Operational Foundation of Einstein's General Theory of Relativity*

SAUL A. BASRI
Department of Physics, Colorado State University, Fort Collins, Colorado

The measurement of time, space, and space–time intervals, and the related geometries in any gravitational field are analyzed carefully. It is pointed out that synchronization is arbitrary and very complicated operationally; and a method for doing away with it is given. Then it is shown how space geodesics can be used to construct several useful coordinate systems in an arbitrary field, and the associated coordinate conditions are derived. These coordinate systems are then used to find a *unique* physically meaningful solution to the field equations in the cases of zero, linear, rotation, and spherically symmetric fields.

1. INTRODUCTION

In spite of the fact that Einstein's general theory of relativity (GR) is about half a century old, its operational meaning is still not completely clear. This is due mainly to the fact that a rigid coordinate system (CS) is not possible in a time-varying gravitational field, and that standard clocks may have different relative rates at different places and times as revealed by a succession of light pulses from one clock to another. Moreover, synchronization of clocks may not even be transitive.

In the last five years or so, there has been a revival of interest in GR, as evidenced in many recent articles, conferences, and books [(1)–(4), (6), (7)]. The purpose of this article is to review, clarify, and present some new material concerning the *operational foundation* of GR. The recent areas of research pertaining to the definition of energy and momentum, gravitational radiation, and quantization of GR, are adequately reviewed in the books referred to above, and are therefore not discussed here.

Time and space measurements are thoroughly discussed in Secs. 2 and 3; and precise definitions are given of a standard clock in Sec. 2B, and length-measuring instrument in Secs. 3A, C. To my knowledge, neither these definitions nor the definition of rigidity given in Sec. 3B, are given elsewhere in the literature.

There are in the literature several formulations of the principle of equivalence, and different measures of the line element. These are reviewed and discussed in Secs. 4 and 5B. This material and that developed in Secs. 2 and 3 are used to discuss the space–time, time, and space geometries in Secs. 5, 6, and 7, respectively.

It is shown in Secs. 2D, and 8 that synchronization of clocks in a gravitational field, is arbitrary, has to be performed continually, and may not even be transitive. Because of this, it seems desirable to eliminate synchronization from GR. An attempt in this direction is made in Sec. 8, and as far as I know this material is new.

Practically all books dealing with GR [(1)–(6), (8)–(12)] emphasize the great freedom one has in the choice of a CS. However, unless a CS is uniquely specified at all times by coordinate conditions (CC) that are operationally meaningful, it is not possible to obtain a solution of the field equations that has a clear physical meaning. The construction of such a CS by means of space geodesics is discussed and carried out in Sec. 9, for rectangular, cylindrical, and spherical coordinates, and all the associated CC are derived. With the help of these CC, a *unique* solution to the field equations is obtained for the cases of the zero, linear, rotation, and spherically symmetric fields in Secs. 10–13, respectively. In this way, a fairly complete illustration is given of the ideas discussed in Secs. 5–9.

* Supported by the National Science Foundation.

Conventions

In referring to pages p to q of Ref. R, we write [(R)p−q]; and in referring to equations, we use "(E.M, N, ···)" instead of "(E.M), (E.N), ···." The radical sign "√" applies only to the first symbol succeeding it.

Unless stated otherwise, particles are denoted by capital letters and events by lower case letters. Moreover, lower case letters are used to denote curvilinear coordinates, whereas capital letters are used to denote coordinates in an inertial CS. Latin and Greek indices range over the values 0, 1, 2, 3, and 1, 2, 3, respectively.

As usual, we use the definitions $x^0=ct$, $X^0=cT$, where c is the speed of light. The summation convention for repeated sub- and super-scripts is adopted, as well as the notation for derivatives: $_{,i}=\partial/\partial x^i$ and $_{,ij}=\partial^2/\partial x^i\partial x^j$, etc.

2. TIME MEASUREMENT

A *clock* may be defined as a physical system that *generates* and *counts* a sequence of events at a particle, called the *output particle*, such as the tip of the hand of a watch or the end of the output lead from an atomic clock. The question now is how to define a *standard clock* (SC) that records the *proper time* used in the theory of relativity? The fundamental importance of an SC is seen in Sec. 5B to lie in the fact that it provides the basis for the definition of *congruence* of space–time intervals.

There seem to be two different types of definition of an SC: (1) A *particular* physical system is adopted as an SC on the basis of a physical theory or hypothesis about its behavior. (2) The SC is chosen as a member of a finite set of clocks as a result of the intercomparison of all the elements of the set. The outstanding example of an SC of the first type (SC_1) is the astronomical standard [(2a) 188–195]. Other examples are a *particular* pendulum clock and a *particular* atomic clock. An SC of the second type (SC_2) is exemplified by a set of at least three atomic clocks or quartz crystal clocks whose beat frequencies are continually analyzed. We now discuss in turn the two types of time standards.

A. Solitary-Type Time Standard

Usually, a primary SC_1 is selected on the basis of a physical theory. However, one cannot use this theory to justify the selection, since the theory itself cannot be verified without the help of the SC. One would have simply to point to a *particular* physical system, assert that it is *the* primary SC, and specify exactly how it is to be used. Any "corrections" included in the instructions are *part* of the definition. Different choices of the SC lead to different forms of the laws of physics, and it is a matter of convenience and taste as to which form is preferable [(13) 66–77].

In addition to the fact that an SC_1 is arbitrary, its basic properties cannot be examined in principle except by comparison with another clock, which is presumably inferior; otherwise it should be selected itself as the primary standard. The situation with regard to the astronomical standard is transitory, since it is compared with atomic clocks which are recognized to be superior to it; and this is the reason why there is a plan to adopt an atomic SC_2.

Moreover, it is usually not possible to use a *single* primary standard to calibrate secondary standards at different locations without some uncertainty and sacrifice in accuracy. It would not do to set up several primary standards and declare that they are all equivalent on the basis of a physical theory; because as mentioned above, this would lead to a circular definition.

An interesting SC_1 is the *geodesic clock* (GC) proposed recently by Marzke [(1a) 50–53]. This clock is constructed from two particles P, Q, moving along *parallel* world lines in a locally *flat* space–time region, and a pulse of light reflected back and forth between them. The parallelism of the world lines is defined in terms of intersecting world lines or other particles and photons. Although this definition was "derived" from SR, it has to be considered simply as part of the definition of the GC, if circularity is to be avoided. Similarly, the instruments measuring the deviation from *flatness* of space–time would have to be considered as part of the GC.

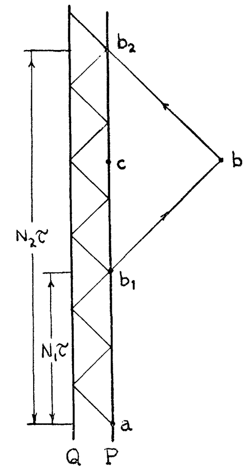

FIG. 1. Geodesic clock.

The space–time interval between any two events a, b is measured by a GC as follows (Fig. 1): Let a occur at one of the particles P of the GC, and let a photon start from P at event b_1, be reflected at b, and return back to P at b_2. Let τ denote the round-trip time of the photon bouncing between the two particles of the GC; and suppose that the number of round trips between a and b_1 is N_1, and between a and b_2 is N_2. Then the space–time interval, (ab), between a and b is defined to be $(N_1N_2)^{\frac{1}{2}}\tau$. One feature of this definition, is that if a, b can be connected by a photon, i.e., if a coincides with either b_1 or b_2, then N_1 or N_2 is zero and $(ab)=0$ *by definition.* Another feature, is that if c is an event on P halfway between b_1 and b_2, and the speed of light is defined by $u=(bc)/\frac{1}{2}(b_1b_2)$, then if

$$(b_1c)=N_1'\tau \text{ and } (cb_2)=N_2'\tau, \text{ we have } N_1'=N_2'=N,$$

$$(bc)=(N_1'N_2')^{\frac{1}{2}}\tau=N\tau, \qquad \tfrac{1}{2}(b_1b_2)=\tfrac{1}{2}(N_1'+N_2')\tau=N\tau,$$

and thus $u=1$, i.e., the local *speed of light is constant by definition.*

If the length of (ab) is such that the deviation of the world lines of P, Q from parallelism is appreciable, then Q would have to be replaced repeatedly by other particles to preserve parallelism [(1a) 56–58].

In order to get a feeling for a GC, let a, b be two events at a particle A in a gravitational field. In order to measure (ab) one would have to literally throw the GC from A at a with such a velocity that it falls back freely to A at b. In view of this, I doubt that anyone would relish using a GC!

B. Comparison-Type Time Standard

A *necessary* condition that any two SC_2's must satisfy is that if the output particles of the two clocks coincide, then the ratio of the number of events generated by one clock to the number of events generated by the other must be a constant independent of the location of the output particles and the initial or final events [(11)63]. This condition was called by Synge [(6)106–107] the *hypothesis of consistency.* Any two clocks that satisfy this condition are said to be *eligible standard clocks* (ESC).

Even if two clocks are ESC, there is a possibility that their rate may be changing in the same way, as for example two quartz crystals aging at the same rate, or two pendulum clocks whose pendulums have the same thermal coefficient of expansion. Such *differential effects* can be discovered by comparing two ESC's constructed from different materials or having different constructions. Upon their discovery, one can set out to devise methods to compensate for them by controlling the environment of the clocks or incorporating servomechanisms into the clocks. In accomplishing this, one can use any detection instruments, physical theory, and experience without fear of circularity, because the ultimate criteria in deciding what clocks are SC's, are based purely on the *intercomparison of the clocks treated as black boxes.*

It is conceivable that the inhomogeneity of a gravitational field may be such that even two clocks A, B of the same construction with coincident output particles, may be in regions where the field is different, and may be differently affected. In order to decide in this case whether the discrepancy between A, B is due to an inhomogeneity of the field over the space region occupied by them or due to the fact that they are not ESC's, one may interchange the locations of A and B. If the discrepancy is exactly the same as before except for the interchange in the roles of A and B, one may assume that the discrepancy is caused by inhomogeneity of the field.

After obtaining ESC's of different materials and construction, there may still be an appreciable probability that due to statistical fluctuation, or unknown factors, the rates of any two clocks may vary together. This probability can be made practically negligible by *defining an SC as an element of at least r different clocks, any two of which are ESC,* where r is a small integer, perhaps 3. In this definition, statistical fluctuations in the comparison of any two clocks have been neglected. A careful statistical treatment of this problem is presented by the author elsewhere [14].

An important question which has been of interest in the last three years or so is the following: It is known that physical interactions may be classified as strong, electromagnetic, weak, and gravitational. Suppose now that we have two sets of SC's each constructed on the basis of a different type of interaction; will the clocks also behave as SC's with respect to intercomparison between clocks belonging to two different sets [(1a)46–48; (1b)139–141; (2a)34–35; (2b)180]? Dicke [(1b)14] and Finzi [15] have shown that as far as strong and electromagnetic interactions are concerned, the answer is yes; but no answer can be given as yet to the cases of weak and gravitational interactions.

C. Proper and Coordinate Time

The time given by SC_2's will now be assumed to be the *proper time* of general relativity. This assumption can be verified by using these SC's to test any of the consequences of the theory. Up to now, no evidence against this assumption has been discovered.

At any particle it is permissible to use a clock that runs at an arbitrary *specified* rate relative to a coincident SC. Such a clock is usually called a *coordinate clock,* and the time it records is called *coordinate time.*

D. Synchronization of Clocks

So far, we have been concerned only with time intervals between events occurring at one particle. In order to correlate the time of events at separate particles, it is necessary to synchronize clocks. Two clocks A, B are synchronized by sending a light signal from A reflecting it at B and then receiving it back at A. If the events of sending, reflecting, and receiving back the signal are denoted by a, b, c, respectively, then *any* event d between a and c may be considered to be simultaneous with b [(16) Secs. 19, 22], i.e., if t_e denotes the time of event e, then

$$t_b=t_d=t_a+\epsilon(t_c-t_a), \qquad 0<\epsilon<1. \tag{2.1}$$

Moreover, if b, d occur at another two clocks A', B' moving relative to A, B, and the same value of ϵ is used for synchronization, b, d will not be simultaneous relative to A', B', as is well known in SR. Einstein suggested for convenience the synchronization *convention* $\epsilon=\frac{1}{2}$. Since only the round-trip average velocity of light can be measured, it is impossible to verify whether this or any other value of ϵ should be adopted. However, in SR it is *assumed* that space is isotropic, which implies that the one-way velocity of light is the same in any direction, and thus $\epsilon=\frac{1}{2}$ may be motivated in this way. But in GR, this motivation does not exist.

In SR if two clocks are synchronized, they will remain synchronized. This is not true in a time varying gravitational field, and consequently synchronization has to be performed continually. Moreover, synchronization is transitive in SR, i.e., if two clocks B, C are synchronized with a clock A, then B, C will be synchronous with each other. This is not necessarily true in an arbitrary CS, as is demonstrated in Sec. 12 for the rotating disk. This poses an important question, namely that if B, C are synchronized with A, and a light signal is sent from B to C, could the time of arrival of the signal be *earlier* than its time of departure? If it could, then the procedure of synchronizing all clocks with one clock will be unacceptable. Reichenbach [(17) Sec. 7, Thm. 6] answered this question in the *negative.* Since his book is out of print and I have not seen this proof elsewhere, his proof is presented here with slight modifications.

Theorem. It is possible to synchronize all (coordinate) clocks with a single (coordinate) clock so that if a light signal leaves any clock at time t and arrives at any other clock at time t', then $t'>t$.

Proof. Let "$[A(t_1)\ B(t_2)\ C(t_3)\cdots]$" mean that a light signal leaves clock A at time t_1, arrives at clock B at time t_2, and then clock C at time t_3, $\cdots$. Suppose that $[A(t_1)\ B(t_2)\ C(t_3)\ A(t_4)]$, $[A(t_1)\ B(t_2)\ A(t_5)]$, and $[A(t_6)\ C(t_3)\ A(t_4)]$, as shown in Fig. 2. Since the last two signal chains are more direct than the first, we have $t_5-t_1\leq t_4-t_1$ and $t_4-t_6\leq t_4-t_1$, which imply

$$t_5\leq t_4,\qquad t_1\leq t_6. \tag{2.2}$$

Notice that only time intervals at the *single* clock A have been considered so far. If we now assume that B and C are synchronized with A, then

$$t_2=t_1+\epsilon(t_5-t_1),\qquad t_3=t_6+\epsilon(t_4-t_6),\qquad 0<\epsilon<1. \tag{2.3}$$

Since $t_5>t_1$, it follows that $t_2>t_1$, which proves the theorem for the signal from A to B. Moreover, from (2.3) we deduce $t_5-t_2=(1-\epsilon)(t_5-t_1)>0$, or $t_5>t_2$, which provides the proof for the signal from B to A. It remains to prove the theorem for the signal from B to C, i.e., to prove $t_3>t_2$. From (2.3, 2) we have

$$t_3-t_2=(1-\epsilon)(t_6-t_1)+\epsilon(t_4-t_5)\geq 0.\quad \text{Q.E.D.}$$

Even though this theorem is true, it can be seen that the experimental procedure of keeping all clocks synchronized with a central clock is very complicated. One of the main purposes of synchronization of clocks is to make it possible to describe the motion of a particle. A simpler and less arbitrary procedure of accomplishing this without clock synchronization, is given in Sec. 8.

3. SPACE MEASUREMENT

There are essentially two popular ways of measuring spacial distance: (1) by means of "rigid rods," (2) by the use of light signals and clocks [(6) 108–109, 112]. In the absence of a gravitational field the results obtained by the two methods agree, but in an arbitrary gravitational field, they may not.

By using light signals alone, one can learn that light signals are *first-signals*, i.e., if a race is conducted between a light signal and any other kind of signal, the light signal will never come out second [(16) 143]. Nothing else can be found about the velocity of light, neither about its constancy nor about its magnitude. To be sure, one can adopt the round-trip time of a light signal between two particles, as a measure of the distance between them [(16) 112] without worrying about the constancy of the velocity of light. In effect, this makes the velocity of light constant by definition [(1b) 11]. If this leads to a convenient description of physical phenomena, then well and good; if not, one would have to seek another method of measuring

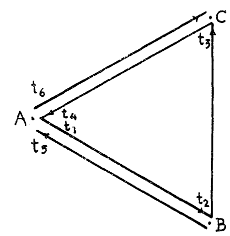

FIG. 2. Synchronization of clocks.

distance. By hindsight, we know that for most practical purposes the use of light signals to measure distance is acceptable, provided it can be carried out (light takes about 10^{-10} sec to travel few centimeters).

However, on the laboratory scale, it would be much more convenient to measure distance by means of solid objects, if a measure can be defined that will give a reasonable description of nature. Moreover, since man has been using this type of measure since time immemorial, it would be highly desirable to see if it can be defined in the presence of an arbitrary gravitational field, and how it is related to the measure obtained by means of light signals. It is shown in Sec. 7A that such a measure can be defined independently of the time measure, and can then be used to study the properties of light signals. In this way, more physical information may be obtained, which is a decisive advantage.

A. Length-Measuring Instruments (LMI)

In formulating the theory of relativity, Einstein made use of the concept of a rigid rod as an LMI. If one attempts to define a rigid rod, he will find the way full of vicious circles. Perhaps one of the outstanding attempts in this direction has been made by Reichenbach [(16) 19–28]. He first defines *solid bodies* [(16) 22] as "bodies having a certain physical state which can be defined ostensibly; it differs from the liquid and gaseous state in a number of observable ways." Then immediately after, he introduces the definition: *Rigid bodies* are solid bodies which are not affected by differential forces, or concerning which the influence of differential forces has been eliminated by corrections; universal forces are disregarded. It was shown by Grünbaum [(13) 81–97] that the reference to universal forces is unnecessary and may only serve to confuse the issue, so in the following only the first part of the definition is analyzed. As pointed out in the previous section, the method by which differential forces are detected, corrected or eliminated is of crucial importance. One method Reichenbach gives [(16) 23], is to identify differential forces with exterior forces, and assume that the rigidity condition is approached as the ratio of exterior to interior forces approaches zero. It is difficult to see how such a ratio can be estimated without the use of a length standard; and thus this method is circular. Another method he prescribes [(16) 24–28] is to construct devices which detect the presence of differential forces by the separation of particles that were originally coincident. However, the operation of the devices depend upon the inhomogeneity of the differential force over different parts of a device, which frequently may not be realized. To use special instruments such as thermometers to detect differential forces would lead to circularity, as these instruments already make use of length standards.

In order to overcome the last objection, Lenzen [(18)292] and Grünbaum [(13)145] suggested a method of successive approximations, which may well work in practice, but which is rather difficult to formulate theoretically. In the following, a simpler method will be described which is similar to that used in defining SC's, and which is free of circularity. As far as I know, this method has not been mentioned elsewhere in the literature.

Basically, an LMI is a physical system which associates with any pair of particles a number, called the *distance* between them which determines whether two pairs of particles can be made to coincide or not, i.e., whether there exists a time interval during which it is known that the two pairs are coincident.

More precisely, we may define *coincidence of pairs of particles* as follows: Let p be the event of coincidence of particles P_1, P_2 into the particle P, and q the event of coincidence of particles Q_1, Q_2 into particle Q. Let p', q' be the events of dissociation of P and Q back into P_1, P_2 and Q_1, Q_2. We say that P_1, Q_1 coincide with P_2, Q_2, uf there exists a clock C and two events a, b on C, with a before b, such that signals can be sent at p, q and arrive at C before the occurrence of a, and from C after b to arrive at P, Q before p', q'. This definition is independent of any CS.

In order to see how to define an LMI, it is helpful to analyze a concrete realization of an LMI, such as the vernier caliper. This instrument consists of a metal ruler with a jaw fixed at one end, and another jaw with a mark on it that slides along the ruler. Imagine that the fixed jaw is to the left of the ruler and the moving jaw is coincident with it. As the moving jaw moves to the right and the distance between the two jaws increases, the mark on it coincides successively with the left sides of the marks fixed on the ruler. On the other hand, as the distance between the two jaws decreases the moving mark coincides with the right sides of the fixed marks. Let coincidences on the left and right sides of the fixed marks be called events of type L and R, respectively. If we count L and R events from the time ($t=0$) when the two jaws were coincident, then clearly the number of L events minus the number of R events that have occurred at any time t gives the number of fixed marks and hence the distance between the two jaws at time t. A length interferometer operates on the same principle, with coincidences between the moving and fixed marks replaced by the appearance and disappearance of interference fringes.

Keeping these examples in mind, we may think of the raw material for an LMI as a physical system, called an L *instrument* (LI), which contains two particles, called its *end particles*, and a clock. On the world line of the clock, two types of events occur, such that at any time t of the clock there exists a definite number $n_1(t)$ of events of one type that have occurred before t, and a number $n_2(t)$ of events of another type that have also occurred before t. The number $s(t)=n_1(t)-n_2(t)$ is called the *scale value* at t.

The elements of a *set* of different LI's are called LMI's if, and only if, the following condition is fulfilled: If the end particles of any LI coincide (according to the above definition of "coincidence") with the end particles of any other LI, their scale values must be equal to within an acceptable margin of error at all times, all places, and all possible scale values [(11)63]. Notice that if the LI's were not rigid, then it would be possible to have coincidence of their end particles with unequal values of their scales, which is excluded by the above definition. The assumption that two LMI's with coincident end particles will have the same scale values anywhere they are compared, regardless of their space-time history, is known as Riemann's postulate [(1a)58–62].

This kind of length standard is of the comparison type, according to the classification given in Sec. 2; and all the comments that were made there about such a definition applies here also. The same is true of the comparison of LMI's operating on the basis of different interactions.

The scale of an LMI as defined so far is not necessarily linear; and it is necessary to define *collinearity* before a linear scale can be defined. This will be accomplished in Sec. 3C. Moreover in order to avoid having two LMI's with coincident end particles affected by inhomogeneities of the gravitational field, it will be assumed that the spatial extension of an LMI is "infinitesimal." This can be checked by the interchange of the two LMI's, as was done with SC's. In view of this, the maximum separation of the end particles of any LMI may be considered as "infinitesimal." Measurement of finite distances can only be achieved after collinearity has been defined, and will be accomplished by the "transport" or "lining up" of LMI's along a space geodesic.

B. Rigidity

The distance between two particles is said to be *constant*, or the particles are said to be *rigidly connected*, if and only if the scale value of an LMI has the *same* value any time its end points coincide with the two particles. Notice that rigidity is defined *after* an LMI has been defined. No mention is made in the definition of solid objects, forces, or geometry. Only the coincidence properties of pairs of particles and clocks are employed.

In experiments concerned with the measurement of the (round-trip) velocity of light in any gravitational field, it is found that for distances over which the inhomogeneities of the gravitational field are negligible, the velocity is constant. This means that if the distance between two particles is judged to be constant by means of an LMI, then the round-trip time of a light signal traveling between the two particles is constant. This result is deduced again in Sec. 7A from the comparison of time and length standards in noninertial and inertial reference frames. Notice that it would not have made much sense to perform these experiments, if the velocity of light was taken to be constant by definition, and distances were measured by the round-trip time of light signals.

Two rigidly connected pairs of particles will be called *congruent* if they are assigned the same distance by an LMI (not necessarily linear), within an acceptable margin of error. The value of this definition is due to the fact that when two separate pairs of particles are judged to be congruent, it is possible to make them coincide when they are brought together, i.e., congruence is *transitive*.

C. Collinearity

Reichenbach [(16) 169] essentially defines three particles to be collinear if they lie along the path of a light-signal. Another definition of collinearity is given here with the help of congruent pairs of rigidly connected particles. If such pairs are connected together into a linear chain, then the minimum number of such pairs linking two particles A, B will form a linearly ordered set of particles between A and B, which will be called the *space geodesic* (SG) between A and B. Any three or more particles on the same SG are said to be *collinear*.

An SG does not always coincide with the path of a light signal. For instance, on a rotating disk, a radial line is an SG [(8) 242], whereas a light signal which starts in the radial direction will deflect away from this direction. Consequently, the above two definitions of collinearity are not equivalent. From now on we adopt the definition of collinearity determined by an SG, since collinearity of particles is basically a property of the space (not space–time) geometry, and SG's are the "straight lines" of the space geometry. One realization of an SG, is a taut string.

If A, B, C are collinear, with B *between* A and C, we shall write $[ABC]$. The *directed* geodesic segment from A to B will be denoted by (AB). We assume that any *SG* can be *linearly* extended on either side; and by *extension* of an *SG*, we always mean *linear* extension.

If the distance measured by an LMI *between* any two particles A, B, on an SG is proportional to the number of congruent pairs of particles between A and B, the LMI is said to be a *linear* LMI (LLMI). The length of (AB) measured by an LLMI is denoted by $|AB|$, and called the *proper length.*

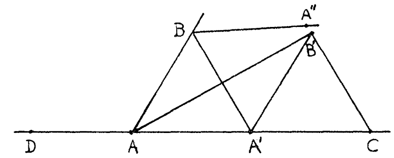

FIG. 3. Parallel displacement.

D. Orthogonality

Two SG's a, b intersecting at A (having a common particle A) are said to be *orthogonal* at A (in symbols, $a \perp_A b$) if and only if there exist two particles B, D on a with $[BAD]$ and $|AB| = |AD|$, and a particle C on b such that $|CB| = |CD|$. Operationally, we may think of b as being rotated about A until $|CB| = |CD|$. In effect, this defines *right* angles, as congruent adjacent angles.

If $a \perp_A b$, it is always possible to construct an SG c orthogonal to both a and b at A. This can be done by first constructing $c \perp_A a$, and then rotating c about a until $c \perp_A b$. The SG's a, b, c, are then said to form an *orthogonal triad* at A.

E. Parallel Displacement

In Euclidean geometry, two lines are said to be *parallel* if they lie in the same plane and make equal angles with a straight line in that plane. The space geometry in an arbitrary gravitational field is general Riemannian (having variable space curvature), and consequently it is necessary to generalize the above definition of parallelism to Riemannian geometry. The two best known generalizations have been given by Levi-Civita [(19) 101–111, 137–140; (20) 62–65, 72–74; (21) II 98–101] and Severi [(22) Sec. 11; (19) 171; (21) II 101–107]. The two definitions agree to the first order infinitesimals, but not to the second order [(21) II 124–127]. The mathematical condition for parallelism is given in Sec. 7E, and we restrict ourselves here to the operational meaning of parallelism.

The Levi–Civita definition for a space of more than two dimensions (we are interested here in three-dimensional space) is given in terms of an imbedding Euclidean space, and is not suitable for an operational interpretation. However, the Severi definition is based upon the intrinsic properties of the space by making use of geodesics. The definition we give in the following is the concrete realization of this definition.

Suppose that we have three *neighboring* rigidly connected particles A, B, A' (Fig. 3), and we wish to find a particle B' such that (AB) is parallel to $(A'B')$. The first point to notice is that the direction of $(A'B')$ depends upon the path from A to A' along which (AB) is displaced. It is assumed that this path is always the (unique) SG connecting A and A'. Now the two directions of the SG's (AB), (AA') determine a surface S_A called the *geodesic surface at* A as follows: Extend (AB) and (AA') on both sides and connect B with A' by an SG. Then connect A with all the particles on (BA') with SG's to obtain the first quadrant of S_A. If D is a neighboring particle such that $[DAA']$, the second quadrant may be obtained by connecting A with the particles of (BD). The other two quadrants may be constructed in the same way. The second thing to notice is that if (AP) and (AQ) are any SG's on S_A and P, Q are a finite *distance* away from A, then (PQ) does not necessarily lie on S_A, i.e., it does not necessarily intersect the SG's between (AP) and (AQ) [(21)II 135–136]. Consequently, it is not possible in general to define a *totally geodesic surface*, and is one of the reasons why A, A' have been chosen to be neighboring particles.

The best thing we can do, is to start $(A'B')$ tangent to S_A, by requiring that (AB') intersect $(A'B)$; the rest of $(A'B')$ may not lie on S_A. In the Euclidean case, this requirement would insure that $(A'B')$ lies in the plane S_A determined by (AB) and (AA').

Finally, by analogy with the Euclidean case, we choose the direction of $(A'B')$ so that $\angle B'A'C = \angle BAA'$, where C is a neighbor of A' with $[AA'C]$, as shown in Fig. 3. This is accomplished by taking $|A'B'| = |AB|$, $|A'C| = |AA'|$, and $|B'C| = |BA'|$, i.e., by requiring the congruence of triangles $A'B'C$ and ABA'. If all the above conditions are satisfied, we say that (AB) *is parallel displaced along* (AA') to $(A'B')$, in symbols $AB \parallel A'B'$ (the order of the four letters is important). Notice that if $AA' \parallel BA''$, then (BA'') and $(A'B')$ do not necessarily interest [(21)II 138–139].

If $AB \perp AA'$ and $AB \parallel A'B'$, then it follows from the above definition that $A'B' \perp A'A$. Whether (AB) and $(A'B')$ intersect or not depends upon the nature of the space geometry. For instance, if the space has *constant curvature*, then they will meet in zero, one (absolute polar), or two points (antipodal points), depending upon whether the geometry is Euclidean or hyperbolic, elliptic, or spherical. However, since the radius of curvature is of cosmical dimensions, we need not worry about the intersection of two SG's orthogonal to another SG within the region of any concrete CS.

4. PRINCIPLE OF EQUIVALENCE

A principle which plays a fundamental part in GR as a whole, and in space–time geometry in particular, is *the principle of equivalence* (PE). There are several different formulations of this principle in the literature, and we now review these.

According to the first formulation by Einstein [(12)118], PE may be stated as follows: *For every infinitesimal space—time region there always exists a physical CS, I, in which SR is applicable.* This CS will be called a *local ICS* (LICS).

Fock [(7)228–230] took issue with Einstein's formulation of PE because of its *local* character. According to Fock, the important principle is the *equality of inertial and passive gravitational mass*, which has *nonlocal* character. By *inertial mass* (IM) is meant the intrinsic property of a body that determines its acceleration under the influence of a nongravitational force; a *passive gravitational mass* (PGM) is the mass on which a gravitational field acts, and an *active gravitational mass* (AGM) is the mass that is the source of a gravitational field [(23)423].

Synge [(6)IX] is not sure what PE is; but according to him, if it means anything, it is perhaps that the signature of the space—time metric is $+2$. The meaning of this statement is explained in Sec. 5B, and is shown to follow from Einstein's PE.

This controversy was somewhat clarified by Dicke [(1b)13; (2b)16–19] by breaking PE into *two* principles, a weak and a strong principle. According to Dicke *the weak principle of equivalence* (WPE) is the statement that all bodies fall with the same acceleration at a given point in space, and the strong principle of equivalence (SPE) is the statement that in a freely falling laboratory all the laws of physics have the same form and numerical content (values of dimensionless "constants"), as in a gravity-free region, independent of the position of the laboratory in space and time. As pointed out by Dicke, WPE is very well supported by the experiments of Eötvös [24] and Dicke [(25); (26)18–19], but SPE has been verified only for electromagnetic and strong interactions, and its status is unknown for weak and gravitational interactions [(1b)14–15; (2b)19–31]. Notice that WPE is a nonlocal statement, whereas SPE is local, and it is SPE that corresponds to Einstein's PE.

The relation of WPE to Fock's statement "IM=PGM" was derived by Einstein himself [(11)57] as follows: For particles with speeds negligibly small compared to the speed of light, Newtonian mechanics provides a valid approximation. In accordance with this theory, the equation of motion of a particle in a gravitational field is given by $m_I\mathbf{a} = -m_{PG}\nabla\phi$, where m_I is the IM, m_{PG} is the PGM, $\mathbf{a}$ is the acceleration, and ϕ is the gravitational potential. According to WPE, we have $\mathbf{a} = -\nabla\phi$, and thus $m_I = m_{PG}$.

Inasmuch as the statement "IM=PGM" derives its meaning from concepts in dynamics and gravitation, whereas Dicke's statement of WPE makes use only of kinematical and geometrical concepts, it seems to me that Dicke's statement is more preferable. Due to the importance of WPE and its acceptance by *all* authors in some form or another, it seems worth while to make it more precise as follows: By WPE we shall mean that *any* two particles which are acted upon by the same electromagnetic, strong, and weak forces, and which are coincident for two neighboring events (have the same initial position and velocity) will move along exactly the same trajectory. The nonlocal character of this statement is clear, and the presence of the listed forces make it unnecessary for the particles to be "falling" to test this statement experimentally. Since the bodies are suspended by wires in the Eötvös and Dicke experiments, it is the statement in the above form which is actually verified. From now on, we shall mean by WPE the above statement.

If the forces listed in WPE are zero, either because their sources are zero or the charge, magnetic moment, etc. of the particles are zero, the particles are said to be *falling freely*. Notice that this concept would be meaningless if it were not for the validity of WPE. A *local ICS* (LICS) may now be defined as a set of freely falling particles confined to within a space region R during a proper-time interval ΔT, such that if D is the dimension of R in any particular direction, and ΔX is the change during ΔT of the proper distance along the same direction of any two particles initially at rest, then $\Delta X/D$ may be considered negligible compared to the smallest dimensionless number under consideration. This is the operational meaning of the adjective "infinitesimal" describing the LICS in Einstein's PE. In Sec. 5C "infinitesimal" will mean that the second derivatives of the metric coefficients are negligible; but it seems circular to define "infinitesimal" in terms of the metric if PE plays a role in the introduction of the metric.

Having defined an LICS, we may express SPE more precisely by the brief statement that *SR is valid in an LICS*. Notice that the validity of WPE is necessary for the formulation of SPE, and thus Einstein's PE implicitly assumes WPE.

Another version of PE was given by Rohrlich [(26)183] in terms of the two statements: (A) IM= PGM, (B) PGM=AGM. Statement (A) was shown above to be related to WPE, whereas statement (B) seems to be a new proposal. Rohrlich noted that (B) is a consequence of Newton's third law and the law of gravitation, and that since Newton's third law must be abandoned in relativity due to the finite propagation speed of interactions, (B) is not trivial. However, it is not clear what role (B) plays in the postulates of GR.

Since according to WPE the trajectories of freely falling particles are the same for all particles, the *dynamical* trajectories may be identified with *geometrical* paths. This fact makes it possible to identify the gravitational field with the metric. In Secs. 5B, C it is seen how Einstein made use of SPE to derive the expression for the line element and equations of motion; but because of the adjective "infinitesimal," it is also seen in Sec. 5D that the applicability of SPE is limited.

5. SPACE–TIME GEOMETRY

Since the points of space–time are *events*, and *four* coordinates are necessary to specify an event, space–time is *four dimensional*. In order to develop space–time geometry, it is necessary to introduce a *metric* or a *measure ds* of space–time intervals. Congruence of intervals can then be defined in terms of the value of such a measure. The type of measure determines the nature of the geometry [(16)35; (27)64–71; (13)11, 115–117]. In this section we discuss the three principal measures used in the literature.

A. Coordinates

For the study of space–time geometry it is convenient (but not necessary) to introduce coordinates [1(c)19]. The main function of coordinates is to catalog events [1(a)40–45]. One rather unconventional way of assigning coordinates to an event which illustrates very well the great freedom one has in such an assignment, was given by Synge [(5)7]. It consists essentially of four "old battered" but hardy clocks carried by flying aeroplanes that turn, dive, and climb in an arbitrary way. If the event is an explosion, the times of arrival of the sound of explosion recorded by the four clocks may be used as the four coordinates of the event. The essential point here is that *four* numbers are needed to identify an event, not three or five; and in this lies the meaning of the statement that the totality of all possible events form a four-dimensional continuum [(5)6].

More conventionally, a coordinate system (CS) in GR may be described [(16)263–264; (28)] as a scaffolding constructed from an elastic material and having coordinate clocks at its intersections which are synchronized with a central clock. The use of *coordinate* instead of *standard* clocks is purely a matter of taste or convenience. However, the use of an *elastic* material *is* necessary, because a rigid structure is not possible in a time-varying gravitational field, as becomes apparent in Sec. 7B.

B. Line Element

As already mentioned, coordinates serve simply as *labels* of events, and differences of coordinates dx^i give no information by themselves about time or space intervals as measured by SC's or LLMI's, i.e., *proper* time and length intervals. In order to obtain such information, it is necessary to state how the space–time interval ds between two events is to be *measured*. There seem to be three different methods in the literature of accomplishing this, which will be shown to be equivalent. We now discuss these methods.

1. *Einstein's Method*

To measure ds, Einstein [(12)118–120; (11)63] made use of SPE as follows: Let dx^i be the coordinate differences of two events in an arbitrary CS and dX^i the rectangular coordinates of the same two events relative to an LICS, I. According to SPE, we may apply SR to I and conclude that

$$ds^2 = -\eta_{ij} dX^i dX^j \tag{5.1}$$

$$= c^2 dT^2 - dL^2, \tag{5.1'}$$

is an invariant, where

$$\eta_{i0} = -\delta_{i0}, \qquad \eta_{\alpha\beta} = \delta_{\alpha\beta},$$

and

$$dX^0 = c\,dT, \qquad dL^2 = \delta_{\alpha\beta} dX^\alpha dX^\beta. \tag{5.2}$$

Since one can measure dT by synchronized SC's in I and dL by an LLMI, it follows that ds is measurable, which is what was desired.

In order to find the expression of ds in terms of dx^i, we recall that coordinates are only labels, and thus there must exist a nonsingular transformation between X^i and x^i, i.e., we can write

$$X^i = X^i(x^0, x^1, x^2, x^3), \qquad dX^i = X^i{}_{,j}\, dx^j, \tag{5.3}$$

where X^i are the coordinates within the "infinitesimal" space–time region of I. Substituting (5.3) into (5.1), we obtain the desired expression

$$ds^2 = -g_{ij} dx^i dx^j, \tag{5.4}$$

where the *metric coefficients* g_{ij} are functions of x^i.

Since (5.3) has an inverse, it is always possible to reduce (5.4) *locally* to (5.1). The reduction to a diagonal form is always possible for a nonsingular quadratic form (5.4), i.e., if $\det(g_{ij}) \neq 0$. The important thing here is the number of positive and negative signs in (5.1), which is characteristic of the form and cannot be changed by a coordinate transformation. The number of negative signs minus the number of positive signs is called the *signature* of the form. In (5.1), there are 3 negative signs and 1 positive sign, and thus the signature is $+2$. The physical significance of the number of positive and negative signs was very well explained by Synge [(5) 17–18], and is worth outlining here.

If ds represents the interval between a particular event a and an arbitrary event b in the neighborhood of a, then b can either be connected with a by a particle, or by a photon, or cannot be connected at all. Since the speed of a particle is always less the speed c of a photon, it follows from (5.1′) that

$$ds \begin{matrix} > \\ = \\ < \end{matrix} 0 \quad \text{for events} \quad \begin{matrix} \text{inside} \\ \text{on} \\ \text{outside} \end{matrix} \quad \text{the light cone.} \tag{5.5}$$

Events b *inside* and *on* the light cone are either in the past or future of a and those outside the cone cannot be connected with a by any signal, and any one of these events may be considered simultaneous with a in a properly chosen CS. Since ds is an invariant, the classification of events relative to any particular event is independent of the CS.

Now if we write (5.1) in the form $ds^2 = \epsilon_\mu (dX^\mu)^2$, where $\epsilon_\mu = \pm 1$, we notice that there are five possibilities for the signs of $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3$, namely:

(a) $----$, (b) $+---$, (c) $++--$,
(d) $+++-$, (e) $++++$.

Since all values of ds in (5.5) must exist, and case (a) excludes $ds \geq 0$, while case (e) excludes $ds \leq 0$, these two cases must be ruled out. In cases (c) and (d), if we let $dX^2 = dX^3 = 0$, we see that along X^1, $ds \leq 0$ is excluded, and thus these two cases must also be ruled out. This leaves case (b), which coincides with (5.1), as the only case possible.

Since ds^2 is an *invariant*, it follows that dx^i are the components of a *vector* and g_{ij} are the components of a *tensor* of the second rank, i.e., if we perform a transformation $x'^i = x'^i(x^0, x^1, x^2, x^3)$, then $dx'^i = x'^i{}_{,j} dx^j$, and

$$g_{ij} = x'^k{}_{,i} x'^l{}_{,j} g'_{kl}. \tag{5.6}$$

For this reason g_{ij} are also referred to as components of the *metric tensor*.

We may always decompose g_{ij} into *symmetric* and *antisymmetric* parts as follows: $g_{ij} = g^S{}_{ij} + g^A{}_{ij}$, where

$$g^S{}_{ij} = \tfrac{1}{2}(g_{ij} + g_{ji}) = g^S{}_{ji}, \qquad g^A{}_{ij} = \tfrac{1}{2}(g_{ij} - g_{ji}) = -g^A{}_{ji}.$$

Since $g^A{}_{ij} dx^i dx^j = 0$, it follows that $g^A{}_{ij}$ contributes nothing to ds^2, and we may assume g_{ij} to be symmetric without any loss of generality, i.e., we may assume

$$g_{ij} = g_{ji}.$$

Consequently, there are only 10 independent metric coefficients instead of 16.

2. *Synge's Method*

Instead of making use of SPE to measure ds, Synge [(6) 105–107] defines first the measure on a time-like interval by the time $d\tau$ recorded by a freely falling (inertial) SC (ISC), from the start-event of the interval to the end-event. The question immediately arises as to the relation of $d\tau$ to the coordinate differences dx^i of the two events [(5) 15–19]. In general, one may write $d\tau = f(x, dx)$, where "x" is an abbreviation for "x^0, x^1, x^2, x^3" and f must be an *invariant* since $d\tau$ is a measurable quantity. If one assumes that $f(x, k\,dx) = kf(x, dx)$ for $k > 0$, then f is positive homogeneous of the first degree and the space–time geometry is said to be a *Finsler space*. This allows many possibilities for the invariant function f. Two such possibilities may be (5.4) with

$$ds = c\,d\tau \qquad \text{and} \qquad d\tau = (g_{ijkl} dx^i dx^j dx^k dx^l)^{\frac{1}{4}}.$$

The choice of f must be decided experimentally. One way of accomplishing this [(16) 249–250], is to assume a particular form for f and find out the number of different time-like intervals emanating from a given event that are necessary to determine the metric coefficients uniquely. For instance, if (5.4) is the correct choice, then 10 measurements of $d\tau$ will be necessary and sufficient to determine the 10 g_{ij}; and it is then possible to *calculate* the value for any other time-like interval and check the result experimentally. On the other hand, if the other possibility of f is correct, then many more measurements of $d\tau$ than 10 are necessary to determine g_{ijkl}. It turns out experimentally that (5.4) is the correct choice, and this fact may be adopted as a postulate. In effect, this states that space–time geometry is a *Riemannian* space, which is a particular case of a Finsler space. However, the g_{ij} obtained in this way do not *a priori* have to be the same as those obtained previously, because the measures of ds are different. But it is shown in Sec. 6A that they are the same.

So far, a measure of ds has only been defined for time-like intervals. However, since this is sufficient to determine g_{ij} completely, one may use these values of g_{ij} to *calculate* ds from (5.4) for any other kind of interval. It turns out experimentally that $ds=0$ for events connected by photons. Again we must introduce this fact as a *postulate* in the theory, and it amounts to the statement that photons have the maximum speed of any possible physical signal. For separated events which cannot be connected by any signal, $ds<0$. In order to express these facts mathematically, one *assumes* that ds^2 is nonsingular ($\det g_{ij}\neq 0$) with signature $+2$[(5)17–19], an assumption which is closely related to Einstein's *PE*.

3. *The Method of Marzke and Wheeler*

The method of Marzke and Wheeler [(1a)45, 53–56] is the same as that of Synge with the exception that they measure *any type* of interval ds by using *geodesic clocks* constructed from freely falling particles and photons, as discussed in Sec. 2A. It was shown there that *by definition*, $ds=0$ for events connected by a photon. Since the same statement was found to be true by the use of SC's, it follows that a geodesic clock is equivalent to an SC. Therefore, the method of Marzke and Wheeler is *equivalent* to that of Synge.

C. Space–Time Geodesics

Before presenting how Einstein identified the paths of freely falling particles (FP) with space–time geodesics by the use of SPE, we shall investigate the approximation implicit in the adjective "infinitesimal" applied to an LICS. For this purpose, let us transform from x^i to x'^i. Then g_{ij} transforms to g'_{ij} according to (5.6). Expanding g'_{ij} about any point P, we get

$$g'_{ij}(x')=g_{ij}(x'_P)+(\partial g'_{ij}/\partial x'^k)_P(x'^k-x'^k_P)$$
$$+(\partial^2 g'_{ij}/\partial x'_k\partial x'_l)_P(x'^k-x'^k_P)(x'^l-x'^l_P)+\cdots.$$

Moller [(8)274–275] showed that it is always possible to find a transformation so that $(\partial g'_{ij}/\partial x'^k)_P=0$. Moreover, because the signature of ds^2 is $+2$, it is also possible to set $g'_{ij}(x'_P)=\eta_{ij}$. Consequently, by choosing $x'^i-x'^i_P$ sufficiently small, we may neglect the remaining terms in the above expansion and obtain an LICS. *Thus, "infinitesimal" means that the extension of the LICS is such that terms involving derivatives of the metric coefficients of the second order and higher may be neglected.*

Following Einstein [(12)142–143], we notice that the path of an FP in an LICS is a straight line, which is a geodesic in this case. As is well known in Riemannian geometry [(20)48–51; (8)228–230], the geodesic equations may be derived from the variational principle

$$\delta\int_a^b ds=0,\qquad \delta x^i_a=\delta x^i_b=0. \tag{5.7}$$

Since ds is invariant, (5.7) is valid in any CS, and by SPE it may be assumed tentatively that (5.7) is the equation of motion of an FP.

Making use of (5.4), the geodesic equations resulting from (5.7) may be written in the form [(9)268–270]

$$a^i=du^i/ds=-\Gamma^i{}_{jk}u^ju^k=A^i \tag{5.8}$$

$$u^i=dx^i/ds, \tag{5.9}$$

where

$$\Gamma^i{}_{jk}=g^{il}\Gamma_{l\cdot jk},\qquad 2\Gamma_{i\cdot jk}=g_{ij,k}+g_{ik,j}-g_{jk,i}, \tag{5.10}$$

and g^{ij} are defined by

$$g^{ij}g_{jk}=\delta^i{}_k. \tag{5.11}$$

$\Gamma_{i\cdot jk}$ and $\Gamma^i{}_{jk}$ are known as the Christoffel symbols of the first and second kind, respectively. From (5.8, 10) we see that only first derivatives of g_{ij} are involved, and thus the use of SPE to obtain (5.7) is justified.

Since Synge did not make use of SPE, he had to *assume* that (5.8) are the equations of motion [(6)110]. The necessity of this assumption stems from the fact that once the measure and form of ds have been decided, then g_{ij} and hence the geodesics are uniquely determined; and there is no guarantee that the paths of FP will be geodesics. After selecting the form of ds, one can either *define* the measure of ds to be given by an ISC and *assume* that the paths of FP are geodesics, or he can define the geodesics by the paths of FP and *assume* that the measure of ds is given by an ISC.

D. Field Equations

Up to now no mention was made of how to calculate g_{ij}. From WPE, i.e., the identification of FP trajectories with geometrical paths, it became evident to Einstein that the gravitational field must be given by the metric. But since the gravitational field is determined by the environment, this meant that equations must be found that relate the metric to the environment. The derivation of these equations is given in every book on GR and is not repeated here. For our purposes we only need the field equations *outside* the sources of the field, namely [(8)310–312, 287]

$$R_{ij}=(\ln\sqrt{-g})_{,ij}-\Gamma^k{}_{ij,k}-(\ln\sqrt{-g})_{,k}\Gamma^k{}_{ij}$$
$$+\Gamma^k{}_{il}\Gamma^l{}_{jk}=0, \tag{5.12}$$

where $g=\det(g_{ij})$, and $R_{ij}=R_{ji}$ are the components of the curvature tensor of the second rank. Since (5.12) involves second-order derivatives of g_{ij}, SPE cannot be used to derive them.

Due to the general covariance of (5.12), i.e., the fact that (5.12) does not change its form under arbitrary transformations of the four coordinates, only six out of these ten equations are independent [(8)311]. Consequently, it is necessary to impose four additional conditions, called the *coordinate conditions* (CC), on g_{ij} in order to determine the 10 g_{ij} uniquely. The CC are related to the choice of the CS and rates of clocks.

E. Matter and Transformation Fields

A gravitational field generated by matter or energy will be called a *matter field*, and one that can be obtained by a coordinate transformation from a (finite) ICS will be called a *transformation field*. The presence of a matter field can always be detected by verifying whether the following condition is fulfilled:

$$2R_{ijkl}=g_{ik,jl}+g_{jl,ik}-g_{il,jk}-g_{jk,il}+2g^{mn}(\Gamma_{m\cdot ik}\Gamma_{n\cdot jl}-\Gamma_{m\cdot il}\Gamma_{n\cdot jk})=0. \quad (5.13)$$

If $R_{ijkl}\neq 0$, then at least part of the field is a matter field; whereas if $R_{ijkl}=0$, then the field is *purely* a transformation field. Since R_{ijkl} is the *curvature* tensor, space–time is said to be *curved* if $R_{ijkl}\neq 0$, and *flat* if $R_{ijkl}=0$. In the latter case, it is possible to find a transformation (5.3) that will reduce (5.4) to (5.1) in a *finite* region, since (5.13) is the necessary and sufficient condition for the existence of such a transformation. By substituting (5.3) into (5.1), it can be seen in this case that

$$g_{ij}=X^k{}_{,i}X^l{}_{,j}\eta_{kl}. \quad (5.14)$$

If g_{ij} are known, (5.14) can be used to derive the transformation (5.3).

The question now is whether it is possible to separate a matter field from a transformation field in case $R_{ijkl}\neq 0$? This can be done if the asymptotic behavior of the field at infinity is examined. According to GR in its present form, a matter field should be asymptotically flat [(7)3, 230; (9) 244–245], i.e., as the proper distance from the sources of the field approach infinity, g_{ij} should approach the values η_{ij}. However, if GR is modified or reformulated [(1b)121–125; (1d)303–344; (2b)31–48; (29)] so as to take Mach's principle into account, g_{ij} should vanish asymptotically. In either case, the asymptotic behavior of a transformation field is different, as will be demonstrated in Secs. 11 and 12.

The condition (5.13) is coordinate independent, since if *all* the components of a tensor vanish in one CS, they will vanish in any CS. Moreover, the curvature tensor R_{ijkl} has the following properties:

$$R_{ij}=g^{kl}R_{ikjl}, \quad (5.15)$$

$$R_{ijkl}=-R_{jikl}=-R_{ijlk}=R_{klij}. \quad (5.16)$$

6. TIME GEOMETRY

The true nature of *time* becomes apparent when one restricts one's self to events at a *single* particle. The subclass of events obtained in this way form a *one-dimensional* space ordered by the *binary* relation "*before*," which is irreflexive, asymmetric, and transitive. A metric on this space is introduce by means of clocks, as was shown in Sec. 2. In order to extend the concept of time over space, it is necessary to synchronize clocks. In the following we go into the formulation of the important properties of time intervals in GR.

A. Relation between Proper and Coordinate Time

Consider a standard and a coordinate clock at rest ($dx^\alpha=0$) at the same point in an arbitrary CS, R. For two neighboring events a, b at this point, it follows from (5.4) in Synge's formulation that

$$d\tau=\sqrt{-g_{00}}dt, \quad (6.1)$$

where $d\tau$ is the proper time and dt is the coordinate time. This shows that $\sqrt{-g_{00}}$ *is the ratio of the rate of a coordinate clock to the rate of a coincident SC*. If it is desired, SC's may be used throughout R, in which case $g_{00}=-1$. By squaring (6.1), it can also be seen that we must always have

$$g_{00}<0. \quad (6.2)$$

Suppose now that an ISC is momentarily at rest relative to the SC under consideration, during the occurrence of a, b, and dT is the time interval between a, b that it records. It is an experimental fact, the validity of which is assumed in GR [(8)49], that

$$d\tau=dT. \quad (6.3)$$

This assumption was thoroughly examined by Romain [30] from the point of view of transformation from an ICS to a "uniformly" accelerating CS.

In the LICS attached to the ISC, $ds_E=cdT$; and in R, $ds_S=cd\tau$. The subscripts E and S are used to differentiate between the measures assigned to ds by Einstein and Synge, respectively. It thus follows from (6.3) that $ds_E=ds_S$, i.e., the two measures of ds are equivalent. Consequently the g_{ij} obtained from the two different measures are the same.

B. Clock Synchronization

Let A, B be two synchronized coordinate clocks in R, and let a photon depart from A at time $(t-dt_1)$, be reflected by B at time t, and return to A at time $(t+dt_2)$. The time intervals dt_μ ($\mu=1$, 2) can be calculated from the equation of motion of a photon [(9)257].

$$-ds^2=g_{00}(cdt)^2+2g_{0\alpha}dx^\alpha(cdt)+g_{\alpha\beta}dx^\alpha dx^\beta=0. \quad (6.4)$$

With the help of the definitions

$$\gamma_\alpha=g_{0\alpha}/\sqrt{-g_{00}}, \qquad \gamma_0=g_{00}/\sqrt{-g_{00}}=-\sqrt{-g_{00}}, \quad (6.5)$$

$$\gamma_{\alpha\beta}=g_{\alpha\beta}+\gamma_\alpha\gamma_\beta, \quad (6.6)$$

the solution of the quadratic equation (6.4) may be written as

$$c\sqrt{-g_{00}}dt_\mu = (\gamma_{\alpha\beta}dx^\alpha dx^\beta)^{\frac{1}{2}} \pm \gamma_\alpha dx^\alpha, \qquad (6.7)$$

where the positive sign is associated with $\mu=1$ and the negative sign with $\mu=2$.

By Einstein's convention [$\epsilon=\frac{1}{2}$ in (2.1)], the event at A which is simultaneous with the reflection event at B occurs at time

$$t' = (t-dt_1)+\tfrac{1}{2}(dt_1+dt_2) = t+\tfrac{1}{2}(dt_2-dt_1).$$

Making use of (6.7) we get

$$c\delta\tau = c\sqrt{-g_{00}}(t'-t) = c\sqrt{-g_{00}}(dt_2-dt_1) = \gamma_\alpha dx^\alpha.$$

Thus, if clocks are synchronized along a closed path, the last clock will be found to be off from the first clock by [(9)259, 276]

$$c\oint \delta\tau = \oint \gamma_\alpha\, dx^\alpha = \int \tau_{\alpha\beta}\, df^{\alpha\beta}, \qquad (6.8)$$

where $df^{\alpha\beta}$ is an area element on the $x^\alpha x^\beta$ surface ($\alpha\neq\beta$) and

$$\tau_{ij} = \gamma_{i,j} - \gamma_{j,i}. \qquad (6.9)$$

The last step was obtained by the use of Stokes' theorem [(9)22]. The necessary and sufficient condition for transitive synchronization is

$$\oint \delta\tau = 0,$$

which in view of (6.8) is equivalent to

$$\tau_{\alpha\beta} = 0. \qquad (6.10)$$

One way of testing experimentally whether synchronization is transitive or not, is to let two beams of light travel around a closed path P of *nonzero* area in two opposite directions and measure the phase difference between the two beams. The last quantity is related to the proper time difference $\Delta\tau$ between the travel times of the two beams. Making use of (6.7), we get as in (6.8)

$$\Delta\tau = 2c^{-1}\oint_P \gamma_\alpha\, dx^\alpha = 2\oint \delta\tau. \qquad (6.11)$$

Such an experiment was performed on a rotating disk first by Sagnac [31] and more recently with the help of lasers by Macek [32].

C. Gravitational Frequency Shift

One of the basic experiments that can be performed with photons, is the comparison of the rate of two clocks at separate points A, B. This can be done by sending two photons from A at times t and $t+dt$, and receiving them at B at times t' and $t'+dt'$. If A and B are neighbors, then according to (6.7) we have

$$c\sqrt{-g_{00}}(t'-t) = (\gamma_{\alpha\beta}dx^\alpha dx^\beta)^{\frac{1}{2}} + \gamma_\alpha dx^\alpha,$$

$$c(-g_{00}-cg_{00,0}dt)^{\frac{1}{2}}(t'+dt'-t-dt)$$
$$= [(\gamma_{\alpha\beta}+c\gamma_{\alpha\beta,0}dt)dx^\alpha dx^\beta]^{\frac{1}{2}} + (\gamma_\alpha + c\gamma_{\alpha,0}dt)dx^\alpha.$$

Subtracting the first equation from the second, we get to the first order infinitesimals

$$dt'/dt = 1+d\epsilon, \qquad (6.12)$$

where

$$\sqrt{-g_{00}}d\epsilon = \gamma_{\alpha,0}dx^\alpha + \tfrac{1}{2}(\gamma_{\alpha\beta}dx^\alpha dx^\beta)^{-\frac{1}{2}}(\gamma_{\alpha\beta,0}dx^\alpha dx^\beta). \qquad (6.13)$$

If B is a finite distance away from A, we can calculate dt'/dt by subdividing the path of the photon from A to B into infinitesimal segments, say n in number. If the time intervals between the two photons along this path are $dt=dt_1, dt_2, \cdots, dt_n=dt'$, then

$$\frac{dt'}{dt} = \frac{dt_n}{dt_{n-1}}\frac{dt_{n-1}}{dt_{n-2}}\cdots\frac{dt_2}{dt_1} = \prod_1^n{}_i(1+d\epsilon_i) \approx 1+\sum_i d\epsilon_i.$$

Thus,

$$dt'/dt = 1+\int d\epsilon. \qquad (6.14)$$

For a stationary field, defined by $g_{ij,0}=0$, $d\epsilon=0$ and $dt'/dt=1$. Making use of (6.1), we get the familiar result [(4)60–61]

$$\nu/\nu' = d\tau'/d\tau = (g'_{00}/g_{00})^{\frac{1}{2}}, \qquad (6.15)$$

where ν, ν' are the proper frequencies of the clocks. This formula is used to explain the gravitational red shift and recent experiments [(33), (34)] as will be seen in Secs. 11–13.

7. SPACE GEOMETRY

The points of space geometry are *particles* not *events*. The most basic relations between particles are those of *collinearity* and *betweenness*, whose operational meaning were discussed in Sec. 3C. When the way these concepts were defined is examined, it is seen that they have an absolute meaning independent of any CS, just as the relation "before" has an absolute meaning. However, whereas *before* is a *binary* relation between events, *betweenness* is a *ternary* relation between particles. This points out one of the important differences between the topological structure of time and space. The properties of *betweenness* are discussed in books on the foundation of geometry [(35)26].

The dimensionality of space may be established with the help of collinearity as follows: One starts with a set L of collinear particles, and discovers that there exists at least one particle which does not lie on L. This establishes that space is at least two-dimensional. Then after defining a geodesic surface S at a particle, as in Sec. 3E, one finds that there is at least one particle

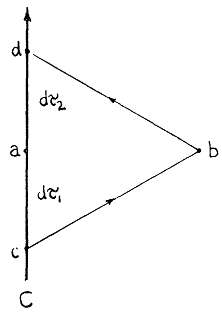

FIG. 4. Space-interval measurement.

not on S, which shows that space is at least three dimensional. Finally, by setting up a three-dimensional space CS, one finds that *any* particle can be located by means of three coordinates, which proves that space is at most three-dimensional.

A metric on space may be introduced with the help of an LLMI, as shown in Sec. 3C. The relation of this measure on intervals between *particles* to the measure on space-like intervals between events, will now be discussed.

A. Measures of Space Intervals

In Sec. 5B2, we have seen that a space-like interval ds between separate events a, b may be *calculated* from (5.4). Synge [(5)24–26; (6)112–113] has given another way of *measuring* such an interval as follows: Suppose that the event a occurs on an SC, C, and a photon starts from C is reflected at b, and returns back to C (Fig. 4). Let the start and return events of the photon be denoted by c and d, respectively. Since ds is space-like, a must occur between c and d. Let the time intervals on C between c and a be $d\tau_1$, and between a and d be $d\tau_2$, Synge proved that (aside from a factor of $i=\sqrt{-1}$, which he takes care of by the use of an indicator [(5)23])

$$ds=ic(d\tau_1 d\tau_2)^{\frac{1}{2}}. \tag{7.1}$$

By definition, a is simultaneous with b if $d\tau_1=d\tau_2=\frac{1}{2}d\tau$, where $d\tau$ is the round-trip time. In this case (7.1) becomes

$$ds=\tfrac{1}{2}icd\tau. \tag{7.2}$$

An interesting question is, if a, b occur on a rigid rod (LLMI), A, at rest in an arbitrary CS, R, and the measured distance is dl, what is the relation between dl and ds given by (7.2)? To investigate this question, we let another rigid rod, B, fall freely and assume that its end points coincide with those of A at the events a, b while B is momentarily at rest. Another assumption in GR [(8)223] is that if the distance between a, b measured by B is dL, then

$$dl=dL. \tag{7.3}$$

Because of (6.3), event a also occurs halfway between c and d on an SC attached to B. Consequently, a, b are simultaneous in the LICS attached to B, and it follows from (5.1′) and (7.3) that

$$ds=idL=idl. \tag{7.4}$$

Comparing (7.4) with (7.2), we see that

$$dl=\tfrac{1}{2}cd\tau. \tag{7.5}$$

Therefore, the chronometric measurement of a space-like interval, described by (7.2), is equivalent to the measurement of the same interval by an LLMI at rest in R. Moreover, (7.5) shows that the *local* speed of light as measured by an SC and LLMI at rest in R is always the same constant c as in an LICS. These significant conclusions could not have been reached by *purely* chronometric measurements.

However, the speed of light may be different than c, if measured over a *finite* distance. For instance, in Fig. 5, the speed of light c' measured over the total distance (dl_1+dl_2) is given by

$$c'=2(dl_1+dl_2)/(d\tau_1+d\tau'_2).$$

If dl_1 and dl_2 are infinitesimal, then it follows from (7.5) that $2dl_\mu=cd\tau_\mu$ ($\mu=1, 2$). Thus

$$c'/c=(d\tau_1+d\tau_2)/(d\tau_1+d\tau'_2).$$

$d\tau_2$ and $d\tau'_2$ are related by (6.12), and are not equal in general. Consequently c' may have a different value than c.

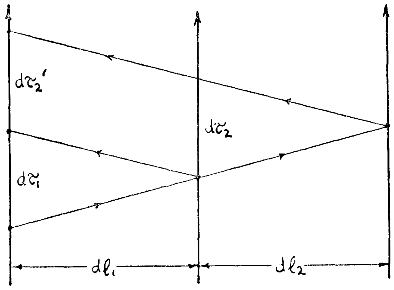

FIG. 5. Measurement of speed of light.

B. Space Metric

We now express the right-hand side of (7.5) in terms of $\gamma_{\alpha\beta}$. Let dt_1 and dt_2 be the coordinate time intervals that the photon takes to travel from C and back to C, respectively. According to (6.1) and (7.5),

$$dl=\tfrac{1}{2}c\sqrt{-g_{00}}(dt_1+dt_2) \tag{7.6}$$

Substituting (6.7) into (7.6), we get the desired result [(8)238; (9)257–258]

$$dl=(\gamma_{\alpha\beta}dx^\alpha dx^\beta)^{\frac{1}{2}}. \tag{7.7}$$

Here dl is determined by $\gamma_{\alpha\beta}$ and *not* by $g_{\alpha\beta}$. Moreover, even if dx^α is constant, dl varies with time if $\gamma_{\alpha\beta}$ varies with time. Thus a necessary and sufficient condition for a spatial structure to be *rigid*, is that in the whole space region occupied by the structure, $\gamma_{\alpha\beta}$ must not vary with the time [(36) Eq. 4.3, p. 161; (3)71]

$$\gamma_{\alpha\beta,0}=0, \tag{7.8}$$

even though $g_{\alpha\beta}$ may depend on time.

From (5.11) we have

$$g^{\alpha\mu}g_{\mu\beta}+g^{\alpha 0}g_{0\beta}=\delta^{\alpha}{}_{\beta}, \qquad g^{\alpha\beta}g_{\beta 0}+g^{\alpha 0}g_{00}=0.$$

Eliminating $g^{0\alpha}$ between these two equations, we get [(9)258]

$$g^{\alpha\mu}\gamma_{\mu\beta}=\delta^{\alpha}{}_{\beta}. \tag{7.9}$$

Thus if we define $\gamma^{\alpha\beta}$ by

$$\gamma^{\alpha\mu}\gamma_{\mu\beta}=\delta^{\alpha}{}_{\beta}, \tag{7.10}$$

we get

$$\gamma^{\alpha\beta}=g^{\alpha\beta}. \tag{7.11}$$

Moreover, it can be shown [(8)381–382] that

$$g=g_{00}\gamma,$$

where

$$g=\det(g_{ij}), \qquad \gamma=\det(\gamma_{\alpha\beta}). \tag{7.12}$$

Møller [(8)374] showed that dl^2 is invariant and $\gamma_{\alpha\beta}$ are the components of a three-tensor with respect to coordinate transformations of the type

$$x'^0=x'^0(x^i), \qquad x'^\alpha=x'^\alpha(x^\beta). \tag{7.13}$$

A special case of (7.13) is the transformation

$$x'^0=x'^0(x^i), \qquad x'^\alpha=x^\alpha. \tag{7.14}$$

As might be expected, (7.14) does not affect the space metric, i.e., $\gamma'_{\alpha\beta}=\gamma_{\alpha\beta}$, a fact which was proved by Møller [(8)374–375]. The class of all coordinate systems related to each other by (7.13) are said to belong to the same *reference system* (RS) [(8)236].

The transformation $x'^0=x'^0(x^i)$ is simply a change in the kind of clocks that are used, and $x'^\alpha=x'^\alpha(x^\beta)$ is nothing but a relabelling of the space coordinates of particles. In view of this, (7.13) does not involve a change in the physical situation, but only a different description of the same situation. For example, on a rotating disk, (7.13) may be a change from cylindrical to rectangular coordinates and from standard to coordinate clocks, whereas a general transformation

$$x'^i=x'^i(x^j), \tag{7.15}$$

involves a transformation to a different RS, such as an ICS, in which the space and time geometry may be completely different.

From the above and (7.7) it can be seen that dl^2 and $\gamma_{\alpha\beta}$ play the same role in space geometry that ds^2 and g_{ij} play in space–time geometry. Consequently, the names *space line element* for dl^2 and *space metric* for $\gamma_{\alpha\beta}$ are fully justified; and many of the equations in space–time geometry can be taken over to space geometry by simple substitutions, as is done in the following. However, dl^2 is *positive definite*, whereas ds^2 is *not*.

Since the quadratic form dl^2 given by (7.7) is positive definite, it follows that for $\alpha\neq\beta$ [(9)258]

$$\gamma_{\alpha\alpha}>0, \qquad \begin{vmatrix}\gamma_{\alpha\alpha} & \gamma_{\alpha\beta}\\ \gamma_{\beta\alpha} & \gamma_{\beta\beta}\end{vmatrix}>0, \qquad \gamma>0. \tag{7.16}$$

From (7.16) and (6.2, 5, 6) it follows that

$$g_{00}<0, \qquad \begin{vmatrix}g_{00} & g_{0\alpha}\\ g_{\alpha 0} & g_{\alpha\alpha}\end{vmatrix}<0, \qquad \begin{vmatrix}g_{00} & g_{0\alpha} & g_{0\beta}\\ g_{\alpha 0} & g_{\alpha\alpha} & g_{\alpha\beta}\\ g_{\beta 0} & g_{\beta\alpha} & g_{\beta\beta}\end{vmatrix}<0, \quad g<0, \tag{7.17}$$

$$g_{\alpha\alpha}>0, \qquad \begin{vmatrix}g_{\alpha\alpha} & g_{\alpha\beta}\\ g_{\beta\alpha} & g_{\beta\beta}\end{vmatrix}>0, \quad \det(g_{\alpha\beta})>0. \tag{7.18}$$

D. Space Geodesics

An SG was defined operationally in Sec. 3C. Since $\gamma_{\alpha\beta}$ plays the same role in *space* geometry that g_{ij} plays in *space time* geometry, we may obtain the equations of an SG from (5.8) by replacing g_{ij} by $\gamma_{\alpha\beta}$, and ds by dl. Thus if we define the "space" Christoffel symbols by

$$C^{\mu}{}_{\alpha\beta}=g^{\mu\delta}C_{\delta\cdot\alpha\beta}, \qquad 2C_{\mu\cdot\alpha\beta}=\gamma_{\mu\alpha,\beta}+\gamma_{\mu\beta,\alpha}-\gamma_{\alpha\beta,\mu}, \tag{7.19}$$

the equations of an SG may be written as

$$d^2x^\mu/dl^2+C^{\mu}{}_{\alpha\beta}(dx^\alpha/dl)(dx^\beta/dl)=0. \tag{7.20}$$

Another useful form of this equation is [(8)241]

$$\frac{d}{dl}\left(\gamma_{\mu\alpha}\frac{dx^\alpha}{dl}\right)\equiv\tfrac{1}{2}\gamma_{\alpha\beta,\mu}\frac{dx^\alpha}{dl}\frac{dx^\beta}{dl}. \tag{7.21}$$

E. Parallel Displacement

In Sec. 3E, the different methods of defining parallel displacement were discussed. It was pointed out there that up to the first order the different definitions are equivalent. The first-order equation they lead to, may be derived in a relatively simpler way as follows: Suppose that a vector A is parallel displaced by an amount dx^μ, and δA^μ is the change in the component A^μ due to the displacement. Intuitively we expect that δA^μ should be proportional to dx^μ. Moreover δA^μ should also depend linearly on A^μ, since the sum of two vectors should change the same way as each of its parts [(9)261]. Therefore, we may expect that $\delta A^\mu=K^{\mu}{}_{\alpha\beta}A^\alpha dx^\beta$, where $K^{\mu}{}_{\alpha\beta}$ is the coefficient of proportionality. It can be shown [(9)261–266] that $K^{\mu}{}_{\alpha\beta}=-C^{\mu}{}_{\alpha\beta}$, and the equation of parallel displacement becomes

$$\delta A^\mu=-C^{\mu}{}_{\alpha\beta}A^\alpha dx^\beta. \tag{7.22}$$

F. Euclidean Geometry

It was stated in Sec. 5D that the necessary and sufficient condition for space–time to be flat, i.e., to

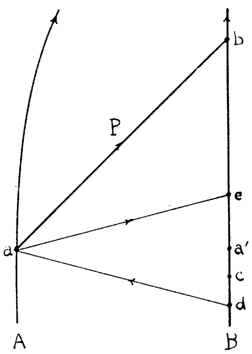

Fig. 6. Lag time and speed measurement.

be able to reduce (5.4) to (5.1) over a finite region, is (5.13). In complete analogy with this, we may conclude that the necessary and sufficient condition to reduce (7.7) over a finite region to the Euclidean form

$$dl^2=\delta_{\alpha\beta}dx^\alpha dx^\beta, \tag{7.23}$$

is that

$$2S_{\alpha\beta\delta\mu}=\gamma_{\alpha\delta,\beta\mu}+\gamma_{\beta\mu,\alpha\delta}-\gamma_{\alpha\mu,\beta\delta}-\gamma_{\beta\delta,\alpha\mu}$$
$$+2\gamma^{\kappa\lambda}(C_{\kappa\cdot\alpha\delta}C_{\lambda\cdot\beta\mu}-C_{\kappa\cdot\alpha\mu}C_{\lambda\cdot\beta\delta})=0. \tag{7.24}$$

This is the condition that determines whether the space geometry is Euclidean in any particular RS.

8. RELATIVITY WITHOUT SYNCHRONIZATION

In GR, a *single* time over all space is established by synchronizing all clocks with a central clock (See Sec. 2D). Let a free particle P depart from a particle A at event a and arrive at a particle B at event b. Let c be the event at B which is simultaneous with a (Fig. 6), i.e., c has the same time according to the clock at B that a has according to the clock at A, where the clocks at A and B are *continually* synchronized with a central clock. Since synchronization may *not* be transitive, A and B are not necessarily synchronous with each other. In other words, if A and B were synchronized with each other, the event a' which is now simultaneous with a may be different than c. Thus, the event at B which is simultaneous with a, not only depends upon the synchronization convention, but also upon which clock, B is synchronized with; which points out clearly the great arbitrariness of this event. Consequently, the time interval between c and b is arbitrary, and so is the velocity of P from A to B, defined in terms of this interval. In view of this arbitrariness, it seems highly desirable to find a less arbitrary way of describing the motion of a particle. Such a way is described in this section.

A. The Basic Relation between Proper Times

For convenience, let $\tau_A(ab)=\tau_A(a)-\tau_A(b)$, where $\tau_A(a)$ denotes the time of event a according to the SC, A. Referring to the same particles and events described above, let a photon depart from B at event d, be reflected by A at a, and return to B at event e (Fig. 6). We shall now derive a relation between the proper time intervals $\tau_P(ba)$, $\tau_B(ed)$, and $\tau_B(be)$, assuming the validity of GR.

Before we do this, the following important points should be noted: (1) According to Secs. 5B2, 6A, if ds is the line element, and c is the speed of light, then

$$c\tau_P(ba)=ds. \tag{8.1}$$

(2) If dl is the proper distance between A, B measured by an LLMI whose end particles coincide with A, B at the events a, a', and dx^α are the components of space coordinate differences between A, B, then according to (7.5, 7),

$$\tfrac{1}{2}c\tau_B(ed)=dl=(\gamma_{\alpha\beta}dx^\alpha dx^\beta)^{\frac{1}{2}}. \tag{8.2}$$

$\gamma_{\alpha\beta}$ is evaluated at event a', where the time of a' is given by Einstein's synchronization convention

$$\tau_B(a')=\tau_B(d)+\tfrac{1}{2}\tau_B(ed). \tag{8.3}$$

(3) $\tau_B(be)$ is the time that P arrives at B behind the photon that starts with P from A, and travels to the same particle B. Thus $\tau_B(be)$ will be called the *lag time* of P from A to B. Since the speed of a photon (not to be confused with phase or group velocity of electromagnetic waves) is independent of the energy (frequency) of the photon, the lag time of a photon is always zero.

From (5.4) and (8.1) we have

$$c^2\tau_P(ba)^2=-g_{00}(dx^0)^2-2g_{0\alpha}dx^\alpha dx^0-g_{\alpha\beta}dx^\alpha dx^\beta, \tag{8.4}$$

where according to (6.1),

$$c^{-1}\sqrt{-g_{00}}dx^0=\tau_B(bc)=\tau_B(be)+\tau_B(ec). \tag{8.5}$$

It thus follows from (8.4, 5) that

$$\tau_P(ba)^2=\tau_B(bc)^2-2\tau_B(bc)c^{-1}\gamma_\alpha dx^\alpha-c^{-2}g_{\alpha\beta}dx^\alpha dx^\beta. \tag{8.6}$$

Moreover, from (6.7, 1) and (8.2) we get

$$\tau_B(ec)=\tfrac{1}{2}\tau_B(ed)+c^{-1}\gamma_\alpha dx^\alpha. \tag{8.7}$$

Substituting (8.7) into (8.5), and (8.5) into (8.6), we find with the help of (6.6) the desired relation

$$\tau_P(ba)^2=\tau_B(be)^2+\tau_B(be)\tau_B(ed). \tag{8.8}$$

With the help of (8.2), we may also write (8.8) in the form

$$\tau_P(ba)^2=\tau_B(be)[\tau_B(be)+2c^{-1}dl_{AB}]. \tag{8.9}$$

B. Proper, Coordinate, and Lag Velocities

In GR, there are two different types of velocity: The *proper velocity*

$$u^\alpha=dx^\alpha/ds=c^{-1}dx^\alpha/\tau_P(ba), \tag{8.10a}$$

$$u=(\gamma_{\alpha\beta}u^\alpha u^\beta)^{\frac{1}{2}}=dl/\tau_P(ba), \tag{8.10b}$$

and the *coordinate velocity*

$$v^\alpha = dx^\alpha/\tau_B(bc) \quad \text{or} \quad v^\alpha = dx^\alpha/\tau_B(ba'), \qquad (8.11a)$$

$$v = (\gamma_{\alpha\beta} v^\alpha v^\beta)^{\frac{1}{2}} = dl/\tau_B(ba'). \qquad (8.11b)$$

As already pointed out, v^α is rather arbitrary, and it seems better to use instead the *lag velocity*

$$w^\alpha = dx^\alpha/\tau_B(be), \qquad w = (\gamma_{\alpha\beta} w^\alpha w^\beta)^{\frac{1}{2}} = dl/\tau_B(be), \qquad (8.12)$$

The measurement of w involves one e.m. signal from A to B, whereas the measurement of v involves in addition another signal from B to A for the purpose of synchronization. Since $\tau_B(be)$ is the lag time, and the lag time of a photon is zero, it follows that the lag velocity of a photon is *infinite*. Thus the maximum lag velocity of any interaction is infinite. In view of these features, it seems that the lag velocity is both operationally simpler, more natural, and does not involve any arbitrariness.

The relation between v and w may be obtained from (8.5, 7) as follows:

$$\tau_B(bc) = \tau_B(be) + \tfrac{1}{2}\tau_B(ed) + c^{-1}\gamma_\alpha dx^\alpha. \qquad (8.13)$$

Dividing this equation by dl, and making use of (8.11b, 12, 2), we get

$$v^{-1} = w^{-1} + c^{-1}(1 + \gamma_\alpha dx^\alpha/dl). \qquad (8.14)$$

C. Equations of Motion

We now derive the equations of motion without synchronization from the variational principle (5.7) and the basic relations (8.1, 8). In order to get a better understanding of the situation, we consider first finite differences as shown in Fig. 7. The actual trajectory is represented by a solid line, and the varied path by a dashed line.

If we introduce the definitions

$$d\tau = \tau_P(a_n a_{n-1}), \qquad \tau_n = \tau_{A_n}, \qquad \tau'_n = \tau_{A'_n}, \qquad \lambda = \tau_n(b_n), \qquad (8.15a)$$

$$d\lambda = \tau_n(a_n b_n), \qquad \eta = d\lambda/d\tau, \qquad \gamma_{,\bar{0}} = \partial\gamma/\partial\lambda, \qquad dl_n = dl, \qquad (8.15b)$$

and apply (8.9) to Fig. 7, we get

$$d\tau^2 = d\lambda^2 + 2c^{-1} d\lambda\, dl_{n-1}, \qquad d\tau'^2 = d\lambda'^2 + 2c^{-1} d\lambda' dl'_{n-1}. \qquad (8.16)$$

Moreover, if we let

$$\delta d\tau = d\tau' - d\tau, \qquad \delta d\lambda = d\lambda' - d\lambda, \qquad \delta dl = dl' - dl,$$

then it follows from (8.16, 15, 10b) that

$$\delta d\tau = (\eta + u)\,\delta d\lambda + c^{-1}\eta\,\delta dl_{n-1}. \qquad (8.17)$$

Referring to Fig. 7, it can be seen that

$$\delta d\lambda = \tau'_n(a'_n b'_n) - \tau_n(a_n b_n) \approx \tau_n(c_n d_n) - \tau_n(a_n b_n)$$
$$= \tau_n(c_n a_n) - \tau_n(d_n b_n) = \delta\tau_n(a_n) - \delta\tau_n(b_n). \qquad (8.18)$$

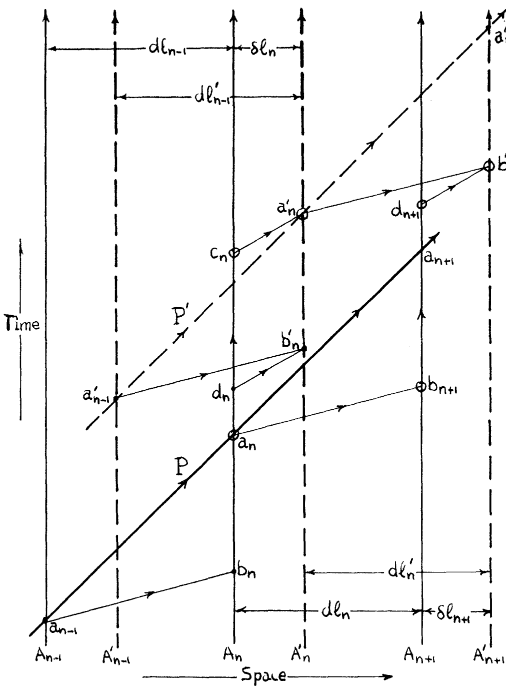

FIG. 7. Variation of trajectory.

The variations $\delta\tau_m(a_m)$ and $\delta\tau_n(b_n)$ are not independent. To find the relation between them, we notice that the times for the paths $(a_n c_n a'_n b'_{n+1})$ and $(a_n b_{n+1} d_{n+1} b'_{n+1})$, marked by circles in Fig. 7 are equal. Thus

$$\tau_n(c_n a_n) + c^{-1}\delta l_n + c^{-1} dl'_n = c^{-1} dl_n + \tau_{n+1}(d_{n+1} b_{n+1}) + c^{-1}\delta l_{n+1},$$

or

$$\delta\tau_n(a_n) = \delta\tau_{n+1}(b_{n+1}) + c^{-1}(\delta l_{n+1} - \delta l_n) - c^{-1}(dl'_n - dl_n)$$
$$= \delta\tau_{n+1}(b_{n+1}) + c^{-1}(d\delta l_n - \delta dl_n).$$

Substituting this value into (8.18), we find with the help of (8.15) that

$$\delta d\lambda = [\delta\tau_{n+1}(b_{n+1}) - \delta\tau_n(b_n)] + c^{-1}(d\delta l_n - \delta dl_n)$$
$$= d\delta\lambda + c^{-1}(d\delta l_n - \delta dl_n).$$

If we use this result in (8.17), and observe that

$$d\delta l_n - \delta dl = 0$$

is permissible by virtue of the fact that $\delta\lambda$ is independent of δx^α, we find

$$\delta d\tau = (\eta + u)\, d\delta\lambda + c^{-1}\eta\,\delta dl. \qquad (8.19)$$

Making use of (8.2), we have

$$\delta dl = \delta(\gamma_{\alpha\beta} dx^\alpha dx^\beta)^{\frac{1}{2}} = (\tfrac{1}{2}\delta\gamma_{\alpha\beta} dx^\beta + \gamma_{\alpha\beta}\delta dx^\beta)\, dx^\alpha/dl$$
$$= [\tfrac{1}{2}c(\gamma_{\alpha\beta,\mu}\delta x^\mu + \gamma_{\alpha\beta,\bar{0}}\delta\lambda)\, u^\beta d\tau + \gamma_{\alpha\beta} d\delta x^\beta] u^\alpha/u, \qquad (8.20)$$

since

$$\delta dx^\beta = \delta(x^\beta{}_{n+1} - x^\beta{}_n) = \delta x^\beta{}_{n+1} - \delta x^\beta{}_n = d\delta x^\beta{}_n$$

and (see Fig. 7)

$$\begin{aligned}\delta\gamma_{\alpha\beta} &= \gamma_{\alpha\beta}(b'_n) - \gamma_{\alpha\beta}(b_n) \\ &= [\gamma_{\alpha\beta}(b'_n) - \gamma_{\alpha\beta}(d_n)] \\ &= \gamma_{\alpha\beta,\mu}\delta x^\mu + \gamma_{\alpha\beta,\bar{o}}\delta\lambda + [\gamma_{\alpha\beta}(d_n) - \gamma_{\alpha\beta}(b_n)]. \qquad (8.20)\end{aligned}$$

With the help of all these results, we may finally write (8.19) in the form

$$\delta d\tau = [(\eta+u)d\delta\lambda - \tfrac{1}{2}u^{-1}\eta\gamma_{\alpha\beta,\bar{o}}u^\alpha u^\beta d\tau\delta\lambda] + (cu)^{-1}\eta\gamma_{\alpha\mu}u^\alpha d\delta x^\mu + \tfrac{1}{2}u^{-1}\eta\gamma_{\alpha\beta,\mu}\delta x^\mu.$$

If we substitute this expression into (5.7), we can deduce in the usual way the *equations of motion*

$$d(\eta+u)/ds - \tfrac{1}{2}u^{-1}\eta\gamma_{\alpha\beta,\bar{o}}u^\alpha u^\beta = 0, \qquad (8.21)$$

$$d(u^{-1}\eta\gamma_{\mu\alpha}u^\alpha)/ds - \tfrac{1}{2}u^{-1}\eta\gamma_{\alpha\beta,\mu}u^\beta u^\alpha = 0. \qquad (8.22)$$

If (8.21) is multiplied by η and (8.22) by u^μ and the two equations are added, it can be verified that the result is $d(\eta^2+2\eta u)/ds=0$. This equation also follows from the division of (8.16) by $d\tau$ and the use of (8.15b, 10b), i.e.,

$$\eta^2+2\eta u=1, \quad \text{or} \quad (\eta+u)^2=1+u^2. \qquad (8.23)$$

Thus the equations (8.21, 22) are not independent. Moreover, since they lead to (8.16), which is equivalent to (5.4), they are together equivalent to (5.8). However, (8.22) is not equivalent to (5.8) for $i=\alpha$, since the variation $\delta\lambda$ was used instead of δx^0. Notice that (8.22) involves only $\gamma_{\alpha\beta}$, whereas (5.8) for $i=\alpha$ involves all the components g_{ij}, [(8) Sec. 110, p. 290] which seems to be an important simplification.

D. Concluding Remarks

By working with proper times only, the Lorentz contraction and time dilation of SR cannot even be formulated, since they depend upon synchronization. Since synchronization is conventional and arbitrary, there is no loss of physical information. In fact it seems from this, that these effects have received altogether too much attention in the literature. However, the twin paradox is still meaningful, since it does not involve synchronization.

In order to complete the reformulation of GR without the use of synchronized time, it still remains to derive the field equations. This is an interesting problem, which the author has not had sufficient time yet to tackle.

9. SPACE GEODESIC COORDINATES

In a CS such as that described in Sec. 5A it *is* possible to (1) specify an event uniquely, (2) measure a unique set of values of g_{ij}, (3) calculate any field invariant such as the scalar curvature $R(=g^{ij}R_{ij})$, and (4) verify whether *all* of the components of R_{ijkl} vanish or not, i.e., whether the field can be obtained by transformation from an ICS. However, it is not possible to determine all the four CC associated with the CS, and thus it is not possible to separate the physical from coordinate contributions to g_{ij}. Another way of expressing this is, if two different values of g_{ij} are measured at different times, it is not possible to tell whether they represent two different physical situations, or the same physical situation in two different states of the CS.

Bergmann [37] and Komar [38] have tried to overcome this difficulty by specifying events in terms of four scalar invariants (observables) of the field, whose values do not depend upon the particular choice of the CS. In a field which has *no* symmetry, a set of four independent observables can be found; otherwise, such a set does not exist. This can be understood as follows: Consider a field in one spatial direction x. If the field varies along x, the position of an event can be specified by the value of the field at the place of occurrence of the event; on the other hand, if the field is uniform, this cannot be done. In SR where space is both isotropic and homogeneous, one would be at a complete loss if he wished to specify the spatial position of an event by observables of the gravitational field, for no such observables exist. In spite of this, it is possible to specify uniquely the position of an event by means of a CS, and physical information can be obtained from such a specification.

In this section we show how this method of specifying events by means of a CS, which is the usual procedure in physics, can be extended to the case of an arbitrary gravitational field. The CS in such a case cannot be rigid and must be continually adjusted, but it can be constructed just the same. Basically, we are going to set up a CS in which g_{ij} will reduce to known values, called the *zero-field values*, if the space–time region in which the CS is set up were inertial. Any deviation from these values can be attributed directly to the physical situation. For instance, it is shown in Sec. 10A that in rectangular coordinates the zero field values are η_{ij}.

Although the CS's that we construct can be used in any situation, it is instructive to distinguish three different situations: (1) If $R_{ijkl}=0$ holds throughout a CS, the values of g_{ij} with the help of (5.14) will determine a transformation to an ICS, and the CS can be specified uniquely by means of this transformation. (2) If $R_{ijkl}\neq 0$ for some values of the indices, and the field has no symmetry, a set of four independent observables exist, which can be used as the four coordinates. (3) If $R_{ijkl}\neq 0$ and the field has some symmetry, then it is necessary to use a specific CS such as described in the following.

A. Rectangular Coordinates

To construct rectangular coordinates, we start with an orthogonal triad of SG's, x^2, $x^3(0)$, $x^1(0, 0)$, at a particle A. The orthogonality conditions at A are

$$\gamma_{12}(x^0, 0, 0, 0)=0,$$
$$\gamma_{13}(x^0, 0, 0, 0)=0,$$
$$\gamma_{23}(x^0, 0, 0, 0)=0. \tag{9.1}$$

If we agree to let the value of x^2 give the *proper* distance from A, then

$$\gamma_{22}(x^0, 0, x^2, 0)=1. \tag{9.2}$$

The condition that x^2 is an SG may be obtained from (7.21) by setting $dx^1=dx^3=0$, and making use of (9.2). Thus

$$[\gamma_{12,2}-\tfrac{1}{2}\gamma_{22,1}]_{0,0}=0, \tag{9.3a}$$
$$[\gamma_{32,2}-\tfrac{1}{2}\gamma_{22,3}]_{0,0}=0, \tag{9.3b}$$

where "$[\ ]_{0,0}$" denotes "$[\ ]_{x^1=x^3=0}$."

Next, we parallel displace the unit vector A^μ ($\gamma_{\alpha\beta}A^\alpha A^\beta=1$) tangent to $x^3(0)$, parallel to itself along x^2 to get the $x^3(x^2)$ axes. If we assume that x^3 gives the proper distance from x^2, then

$$\gamma_{33}(x^0, 0, x^2, x^3)=1. \tag{9.4}$$

Consequently, $A^1=A^2=0$, $A^3=\gamma_{33}{}^{-\frac{1}{2}}=1$, $dx^1=dx^3=0$, and the condition (7.22) for parallel displacement becomes $C^\mu{}_{32}=0$. Making use of (7.19), this implies

$$[\gamma_{22,3}]_{0,0}=0, \tag{9.5a}$$
$$[\gamma_{12,3}+\gamma_{13,2}-\gamma_{23,1}]_{0,0}=0. \tag{9.5b}$$

Moreover, if we require the $x^3(x^2)$ axes to be SG's, we may set $dx^1=dx^2=0$, $dl=\sqrt{\gamma_{33}}dx^3=dx^3$ in (7.21) and get

$$[\gamma_{13,3}-\gamma_{33,1}]_0=0, \tag{9.6a}$$
$$[\gamma_{23,3}]_0=0, \tag{9.6b}$$

where "$[\ ]_0$" denotes "$[\ ]_{x^1=0}$."

Finally, we parallel displace $x^1(0, 0)$, first along x^2, and then along $x^3(x^2)$ to get $x^1(x^2, x^3)$ axes, assume that these axes are SG's, and let x^1 give the proper distance from $x^3(x^2)$. The last requirement means

$$\gamma_{11}=1, \tag{9.7}$$

and the assumption that $x^1(x^2, x^3)$ are SG's implies with the help of (7.21) that

$$\gamma_{12,1}=0, \qquad \gamma_{13,1}=0. \tag{9.8}$$

Making use of (7.22), the parallel displacement of $x^1(0, 0)$ along x^2 yields the conditions,

$$[\gamma_{22,1}]_{0,0}=0, \tag{9.9a}$$
$$[\gamma_{23,1}+\gamma_{13,2}-\gamma_{12,3}]_{0,0}=0, \tag{9.9b}$$

and the parallel displacement of $x^1(x^2, 0)$ along x^3 yields

$$[\gamma_{33,1}]_0=0, \tag{9.10a}$$
$$[\gamma_{32,1}+\gamma_{12,3}-\gamma_{13,2}]_0=0. \tag{9.10b}$$

In order to specify the x^2 curves uniquely, we assume that the angles they make with the x^1 axes do not vary with x^3 at $x^1=0$, i.e.,

$$[\gamma_{12,3}]_0=0. \tag{9.11}$$

We now deduce the CC pertaining to rectangular coordinates. From (9.3a, 9a) it follows that

$$[\gamma_{12,2}]_{0,0}=0. \tag{9.12}$$

It then follows from (9.12, 1) that $\gamma_{12}(x^0, 0, x^2, 0)=0$, which with (9.11) implies $\gamma_{12}(x^0, 0, x^2, x^3)=0$. In view of (9.8), this means that

$$\gamma_{12}=0. \tag{9.13}$$

Similarly, (9.5b, 9b) imply $[\gamma_{13,2}]_{0,0}=0$, and (9.6a, 10a) imply $[\gamma_{13,3}]_0=0$. From these results and (9.1, 8) we can conclude

$$\gamma_{13}=0. \tag{9.14}$$

Finally from (9.3b, 5a) we get $[\gamma_{23,2}]_{0,0}=0$. Combining this result with (9.1, 6b) we find

$$\gamma_{23}(x^0, 0, x^2, x^3)=0. \tag{9.15}$$

Moreover, (9.10b, 13, 14) imply

$$[\gamma_{23,1}]_0=0. \tag{9.16}$$

Summarizing, we have:

$$\gamma_{11}=1, \qquad \gamma_{12}=\gamma_{13}=0; \tag{9.17a}$$
$$\begin{gathered}\gamma_{22}(x^0, 0, x^2, 0)=1,\\ [\gamma_{22,1}]_{x^1=x^3=0}=0,\\ [\gamma_{22,3}]_{x^1=x^3=0}=0;\end{gathered} \tag{9.17b}$$
$$\gamma_{33}(x^0, 0, x^2, x^3)=1, \qquad [\gamma_{33,1}]_{x^1=0}=0; \tag{9.17c}$$
$$\gamma_{23}(x^0, 0, x^2, x^3)=0, \qquad [\gamma_{23,1}]_{x^1=0}=0. \tag{9.17d}$$

According to (9.17a),

$$\gamma=\det(\gamma_{\alpha\beta})=\gamma_{22}\gamma_{33}-\gamma_{23}{}^2. \tag{9.18}$$

In addition, we find with the help of (7.10),

$$\gamma^{11}=1, \qquad \gamma^{12}=\gamma^{13}=0, \tag{9.19a}$$
$$\gamma\gamma^{22}=\gamma_{22}, \qquad \gamma\gamma^{33}=\gamma_{33}, \qquad \gamma\gamma^{23}=-\gamma_{23}. \tag{9.19b}$$

The order in which a rectangular CS is constructed is important, and the space coordinates assigned to an event will in general be different if, say, x^2 is parallel displaced along x^3 instead of the reverse. Moreover, in general the curves $x^3(x^2)=$ const are not geodesics, nor are they parallel to x^2. In addition, the distance along these curves (determined by γ_{22}) may vary with time. Similar statements can be made about the curves $x^1(x^2, x^3) =$ const. Thus our CS is not rigid, but it leads to the specific CC (9.17).

B. Cylindrical Coordinates

We start with an SG called the $z(0, 0)$ axis and construct at some point A on it, SG's called the $r(\theta)$ axes, orthogonal to it at A. Then we parallel displace $z(0, 0)$ along $r(\theta)$ to get the $z(r, \theta)$ axes. If we let z and r give the proper distance, then

$$\gamma_{zz}=1, \qquad \gamma_{rr}(x^0, r, \theta, 0)=1. \tag{9.20}$$

The condition that $z(r, \theta)$ are SG's implies

$$\gamma_{rz,z}=0, \qquad \gamma_{z\theta,z}=0, \tag{9.21}$$

and that $r(\theta)$ are SG's implies

$$[\gamma_{r\theta,r}]_0=0, \tag{9.22}$$

$$[\gamma_{rz,r}-\gamma_{rr,z}]_0=0, \tag{9.23}$$

where "$[\ ]_0$" denotes "$[\ ]_{z=0}$."

Moreover, from the parallel displacement of $z(0, 0)$ along $r(\theta)$, we find

$$[\gamma_{rr,z}]_0=0, \tag{9.24}$$

$$[\gamma_{z\theta,r}+\gamma_{r\theta,z}-\gamma_{rz,\theta}]_0=0. \tag{9.25}$$

The θ coordinate can be partly specified by requiring that the θ direction be orthogonal to z on $r(\theta)$,

$$\gamma_{\theta z}(x^0, r, \theta, 0)=0. \tag{9.26}$$

Further specification is achieved by requiring that the r directions at A be continuously aligned with equally spaced radial directions on an LICS. This implies that at A, g_{ij} must equal to the corresponding quantities in the ICS [(8) p. 274, Eq. 57]. Thus, we assume [(39) 181]

$$\gamma_{rz}(x^0, 0, \theta, 0)=0, \qquad \gamma_{r\theta}(x^0, 0, \theta, 0)=0, \tag{9.27a}$$

$$\lim_{r\to 0} r^{-2}[\gamma_{\theta\theta}/\gamma_{rr}]_{z=0}=1, \qquad \lim_{r\to 0} r^{-1}[\gamma_{\theta\theta,r}/\gamma_{rr}]_{z=0}=2. \tag{9.27b}$$

From (9.26, 21) we conclude $\gamma_{\theta z}=0$, and from (9.23, 24) we obtain $[\gamma_{rz,r}]_0=0$. Thus, in view of (9.27a, 21), $\gamma_{rz}=0$. Similarly, (9.22, 27a) imply $\gamma_{r\theta}(x^0, r, \theta, 0)=0$.

Summarizing, we have

$$\gamma_{zz}=1, \qquad \gamma_{rz}=0, \qquad \gamma_{\theta z}=0; \tag{9.28a}$$

$$\gamma_{rr}(x^0, r, \theta, 0)=1, \qquad [\gamma_{rr,z}]_{z=0}=0; \tag{9.28b}$$

$$\gamma_{r\theta}(x^0, r, \theta, 0)=0, \qquad [\gamma_{r\theta,z}]_{z=0}=0, \tag{9.28c}$$

and (9.27b). From this we see that for $z\neq 0$, the r and θ directions are not necessarily orthogonal, and the distance along r may vary with time.

Making use of (9.28a) and (7.10), we find

$$\gamma= \det(\gamma_{\alpha\beta})=\gamma_{rr}\gamma_{\theta\theta}-\gamma_{r\theta}^2, \tag{9.29}$$

$$\gamma^{zz}=1, \qquad \gamma^{zr}=\gamma^{z\theta}=0, \tag{9.30a}$$

$$\gamma\gamma^{rr}=\gamma_{\theta\theta}, \qquad \gamma\gamma^{\theta\theta}=\gamma_{rr}, \qquad \gamma\gamma^{r\theta}=-\gamma_{r\theta}. \tag{9.30b}$$

C. Spherical Coordinates

Let many SG's, called the $r(\theta, \phi)$ axes, originate from the same particle A. The directions of these SG's at A may be controlled by continually aligning them with radial directions in an LICS. As above, this implies

$$\gamma_{r\theta}(x^0, 0, \theta, \phi)=0, \qquad \gamma_{r\phi}(x^0, 0, \theta, \phi)=0. \tag{9.31a}$$

$$\lim_{r\to 0} r^{-2}\gamma_{\theta\theta}=\gamma_{rr},$$

$$\lim_{r\to 0} r^{-2}\gamma_{\phi\phi}=\gamma_{rr}\sin^2\theta,$$

$$\lim_{r\to 0} r^{-2}\gamma_{\theta\phi}=0, \tag{9.31b}$$

$$\lim_{r\to 0} r^{-1}\gamma_{\theta\theta,r}=2\gamma_{rr},$$

$$\lim_{r\to 0} r^{-1}\gamma_{\phi\phi,r}=2\gamma_{rr}\sin^2\theta,$$

$$\lim_{r\to 0} r^{-1}\gamma_{\theta\phi,r}=0. \tag{9.31c}$$

From the condition that $r(\theta, \phi)$ are geodesics, it follows that

$$\gamma_{r\theta,r}=0, \qquad \gamma_{r\phi,r}=0, \tag{9.32}$$

We thus conclude from (9.31, 32) that

$$\gamma_{r\theta}=\gamma_{r\phi}=0. \tag{9.33}$$

If we wish, we may agree to let r give the proper distance from A and write

$$\gamma_{rr}=1. \tag{9.34}$$

Then (9.33, 34, 31b, c) will constitute the three CC.

D. Radar Method

So far we have shown how to specify events within the space-time region of a CS. In many cases, this method has to be supplemented by a radar type of detection. For this method it is necessary to specify the directions of emission and reception [(6) 118–119, 123–124] and the round-trip time of a photon. But in order to accomplish this, it is necessary to have a "photon gun," as Synge puts it, whose direction must be referred to a CS such as we have been describing. One difficulty with this method is that the path of the photon may not coincide with an SG during its round trip. Thus to get full information about the reflecting object, it is necessary to know something about the environment through which the photon travels. If this knowledge is not available, then one has to be satisfied with some ambiguity of information from this method.

E. Coordinate Conditions

Choice of the space coordinates, as was done above, determines three γ's and some boundary conditions on the other γ's, and thus provides three CC. The fourth CC may be determined by specifying the clock rates and synchronization convention. Mathematically this may be accomplished by the transformation

$$x^\alpha = x'^\alpha, \qquad x^0 = f(x'^\alpha, x'^0), \tag{9.35}$$

since such a transformation does not affect the values of $\gamma_{\alpha\beta}$[(8)374].

Substituting (9.35) into (5.4) and making use of the invariance of ds^2, we get

$$g'_{00} = g_{00}(f_{,0})^2, \qquad g'_{0\alpha} = (g_{0\alpha} + g_{00} f_{,\alpha}) f_{,0}, \tag{9.36a}$$

$$g'_{\alpha\beta} = g_{\alpha\beta} + 2g_{0\alpha} f_{,\beta} + g_{00} f_{,\alpha} f_{,\beta}. \tag{9.36b}$$

It is always possible to use SC's throughout; in which case, we must set $g'_{00} = -1$. On the other hand, it may be more convenient to set $g'_{0\alpha} = 0$ for one of the values of α.

The values of $\tau_{\alpha\beta}$ limit the choice of the function f. For instance, if $\tau_{\alpha\beta} = 0$, then it is possible by the synchronization convention, to set $g'_{01} = g'_{02} = g'_{03} = 0$ by taking $f_{,\alpha} = -g_{0\alpha}/g_{00}$. This is not permissible, if for some values of α and β, $\tau_{\alpha\beta} \neq 0$.

Another restriction on the choice of f is exerted by the value of the acceleration function [see Eq. (5.8)]

$$A^\alpha = -\Gamma^\alpha{}_{ij} u^i u^j. \tag{9.37}$$

For a particle at rest, $u^\alpha = 0$, and

$$A^\alpha = -\Gamma^\alpha{}_{00}(u^0)^2 = -g^{\alpha i}(g_{0i,0} - \tfrac{1}{2} g_{00,i})(u^0)^2. \tag{9.38}$$

Thus, if $A^\alpha \neq 0$, (9.38) sets a limitation on the possible values of g_{00} and $g_{0\alpha}$. If $g_{00} = -1$, then $g_{0\alpha,0}$, and hence $g_{0\alpha}$ cannot be zero for all values of α. On the other hand, if $g_{0\alpha} = 0$ for all α, then at least for some α, $g_{00,\alpha} \neq 0$. The fourth CC is not simply one condition on the g_{ij}, but rather a transformation (9.35), which may determine several g's at once.

In view of (9.35), the above discussion is limited to transformations within the *same* RS (see Sec. 7C). If an arbitrary transformation (7.15) to another RS is allowed, then one can always find a RS in which $g_{i0} = -\delta_{i0}$ [(8)296–298]. Such a RS consists of freely falling particles, but is in general not rigid. However, once the RS is fixed, then in general it is no longer possible to make $g_{i0} = -\delta_{i0}$. In other words, the question of transitivity of synchronization is only relevant to a particular RS (*not* CS).

10. ZERO FIELD

A zero field $g^{(0)}{}_{ij}$ is the field in any ICS. For example, in an ICS in which rectangular coordinates, standard clocks, and the Einstein synchronization convention [$\epsilon = \frac{1}{2}$ in Eq. (2.1)] are used, $g^{(0)}{}_{ij} = \eta_{ij}$, where η_{ij} is defined by (5.2). A zero field is characterized by the fact that it is the *most uniform* field possible [(7)1]. We now explain what this statement means.

A transformation,

$$x'^i = x^i + \xi^i(x), \tag{10.1}$$

is said to be a *motion* if at any point P, $g'_{ij}(P) = g_{ij}(P)$, where g'_{ij} is calculated by means of (5.6), and ξ^i is infinitesimal. The necessary and sufficient condition for (10.1) to be a motion is that *Killing's equation*,

$$\xi_{i;j} + \xi_{j;i} = 0, \tag{10.2}$$

is satisfied, where $\xi_{i;j}$ is the covariant derivative of $\xi_i = g_{ij}\xi^j$[(20)233–234; (1c)35–38]. The larger the number of arbitrary parameters in ξ^i the more motions are possible, i.e., the more symmetric or uniform the field is. It can be shown [(20) Sec. 27] that in a space of n dimensions, the maximum possible number of parameters in ξ^i is $\frac{1}{2}n(n+1)$. In our case [$n=4$, $\frac{1}{2}n(n+1) = 10$], the most uniform field is the one that permits motions with 10 parameters.

By making use of $g^{(0)}{}_{ij} = \eta_{ij}$ in (10.2), it is found [(1c)38] that the most general solution of (10.2) is

$$\xi_i = a_{ij}x^j + b_i, \qquad a_{ij} = -a_{ji}. \tag{10.3}$$

This generates the infinitesimal inhomogeneous Lorentz transformations, and does contain 10 arbitrary parameters. In effect (10.3) states that for a zero field, space is homogeneous, isotropic, and all ICS's moving with constant velocity relative to each other are equivalent [(7)1]. From this we see that a zero field is indeed the most uniform field possible in four dimensions. Moreover, it is possible to check whether a field g_{ij} is a zero field or not by making use of g_{ij} in (10.2), and verifying whether the most general solution of (10.2) contains 10 arbitrary parameters or not.

If in any particular CS, g_{ij} and $g^{(0)}{}_{ij}$ are known, then $g^{(P)}{}_{ij} = g_{ij} - g^{(0)}{}_{ij}$ is the contribution to the field from either acceleration of the CS relative to an ICS, or the presence of matter or both. If $R_{ijkl} = 0$, we know that $g^{(P)}{}_{ij}$ is due to acceleration of the CS. However, if not all of the components of R_{ijkl} are zero, then at least part of the field is due to the presence of matter. To separate this part from the rest, the asymptotic behavior of $g^{(P)}{}_{ij}$ should be examined, as discussed at the end of Sec. 5D.

The importance of knowledge of $g^{(0)}{}_{ij}$ in any given CS is clear from this. A unique solution for $g^{(0)}{}_{ij}$ may be obtained with the help of the following four necessary and sufficient conditions for a *reference* system to be inertial:

(i) Space–time is flat (5.13): $R_{ijkl} = 0.$ (10.4)

(ii) Space geometry is Euclidean (7.24): $S_{\alpha\beta\gamma\delta} = 0.$ (10.5)

(iii) Synchronization is transitive (6.10): $\tau_{\alpha\beta} = 0.$ (10.6)

(iv) Acceleration function is zero: $A^\alpha = 0.$ (10.7)

To Synge [(7)IX], the most important of these conditions is (i). However, (i) can be satisfied without any of the other three conditions being satisfied, as will be seen in Sec. 12, which constitutes a considerable deviation from the space–time geometry in an ICS.

It was shown in Sec. 9 that the specification of the CS determines three $\gamma_{\alpha\beta}$'s, and specification of the coordinate clocks determine g_{00}. In particular, one may use SC's throughout and get

$$g_{00}=-1. \tag{10.8}$$

Since this leaves only *six* of the metric coefficients to be determined by the conditions (i)–(iv), it is clear that these conditions must be highly interdependent.

It will be seen in the following that (ii)–(iv) imply (i). Moreover, out of the six equations contained in (iii) and (iv), only three are independent; and according to Sec. 9E, they make it possible to choose the synchronization convention so that

$$g_{0\alpha}=0. \tag{10.9}$$

The remaining three $\gamma_{\alpha\beta}$'s are then determined by three out of the six equations in condition (ii), as will now be demonstrated for the CS's discussed in Sec. 9.

A. Rectangular Coordinates

The decision to use rectangular coordinates amounts to the adoption of the CC (9.17). Since these conditions already specify three of the γ's, (10.5) may be used to determine the remaining three γ's. Making use of (7.24, 19), we get

$$2S_{1212}=\gamma_{22,11}-\tfrac{1}{2}[\gamma_{22,1}(\gamma^{22}\gamma_{22,1}+2\gamma^{23}\gamma_{23,1})+\gamma^{33}\gamma_{23,1}{}^{2}]=0,$$

$$2S_{1313}=\gamma_{33,11}-\tfrac{1}{2}[\gamma_{33,1}(\gamma^{33}\gamma_{33,1}+2\gamma^{23}\gamma_{23,1})+\gamma^{22}\gamma_{23,1}{}^{2}]=0,$$

$$2S_{1213}=\gamma_{23,11}-\tfrac{1}{2}[\gamma_{23,1}(\gamma^{22}\gamma_{22,1}+\gamma^{23}\gamma_{23,1}+\gamma^{33}\gamma_{33,1}) +\gamma^{23}\gamma_{22,1}\gamma_{33,1}]=0.$$

Differentiating (9.18) once, we find

$$\gamma_{,1}=\gamma_{33}\gamma_{22,1}+\gamma_{22}\gamma_{33,1}-2\gamma_{23}\gamma_{23,1}.$$

With the help of this equation, (9.19), and the definition

$$K=\gamma_{22,1}\gamma_{33,1}-\gamma_{23,1}{}^{2}, \tag{10.10}$$

we may rewrite the preceding three equations in the form

$$2\gamma^{\frac{3}{2}}(\gamma^{-\frac{1}{2}}\gamma_{\alpha\beta,1})_{,1}+K\gamma_{\alpha\beta}=0, \qquad \alpha,\beta=2,3. \tag{10.11}$$

With the help of (10.11), the equation "$(\gamma^{-\frac{1}{2}}K)_{,1}=0$" may be derived, whose solution is $\gamma^{-\frac{1}{2}}K=f(x^0,x^2,x^3)$. Making use of (9.17b–d), it can be seen that at $x^1=0$, $f=0$. Thus, $K=0$ and $(\gamma^{-\frac{1}{2}}\gamma_{\alpha\beta,1})_{,1}=0$. Again, use of (9.17) leads to $\gamma_{\alpha\beta,1}=0$, and finally to $\gamma_{22}=\gamma_{33}=1$, $\gamma_{23}=0$. Therefore, it follows from (9.17) and (10.5) that $\gamma_{\alpha\beta}=g_{\alpha\beta}=\delta_{\alpha\beta}$. These results and (10.8, 9) may be summarized in the statement that the zero-field for the rectangular coordinates defined in Sec. 9A is given by

$$g^{(0)}{}_{ij}=\eta_{ij}. \tag{10.12}$$

B. Cylindrical Coordinates

Cylindrical coordinates may be treated in close analogy with rectangular coordinates by giving z the role of x^1. Using the CC (9.28), we find as above that $S_{z\alpha z\beta}=0$ imply $K_c=\gamma_{rr,z}\gamma_{\theta\theta,z}-\gamma_{r\theta,z}{}^2=0$, and $\gamma_{rr}=1$, $\gamma_{r\theta}=0$. Moreover, we deduce from $S_{r\theta r\theta}=0$, and (9.27) that $\gamma_{\theta\theta}=r^2$. Summarizing, we have

$$\gamma^{(0)}{}_{zz}=\gamma^{(0)}{}_{rr}=r^{-2}\gamma^{(0)}{}_{\theta\theta}=1, \quad \gamma^{(0)}{}_{zr}=0, \quad \gamma^{(0)}{}_{z\theta}=\gamma^{(0)}{}_{r\theta}=0. \tag{10.13}$$

C. Spherical Coordinates

Here again, by using the CC (9.33, 34), and letting r take the role of x^1, we deduce from $S_{r\alpha r\beta}=0$ and (9.31b, c) that $K_s=\gamma_{\theta\theta,r}\gamma_{\phi\phi,r}-\gamma_{\theta\phi,r}{}^2=4\sqrt{\gamma}\sin\theta$, where $\gamma=\gamma_{\theta\theta}\gamma_{\phi\phi}-\gamma_{\theta\phi}{}^2$. In addition, we have in analogy with (5.7),

$$(\gamma^{-\frac{1}{2}}\gamma_{\alpha\beta,r})_{,r}+2\sin\theta\,\gamma^{-1}\gamma_{\alpha\beta}=0, \qquad \alpha,\beta=\theta,\phi. \tag{10.14}$$

Differentiating γ with respect to r and making use of (10.14), we find $(\gamma^{-\frac{1}{2}}\gamma_{,r})_{,r}=4\sin\theta$, whose solution is

$$\sqrt{\gamma}=r^2\sin\theta. \tag{10.15}$$

If we substitute (10.15) into (10.14), and use (9.31b, c), we get finally

$$\gamma^{(0)}{}_{\theta\theta}=r^2, \qquad \gamma^{(0)}{}_{\phi\phi}=r^2\sin^2\theta, \qquad \gamma^{(0)}{}_{\theta\phi}=0. \tag{10.16}$$

This, in conjunction with

$$\gamma^{(0)}{}_{rr}=1, \qquad \gamma^{(0)}{}_{r\theta}=0, \qquad \gamma^{(0)}{}_{r\phi}=0, \tag{10.17}$$

obtained from (9.33, 34), completes the solution.

11. LINEAR FIELD

The linear acceleration field has been discussed by many authors from the point of view of transformation from an ICS, e.g., [(8)118–123, 253–258]. In the following, the same problem is solved by starting with the "rectangular" coordinates defined in (9.17), and the field equations, and then imposing *physical conditions* (PC) that limit the field to the one of interest. In this way, we obtain an illustration of the ideas of Sec. 9 and a useful supplement to the usual treatment of the problem.

CC 1–3. (9.17).

PC 1. The *space geometry is Euclidean*: $S_{\alpha\beta\gamma\delta}=0$.

As was shown in Sec. 10B, this implies

$$\gamma_{\alpha\beta}=\delta_{\alpha\beta}, \qquad \gamma^{\alpha\beta}=g^{\alpha\beta}=\delta^{\alpha\beta}, \qquad g_{00}g^{0\alpha}=-g_{0\alpha}. \tag{11.1}$$

PC 2. The *acceleration function has zero components only along x^2 and x^3*:

$$A^1 \neq 0, \qquad A^2 = A^3 = 0, \tag{11.2}$$

where A^α is defined in (5.8). Since (11.2) holds for any value of the velocity components u^α, it follows with the help of (11.1) that

$$g_{0\alpha}\Gamma_{0,ij} = g_{00}\Gamma_{\alpha,ij} \quad \text{for} \quad \alpha = 2, 3. \tag{11.3}$$

Making use of the definitions (6.5, 9), we may deduce from (11.3) for $i=0$, and $j=0$; 2, 3; 1 the following results, respectively:

$$\tau_{02} = \tau_{03} = 0; \tag{11.4}$$

$$\tau_{23} = 0; \tag{11.5}$$

$$\gamma_0\tau_{12} = \gamma_2\tau_{10}, \qquad \gamma_0\tau_{13} = \gamma_3\tau_{10}. \tag{11.6}$$

PC 3. *Synchronization is transitive in the* x^1x^2 and x^1x^3 planes:

$$\tau_{12} = \tau_{13} = 0. \tag{11.7}$$

Making use of (11.4–6), we may write

$$A^1 = \gamma_0^{-1}\tau_{01}\gamma_i\gamma_j u^i u^j = \gamma_0^{-1}\tau_{01}(1 + \delta_{\alpha\beta}u^\alpha u^\beta). \tag{11.8}$$

Since, according to (11.2), $A^1 \neq 0$, then $\tau_{01} \neq 0$, and it follows from (11.7, 8) that

$$\gamma_2 = \gamma_3 = 0, \quad \text{or} \quad g_{02} = g_{03} = 0, \tag{11.9}$$

because $g_{00} = 0$. Consequently, (11.4) implies $g_{00,2} = g_{00,3} = 0$, (11.7) implies $g_{01,2} = g_{01,3} = 0$, and thus

$$g_{ij,2} = g_{ij,3} = 0. \tag{11.10}$$

Because $A^1 \neq 0$, (11.9) is dictated rather than allowed by (11.7). Moreover, since

$$-\tau_{01} = (\sqrt{-g_{00}})_{,1} + \gamma_{1,0} \tag{11.11}$$

we cannot set both $g_{01} = 0$ and $g_{00} = -1$. However, because of (11.7), we may take

CC 4. $$g_{01} = 0, \tag{11.12}$$

but *not* $g_{00,0} = 0$, if we are to allow A^1 to vary with x^0 in case of nonuniform acceleration.

In view of these results, we get from (5.10)

$$2g_{00}\Gamma^0{}_{00} = g_{00,0}, \qquad 2g_{00}\Gamma^0{}_{01} = g_{00,1}, \qquad 2\Gamma^1{}_{00} = -g_{00,1}, \tag{11.13}$$

and all other Γ's vanish. We then find that the field equations (5.12) are identically satisfied for $i, j = 2, 3$; and for $i, j = 0, 1$, they yield the single equation

$$2g_{00}g_{00,11} - (g_{00,1})^2 = 0.$$

This equation is equivalent to $\{\ln[\sqrt{-g_{00}})_{,1}]\}_{,1}$, and thus its solution is

$$\sqrt{-g_{00}} = 1 + a(x^0)x^1; \tag{11.14}$$

since for zero field we must have $\sqrt{-g_{00}} = 1$.

By examining the nonrelativistic form of the equations of motion (5.8), $c^2a(x^0)$ may be identified with the nonrelativistic acceleration of the CS relative to an ICS. As $x^1 \to -a^{-1}$, $g_{00} \to 0$, ds^2 becomes negative, and no time-like intervals or signals are possible. This imposes a restriction on the spatial extension of the CS, and is a consequence of the fact that the field is an acceleration field.

From (11.1, 9, 12, 14) it follows that

$$ds^2 = (1 + ax^1)^2(dx^0)^2 - \delta_{\alpha\beta}dx^\alpha dx^\beta \tag{11.15}$$

which is in exact agreement with the line element obtained by Møller [(8) p. 255, Eq. 137] from a different approach.

If we wish to use SC's instead of the coordinate clocks having the rate (11.14), we will have to make the transformation

$$x'^0 = \int \sqrt{-g_{00}}\, dx^0, \qquad x'^\alpha = x^\alpha. \tag{11.16}$$

For the case $a =$ const,

$$x'^0 = \int (1 + ax^1)\, dx^0 = \sqrt{-g_{00}}x^0.$$

Thus, $dx^\alpha = dx'^\alpha$, $\sqrt{-g_{00}}dx^0 = dx'^0 - (1 + ax'^1)^{-1}ax'^0dx'^1$. Substituting these values into (11.15), and making use of the invariance of ds^2, we find

$$g'_{00} = -1, \qquad g'_{01} = (1 + ax'^1)^{-1}ax'^0, \qquad g'_{02} = g'_{03} = 0, \tag{11.17a}$$

$$g'_{11} = 1 - g'^2_{01}, \qquad g'_{22} = g'_{33} = 1, \qquad g'_{12} = g'_{13} = g'_{23} = 0. \tag{11.17b}$$

As was stated after (7.9), this transformation does not affect the space geometry, i.e., $\gamma'_{\alpha\beta} = \gamma_{\alpha\beta}$.

It is interesting to note that all the components R_{ijkl} are zero except R_{0101}, and that $R_{0101} = 0$ is equivalent to $R_{00} = 0$. Thus, the conditions that have been imposed on the field already insure that $R_{ijkl} = 0$, i.e., that the field can be produced by motion of the CS relative to an ICS, I. The transformation between x^i and the coordinates X^i of I can be obtained with the help of (5.14), supplemented by the condition that a particle at rest in I, will be falling freely along x^i, i.e.,

$$dX^1/dx^0 = X^1{}_{,0} + \beta^1 X^1{}_{,1} = 0, \tag{11.18}$$

where the velocity $\beta^1 = dx^1/dx^0$ has to be obtained from the equations of motion. It can be shown that for $a =$ const, the transformation obtained in this way is in complete agreement with the transformation obtained by Møller [(8) p. 256, Eq. (140)] in a different way.

The equations of motion are solved by Møller [(8) 257], and the solution need not be repeated here. However, the following two observations can be made, which are not usually mentioned in the literature: (1) The motion in the x^2 and x^3 directions is coupled to the x^1 motion through the factor $(-g_{ij}\beta^i\beta^j)^{-\frac{1}{2}}$, in contrast to the nonrelativistic motion which is not coupled. (2) The motion of a particle in a constant gravitational field obtained here, is different than the motion under the influence of a constant nongravitational force in an ICS [(8)75], such as the motion of a charge in a constant electric field.

The earth's gravitational field may be considered to be a uniform linear field in a sufficiently small region of space. The gravitational frequency shift was recently measured in such a region [33], and the results confirmed the theory. We now use (6.14) to calculate this shift in two ways, the first and usual way is by means of (11.14), and the second way is by means of (11. 17). Since the solution of (11.15) is stationary, we may use (6.15) and (11.14), and get for the ratio of proper frequencies at two points separated by a vertical distance Δx, the expression

$$\nu_1/\nu_2=(g^{(2)}{}_{00}/g^{(1)}{}_{00})^{\frac{1}{2}}$$
$$=[1+a(x+\Delta x)]/(1+ax)\approx 1+a\Delta x, \quad (11.19)$$

where we let $x^1=x$. On the other hand, since (11.17) is not stationary, we have to use (6.14, 13). Noting that

$$\gamma'_{\alpha\beta,0}=\delta_{\alpha\beta,0}=0, \qquad \gamma'_{\alpha,0}=\delta_{\alpha 1}g'_{01,0}=a(1+ax')^{-1}\delta_{\alpha 1},$$

we get

$$\nu_1/\nu_2-1=\int d\epsilon=\int(1+ax')^{-1}\,d(ax')$$
$$=[\ln\,(1+ax')]_x^{x+\Delta x}\approx a\Delta x,$$

in agreement with (11.19).

12. ROTATION FIELD

In the previous section we discussed a transformation field in which the space geometry is *Euclidean*, synchronization is *transitive*, and a rigid CS can be constructed regardless whether the acceleration is constant or not. In this section, we discuss a transformation field in which none of these properties hold. This field can be produced by the rotation of a CS about an axis fixed relative to an ICS. The rotating CS can be rigid only in the special case where the angular velocity is *constant*. Thus by studying this field, we can get a feeling for all the relativistic pecularities arising in an arbitrary gravitational field, except for the property $R_{ijkl}\neq 0$. A field with the latter property is taken up in Sec. 13.

Our approach is again to start with the field equations and find a unique solution with the help of coordinate and physical conditions. The approach by transformation from an ICS is adequately treated elsewhere [(8)123–125, 222–226, 240–245].

A. Metric Tensor

CC 1–3. Cylindrical coordinates: (9.28, 27b).

PC 1. No acceleration along z: $A^z=0$. (12.1)

As in (11.3), this implies $g_{0z}\Gamma_{0,ij}=g_{00}\Gamma_{z,ij}$ and hence

$$\tau_{0z}=0, \quad (12.2)$$

$$\gamma_z\tau_{0\alpha}=\gamma_0\tau_{z\alpha}, \quad (12.3)$$

where γ_i and τ_{ij} are defined in (6.5, 9).

PC 2. Synchronization is transitive in the zr plane:

$$\tau_{zr}=0. \quad (12.4)$$

Here again, A^r is proportional to τ_{0r}, and if $A^r\neq 0$, then $\tau_{0r}\neq 0$, and we conclude from (12.3, 4) and (9.28a) that

$$\gamma_z=0, \qquad g_{zi}=\delta_{zi}, \qquad \tau_{z\theta}=0. \quad (12.5)$$

Therefore, (12.2, 4, 5) implies

$$g_{0i,z}=0.$$

PC 3. $\gamma_{rr,z}=0$, $\gamma_{r\theta,z}=0$, $\gamma_{\theta\theta,z}=0$.

This, in conjunction with (9.28b, c), implies

$$\gamma_{rr}=1, \qquad \gamma_{r\theta}=0, \quad (12.6)$$

and

$$g_{ij,z}=0. \quad (12.7)$$

PC 4. Cylindrical symmetry:

$$\gamma_{\theta\theta,\theta}=0, \qquad \tau_{r\theta,\theta}=0, \qquad A^\alpha{}_{,\theta}=0$$

when

$$u^\beta=0.$$

Consequently,

$$g_{ij,\theta}=0. \quad (12.8)$$

Because of (12.4), we may adopt

CC 4. $\qquad g_{0r}=0.$ (12.9)

However, we cannot set in addition $g_{0\theta}=0$, since $\tau_{r\theta}\neq 0$.

Instead of continually lining up the radial directions with the help of a local ICS at the origin, as indicated in (9.27), property (12.8) allows us to accomplish the same thing much easier as follows: Construct a rigid circular ring with 360 equally spaced holes around the circumference, then let the radial coordinates be marked along rigid wires that slide freely through the holes. The ring will move radially in or out as the angular speed increases or decreases, respectively, as is shown later in this section. But the radial wires will all be maintained at equal angular spacings.

Summarizing, we have from (9.28) and (12.5, 6, 9),

$$\gamma_{zi}=g_{zi}=\delta_{zi}, \qquad \gamma_{ri}=g_{ri}=\delta_{ri}, \qquad g^{zi}=\delta^{zi}, \qquad g^{ri}=\delta^{ri}, \tag{12.10}$$

$$gg^{00}=g_{\theta\theta}, \qquad gg^{\theta\theta}=g_{\theta\theta}, \qquad gg^{0\theta}=-g_{0\theta}, \tag{12.11a}$$

where

$$g=\det(g_{ij})=g_{00}g_{\theta\theta}-g_{0\theta}{}^2=g_{00}\gamma_{\theta\theta}. \tag{12.11b}$$

With the help of these values, we find from (5.10) that

$$2\Gamma^i{}_{00}=g^{i0}g_{00,0}+2g^{i\theta}g_{0\theta,0}-\delta^{ir}g_{00,r}, \qquad \Gamma^i{}_{rr}=0, \tag{12.12a}$$

$$2\Gamma^i{}_{0\theta}=g^{i\theta}g_{\theta\theta,0}-\delta^{ir}g_{0\theta,r}, \qquad 2\Gamma^i{}_{\theta\theta}=-g^{i0}g_{\theta\theta,0}-\delta^{ir}g_{\theta\theta,r}, \tag{12.12b}$$

$$2\Gamma^i{}_{0r}=g^{i0}g_{00,r}+g^{i\theta}g_{0\theta,r}, \qquad 2\Gamma^i{}_{r\theta}=g^{i0}g_{0\theta,r}+g^{i\theta}g_{\theta\theta,r}. \tag{12.12c}$$

Substituting these values into the field equations (5.12), and using the definition

$$K=g_{00,r}g_{\theta\theta,r}-(g_{0\theta,r})^2, \tag{12.13}$$

we get from $R_{rr}=0$,

$$A=g^{00}g_{00,rr}+g^{\theta\theta}g_{\theta\theta,rr}+2g^{0\theta}g_{0\theta,rr}-2(\ln\sqrt{-g})_{,r}{}^2=-K/g. \tag{12.14}$$

From the vanishing of R_{00}, $R_{0\theta}$, and $R_{\theta\theta}$ we obtain

$$g_{ij,rr}-g_{ij,r}(\ln\sqrt{-g})_{,r}=g_{ij}(B-K/g), \qquad i,j=0,\theta, \tag{12.15}$$

where

$$B=(-g)^{-\frac{1}{2}}[(-g)^{-\frac{1}{2}}g_{\theta\theta,0}]_{,0}. \tag{12.16}$$

The other field equations do not lead to anything new.

From (12.15) we can deduce $A=2(B-K/g)$. Comparing this equation with (12.14), we find

$$[\ln(K/\sqrt{-g})]_{,r}=0.$$

Thus,

$$K/\sqrt{-g}=f(x^0), \quad \sqrt{-g}B=(g_{\theta\theta,0}/\sqrt{-g})_{,0}=-\tfrac{1}{2}f(x^0),$$

and consequently $g_{\theta\theta,0}/\sqrt{-g}=h(x^0)$. With the help of (9.27b), we conclude that $h(x^0)=0$, and therefore

$$g_{\theta\theta,0}=0, \qquad K=0, \qquad (g_{ij,r}/\sqrt{-g})_{,r}=0 \quad \text{for} \quad i,j=0,\theta. \tag{12.17}$$

The solution of these equations may be written in the form

$$g_{00}=-1+a(x^0)f, \qquad g_{\theta\theta}=b^2f, \qquad g_{0\theta}=\sqrt{abf}, \tag{12.18}$$

where

$$f(r)=\int\sqrt{-g}\,dr. \tag{12.19}$$

Substituting (12.18) into (12.11b), and using (12.19), we find $-g=b^2f=f_{,r}{}^2$. Thus $2f^{\frac{1}{2}}{}_{,r}=b$, $2\sqrt{f}=br+d$, and

$$\sqrt{-g}=\sqrt{-g_{00}}\sqrt{\gamma_{\theta\theta}}=b\sqrt{f}=\tfrac{1}{2}b(br+d).$$

Applying (9.27b) to this equation, we conclude $b=\sqrt{2}$, and $2f=r^2$. If we let $a=2[\omega(x^0)c]^2$, and

$$\mu=\omega(x^0)r/c \tag{12.20}$$

we obtain from (12.18),

$$g_{00}=-1+\mu^2, \qquad g_{\theta\theta}=r^2, \qquad g_{0\theta}=\pm\mu r, \qquad g=-r^2. \tag{12.21}$$

It thus follows from (12.10, 21) that

$$ds^2=(-1+\mu^2)(dx^0)^2+2\mu dx^0 rd\theta+(dr)^2+(rd\theta)^2+(dz)^2, \tag{12.22}$$

in agreement with [(8) p. 240, Eq. (72)].

As $r\rightarrow\infty$, $\mu\rightarrow\infty$, ds^2 becomes negative, and no time-like signals are possible. This sets a limitation on the possible physical extention of the CS, and is to be expected from the fact that the field is an acceleration field.

To find what the metric tensor would look like if we use SC's throughout, we have to perform the transformation (11.16). The result for the special case of constant ω is

$$g'_{00}=-1, \qquad g'_{0r}=-(1-\mu'^2)^{-1}\mu'^2x'^0/r,$$
$$g'_{0\theta}=(1-\mu'^2)^{-\frac{1}{2}}\mu'r', \qquad g'_{0z}=0, \tag{12.23a}$$

$$g'_{rr}=1-g'_{0r}{}^2,\ g'_{\theta\theta}=r'^2, \qquad g'_{r\theta}=-g'_{0r}g'_{0\theta}, \qquad g_{\alpha z}=\delta_{\alpha z}, \tag{12.23b}$$

where $\mu'=\omega r'/c$. As before, this transformation does not alter the values of $\gamma_{\alpha\beta}$. When ω is constant, the solution (12.10, 21) is stationary, i.e., g_{ij} are independent of x^0, whereas the solution (12.23) is not.

B. Transformation to an ICS

Since $R_{ijkl}=0$ is satisfied by (12.10, 21), there exists a transformation to an ICS, I. Instead of deriving the transformation from (5.14) it is much easier to rewrite (12.22) and make use of its invariance. Thus

$$-ds^2=-(cdt)^2+(dr)^2+r^2(d\theta\pm\omega dt)^2+(dz)^2$$
$$=-(cdt)^2+(dR)^2+R^2(d\Theta)^2+(dZ)^2,$$

where T, R, Θ, Z are the coordinates in I. It is clear from this that the desired transformation is [(8) 222, 226]

$$T=t, \qquad R=r, \qquad \Theta=\theta\pm\int_0^t \omega(t')\, dt', \qquad Z=z.$$

The significance of $T=t$ will be discussed shortly. It can be seen from this that the sign of $g_{0\theta}$ in (12.21) is associated with the direction of rotation.

C. Space and Time Measurements

From (12.11b, 21) we see that $\gamma_{\theta\theta}=(1-\mu^2)^{-1}r^2$. Thus, in view of (12.10), the *space* line-element is given by

$$(dl)^2=(dr)^2+(1-\mu^2)^{-1}(rd\theta)^2+(dz)^2, \qquad (12.25)$$

in agreement with [(8) Eq. (7), p. 224]. This shows that the space geometry is not Euclidean, a fact which is not altered by the transformation (11.16).

If l and r are the circumference and radius of a circle on the rotating disk with center at z, measured by an LLMI at rest on the disk, then for $\omega>0$

$$l/r=\int_0^{2\pi}(1-\mu^2)^{-\frac{1}{2}}r\, d\theta \Big/ \int_0^r dr'=2\pi(1-\mu^2)^{-\frac{1}{2}}>2\pi.$$

This result was first derived by Einstein [(12) 116]. A detailed interpretation of his derivation is as follows: Let the circumference and radius of the projection of the rotating circle on I, measured by an LLMI in an ICS, I, be denoted by L and R, respectively. According to assumption (7.3), a moving length interval measured in I undergoes a Lorentz contraction determined only by its speed and not influenced by its acceleration. Thus $L=l[1-(\omega R/c)^2]$ and $R=r$, because $v=\omega R$ is the linear speed, and the radius is perpendicular to the direction of motion. Since the space geometry on the ground is Euclidean, then $L=2\pi R$, and

$$l/r=(L/R)[1-(\omega R/c)^2]^{-\frac{1}{2}}=2\pi[1-(\omega r/c)^2]^{-\frac{1}{2}},$$

in agreement with the above result. Consequently it is impossible to rotate a rigid disk from rest without cracking at the circumference. A rotating rigid disk must be assembled in motion like a space-station. Moreover, the angular velocity of a rigid disk cannot be changed without the disk breaking up. For a disk having a radius of few centimeters, an angular frequency of 10^6 turns/sec will produce a fractional change in the circumference of only about 10^{-5}, which can be accommodated by most materials.

The space geodesics on the disk may be determined from (7.21). This problem is solved by Møller [(8) 241–243], and the reader will find it worthwhile to look it up.

For the sake of discussing time measurement in the rotating CS, let C_R be a coordinate clock at rest on the disk, C_S an SC coinciding with C_R, and C_I a set of synchronized SC's in I distributed along the path of C_R and C_S. Let dt, $d\tau$, dT be the time intervals between the same two events recorded by clocks C_R, C_S, C_I, respectively. According to (6.1),

$$d\tau=\sqrt{-g_{00}}dt. \qquad (12.26)$$

Moreover, according to assumption (6.3), the rate of C_S relative to C_I is only determined by its speed and is not influenced by its acceleration. It thus follows from the time dilation of special relativity that

$$d\tau=(1-\mu^2)^{\frac{1}{2}}dT. \qquad (12.27)$$

Eliminating $d\tau$ between the last two equations, we get

$$\sqrt{-g_{00}}dt=(1-\mu^2)^{\frac{1}{2}}dT. \qquad (12.28)$$

By adopting the value of g_{00} in (12.21), we have in effect set $t=T$, i.e., we have used coordinate clocks whose rate is just enough higher than that of local SC's that the time dilation is exactly compensated [(8) 226]. For this reason solution (12.10, 21) is stationary when $\omega=$ const.

If we integrate (6.8) along a closed path starting with a clock on the rim proceeding to the clock in the center, then to a neighboring clock on the rim, and back, we find [using (12.10, 21)]

$$c\oint \delta t=\left(\int_r^0 dr'+\int_0^r dr'+\int_\theta^{\theta+d\theta} d\theta'\right)(g_{0\theta}/-g_{00}) =\mu r\, d\theta\neq 0.$$

Thus, if two neighboring clocks on the rim are synchronized with the center clock, they are not synchronous with each other, i.e., two events on the rim clocks that are judged to be simultaneous by synchronization with the center clock, are found to be off by $(\mu r d\theta)$ when the two rim clocks are synchronized with each other. Similarly, if we synchronize clocks completely around the circumference of the disk, then according to (6.8) the last clock will differ from the first clock by a time interval [(9) 281]

$$c\oint \delta t=\int_0^{2\pi}(g_{0\theta}/-g_{00})\, d\theta=2\pi(1-\mu^2)^{-1}\mu r\neq 0. \qquad (12.29)$$

Moreover, according to (6.11), (12.29) implies that two light signals travelling in opposite directions around the circumference take different times to complete one rotation, a fact which has been verified experimentally [(31), (32)].

To understand (12.29), we apply the general argument used in deriving (6.8) to the disk. Assuming $\mu\ll 1$, we neglect in the following all terms having powers of μ higher than the first. We may therefore, ignore the difference between the values of the time and space intervals on the disk and their values in I. If we send a light signal a distance dl along the circumference of a circle on the disk and reflect it back, then because of the rotation of the disk, it takes a time interval $cdt_1 \approx dl + \omega r dt_1$ to travel in the direction of rotation and $cdt_2 \approx dl - \omega r dt_2$ to travel in the opposite direction. Therefore the difference in time assigned to two events at the ends of dl, which are simultaneous according to Einstein's synchronization convention, is given by

$$c\delta t = c(dt_2 - dt_1) \approx \tfrac{1}{2} dl [(1+\mu)^{-1} - (1-\mu)^{-1}] \approx \mu dl.$$

Consequently,

$$c\oint \delta t \approx \mu \oint dl \approx 2\pi\mu r,$$

in agreement with (12.29) to the first order in μ.

The gravitational frequency shift on a rotating disk can be calculated by means of (12.21) or (12.23), as was done at the end of Sec. 11. From (6.15) and (12.21) we find

$$\nu_1/\nu_2 = (g^{(2)}{}_{00}/g^{(1)}{}_{00})^{\frac{1}{2}} = (1-\omega^2 c^{-2} r_2^2)^{\frac{1}{2}}/(1-\omega^2 c^{-2} r_1^2)^{\frac{1}{2}}$$

$$\approx 1 - \tfrac{1}{2}\omega^2 c^{-2}(r_2^2 - r_1^2). \qquad (12.30)$$

This result has also been confirmed experimentally [34].

D. Equations of Motion

Since the transformation (12.24) is exactly the same as the nonrelativistic transformation, the resulting equations of motion and their solution also have the same form as in the nonrelativistic case. However, these similarities are misleading, since the time t in (12.24) is measured by clocks that run at a higher rate than local SC's. If we were to use SC's we would have to use (12.23), and the equations of motion would become considerably more complicated.

13. SPHERICALLY SYMMETRIC FIELD

The problem of the spherically symmetric field is discussed in practically every book on GR. The only justification for discussing this problem here is to bring out clearly the physical and coordinate conditions involved in the solution.

Since we are interested here in the solution *outside* the sources of the field, and are taking the origin of the CS at the center of the mass distribution, we cannot line up the directions of the radial SG's by (9.31). However, we can start as before with radial SG's, i.e., we adopt (9.32) as our first CC.

CC 1. $$\gamma_{r\theta,r} = 0, \qquad \gamma_{r0,r} = 0. \qquad (13.1)$$

We may agree to measure the radial distance by an LLMI or an equivalent method, and thus take:

CC 2. $$\gamma_{rr} = 1. \qquad (13.2)$$

In order to specify the nature of the angular coordinates and line up the radial directions we require:

CC 3. As $$r \to \infty, \qquad \gamma_{r\theta} \to 0, \qquad \gamma_{r\phi} \to 0, \qquad (13.3a)$$

$$\gamma_{\theta\phi} \to 0, \qquad \gamma_{\phi\phi} \to \gamma_{\theta\theta} \sin^2\theta; \qquad (13.3b)$$

(13.3a) states that asympotically the θ and ϕ coordinates lie on a surface orthogonal to r, $\gamma_{\theta\phi} \to 0$ requires that they are mutually orthogonal on this surface, and the last relation establishes the angular spacing.

An immediate consequence of (13.1, 3a), is

$$\gamma_{r\theta} = \gamma_{r\phi} = 0, \qquad (13.4)$$

as in (9.33).

PC 1. Synchronization is transitive: $\tau_{\alpha\beta} = 0$. (13.5)

PC 2. Acceleration function is static:

$$A^{\alpha}{}_{,0} = 0 \quad \text{when} \quad u^{\beta} = 0. \qquad (13.6)$$

According to (9.36) and the discussion following it, (13.5, 6) permit us to adopt the following CC:

CC 4. $$g_{0\alpha} = 0, \qquad g_{00,0} = 0. \qquad (13.7)$$

$g_{0\alpha} = 0$ is allowed by (13.5) and $g_{00,0} = 0$ by (13.6). In view of the definitions (6.5, 6), (13.7) implies

$$\gamma_{\alpha\beta} = g_{\alpha\beta}. \qquad (13.8)$$

Frequently, the condition of spherical symmetry is imposed before the coordinates have been specified. In that case it is partly a condition on the coordinates, which is purely conventional, and partly a physical requirement. However, after the coordinates have been specified, as in CC 1–4, it becomes purely a physical condition. One of the very few authors who brought out this point is Synge [(6) 266].

PC 3. Spherical symmetry:

$$\gamma_{\theta\theta} = h^2(r, x^0), \qquad \gamma_{\phi\phi} = h^2 k(\theta), \qquad \gamma_{\theta\phi} = h^2 m(\theta), \qquad (13.9a)$$

$$A^{\alpha}{}_{,\theta} = A^{\alpha}{}_{,\phi} = 0 \quad \text{when} \quad u^{\beta} = 0. \qquad (13.9b)$$

It follows from (13.9a, 3b) that

$$\gamma_{\phi\phi} = h^2(r, t) \sin^2\theta, \qquad \gamma_{\theta\phi} = 0, \qquad (13.10)$$

and from (13.9b, 7) that

$$g_{00,\theta} = g_{00,\phi} = 0. \qquad (13.11)$$

It is important to make sure that we do not use a rotating CS, since this vitiates the physical meaning of measurement of quantities such as the precession of perihelion of Mercury. We have seen in Sec. 12C that in a rotating CS, $\tau_{\alpha\beta} \neq 0$ and g_{00} depends on the angular speed, [see (12.20, 21)] if g_{0r} is taken to be zero, as in (12.9). Thus by assuming (13.5), we have already partly insured that our CS is not rotating. To make completely sure, we also impose the following conditions [(10) 165, 170):

PC 4. As $r\rightarrow\infty, \quad g_{00}\rightarrow -1, \quad h\rightarrow r.$ (13.12)

This does not agree with Mach's principle, which requires $g_{ij}\rightarrow 0$.

In order to relate the field to the properties of the source, we need one more condition, namely:

PC 5. As $r\rightarrow\infty$, A^r *must approach the Newtonian value*, i.e.,

$$c^2A^r\rightarrow -GMr^{-2} \quad \text{when} \quad u^\alpha=0, \qquad (13.13)$$

where G is the universal gravitational constant, and M is the mass of the source.

From (13.2, 4, 7–10) we have

$$ds^2=g_{00}(r)(dx^0)^2+dr^2+h^2(r,x^0)(d\theta^2+\sin^2\theta d\phi^2). \qquad (13.14)$$

The coordinates we have chosen are called by Synge [(6) 266, 270] *polar Gaussian coordinates*. If we let

$$-g_{00}=e^{2a}, \qquad h^2=e^b \qquad (13.15)$$

(Synge uses β for b and γ for $2a$), then the field equations can be shown to be [(6) p. 271, Eq. (77)],

$$R_{rr}=a_{,rr}+a_{,r}{}^2+b_{,rr}+\tfrac{1}{2}b_{,r}{}^2=0. \qquad (13.16a)$$

$$2R_{\theta\theta}=-2+e^b(b_{,rr}+b_{,r}{}^2+a_{,r}b_{,r})-e^{b-2a}(b_{,00}+2b_{,0}{}^2)=0, \qquad (13.16b)$$

$$R_{00}=b_{,00}+\tfrac{1}{2}b_{,0}{}^2-e^{2a}(a_{,rr}+a_{,r}{}^2+a_{,r}b_{,r})=0 \qquad (13.16c)$$

$$2R_{0r}=2b_{,0r}+b_{,0}(b_{,r}-2a_{,r})=0. \qquad (13.16d)$$

$R_{\phi\phi}=R_{\theta\theta}\sin^2\theta$, and the other equations are identically satisfied. Aside from the factor "2" of $b_{,0}{}^2$ in (13.16b), these equations are the same as those given by Synge.

Noting that a depends only on r, differentiating (13.16*a*) with respect to x^0, and (13.16d) with respect to r, and eliminating $b_{,0rr}$ between them, we get

$$(a_{,rr}+a_{,r}{}^2)b_{,0}=0.$$

For a nonzero field ($A^r\neq 0$), $a_{,rr}+a_{,r}{}^2\neq 0$, and thus

$$b_{,0}=0, \qquad (13.17)$$

which shows that the field is static, since $g_{ij,0}=0$.

From (13.16c, 17), we find $[\ln(a_{,r}e^{a+b})]_{,r}=0$, or $a_{,r}e^{a+b}=$ const. Similarly, we find with the help of (13.16a), $b_{,r}e^{\frac{1}{2}b-a}=$ const. Making use of (13.15, 12, 13), we get

$$h^2(\sqrt{-g_{00}})_{,r}=GMc^{-2}=\tfrac{1}{2}\alpha, \qquad h_{,r}=\sqrt{-g_{00}}. \qquad (13.18)$$

Finally from (13.16b, 17, 18) we deduce

$$-g_{00}=1-\alpha/h=(h_{,r})^2. \qquad (13.19)$$

Solving this equation for h, we get

$$r-r_0=[h(h-\alpha)]^{\frac{1}{2}}+\alpha\ln[(h-\alpha)^{\frac{1}{2}}+h^{\frac{1}{2}}], \qquad (13.20)$$

which is rather complicated. This is the reason why

CC 2′. $\qquad h=r \qquad$ (13.21)

is usually preferred over (13.2).

If one adopts (13.21) and solves for $g_{00}(r)$, $g_{rr}(r,x^0)$, then one would get the Schwartzschild exterior solution [(8) 323–325]

$$-g_{00}=1-\alpha/r, \qquad g_{rr}=-1/g_{00}. \qquad (13.22)$$

14. SUMMARY

We have finally reached the end of our task, and it is hoped that at this point an adequate, sound, operational foundation of GR has been laid. In particular, we have tried to show how a physical CS can actually be constructed in the presence of an arbitrary gravitational field, and how the coordinate conditions are linked to the CS, choice of time and length standards, and the synchronization convention. The operational meaning of all the metric coefficients, and their role in the time, space, and space–time geometries were thoroughly discussed. It was also demonstrated by several examples how a unique solution for the metric can be obtained from the field equations, coordinate conditions, and other physical conditions. Since many of these points are not adequately discussed in standard books on GR, this article should be a useful supplement to these books.

ACKNOWLEDGMENTS

I am grateful to Professor N. Ashby, Professor A. Barut, Professor P. Bergmann, Professor W. Brittin, Professor D. Burkhard, Professor A. Grünbaum, and Professor G. Hudson for discussions that helped sharpen some of the ideas in this article.

REFERENCES

[1] (a) R. Marzke and J. Wheeler, (b) R. Dicke, (c) J. Anderson, (d) J. Wheeler, *Gravitation and Relativity* (W. A. Benjamin, Inc., New York, 1964).
[2] (a) B. Bertotti, (b) R. Dicke, "Evidence for Gravitational Theories," Course 20, *Proceedings of the International School of Physics "Enrico Fermi"* (Academic Press Inc., New York, 1961).
[3] A. Arzelies, *Relativité Generalisée Gravitation* (Gauthier-Villars, Paris, 1961).
[4] J. Weber, *General Relativity and Gravitational Waves* (Interscience Publishers, Inc., New York, 1961).
[5] J. Synge, *Relativity: The Special Theory* (North-Holland Publishing Company, Amsterdam, 1956).

[6] J. Synge, *Relativity: The General Theory* (North-Holland Publishing Company, Amsterdam, 1960).
[7] V. Fock, *The Theory of Space, Time and Gravitation*, translated from the Russian by N. Kemmer (The Macmillan Company, Inc., New York, 1964).
[8] C. Møller, *The Theory of Relativity* (Oxford University Press, Oxford, 1955).
[9] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1951).
[10] W. Pauli, *Theory of Relativity*, translated from the German by G. Field (Pergamon Press, Inc., New York, 1958).
[11] A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, New Jersey 1953).
[12] A. Einstein and others, *The Principle of Relativity* (Dover Publications, Inc., New York).
[13] A. Grünbaum, *Philosophical Problems of Space and Time* (A. A. Knopf, Inc., New York, 1963).
[14] S. Basri, IEEE Trans. Instr. Meas. (to be published).
[15] A. Finzi, Nuovo Cimento **20,** 1079 (1961).
[16] H. Reichenbach, *The Philosophy of Space and Time* (Dover Publications, Inc., New York, 1958).
[17] H. Reichenbach, *Axiomatik der Relativistischen Raum-Zeit-Lehre* (Friedrich Vieweg and Son, Braunschweig, 1924).
[18] V. Lenzen, International Encyclopedia of Unified Science (University of Chicago Press, Chicago, 1955) Vol. 1, Part 1.
[19] T. Levi-Civita, *The Absolute Differential Calculus*, translated from the Italian by M. Long (Blackie & Sons, Ltd., London, 1947).
[20] L. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, New Jersey, 1949).
[21] A. Forsyth, *Intrinsic Geometry of Ideal Space* (MacMillan and Company, Ltd., London, 1935), Vols. I and II.
[22] F. Severi, Rendiconti di Palermo **42,** 227 (1917).
[23] H. Bondi, Rev. Mod. Phys. **29,** 423 (1957).
[24] R. Eötvös, D. Pekár, and E. Fekete, Ann. Physik **68,** 11 (1922).
[25] P. Roll, R. Krotkov, and R. Dicke, Ann. Phys. (N.Y.) **26,** 442 (1964).
[26] F. Rohrlich, Ann. Phys. (N.Y.) **22,** 169 (1963).
[27] H. Poincaré, *Science and Hypothesis* (Dover Publications, Inc., New York, 1952).
[28] B. Hoffmann, Rev. Mod. Phys. **4,** 173 (1932).
[29] F. Gürsey, Ann. Phys. (N.Y.) **24,** 211 (1963).
[30] J. Romain, Rev. Mod. Phys. **35,** 376 (1963).
[31] G. Sagnac, Compt. Rend. **157,** 708, 1410 (1913).
[32] W. Macek and D. Davis, Appl. Phys. Letters **2,** 67 (1963); W. Macek and G. White, IEEE Trans. Instr. Meas. (to be published).
[33] R. Pound and G. Rebka, Phys. Rev. Letters **4,** 337 (1960).
[34] W. Kundig, Phys. Rev. **129,** 2371 (1963).
[35] K. Borsuk and W. Szmielew, *Foundations of Geometry* (North-Holland Publishing Company, Amsterdam, 1960).
[36] E. Newman and A. Janis, Phys. Rev. **116,** 1610 (1959).
[37] P. Bergmann, Rev. Mod. Phys. **33,** 510 (1961).
[38] A. Komar, Phys. Rev. **111,** 1182 (1958).
[39] L. Eisenhart, *An Introduction to Differential Geometry* (Princeton University Press, Princeton, New Jersey, 1947).