Sequential Decay Theory for the B^{11} $(p, 3\alpha)$ Reaction*

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I. INTRODUCTION

For bombarding energies of only a few MeV, many nuclear reactions proceed via a compound-nucleus intermediate state of width $\lesssim 1$ MeV. In other cases, particularly at higher energies, a direct interaction may occur with level width ~ 10 MeV. For compound nuclei, the cluster model[1] predicts one or more pairs of product nuclei, according to the purity of the cluster state, as does likewise the direct-interaction model for direct-interaction states.

If one of the product nuclei is a virtual particle, it will decay on the cluster model into two nuclei, the energy–angular distribution of the three fragments being strongly influenced by the strong "final-state interaction" between the two second-emitted particles. For compound nuclei, one can get "sequential decay," which in some cases can reoccur a number of times.

Experimental evidence for sequential decay falls into two classes. The less sophisticated group includes measurements of total yield distributions of particles and energy distributions of the first-emitted particle. These include the $p(d, n)2p$ experiment,[2] in which the neutron energy distribution points to a probable singlet n–p resonance, and the p+d reaction, where the proton spectra are more peaked than a phase-space argument alone would predict.[3] In the latter case, the final-state interaction theory of Watson[4] (see below) gives qualitative agreement with the experimental results for $d^2\sigma/dE_p d\Omega_p$. Measurements of total n–D cross sections and deuteron recoil spectra[5] yield inconclusive results regarding the existence of the di-neutron, due to the two alternative sets of doublet–quartet scattering lengths. However, other data[6] on the proton energy spectra for n–d collisions shows a strong final-state interaction peak at the upper end, indicating the presence of the di-neutron. Another very nice experiment on the electrodisintegration of the deuteron by 146.9-MeV electrons[7] measured the distribution $d^2\sigma/dE\,d\Omega$ of scattered electrons. It showed, besides a broad quasi-elastic peak centered at 112 MeV and a narrow elastic peak at 128 MeV, a small, narrow, final-state interaction peak above 2 MeV below the elastic peak, attributed to the n–p singlet state.

Finally, in the first group, there is a large number of measurements of the $B^{11}(p, \alpha)Be^8$ reaction, usually directed towards measurement of the C^{12}* intermediate-state level parameters,[8] with little attempt to interpret the distributions in more detail.

Interpretation of the above experiments is often made ambiguous in that the measurements do not distinguish between first- and second-emitted particles, if both are charged. This can only be resolved by triple-coincidence data, in which the energy and angle of both first- and second-emitted particles are measured simultaneously (one of the energies is actually redundant). Energy–momentum conservation relations reduce the number of measured variables needed from the nine involved in three vector momenta variables to five, supplied by the solid-angle elements of the first- and second-emitted particles and the energy of the first.

The second group of experiments thus measures the triple correlation distribution $d^3\sigma/dE_1 d\Omega_1 d\Omega_2$ for the first- and second-emitted particles. Recent data[9,10] on the $B^{11}(p, 3\alpha)$ reaction have shown it to be an almost entirely sequential decay process for proton energies of a few MeV. Plots of coincidence yields versus lab energies E_1 and E_2 show strong peaks at RCM (recoil center of mass) energies checking in position and width with the 0^+ and 2^+ levels of Be^8. There is also strong evidence for ^1S deuteron and di-proton final-state interactions in the d+p and p+d triple-coincidence

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[1] G. C. Phillips and T. Tombrello, Nucl. Phys. **19**, 555 (1960); T. A. Tombrello and G. C. Phillips, *ibid.* **20**, 648 (1960); Phys. Rev. **122**, 224 (1961).
[2] L. Cranberg and R. K. Smith, Phys. Rev. **113**, 587 (1959).
[3] J. L. Friedes and M. K. Brussel, Phys. Rev. **131**, 1194 (1963).
[4] K. M. Watson, Phys. Rev. **88**, 1163 (1952).
[5] H. B. Willard, J. K. Bair, and C. M. Jones, Phys. Letters **9**, 339 (1964).
[6] I. Slaus, *Progress in Fast Neutron Physics* (University of Chicago Press, Chicago, 1962).
[7] M. R. Yearian and E. B. Hughes, Phys. Letters **10**, 234 (1964).
[8] O. Beckman, T. Huus, and Č. Zupaničič, Phys. Rev. **91**, 606 (1953); P. B. Treacy, Proc. Phys. Soc. (London) **68**, 204 (1955); R. E. Holland, D. R. Inglis, R. E. Malm, and F. P. Mooring, Phys. Rev. **99**, 92 (1955); E. H. Geer, E. B. Nelson, and E. A. Wolicki, *ibid.* **100**, 215 (1955); G. Dearnaley, G. A. Dissanaike, A. P. French, and G. L. Jones, *ibid.* **108**, 743 (1957); D. Kamke, Z. Physik **156**, 603 (1959); **156**, 621 (1959); R. Bouchez, H. Beaumevielle, J. Fluery, P. Perrin, R. de Swiniarski, and M. Chabre, J. Phys. Radium **21**, 819 (1960); R. E. Segal and M. J. Bina, Phys. Rev. **124**, 814 (1961); D. Dehnhard, D. Kamke, and P. Kramer, Z. Naturforsch. **16A**, 1245 (1961); Phys. Letters **3**, 52 (1962); G. D. Symons and P. B. Treacy, Nucl. Phys. **46**, 93 (1963).
[9] J. D. Bronson, W. D. Simpson, and G. C. Phillips, Bull. Am. Phys. Soc. **8**, 124 (1963), and work to be published; J. D. Bronson, Ph.D. thesis, William Marsh Rice University, 1964 (unpublished).
[10] J. P. Longequeue, N. Longequeue, and H. Beaumevieille, Phys. Letters **9**, 171 (1964).

measurements at Rice.[11] Similar measurements[12] on the $He^3(He^3, \alpha)2p$ reaction show the predominant sequential decay mode to be via the intermediate state $p+Li^5$. Finally, an experiment using 40-MeV α's in $O^{16}(\alpha, 2\alpha)C^{12}_{g.s.}$ gave triple-correlation spectra,[13] interpreted as a mixture of sequential decay via intermediate-state $O^{16*}+\alpha$, and direct knock-out scattering. The latter effect might be expected at the comparatively high energy employed.

An early approximate treatment of final-state interactions was given by Watson,[4] the total interaction being split into primary and final-state components. The primary reaction occurs within a certain interaction volume and the final-state interaction over a somewhat larger volume of configuration space. Watson considered the reverse reaction proceeding backwards to produce the initial virtual state particle *inside* the compound-nucleus interaction volume a^3. The probability of this occurring is proportional to the final-state particles' scattering cross section $(\sin^2\eta_0)/k_f^2$, and to the momentum component $d^3k_f=k_f^2dk_fd\Omega_f$ of phase space ($k_f=$ final-state wave number, and $l_f=0$). By detailed balancing, the final-state contribution to the cross section for emission of the first product particle is the extra factor $\sin^2\eta_0dk_f$, so that

$$d\sigma_0 \propto \sin^2\eta_0 dk_f. \tag{1}$$

For arbitrary l_f, this is modified to

$$d\sigma_l \propto (\sin^2\eta_l/k_f^{2l})dk_f. \tag{2}$$

These results are approximate, holding only with the explicit neglect of the hard-sphere scattering phase ϕ_l ($\phi_0=ka$), or spatial localization, valid for $ka\to 0$.

A more general exact treatment has been given by Phillips, Griffy, and Biedenharn,[14] via the generalized density-of-states function in the continuum. The reaction

$$a+A\to D^*\to b+B \tag{3}$$

for sharp states B can be treated by perturbation theory to yield the differential cross section for particles b,

$$d^2\sigma/dE_bd\omega_b=(\mu_a\mu_bk_b/4\pi^2\hbar^2k_a)\cdot|\langle B+b, E_b|H'|A+a, E_a\rangle|^2\rho(E_B), \tag{4}$$

where H' is the interaction Hamiltonian for (3) and $|a+A, E_a\rangle$ represents the initial-state vector, $|B+b, E_b\rangle$ the final-state vector. The initial and final wave numbers are k_a and k_b, and $\rho(E_B)$ is a generalized density-of-states function for sharp states B with eigenenergies E_n:

$$\rho(E_B)=\sum_n\delta(E-E_n). \tag{5}$$

If the incident energy E_a is near a narrow level of D^* well separated from other levels, we may replace (4) by single-level dispersion scattering to give the energy–angle distribution for particle b as

$$d^2\sigma/dE_bd\omega_b=(\pi/k_a^2)g_J\{\Gamma_{\lambda_a}\Gamma_{\lambda_b}/[(E_{\lambda_a}-E_a)^2+\tfrac{1}{4}\Gamma_\lambda^2]\}\cdot|Y_{l_b,0}(\theta_b,\phi_b)|^2\rho(E_B), \tag{6}$$

where Γ_{λ_a} and Γ_{λ_b} are the ingoing and outgoing partial channel widths, respectively, E_{λ_a} is the resonance energy, and $\Gamma_\lambda=\sum_\alpha\Gamma_{\lambda\alpha}$, summed over all channels α. Also we have

$$g_J=(2J+1)/(2I_A+1)(2i_a+1), \tag{7}$$

where D^* has total spin quantum numbers J, M, and I_A and i_a are the intrinsic spins of particles "A" and "a," respectively.

If B is not a sharp state, but is in a virtual or resonant state which decays further into two particles:

$$B\to c+C, \tag{8}$$

then the generalized density of states (5) for sharp states must be replaced by that for a semistable particle B (note the transition probability involves the density of final states).

A calculation of the number of eigenstates of B allowed between two spheres, the one of the small interaction radius "a" and the other of an arbitrary large radius R, gives the density of states approximately[14] (the radii "a" and "R" are not strictly multiples of the one de Broglie wavelength):

$$\rho_{l_c}(E_B)=\pi^{-1}(d/dE_B)[\eta_{l_c}(k_c)+\phi_{l_c}(k_c, a)], \tag{9}$$

where $\eta_{l_c}(k_c)$ is the elastic-scattering phase of the final-state particles with relative orbital angular momentum l_c, $\phi_{l_c}(k_c, a)$ is the hard-sphere scattering phase of the final-state particles, $E_B=(\hbar^2/2\mu_B)k_c^2$, and μ_B is the reduced mass of $c+C$.

An exact treatment,[14] based on the renormalization of the final-state wave function between spheres of radius "a" and "R," respectively, gives the result

$$\begin{aligned}\rho_{l_c}(E_B)=&-\frac{\mu_B}{4\pi\hbar^2k_c^2}(F_{l_c}\cos\eta_{l_c}+G_{l_c}\sin\eta_{l_c})^2\frac{\partial^2}{\partial k_c\partial r_c}\cdot[\ln(F_{l_c}\cos\eta_{l_c}+G_{l_c}\sin\eta_{l_c})]|^{r=a}\\=&\left[\pi^{-1}\frac{d}{dE_B}(\eta_{l_c}+\phi_{l_c})-\frac{\mu_B}{2\pi\hbar^2k_c}\cdot\left\{\left(k_c^{-1}-\frac{2}{A_{l_c}}\frac{\partial A_{l_c}}{\partial k_c}\right)\sin 2(\eta_{l_c}+\phi_{l_c})\right.\right.\\&\left.\left.-k_c^{-1}\left(A_{l_c}\frac{\partial^2A_{l_c}}{\partial r\partial k_c}-\frac{\partial A_{l_c}}{\partial r}\frac{\partial A_{l_c}}{\partial k_c}\right)\sin^2(\eta_{l_c}+\phi_{l_c})\right\}\right]^{r=a}.\end{aligned} \tag{10}$$

[11] W. D. Simpson, J. D. Bronson, W. R. Jackson, and G. C. Phillips, William Marsh Rice University, 1964 (to be published).
[12] J. P. Aldrich, B. H. Wildenthal, and D. H. Youngblood, Physics Department, William Marsh Rice University, 1964 (to be published).
[13] P. F. Donovan, J. V. Kane, Č. Zupančič, C. P. Baker, and J. F. Mellenauer, Phys. Rev. **135B**, 61 (1964).
[14] G. C. Phillips, T. A. Griffy, and L. C. Biedenharn, Nucl. Phys. **21**, 327 (1960).

Here, $F_l(k,r)$ and $G_l(k,r)$ are the regular and irregular wave functions for zero short-range potential (Coulomb wave functions in the Coulomb-field case), and

$$A_l=(F_l^2+G_l^2)^{\frac{1}{2}}. \tag{11}$$

In the single-level dispersion scattering approximation, both expressions (9) and (10) for $\rho_{l_c}(E_B)$ reduce to the forms

$$\rho_{l_c}(E_B)=\frac{\sin^2(\eta_{l_c}+\phi_{l_c})}{\pi\gamma_\lambda^2 P_{l_c}}=\frac{\Gamma_\lambda/2\pi}{[(E_\lambda-E)^2+\frac{1}{4}\Gamma_\lambda^2]}, \tag{12}$$

where $\Gamma_\lambda=2P_{l_c}\gamma_{\lambda_c}^2$ and γ_λ^2 is the reduced width of B for decay into channel λ,

$$\begin{aligned}\gamma_{\lambda_c}&=(\hbar^2/2\mu_B a)^{\frac{1}{2}}[F_{l_c}\cos\eta_{l_c}+G_{l_c}\sin\eta_{l_c}]^{r=a}\\&=(\hbar^2/2\mu_B a)^{\frac{1}{2}}[A_{l_c}\sin(\eta_{l_c}+\phi_{l_c})]^{r=a},\end{aligned} \tag{13}$$

and $P_{l_c}=ka/A_{l_c}^2$ is the penetrability.

The approximate expression (12) has been derived by a number of authors[14–16] and commonly assumes the semistable state B decays exponentially, an assumption most nearly valid for narrow levels. Trammell[17] has in fact shown that broad levels lead to correction factors of order $t^{-(2l+3)}$ to exponential decay, deriving[16] from the energy dependence of the numerator of Eq. (12).

Note the generalized density-of-states function $\rho_l(E_B)$ enables the scattering to be described in terms of the experimental phase shifts for scattering of the final-state particles, no knowledge of the interparticle interaction being required.

From Eq. (6), the scattering amplitude for "b" becomes (cf. Appendix II)

$$f_{l_b,0}(E_b;\theta_b,\phi_b)=\frac{\pi^{\frac{1}{2}}}{k_a}g_J^{\frac{1}{2}}\frac{\Gamma_{\lambda_a}^{\frac{1}{2}}\Gamma_{\lambda b}^{\frac{1}{2}}\exp[2i(\eta_{l_b}+\sigma_{l_b})]}{(E_{\lambda_a}-E_a-\frac{1}{2}i\Gamma_\lambda)}\cdot\rho_{l_c}^{\frac{1}{2}}(E_B)\,Y_{l_b,0}(\theta_b,\phi_b), \tag{14}$$

where σ_{l_b} is the Coulomb phase (75) and η_{l_b} the inelastic phase for channel b. The corresponding triple-correlation amplitude for the distribution of both particles "b" and "c" is then

$$f_{l_b,0;l_c,m_c}(E_b;\theta_b,\phi_b;\theta_c,\phi_c)=\frac{\pi^{\frac{1}{2}}}{k_a}g_J^{\frac{1}{2}}\cdot\frac{\Gamma_{\lambda_a}^{\frac{1}{2}}\Gamma_{\lambda_b}^{\frac{1}{2}}\exp[2i(\eta_{l_b}+\sigma_{l_b})]}{(E_{\lambda_a}-E_a-\frac{1}{2}i\Gamma_\lambda)}\rho_{l_c}^{\frac{1}{2}}(E_B)\,i^{l_c}\exp[i(\eta_{l_c}+\sigma_{l_c})]\cdot Y_{l_b,0}(\theta_b,\phi_b)\,Y_{l_c,m_c}(\theta_c,\phi_c)\exp(i\beta_{l_c}), \tag{15}$$

where β_{l_c} is a phase constant, $\beta_{l_c}=\eta_{l_c}+\phi_{l_c}$, which in the single-level dispersion scattering approximation is equal to the resonant phase. Thus if (12) holds for $\rho_{l_c}(E_B)$, then

$$\beta_{l_c}=\eta_{l_c}+\phi_{l_c},\qquad \tan\beta_{l_c}=\tfrac{1}{2}\Gamma_B/(E_{B,l_c}-E_c). \tag{16}$$

In the latter case, we identify

$$\rho_{l_c}^{\frac{1}{2}}(E_B)\exp i\beta_{l_c}=(\Gamma_{B,l_c}/2\pi)^{\frac{1}{2}}/(E_{B,l_c}-E_c-i\tfrac{1}{2}\Gamma_B). \tag{17}$$

In the resonant-scattering approximation, the generalized density-of-states method gives the same amplitude as obtained via the resonant-pole, intermediate-state method (c.f. intermediate-state transition $0\to l\to m$, modified to allow for exponential decay of l, and parameterized assuming exact wave functions. This gives

$$H=\sum_l\frac{H_{ml}'H_{l0}'}{E_0-E_l-\frac{1}{2}i\Gamma_l}$$

as the matrix element), provided we sum over possible alternative semistable states of B.

The simplest case of sequential decay is the reaction

$$p+\mathrm{B}^{11}\to\mathrm{C}^{12*}\to\alpha_1+\mathrm{Be}^8,\qquad \mathrm{Be}^8\to\alpha_2+\alpha_3 \tag{18}$$

at proton energies of a few MeV. The odd parity of B^{11} and low proton energy shows that only $l=1$ protons contribute, since known level diagrams and reaction data indicate that only the 0^+ and 2^+ states of both C^{12} and Be^8 are appreciably involved. This and the zero intrinsic spin of α particles greatly reduces the number of channels normally expected in reactions, making both theory and data interpretation much simpler.

A pilot treatment of the problem was made by Bonnevay,[18] applying field-theoretic methods to the $B_1+B_2\to A_1+A_2+A_3$ process for spinless particles in only one partial wave channel. All two-body interactions are described by resonances, with the first two rescatterings allowed for, the latter giving correcting terms to the scattering obtained using Watson's final-state interaction method.[4,19] The model has in fact been applied to the $C^{12}(\gamma,3\alpha)$ photodisintegration process, giving good agreement with experiment, while Watson's method gives fair results.[19] The reaction actually appears to have been recognized as sequential much earlier.[20]

The generalized density-of-states method takes account of rescattering of the two final-state particles. If the decay is truly sequential, with the first particle emitted well before the second two, then rescattering between first and second, and between first and third particles should be negligible. It is also probably par-

[15] C. W. Cook, W. A. Fowler, C. C. Lauritsen, and T. Lauritsen, Phys. Rev. **111**, 567 (1958); T. A. Griffy and L. C. Biedenharn, Nucl. Phys. **15**, 636 (1960); E. W. Hamburger and J. R. Cameron, Phys. Rev. **117**, 781 (1960).

[16] M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964), p. 450.

[17] G. T. Trammel, Oak Ridge National Laboratory, Physics Division Report, September 1956, p. 7 (unpublished).

[18] G. Bonnevay, Nuovo Cimento **30**, 1325 (1963).

[19] J. Letessier, Phys. Letters **10**, 102 (1964).

[20] V. L. Telegdi, Phys. Rev. **84**, 600 (1952).

tially allowed for by adjustment of the somewhat arbitrary interaction radius "a" of D^* in (3).

Duck[21] has given a field theoretic treatment of the $B^{11}(p, 3\alpha)$ reaction, allowing for all partial-wave channels important at low energies, and treating all two-particle interactions by resonances. In the next section, we employ the generalized density-of-states method along the lines laid down above.

Other calculations which have been reported include estimates of p–p and n–n final-state interaction effects, in particular the energy spectrum of particle X in the reaction $A+B\rightarrow X+2N$. A sharp peak at the upper end is found, with a Coulomb field damping the peak. Watson's method was employed.[22]

Alternative approaches which have been used are a field-theoretic, impulse-type approximation calculation of $d^2\sigma/dE_p d\Omega_p$ for $n+d\rightarrow 2n+p$, giving fair agreement with experiment except at the upper end,[23] and a direct-interaction, Born-approximation treatment allowing for ^1S deuteron and continuum di-neutron.[24] The spectra in the latter case were in fairly good agreement with experiment, although only $l=0$ encounters were considered. In contrast, the Gammel–Thaler potential was employed. Neither of these two calculations attempted the triple-correlation cross section, which is much more sensitively dependent on the final-state interaction.

II. REACTION KINEMATICS FOR THREE FINAL PARTICLES

Both the energy-spectrum function $d^2\sigma/dEd\Omega$ for one final particle, and the triple-correlation distribution $d^3\sigma/dE_1 d\Omega_1 d\Omega_2$ for two final-state particles, are proportional to the corresponding transition probabilities. The transition probability for a reaction has the form

$$w=(2\pi/\hbar)\,|H|^2\rho_F(E), \tag{19}$$

where H is a compound matrix element,[25] involving summation over any intermediate states, and $\rho_F(E)$ is the density of final states. Cross sections are obtained from Eq. (19) merely by changing the normalization of the incident particle "a" in Eq. (3) from one particle in volume L^3 to one particle incident on unit normal area per second.

Two different-reaction mechanisms are possible for three-body decay: (1) direct break-up into three particles and (2) sequential two-body decay. For direct break-up, H in Eq. (19) is commonly assumed to be a constant, so that the decay probability is uniform over the available phase space. For particles of spin zero, there is one final state in each phase volume h^3 for each particle (replaced by $2s+1$ for particles of spin s), which we assume hereafter. We then obtain the cross section

$$d\sigma=C\rho_F(E), \tag{20}$$

where for total energy $E=E_1+E_2+E_3$,

$$\rho_F(E)=\frac{d}{dE}\int_{P_2}\int_{P_3}\delta\Big(E-\sum_{i=1}^{3}P_i^2/2m_i\Big)\,\delta(\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3)\cdot d\mathbf{P}_1(h^3)^{-1}d\mathbf{P}_2(h^3)^{-1}d\mathbf{P}_3$$
$$=\frac{d}{dE}\int_{P_2}\int_{P_3}(h^3)^{-1}\,d\mathbf{P}_2(h^3)^{-1}\,d\mathbf{P}_3, \tag{21}$$

on allowing for conservation of energy and momentum.[26] C is a constant in (20).

Several authors[27–31] have treated the purely statistical decay (direct break-up) of many-body systems. The first two[27,28] deal with decays into three or more particles, measuring the energy spectrum of one final particle only, while Chuan[29] does likewise for the three final particle case alone. Relativistic formulas for the momentum distribution $d\rho_F(E)/dP_1$ have been given by Block[30] for systems of 3–5 particles.

Equation (21) for $\rho_F(E)$ has been integrated to give the results[31] (note errors in reference)

$$d^2\sigma/dE_1 d\Omega_1 = C'E_1^{\frac{1}{2}}[E-((m_1+m_2+m_3)/(m_2+m_3))E_1]^{\frac{1}{2}}, \tag{22}$$

$$d^3\sigma/dE_1 d\Omega_1 d\Omega_2 = CP_1P_2^2/|AP_2+P_1\cos\Delta_{12}-P_0\cos\Theta_2|, \tag{23}$$

where

$$C=(\pi^3)^{-1}\left(\frac{m_1+m_2+m_3}{m_3}\right)^{\frac{3}{2}}\frac{m_3}{m_1}\frac{E_0}{(E_1'+E_2'+E_3')},$$

E_0 being the lab energy of particle "a" and E_i' ($i=1, 2, 3$) the c.m. energy of particle "i." Here $A=1+m_3/m_2$, C' and C are constants, and Δ_{12} is the angle between the directions of emission of particles 1 and 2 in the laboratory system (lab). Writing Θ_1, Φ_1 and Θ_2, Φ_2 as spherical polar angles of 1 and 2, respectively, with respect to the target, we have

$$\cos\Delta_{12}=\cos\Theta_1\cos\Theta_2+\sin\Theta_1\sin\Theta_2\cos(\Phi_1-\Phi_2). \tag{24}$$

Here E_i and P_i are in the lab system.

[21] I. Duck, Nucl. Phys. **57**, 643 (1964).
[22] R. J. N. Phillips, Nucl. Phys. **53**, 650 (1964).
[23] F. Ferroni and V. Wataghin, Nuovo Cimento **28**, 1342 (1963).
[24] D. R. Koehler and R. A. Mann, Phys. **135B**, 91 (1964).
[25] J. McConnell, *Quantum Particle Dynamics* (North-Holland Publishing Company, Amsterdam, 1958), p. 157.
[26] S. DeBenedetti, *Nuclear Interactions* (John Wiley & Sons, Inc., New York, 1964), p. 314.
[27] T. A. Welton, private set of notes distributed by C. B. Moak (unpublished).
[28] G. E. Uhlenbeck and S. Goudsmidt, *Verhandelingen van Dr. P. Zeeman* (Martimus Nijhoff, The Hague, Netherlands, 1935), pp. 201–211.
[29] C. X. Chuan, J. Phys. Radium **23**, 78 (1962).
[30] M. M. Block, Phys. Rev. **101**, 796 (1956).
[31] T. H. Berlin and G. E. Owen, Nucl. Phys. **5**, 669 (1958).

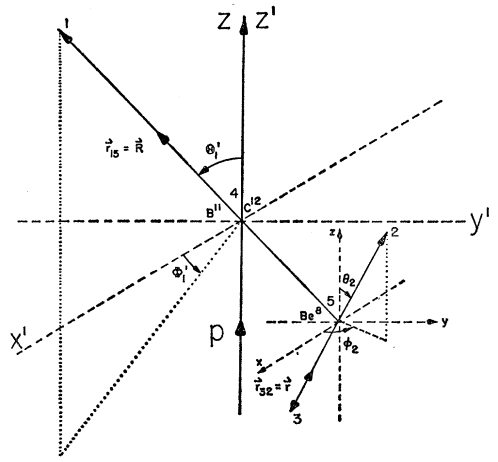

FIG. 1. Coordinate systems for $p+B^{11}\rightarrow C^{12*}\rightarrow\alpha_1+Be^8$, $Be^8\rightarrow\alpha_2+\alpha_3$ reaction. In SCM (system center of mass) coordinate system $X'Y'Z'$, particle α_1 has spherical polar angles (Θ_1', Φ_1') and its vector distance from Be^8 (particle 5) is **R**. In RCM (recoil center of mass) coordinate system xyz, the Be^8 breakup particles have spherical polar angles (θ_2, ϕ_2), with vector distance apart **r**.

The distributions (22) and (23) are in fact subject to modifications due to angular momentum and Coulomb effects, as discussed by Delves.[32] However this does not change the characteristic cross-section shape, which shows broad smooth variations with E_1.

The sequential decay process also involves the same c.m. density-of-states factors $\rho_F(E)$ as in Eqs. (22) and (23), as has been found by Duck[21] via the phase-space method. However, the compound matrix element H in Eq. (19) is no longer a constant, and must be evaluated. One finds strong enhancement of the coincidence yield vs E_1 where the internal cluster energy ϵ_{23} equals the energy of a metastable state in the intermediate nucleus B, the sharpness of the peaking dependent on the intermediate state width. If two or more of the three particles are nonidentical, they may alternatively enter different counters, this being detected either directly or via the different loci obtained when particle yields are plotted against E_1 and E_2. Different cross sections are thus obtained for the several possibilities in a complete experiment. The wave function and hence amplitude should be either symmetrized (if bosons) or antisymmetrized (if fermions) with respect to identical particles, where of course the yield vs E_1 and E_2 loci are degenerate.

Figure 1 illustrates the sequential decay for a beam "p" (particle 0) along the Z axis, incident on a target at 4. The first emitted particle 1 has spherical polar angles (Θ_1', Φ_1') in SCM (system center-of-mass coordinates) and (Θ_1, Φ_1) in lab coordinates. The recoil particle at 5 is vector distance **R** from 1, and decays to final-state particles 2 and 3, vector distance **r** apart.

[32] L. M. Delves, Nucl. Phys. **20**, 275 (1960).

The RCM (recoil center of mass) spherical polar angles of 2 (in an xyz frame with z axis parallel to the Z axis of the lab frame) are (θ_2, ϕ_2), with SCM and lab angles as (Θ_2', Φ_2') and (Θ_2, Φ_2), respectively.

Lab velocities, momenta, and energies are labeled V_i, P_i and E_i $(i=0, 1, 2, 3)$, where $i=0$ denotes the incident particle; SCM quantities are V_i', P_i', and E_i', and relative quantities between i and j in RCM are v_{ij}, p_{ij}, and ϵ_{ij}. Solid-angle elements are correspondingly $d\Omega_i$, $d\Omega_i'$, and $d\omega_i$ in the three systems.

In Fig. 1, we take components of velocity of particle 2 along the Z, X, and Y axes to obtain the relations between lab, RCM and SCM coordinates

$$\begin{aligned}
V_2\cos\Theta_2&=(\mu_{23}/m_2)v_{23}\cos\theta_2\\
&\quad-[m_1/(m_1+m_2+m_3)]V_{1-23}'\cos\Theta_1'+v_D,\\
V_2\sin\Theta_2\cos\Phi_2&=(\mu_{23}/m_2)v_{23}\sin\theta_2\cos\phi_2\\
&\quad-[m_1/(m_1+m_2+m_3)]V_{1-23}'\sin\Theta_1'\cos\Phi_1',\\
V_2\sin\Theta_2\sin\Phi_2&=(\mu_{23}/m_2)v_{23}\sin\theta_2\cos\phi_2\\
&\quad-[m_1/(m_1+m_2+m_3)]V_{1-23}'\sin\Theta_1'\sin\Phi_1',\qquad(25)
\end{aligned}$$

where V_{1-23}' is the relative velocity in SCM of 1 and the recoil system (23) (labeled 5 in Fig. 1), v_D is the SCM velocity of D^* in Eq. (3), and μ_{23} is the reduced mass of 2 and 3. Note $V_1'=(\mu_{1-23}/m_1)V_{1-23}'$, where μ_{1-23} is the reduced mass of 1 and the recoil system 23.

Rewritten in terms of momenta, Eqs. (25) become

$$\begin{aligned}
p_{23}\cos\theta_2&=P_2\cos\Theta_2\\
&\quad+(1/A)P_1'\cos\Theta_1'-(1/B)P_0,\\
p_{23}\sin\theta_2\cos\phi_2&=P_2\sin\Theta_2\cos\Phi_2\\
&\quad+(1/A)P_1'\sin\Theta_1'\cos\Phi_1',\\
p_{23}\sin\theta_2\sin\phi_2&=P_2\sin\Theta_2\sin\Phi_2\\
&\quad+(1/A)P_1'\sin\Theta_1'\sin\Phi_1',\qquad(26)
\end{aligned}$$

where

$$\begin{aligned}
A&=1+(m_3/m_2),\\
B&=1+(m_1+m_3)/m_2,\\
C&=1+(m_2+m_3)/m_1,\qquad(27)
\end{aligned}$$

and $P_0=P_D$ by momentum conservation.

Additional relations are obtained between lab and SCM velocity components:

$$\begin{aligned}
V_1\cos\Theta_1&=V_1'\cos\Theta_1'+V_c,\\
V_1\sin\Theta_1\cos\Phi_1&=V_1'\sin\Theta_1'\cos\Phi_1',\\
V_1\sin\Theta_1\sin\Phi_1&=V_1'\sin\Theta_1'\sin\Phi_1',\qquad(28)
\end{aligned}$$

leading to momentum equations

$$P_1' \cos \Theta_1' = P_1 \cos \Theta_1 - (1/C) P_0,$$

$$P_1' \sin \Theta_1' \cos \Phi_1' = P_1 \sin \Theta_1 \cos \Phi_1,$$

$$P_1' \sin \Theta_1' \sin \Phi_1' = P_1 \sin \Theta_1 \sin \Phi_1. \quad (29)$$

Substitution of Eqs. (29) in (26) leads to

$$p_{23} \cos \theta_2 = P_2 \cos \Theta_2 + (1/A) P_1 \cos \Theta_1 - (1/A) P_0,$$

$$p_{23} \sin \theta_2 \cos \phi_2 = P_2 \sin \Theta_2 \cos \Phi_2 + (1/A) P_1 \sin \Theta_1 \cos \Phi_1,$$

$$p_{23} \sin \theta_2 \sin \phi_2 = P_2 \sin \Theta_2 \sin \Phi_2 + (1/A) P_1 \sin \Theta_1 \sin \Phi_1. \quad (30)$$

The sum of squares of Eqs. (30) gives the internal momentum of the cluster 23 as [defining Δ_{12} in Eq. (24)]

$$p_{23} = [(1/A^2) P_1^2 + P_2^2 + (2/A) P_1 P_2 \cos \Delta_{12} + (1/A^2) P_0^2 - (2/A) P_0 (P_2 \cos \Theta_2 + (1/A) P_1 \cos \Theta_1)]^{\frac{1}{2}}, \quad (31)$$

the internal cluster energy being $\epsilon_{23} = p_{23}^2/2\mu_{23}$. Sequential decay will lead to a peaking of coincidence counts of 1 and 2 for energies ϵ_{23} near the resonance level energies of the recoil cluster $B = (23)$.

In dividing pairs of equations in (26), we get relations between RCM and lab angles of 1 and 2:

$$\tan \theta_2 \sin \phi_2 = \frac{P_1 \sin \Theta_1 \sin \Phi_1 + A P_2 \sin \Theta_2 \sin \Phi_2}{P_1 \cos \Theta_1 + A P_2 \cos \Theta_2 - P_0}, \quad (32)$$

$$\tan \phi_2 = \frac{P_1 \sin \Theta_1 \sin \Phi_1 + A P_2 \sin \Theta_2 \sin \Phi_2}{P_1 \sin \Theta_1 \cos \Phi_1 + A P_2 \cos \Theta_2 \cos \Phi_2}. \quad (33)$$

Similarly, Eqs. (29) give relations between SCM and lab angles of 1:

$$\tan \Theta_1' = P_1 \sin \Theta_1 / [P_1 \cos \Theta_1 - (1/C) P_0],$$

$$\Phi_1' = \Phi_1. \quad (34)$$

The sums of squares of Eqs. (29) gives also the relative momentum of particle 1 and the recoil cluster $B = (23)$:

$$P_1' = [P_1^2 + (1/C^2) P_0^2 - (2/C) P_1 P_0 \cos \Theta_1]^{\frac{1}{2}} = P_{1-23}', \quad (35)$$

the relative energy being $E_{1-23}' = P_1'/2\mu_{1-23}$.

Theoretical energy-spectrum distributions for one particle 1 and triple correlations for two particles 1 and 2 are normally obtained with particle 1 in SCM and particle 2 in RCM coordinates, viz., (Θ_1', Φ_1') and (θ_2, ϕ_2), respectively. Relations (34) and (32), (33) enable the appropriate SCM and RCM quantities to be found from lab values. Note that if the incident beam is unpolarized, then the first-emitted particle 1 has orbital magnetic quantum number $m_1 = 0$, so that the distribution of 1 is independent of Φ_1, and we may put $\Phi_1 = 0$ in Eqs. (31)–(34). The result does not hold for a polarized incident beam, or for a spin–orbit force leading to coupling between channels (e.g., tensor force), but is valid for central and spin–orbit coupling ($\mathbf{L} \cdot \mathbf{S}$) forces.[33] The amplitudes (14) and (15) assume these conditions hold. Actually spin–orbit forces give Φ-independent distributions for unpolarized beams, due to averaging over all directions of polarization (values of m_1).

The lab energy spectrum and triple-correlation cross section for direct breakup were obtained earlier via the phase-space method, the results being summarized in Eqs. (22) and (23). Sequential decay is found to lead to the same density-of-states factor $\rho_F(E)$ as in (22) and (23). It now appears as a conversion factor between the SCM and lab energy spectra for the emission of the first particle, and between the SCM, RCM triple correlation, and the corresponding lab cross section, c.f. Eqs. (40) and (41).

An alternative method which has been used[9,31] is to transform directly from one reference system to the other using the appropriate Jacobian. We get

$$d^2\sigma/dE_1 d\Omega_1 = J_1^L d^2\sigma/dE_1' d\Omega_1', \quad (36)$$

$$d^3\sigma/dE_1 d\Omega_1 d\Omega_2 = J_{12}^L d^3\sigma/dE_1' d\Omega_1' d\omega_2, \quad (37)$$

where

$$J_1^L = \frac{\partial(E_1'; \cos \Theta_1', \Phi_1')}{\partial(E_1; \cos \Theta_1, \Phi_1)}, \quad (38)$$

$$J_{12}^L = \frac{\partial(E_1'; \cos \Theta_1', \Phi_1'; \cos \theta_2, \phi_2)}{\partial(E_1; \cos \Theta_1, \Phi_1; \cos \Theta_2, \Phi_2)}. \quad (39)$$

Evaluation of these Jacobians by Bronson[9] and by Simpson[34] leads to the results

$$J_1^L = C_1' E_1^{\frac{1}{2}} [E - ((m_1+m_2+m_3)/(m_2+m_3)) E_1]^{\frac{1}{2}}, \quad (40)$$

$$J_{12}^L = [1 + (m_1+m_2)/m_3] (1/P_1' p_{23}) \cdot | P_1 P_2^2 / (A P_2 + P_1 \cos \Delta_{12} - P_0 \cos \Theta_2) |, \quad (41)$$

where P_1' and p_{23} are given by Eqs. (35) and (31), respectively, and C' is a constant. The extra variables

[33] J. M. Blatt and L. C. Biedenharn, Rev. Mod. Phys. **24**, 258 (1952).

[34] W. D. Simpson, Physics Department, William Marsh Rice University (private communication).

$P_1'p_{23}$ in (41), as compared to Eq. (23) for direct breakup, are due to particle 2 being in RCM instead of SCM coordinates as in the latter case.

Bronson[9] discusses also the Jacobian transformations for the case where the first-emitted particle enters detector 2 (instead of detector 1), and alternatively the particles actually detected at detectors 1 and 2 are both breakup particles from the cluster $B=(23)$. This order of detection effect need only be taken account of specifically when two or more of the final particles are nonidentical, in which case the cross sections for permutations of the nonidentical particles between detectors 1 and 2 must be considered separately, each with its appropriate Jacobian transformation. If triple coincidence counts are plotted against E_1 and E_2, three nonidentical particles would give six loci and two nonidentical particles three loci. The case of three identical particles gives one degenerate locus, and is taken account of by symmetrization or antisymmetrization of the wave function for bosons and fermions, respectively. This leads to interference effects between the components of the resulting amplitude, as in the $B^{11}(p, \alpha)2\alpha$ correlation.

III. REACTION AMPLITUDE FOR $B^{11}(p, \alpha)2\alpha$ TRIPLE CORRELATIONS

We take the relative wave functions of the α_1–Be^8 and α_2–α_3 systems as $\Phi_1(j_1, m_1; \mathbf{R})$ and $\Phi_2(j_2, m_2; \mathbf{r})$, respectively, so that summing over j_1, j_2 and m_1, m_2, the total wave function of the intermediate C^{12} system in state JM [c.f. Eq. (18)] is

$$\Psi_{JM}=\sum_{j_1m_1}\sum_{j_2m_2}(j_1j_2m_1m_2 \mid JM)\Phi_1(j_1m_1;\mathbf{R})\Phi_2(j_2m_2;\mathbf{r}). \tag{42}$$

We note also that as the α particles have zero intrinsic spin, then $j_1=l_1$ and $j_2=l_2$, with the corresponding magnetic quantum numbers identical. If the particles had intrinsic spin, (42) would have to include summations over intrinsic spins also.

From Eq. (42), the single-channel amplitude becomes

$$f_{JM}(E_1'; \Theta_1', \Phi_1'; \theta_2, \phi_2) = \sum_{l_2}\Big[\sum_{l_1=|J-l_2|}^{J+l_2}(l_1l_20M \mid JM) \cdot f_{JM}(l_1m_1; l_2m_2; E_1'; \Theta_1', \Phi_1'; \theta_2, \phi_2)\Big], \tag{43}$$

where for $E_0\sim$ few MeV, $J^\pi=0^+$, 2^+ for C^{12} and $l_2{}^\pi=0^+$, 2^+ for Be^8. Also $|J-l_2|\leq j_1\leq J+l_2$ and $m_1=0$ gives $m_2=M$, assuming an unpolarized incident proton beam and central or $\mathbf{L}\cdot\mathbf{S}$ type forces (no coupling between different l_1m_1 or l_2m_2 channels as with spin–orbit-type forces).

The amplitude on the right of Eq. (43) is given from Eq. (15) as

$$f_{JM}(l_1m_1; l_2m_2; E_1'; \Theta_1', \Phi_1'; \theta_2, \phi_2) = \exp i[2(\eta_{l_1}+\sigma_{l_1})+\beta_{l_1}]i^{l_2}\exp i[\eta_{l_2}+\sigma_{l_2}+\beta_{l_2}] \times A_{l_0}{}^J(E_0)\,\Gamma_{\alpha_1 l_1{}^{\frac{1}{2}}}Y_{l_1,0}(\Theta_1', \Phi_1')\rho_{l_2{}^{\frac{1}{2}}}(\epsilon_{23})\,Y_{l_2m_2}(\theta_2, \phi_2), \tag{44}$$

where the phase β_{l_1} is defined by Eq. (71) in Appendix II, and

$$A_{l_0}{}^J(E_0)=[\tfrac{1}{8}\pi(2J+1)]^{\frac{1}{2}}(K_0')^{-1} \times\frac{\Gamma_{p,l_0{}^{\frac{1}{2}}}}{[(E_p{}^J-E_0')^2+\frac{1}{4}(\Gamma_{\lambda,l_0}{}^J)^2]^{\frac{1}{2}}}, \tag{45}$$

where only p-state protons ($l_0=1$) contribute, and $J=0$, 2. Note that $K_0'=P_0'/\hbar$ is the proton wave number in SCM, and E_0' the corresponding energy.

IV. SYMMETRIZATION OF THE WAVE FUNCTION

We must symmetrize the wave function with respect to exchange of the three identical bosons 1, 2, 3. The resulting wave function is

$$\Psi(JM)=\Psi_{JM}(1, 23)+\Psi_{JM}(1, 32)+\Psi_{JM}(2, 31) +\Psi_{JM}(2, 13)+\Psi_{JM}(3, 21)+\Psi_{JM}(3, 12). \tag{46}$$

This leads to the symmetrized reaction amplitude

$$\begin{aligned}F_{JM}(E_1'; \Theta_1', \Phi_1'; \theta_2, \phi_2) =& f_{JM}(E_1'; \Theta_1', \Phi_1'; \theta_2, \phi_2)\\ &+f_{JM}(E_1'; \Theta_1', \Phi_1'; \pi-\theta_2, \pi+\phi_2)\\ &+f_{JM}(E_2'; A_2', B_2'; \alpha_3, \beta_3)\\ &+f_{JM}(E_2'; A_2', B_2'; \pi-\alpha_3, \pi+\beta_3)\\ &+f_{JM}(E_3'; M_3', N_3'; \mu_1, \nu_1)\\ &+f_{JM}(E_3'; M_3', N_3'; \pi-\mu_1, \pi+\nu_1),\end{aligned} \tag{47}$$

where (A_2', B_2') and (α_3, β_3) are the SCM and RCM spherical polar angles for particles 2 and 3 entering counters 1 and 2, respectively, and (M_3', N_3') and (μ_1, ν_1) correspond to particles 3 and 1 entering counters 1 and 2, respectively.

Using the relation for integral m_2,

$$Y_{l_2,m_2}(\pi-\theta_2, \pi+\phi_2)=(-1)^{l_2}Y_{l_2,m_2}(\theta_2, \phi_2),$$

we see Eq. (47) simplifies to

$$\begin{aligned}F_{JM}(E_1'; \Theta_1', \Phi_1'; \theta_2, \phi_2)=&[1+(-1)^{l_2}]\\ &\times[f_{JM}(E_1', \Theta_1', \Phi_1'; \theta_2, \phi_2)+f_{JM}(E_1'; A_2', B_2'; \alpha_3, \beta_3)\\ &+f_{JM}(E_1'; M_3', N_3'; \mu_1, \nu_1)],\end{aligned} \tag{48}$$

so that Be^8 must have even spin.

The exchange of particle positions is illustrated in Fig. 2, which we use to relate the exchange angles to (Θ_1', Φ_1') and (θ_2, ϕ_2), and hence, via Eqs. (29) and (30), to the lab angles (Θ_1, Φ_1) and (Θ_2, Φ_2). In terms of α_1–Be^8 and α_2–α_3 relative distances $\mathbf{R}$ and $\mathbf{r}$, respec-

tively, vector analysis gives the relations

$$\tan A_2' \sin B_2' = \frac{3r \sin\theta_2 \sin\phi_2 - 2R \sin\Theta_1' \sin\Phi_1'}{3r \cos\theta_2 - 2R \cos\Theta_1'},$$

$$\tan B_2' = \frac{3r \sin\theta_2 \sin\phi_2 - 2R \sin\Theta_1' \sin\Phi_1'}{3r \sin\theta_2 \cos\phi_2 - 2R \sin\Theta_1' \cos\Phi_1'},$$

$$\tan\alpha_3 \sin\beta_3 = \frac{r \sin\theta_2 \sin\phi_2 + 2R \sin\Theta_1' \sin\Phi_1'}{r \cos\theta_2 + 2R \cos\Theta_1'},$$

$$\tan\beta_3 = \frac{r \sin\theta_2 \sin\phi_2 + 2R \sin\Theta_1' \sin\Phi_1'}{r \sin\theta_2 \cos\phi_2 + 2R \sin\Theta_1' \cos\Phi_1'}. \quad (49)$$

$$\tan M_3' \sin N_3' = \frac{3r \sin\theta_2 \sin\phi_2 + 2R \sin\Theta_1' \sin\Phi_1'}{3r \cos\theta_2 + 2R \cos\Theta_1'},$$

$$\tan N_3' = \frac{3r \sin\theta_2 \sin\phi_2 + 2R \sin\Theta_1' \sin\Phi_1'}{3r \sin\theta_2 \cos\phi_2 + 2R \sin\Theta_1' \cos\Phi_1'},$$

$$\tan\mu_1 \sin\nu_1 = \frac{-r \sin\theta_2 \sin\phi_2 + 2R \sin\Theta_1' \sin\Phi_1'}{-r \cos\theta_2 + 2R \cos\Theta_1'},$$

$$\tan\nu_1 = \frac{-r \sin\theta_2 \sin\phi_2 + 2R \sin\Theta_1' \sin\Phi_1'}{-r \sin\theta_2 \cos\phi_2 + 2R \sin\Theta_1' \cos\Phi_1'}.$$

Now (50)

$$\frac{R}{r} = \frac{V_{1-23}'}{v_{23}} = \frac{K_1'}{\mu_{1-23}} \Big/ \frac{k_{23}}{\mu_{23}} = \frac{\mu_{23}}{\mu_{1-23}} \frac{K_1'}{k_{23}}, \quad (51)$$

so that $R/r = 3K_1'/4k_{23}$ in our case. Substituting for R/r in Eqs. (49) and (50) and also using Eqs. (30) and (29) for $\sin\theta_2 \sin\phi_2$, $\sin\theta_2 \cos\phi_2$, $\cos\theta_2$ and $\sin\Theta_1' \times \sin\Phi_1'$, $\sin\Theta_1' \cos\Phi_1'$, $\cos\Theta_1'$, we obtain the required expressions for the exchange angles in lab coordinates:

$$\tan A_2' \sin B_2' = \frac{3P_2 \sin\Theta_2 \sin\Phi_2}{3P_2 \cos\Theta_2 - P_0},$$

$$B_2' = \Phi_2,$$

$$\tan\alpha_3 \sin\beta_3 = \frac{2P_1 \sin\Theta_1 \cos\Phi_1 + P_2 \sin\Theta_2 \cos\Phi_2}{2P_1 \cos\Theta_1 + P_2 \cos\Theta_2 - P_0},$$

$$\tan\beta_3 = \frac{2P_1 \sin\Theta_1 \sin\Phi_1 + P_2 \sin\Theta_2 \sin\Phi_2}{2P_1 \sin\Theta_1 \cos\Phi_1 + P_2 \sin\Theta_2 \cos\Phi_2}. \quad (52)$$

$$\tan M_3' \sin N_3' = \frac{P_1 \sin\Theta_1 \sin\Phi_1 + P_2 \sin\Theta_2 \sin\Phi_2}{P_1 \cos\Theta_1 + P_2 \cos\Theta_2 - \frac{2}{3}P_0},$$

$$\tan N_3' = \frac{P_1 \sin\Theta_1 \sin\Phi_1 + P_2 \sin\Theta_2 \sin\Phi_2}{P_1 \sin\Theta_1 \cos\Phi_1 + P_2 \sin\Theta_2 \cos\Phi_2},$$

$$\tan\mu_1 \sin\nu_1 = \frac{P_1 \sin\Theta_1 \sin\Phi_1 - P_2 \sin\Theta_2 \sin\Phi_2}{P_1 \cos\Theta_1 - P_2 \cos\Theta_2},$$

$$\tan\nu_1 = \frac{P_1 \sin\Theta_1 \sin\Phi_1 - P_2 \sin\Theta_2 \sin\Phi_2}{P_1 \sin\Theta_1 \cos\Phi_1 - P_2 \sin\Theta_2 \cos\Phi_2}. \quad (53)$$

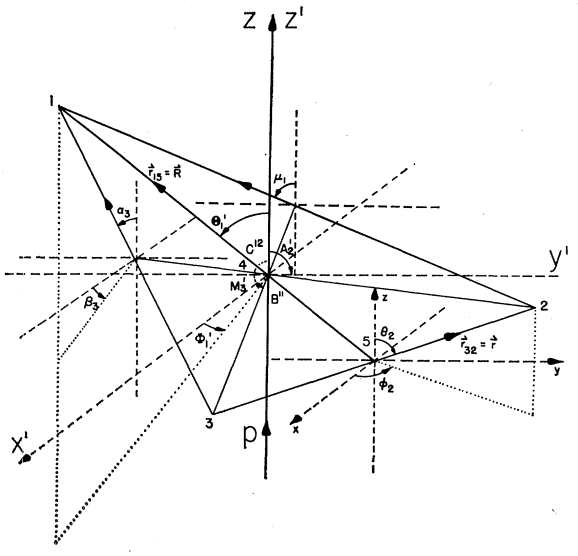

FIG. 2. SCM and RCM coordinate systems as in Fig. 1, but allowing for exchange between identical particles 1, 2, and 3. Taking the first particle in brackets as the first emitted particle, and the second as the second emitted particle, the corresponding SCM and RCM spherical polar coordinate angles are:

(1, 23):	(Θ_1', Φ_1')	and	(Θ_2, ϕ_2).
(2, 31):	(A_2', B_2')	and	(α_3, β_3).
(3, 12):	(M_3', N_3')	and	(μ_1, ν_1).

Not all angles angles are shown in the diagram.

Given lab angles (Θ_1, Φ_1), (Θ_2, Φ_2) and momenta P_1, P_2 for final particles 1 and 2, Eqs. (32)–(34), (52), and (53) enable the exchange amplitude (48) to be evaluated.

V. FRACTIONAL PARENTAGE COEFFICIENTS AND POLARIZATION OF C^{12}

If C^{12} is produced in several excited states of spin J, parity π, and we define fractional, two-body parentage coefficients $a_J{}^\pi$ for each state, then the total wave function becomes

$$\Psi = \sum_{J,\pi} a_J{}^\pi \Psi(J^\pi, M). \quad (54)$$

While the states are not strictly orthogonal, they may be considered so if the clusters are well separated. Thus the asymptotic beam of C^{12} produced via $p+B^{11}$ reaction has orthogonal components, in the sense that the sum of probabilities of producing the various J, π states must be unity:

$$\sum_{J,\pi} |a_J|^2 = 1. \quad (55)$$

The $a_J{}^\pi$ fractional parentage coefficients must be found empirically by fitting theory to experiment, remembering that for a proton energy near a well-separated C^{12} resonance level $J_1{}^\pi$, the C^{12} state produced should be relatively pure, with other $a_J{}^\pi$ coefficients quite small in comparison, $a_{J_1}{}^\pi \approx 1$. Hereafter we omit the parity sign π from a_J. An approximate relation for $|a_J|^2$ has

been noted by Phillips and Tombrello,[1] in terms of the reduced width and two-body interaction radius a_λ of the level, viz.

$$|a_J|^2 \approx \gamma_\lambda^2 [3\hbar^2/2\mu_\lambda a_\lambda^2].$$

For unpolarized C^{12} and a particular J value, all allowed magnetic substates have equal weighting. Assuming partial polarization of the C^{12}, we take $b_{J,|M|}$ as the fractional weighting (population parameter) of the magnetic substate M, subject to

$$\sum_{M=-J}^{J} |b_{J,M}|^2 = 1, \tag{56}$$

where from time reversal invariance,

$$b_{J,-M} = b_{J,M} = b_{J,|M|}. \tag{57}$$

The B^{11} target has spin $\frac{3}{2}^-$ and the lower C^{12} levels spin 0^+, 2^+, so that parity conservation requires the incident proton to have odd parity; thus only $l=1$ protons contribute appreciably at proton energies of a few MeV. The proton beam is along the Z direction, with respect to which magnetic quantum numbers are measured throughout the experiment.

The procedure followed is: (a) We combine proton spin $\frac{1}{2}$ and B^{11} spin $\frac{3}{2}$ to get intrinsic channel spin 1 and 2, respectively; (b) The intrinsic channel spin is combined with the proton orbital angular momentum and orbital magnetic quantum number 0 (the latter as the proton has its orbital angular momentum vector $\mathbf{L}=\mathbf{r}\times\mathbf{p}$ perpendicular to the Z direction). The lowest resultant state is the singlet 0^+ state of C^{12} and $J=M=0$, so that $b_{0,0}=1$ for a pure state. The $J=2^+$ level of C^{12} may be constructed from a mixture of $J=1$ and $J=2$ intrinsic channel spins, leading to polarization of C^{12} in this state.

(a) Denote the intrinsic channel spin function for C^{12} in state J, M by $\chi_{J,M}$, with $J=1(M=\pm1, 0)$ and $J=2$ $(M=\pm2, \pm1, 0)$, respectively.

(b) Employing orbital angular momentum function $Y_{1,0}$, the total spin function for C^{12} becomes, in terms of mixture parameters ϵ_1 and ϵ_2,

$$\begin{aligned}\Psi_{2,M} &= [\epsilon_1 \sum_{m_1=-1}^{1} (1, 1, m_1, 0 \mid 2, M)\chi_{1,M} \\ &\quad + \epsilon_2 \sum_{m_1=-2}^{2} (2, 1, m_1, 0 \mid 2, M)\chi_{2,M}]Y_{1,0} \\ &= \epsilon_1(1, 1, M, 0 \mid 2, M)\chi_{1,M}Y_{1,0} \\ &\quad + \epsilon_2(2, 1, M, 0 \mid 2, M)\chi_{2,M}Y_{1,0}.\end{aligned} \tag{58}$$

In particular, we obtain

$$\begin{aligned}&\Psi_{2,2} = (\tfrac{2}{3})^{\frac{1}{2}}\epsilon_2\chi_{1,2}Y_{1,0}, \qquad \Psi_{2,-2} = -(\tfrac{2}{3})^{\frac{1}{2}}\epsilon_2\chi_{1,-2}Y_{1,0}, \\ &\Psi_{2,1} = 2^{-\frac{1}{2}}\epsilon_1\chi_{1,1}Y_{1,0} + 6^{-\frac{1}{2}}\epsilon_2\chi_{2,1}Y_{1,0}, \\ &\Psi_{2,-1} = 2^{-\frac{1}{2}}\epsilon_1\chi_{1,-1}Y_{1,0} - 6^{-\frac{1}{2}}\epsilon_2\chi_{2,-1}Y_{1,0}, \\ &\Psi_{2,0} = (\tfrac{2}{3})^{\frac{1}{2}}\epsilon_1\chi_{1,0}Y_{1,0}.\end{aligned} \tag{59}$$

We write

$$\Psi_2 = \sum_{M=-2}^{2} b_{2,|M|}\Psi_{2,M}, \tag{60}$$

where $b_{2,|M|}$ satisfies Eq. (56) for $J=2$.

Equations (59) and (56) then give

$$\begin{aligned}|b_{2,2}|^2 &= 4/(5+7\epsilon^2), \\ |b_{2,1}|^2 &= (1+3\epsilon^2)/(5+7\epsilon^2), \\ |b_{2,0}|^2 &= 4\epsilon^2/(5+7\epsilon^2),\end{aligned} \tag{61}$$

where $\epsilon=\epsilon_1/\epsilon_2$ is a parameter solely determining the population parameters $b_{2,|M|}$. It must be found empirically to fit the experimental data.

Finally we have from Eq. (54) that the total reaction amplitude is

$$\begin{aligned}&F(E_1'; \Theta_1', \Phi_1'; \theta_2, \phi_2) \\ &= \sum_J a_J \sum_{M=-J}^{J} b_{J,|M|}F_{J,M}(E_1'; \Theta_1', \Phi_1'; \theta_2, \phi_2),\end{aligned} \tag{62}$$

where the amplitude on the right is defined by Eqs. (48), (43), and (44).

The triple correlation cross section for particle 1 in SCM and 2 in RCM is

$$d^3\sigma/dE_1'd\Omega_1'd\omega_2 = |F(E_1'; \Theta_1', \Phi_1'; \theta_2, \phi_2)|^2, \tag{63}$$

with the lab distribution defined by Eqs. (37), (41), and (63), together with the relations (32), (33), (52), and (53) between coordinate systems.

Actual amplitude components (43) needed in the calculation for $B^{11}(p, \alpha)2\alpha$ are given in Appendix I below.

APPENDIX I: DETAILS OF AMPLITUDE COMPONENTS FOR $B^{11}(p, \alpha)2\alpha$

We assume only $J=0^+$, 2^+ levels of C^{12} are involved, which may decay into the 0^+ and into the 0^+, 2^+ levels of Be^8, respectively. We hereafter omit the angles in the amplitudes of Eq. (43), for the sake of brevity.

(1) $J=0^+$, so $M=0$ and $b_{0,0}=1$. From Eq. (43),

$$\begin{aligned}f_{0,0} &= (0, 0, 0, 0 \mid 0, 0)f(0, 0; 0, 0) \\ &\quad + (2, 2, 0, 0 \mid 0, 0)f(2, 0; 2, 0)\end{aligned}$$

or

$$f_{0,0} = f(0, 0; 0, 0) + 5^{-\frac{1}{2}}f(2, 0; 2, 0). \tag{64}$$

(2) $J=2^+$, so $M=0, \pm1, \pm2$.

(a) $M=0$, so

$$f_{2,0} = \sum_{j_2=0,2} \sum_{j_1=|2-j_2|}^{2+j_2} (j_1, j_2; 0, 0 \mid 2, 0)f(j_1, 0; j_2, 0)$$

or

$$f_{2,0}=f(2,0;0,0)+f(0,0;2,0)-(\tfrac{2}{7})^{\frac{1}{2}}f(2,0;2,0)+(\tfrac{2}{7})^{\frac{1}{2}}f(4,0;2,0). \quad (65)$$

(b) $M=-1$, so

$$f_{2,-1}=\sum_{j_2=0,2}\sum_{j_1=|2-j_2|}^{2+j_2}(j_1,j_2;0,-1\,|\,2,-1)f(j_1,0;j_2,-1);$$

$$f_{2,-1}=f(0,0;2,-1)+6^{-\frac{1}{2}}f(1,0;2,-1)-(14)^{-\frac{1}{2}}f(2,0;2,-1)-(\tfrac{2}{7})^{\frac{1}{2}}f(3,0;2,-1)-(\tfrac{8}{63})^{\frac{1}{2}}f(4,0;2,-1). \quad (66)$$

(c) $M=1$:

$$f_{2,1}=\sum_{j_2=0,2}\sum_{j_1=|2-j_2|}^{2+j_2}(j_1,j_2;0,1\,|\,2,1)f(j_1,0;j_2,1);$$

$$f_{2,1}=f(0,0;2,1)-6^{-\frac{1}{2}}f(1,0;2,1)-(14)^{-\frac{1}{2}}f(2,0;2,1)+(\tfrac{2}{7})^{\frac{1}{2}}f(3,0;2,1)-(\tfrac{8}{63})^{\frac{1}{2}}f(4,0;2,1). \quad (67)$$

(d) $M=-2$:

$$f_{2,-2}=\sum_{j_1=|2-j_2|}^{i+j_2}(j_1,j_2;0,-2\,|\,2,-2)f(j_1,0;j_2,-2);$$

$$f_{2,-2}=f(0,0;2,-2)+(\tfrac{2}{3})^{\frac{1}{2}}f(1,0;2,-2)+(\tfrac{2}{7})^{\frac{1}{2}}f(2,0;2,-2)+(14)^{-\frac{1}{2}}f(3,0;2,-2)+(126)^{-\frac{1}{2}}f(4,0;2,-2). \quad (68)$$

(e) $M=2$:

$$f_{2,2}=\sum_{j_2=0,2}\sum_{j_1=|2-j_2|}^{2+j_2}(j_1,j_2;0,2\,|\,2,2)f(j_1,0;j_2,2)$$

$$f_{2,2}=f(0,0;2,2)-(\tfrac{2}{3})^{\frac{1}{2}}f(1,0;2,2)+(\tfrac{2}{7})^{\frac{1}{2}}f(2,0;2,2)-(14)^{-\frac{1}{2}}f(3,0;2,2)+(126)^{-\frac{1}{2}}f(4,0;2,2). \quad (69)$$

APPENDIX II: NOTE ON COMPLEX AMPLITUDE PHASE FACTORS

The amplitude for emission of a particle b in the reaction (3) for stable final particles $B+b$ is given by Eqs. (14) and (5), where the amplitude factor $(E_{\lambda a}-E_a-\frac{1}{2}i\Gamma_\lambda)^{-1}$ in (14) may be written in the form

$$(E_{\lambda a}-E_a-\tfrac{1}{2}i\Gamma_\lambda)^{-1}=[(E_{\lambda a}-E_a)^2+\tfrac{1}{4}\Gamma_\lambda{}^2]^{-\frac{1}{2}}\exp i\beta_{l_b}. \quad (70)$$

Here

$$\tan\beta_{l_b}=\tfrac{1}{2}\Gamma_\lambda/(E_{\lambda a}-E_a) \quad (71)$$

and $\beta_{l_b}=\eta_{l_b}+\phi_{l_b}$ is the resonant phase for the inelastic process (3), equal to the sum of nuclear reaction phase η_{l_b} and hard sphere scattering phase ϕ_{l_b}. Coulomb effects enter via $\Gamma_\lambda=2P_l\gamma_\lambda{}^2$, $P_l=$ penetrability$=ka/A_l{}^2$, $A_l{}^2=F_l{}^2+G_l{}^2\,|^{r=a}$ (F_l and G_l are regular and irregular coulomb wave functions, respectively), $\gamma_\lambda{}^2=$ reduced width of D^* levels and hard-sphere phase $\phi_l=[\arctan F_l/G_l]^{r=a}$. The phase factor

$$\exp[2i(\eta_{l_b}+\sigma_{l_b})],$$

where $\sigma_l=$ Coulomb phase (75), derives from the theory of rearrangement collisions.[35]

If B is a resonant or virtual state particle, it may then be treated as a semistable bound-state particle, wave function

$$\Phi(\mathbf{r})=(1/r)\phi(r)\,Y_{l,m}(\theta_c,\phi_c) \quad (r\leq a)$$
$$=0, \quad (r>a) \quad (72)$$

where "a" is the interaction radius of D^* for decay into $B+b$. The bound-state normalization condition

$$\int_0^a|\,\Phi(\mathbf{r})\,|^2\,d\mathbf{r}=1 \quad (73)$$

is then satisfied, and $\phi(r)$ is a real function of r, befitting a bound state.

When decay of B into $C+c$ occurs, the wave function of B must be renormalized[14] inside a large spherical box of radius R. The wave function of B becomes[4]

$$\Psi^{(+)}(\mathbf{r})=N(E_B)i^{l_c}\exp i(\eta_{l_c}+\sigma_{l_c})\Phi(\mathbf{r}), \quad (74)$$

being the final-state scattering wave function of $B=C+c$, renormalization factor $N(E_B)$. The complex amplitude factor $i^{l_c}\exp i(\eta_{l_c}+\sigma_{l_c})$ comes from the Faxén–Holtsmark partial-wave scattering expansion, with η_{l_c} the nuclear elastic scattering phase in the final-state interaction of $C+c$, and σ_{l_c} the Coulomb phase, given by

$$\sigma_l=\arg\Gamma(i\alpha+l+1), \qquad \alpha=Z_CZ_ce^2/\hbar v. \quad (75)$$

The amplitude for the decay process $B\rightarrow C+c$ becomes

$$\langle\Psi^{(-)}(\mathbf{r})\,|\,\Phi(\mathbf{r})\rangle=N(E_B)i^{l_c}\exp i(\eta_{l_c}+\sigma_{l_c})\times\langle\Phi(r)\,|\,\Phi(r)\rangle$$
$$=N(E_B)i^{l_c}\exp i(\eta_{l_c}+\sigma_{l_c}) \quad (76)$$

on using Eq. (74).

This gives the usual generalized density-of-states function $\rho(E_B)$ as in Eq. (10), with $\rho(E_B)=N^2(E_B)$, and leads to Eq. (15) for the triple-correlation amplitude.

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[35] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, Oxford, England, 1948), second ed., p. 161.

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Discussion

MEYERHOF: From a theoretical point of view, is it worth while to deal with three distinguishable particles rather than three alpha particles? Here you have to take into account the antisymmetrization; and so you never know which particle comes first. Whereas, if you have three distinguishable particles, you can see on the basis of a model whether a particular particle comes first.

SWAN: The ideal model will involve three different particles. On the other hand, known interference effects between identical particles can be very useful. If you have nonidentical particles, you may have different sorts of diagrams, and these have to be arbitrarily added together in the reaction amplitude to give interference effects. Although you can now distinguish the particles, you may have different processes giving different orders of emission.

DONOVAN: Identity or nonidentity of the particles, of course, has nothing to do with the difficulty of telling which one comes out first. That involves a conceptual time delay measurement, whether the particles are identical or not. In our $(\alpha, 2\alpha)$ paper a few years ago we showed, and more recently Phillips has shown, that by measuring the angle of one α particle as a function of the energy of the other, you can ascertain which particle came out first.

SWAN: Symmetrization really allows for this in the $(C^{12}, 3\alpha)$ reaction.

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Angular Momenta and Pauli Principle in Nuclear Reactions $a+A\rightarrow C^*\rightarrow b_1+b_2+b_3$

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In the breakup of a compound nucleus C* into k clusters $b_1\cdots b_k$, the asymptotic wave function ψ for short-range forces is a product of k functions ϕ_j which depend on internal vectors, spins, and isospins and of a function χ which depends on $k-1$ relative vectors. χ contains the correlations in energy and angle of the clusters emitted. If the reaction proceeds in several steps, χ will separate into functions χ_1, χ_2, $\cdots$ of the corresponding relative vectors.

Due to the Pauli principle, ψ should be replaced by a sum over all nucleon permutations,

$$\psi^A=\sum_P(-1)^P(P\psi).$$

Permutations P' within one cluster b_j will only affect ϕ_j. Permutations P'' of nucleons between clusters can usually be neglected since, due to the nonoverlap of the cluster wave functions, in a term $\psi(P''\psi)^*$ one factor will vanish. If, however, two clusters b_j and b_l are in the same internal state ϕ, the exchange of all particles will result in the replacement of the relative vector ξ_{jl} by $-\xi_{jl}$ in χ and therefore will require symmetry properties of χ. In general, if k-like clusters are emitted at one step of the reaction, it will be necessary to classify the functions according to the group S_k of cluster permutations.

The function χ results from nucleon–nucleon forces in a complicated way. For three clusters, it depends on the relative vector ξ^1 of clusters b_1, b_2 and on the relative vector ξ^2 of cluster b_3. χ may be compared with experiments by means of a Dalitz plot.[1] The Hamiltonian for large distances between the clusters may be written $H=(2\mu)^{-1}[(\pi^1)^2+(\pi^2)^2]$. For many-body breakup, Smith[2] has shown that it is reasonable to classify χ in terms of generalized angular momentum. This arises from the generators of the symmetry group R_6 of H. A three-body breakup is then characterized by low eigenvalues of the Casimir operator Λ^2 of R_6, $\Lambda^2=\hbar^2\lambda(\lambda+4)$, $\lambda=0, 1, 2, \cdots$. The function χ may be further characterized by the relative angular momenta l^1, l^2 corresponding to ξ^1, ξ^2. If b_1 and b_2 are in the same

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[1] D. Dehnhard, D. Kamke, and P. Kramer, Phys. Letters **3**, 52 (1962).

[2] F. T. Smith, Phys. Rev. **120**, 1058 (1960).