $\hat{\mathbf{k}}_p$ and $\hat{\mathbf{k}}_n$. We need for this a good approximation to the wave function $\psi_B{}^{(-)}$ describing neutron scattering on A, preferably at the peak of the resonance level B^*. If the neutron is unobserved (as in usual stripping experiments), or only the energy integral over the resonance is measured, one can perform the appropriate integration over $\mathbf{k}_n$ or E_p.

APPLICATION

Calculations are being made for the O^{16} (d, pn) reaction proceeding via the $d_{\frac{3}{2}}$ resonance in O^{17}. In these we use measured deuteron and proton optical model parameters and a neutron wave function derived from a real potential[3] which reproduces the observed O^{16} (n, n) phase shifts.

ACKNOWLEDGMENT

We would like to express our thanks to Dr. R. Sack for advice on various mathematical problems.

Discussion

ZAMICK: Do you obtain the same result if you use a slightly bound wave function rather than a scattering wave function for the neutron?

MINES: There is no theoretical justification for the calculation you suggest. It might give a similar numerical result but this would not be of interest.

MCCARTHY: Could I ask you the question that Jerry Phillips asked me? It seems that you are more qualified to answer it. This theory seems to have some chance of describing something like $(p, 2p)$ where the final particles would stick together for a while.

MINES: This may be possible. I am not very familiar with the $(p, 2p)$ reaction.

PHILLIPS: It seems to me that in the very nature of your model you have just built in that assumption of the time delay, haven't you?

MINES: Yes.

PHILLIPS: I would just like to remark that it seems to me this is the power of this approach, because in so many stripping reactions, and in the interpretation of stripping measurements, one tries to deduce essentially how much single-particle state there is; so that this work of Mines is important, because from another experiment, namely, from neutron–oxygen scattering you know how much there is, so now you can check up on the theory. Not just your theory, but direct interaction theory.

MINES: This is our intention.

PUGH: The point being that you have one extra particle to study. You have more experimental information and more ways to check your theory.

MINES: We used the potential arrived at in the proceedings in the Kingston Conference.

FOWLER: That's a good one, too!

[3] J. L. Fowler, E. G. Corman, and E. C. Campbell in *Proceedings of the Kingston Conference, Kingston, 1960*, edited by D. A. Bromley and E. W. Vogt (University of Toronto Press, Toronto, 1960), p. 474.