REVIEWS OF

MODERN PHYSICS

VOLUME 37, NUMBER 2 APRIL 1965

Coherence Properties of Optical Fields*

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This article presents a review of coherence properties of electromagnetic fields and their measurements, with special emphasis on the optical region of the spectrum. Analyses based on both the classical and quantum theories are described. After a brief historical introduction, the elementary concepts which are frequently employed in the discussion of interference phenomena are summarized. The measure of second-order coherence is then introduced in connection with the analysis of a simple interference experiment and some of the more important second-order coherence effects are studied. Their uses in stellar interferometry and interference spectroscopy are described. Analysis of partial polarization from the standpoint of correlation theory is also outlined. The general statistical description of the field is discussed in some detail. The recently discovered universal "diagonal" representation of the density operator for free fields is also considered and it is shown how, with the help of the associated generalized phase-space distribution function, the quantum-mechanical correlation functions may be expressed in the same form as the classical ones. The sections which follow deal with the statistical properties of thermal and nonthermal light, and with the temporal and spatial coherence of blackbody radiation. Later sections, dealing with fourth- and higher-order coherence effects include a discussion of the photoelectric detection process. Among the fourth-order effects described in detail are bunching phenomena, the Hanbury Brown–Twiss effect and its application to astronomy. The article concludes with a discussion of various transient superposition effects, such as light beats and interference fringes produced by independent light beams.

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* Work supported by the U.S. Air Force Office of Scientific Research, by the U.S. Army Research Office, Durham, North Carolina, and by the Air Force Cambridge Research Laboratories

1. INTRODUCTION

Although some optical coherence phenomena are known to all physicists, no general agreement exists on the precise meaning of the term "coherence," or on the domain encompassed by coherence theory. This lack of agreement is partly due to the fact that the subject has substantially outgrown its traditional bounds. It has gradually become evident that the notion of coherence is involved in the whole field of statistical optics and more generally in the quantum description of mixed states.

Every optical field found in nature has certain fluctu-

ations associated with it. In its broadest sense, optical-coherence theory is concerned with the statistical description of the fluctuations, and optical coherence phenomena may be said to be manifestations of correlations between them. The concept of optical coherence has long been associated with interference, presumably because interference is the simplest phenomenon that reveals correlation between light beams. However, with the availability of modern light detectors and electronic circuitry of very short resolving time, other types of correlations in optical fields began to be studied. These investigations, as well as the development of some quite new types of light source, lead to questions concerning the systematic classification of optical correlation phenomena and the complete statistical description of optical fields.

The earliest investigations of coherence[1] were those of Verdet (1865, 1869), von Laue (1907, 1915c, p. 405–410), Berek (1926a, b, c, 1927), Lakeman and Groosmuller (1928a, b, c), and Schrödinger (1928). Some early researches of Stokes (1852) and Michelson (1890, 1891a, b, 1892, 1920), although not explicitly concerned with coherence, have also contributed to the understanding of the subject (cf. Zernike, 1948). The work was carried further by Wiener (1928, 1929, 1930), van Cittert (1934, 1939), Zernike (1938), Hopkins (1951, 1953, 1957), Wolf (1954a, b, 1955, 1956, 1959), Blanc-Lapierre and Dumontet (1955), Dumontet (1956b), and Pancharatnam (1956a, b, 1957a, b, 1963a, b). The main outcome of these researches was the introduction of a precise measure of correlation between the field variables at two space–time points, and the formulation of dynamical laws which the correlations (in general second-order tensors) obey. This "second-order" theory also provides a unified framework for the treatment of most commonly occurring coherence and polarization phenomena.

Shortly after the second-order theory was formulated, some important experiments of Hanbury Brown and Twiss (1956a, 1957a, b) demonstrated that correlations can also be measured between quantities which depend quadratically on the field variables. This observation led to the study of higher-order correlation effects in optical fields. As long as the fields considered were produced by thermal sources (e.g., incandescent matter or a gas discharge), second-order correlation functions were adequate for the description of second as well as higher-order effects; for a thermal light wave has the statistical character of a Gaussian random process, and such a process is completely characterized by second-order correlations [cf. Davenport and Root (1958), p. 154]. However, with the development of some new types of light source (e.g., the Smith–Purcell radiator and the optical maser), a more general approach was required, which would provide a complete statistical description of any optical field. Such an approach, based on the concepts of the theory of stochastic processes has been described by Wolf (1963) [cf. also, Mandel (1964a); Wolf (1964)].

These developments which employed an almost entirely classical description of the field,[2] have now been paralleled by a systematic quantum-mechanical description [Glauber (1963b, c)]. A connection between the two descriptions was already apparent in the coherence-matrix representation of the state of polarization of light [Wiener (1928, 1929, 1930); Wolf (1954b, 1959); McMaster (1954); Tolhoek (1956); Fano (1957); Parrent and Roman (1960)], which has the form of a density matrix even in the classical description. Glauber has now introduced the quantum analogs of the correlation functions of the classical theory. These quantum correlation functions are expectation values of normal ordered products of the creation and annihilation operators, and are closely related to quantities that are measured by means of photoelectric detectors.

Following this development, several workers have studied the relation between the classical and the quantum description of coherence. The starting point of some of these investigations was a result found by Sudarshan (1963a, b), according to which the quantum-mechanical correlation functions are expressible in the same form as the classical ones, if a certain generalized phase-space distribution is employed for the description of the statistical properties of the field.

In practice, field correlations are measured with the help of photoelectric detectors. In all cases of practical interest that have been considered so far, it was found that correlations between photon numbers of the quantized field can be determined from correlations between photoelectrons in an appropriate photodetection experiment. Moreover it was shown that such correlations can be calculated from a semiclassical theory in which the electromagnetic field is described classically, and the interaction is treated quantum mechanically [Mandel, Sudarshan, and Wolf (1964); see also, Jaynes and Cummings (1963); Senitzky (1965)].

The development of optical coherence theory has incidentally cleared up a number of old misconceptions surrounding the subject. Among these we might mention the widely held belief that no intereference effects between completely independent light beams are possible, and that photon counting measurements with a photoelectric detector can give no information about the spectrum of the light. In fact the appearance of beats [Forrester, Gudmundsen, and Johnson (1955);

[1] For a fuller historical account, see M. Born and E. Wolf (1964), Chap. X.

[2] Until very recently no systematic treatment of coherence within the framework of quantum mechanics appears to have been given, although a number of publications dealing with special problems were available [cf. Whittaker (1953), p. 96; Pauli (1958), p. 133; Fano (1961)].

Javan, Ballik, and Bond (1962); Lipsett and Mandel (1964a)] and also of interference fringes [Magyar and Mandel (1964)] resulting from the superposition of independent light beams has now been conclusively demonstrated, and can be understood in terms of both the classical and the quantum description. Moreover, the analysis of photoelectric fluctuations is becoming an important technique for the determination of extremely narrow spectral profiles, such as those of laser beams.

In this review we confine ourselves to the discussion of coherence properties of free fields and their measurements. We are not concerned with questions of the origin and growth of coherence within the source itself, which have been treated by Dicke (1954), Senitzky (1959, 1960, 1961a, b, 1962a, b), and other authors, nor with problems of induced coherence and optical pumping which have been discussed at length [Brossel and Kastler (1949); Brossel and Bitter (1952); Kastler (1957); Barrat (1961); Barrat and Cohen-Tannoudji (1961a, b); Bell and Bloom (1961a, b); Brossel (1961); Cohen-Tannoudji (1961a, b); Series (1961); Dodd, Series, and Taylor (1963); Kastler (1963); Cohen-Tannoudji and Kastler (1965)].

We begin in Sec. 2 by summarizing some quite elementary concepts which are frequently used in the analysis of the simplest interference phenomena. The quantitative description of second-order coherence is given in Sec. 3, on the basis of both classical and quantum theories. Section 4 deals with the problem of the complete statistical description of an optical field. This problem is again discussed from the standpoint of classical and quantum theories, and Sudarshan's phase-space representation, which leads to a formal similarity between the two theories, is described. The notion of coherence of different orders is introduced, and the difference between the fluctuation properties of thermal and laser light is emphasized. The most important second-order coherence effects are discussed in Sec. 5,[3] and the most important higher-order effects—including the Hanbury Brown–Twiss effect—in Sec. 6.[4] The final section (Sec. 7) deals with various transient interference and beat effects produced by independent light beams.

2. SOME ELEMENTARY CONCEPTS AND DEFINITIONS[5]

2.1. Temporal Coherence and the Coherence Time

Suppose that a "steady" light beam from a small source σ is divided into two beams in a Michelson interferometer and that the two beams are united after a path delay $\Delta s = c\Delta t$ (c = velocity of light) has been introduced between them (Fig. 1). If Δs is sufficiently small, interference fringes are formed in the plane $\mathfrak{B}$. The appearance of the fringes is said to be a manifestation of *temporal coherence* between the two beams, since the fringe contrast depends on the time delay Δt introduced between them. In general, interference fringes will only be observed if

$$\Delta t \Delta \nu \lesssim 1,$$

where $\Delta\nu$ is the effective bandwidth of the light. The time delay

$$\Delta t \sim 1/\Delta\nu \tag{2.1}$$

is called the *coherence time* of the light and the corresponding path $c\Delta t$ the *coherence length*.

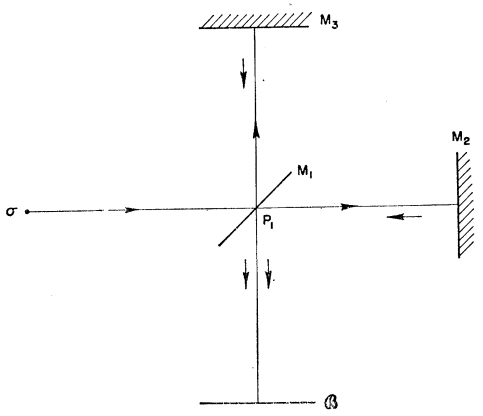

FIG. 1. Temporal coherence illustrated by means of the Michelson interferometer.

A rough elementary derivation of Eq. (2.1) may be obtained by decomposing the total intensity pattern into a sum of contributions of different frequency components, and noting the condition for the different monochromatic contributions to remain "in step" (all intensity maxima to be sufficiently close to each other).

2.2. Spatial Coherence and the Area of Coherence

Let us next consider an interference experiment of the Young type, with quasimonochromatic light from an extended *thermal* source (Fig. 2). The qualifying term *quasimonochromatic* means that the effective bandwidth $\Delta\nu$ of the light is small compared to its mid-frequency ν_0,

$$\Delta\nu/\nu_0 \ll 1. \tag{2.2}$$

For simplicity a symmetrical arrangement is assumed in Fig. 2, with a source in the form of a square of side Δl. If the pinholes P_1 and P_2 are close enough to each other, interference fringes will be observed near the central point P on the screen $\mathfrak{B}$. The appearance of

[3] A. fuller account of some of the second-order coherence effects will be found in Born and Wolf (1964), Chap. X.

[4] A fuller discussion of fourth-order coherence effects has been given by Mandel (1963d).

[5] More precise definitions of some of the quantities introduced heuristically in Secs. 2.1 and 2.2 will be given later (Secs. 5.1 and 5.4). See also A. T. Forrester (1956).

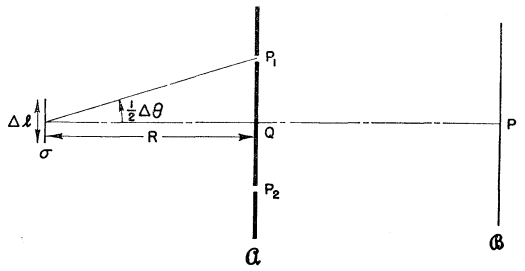

FIG. 2. Spatial coherence illustrated by means of Young's interference experiment.

these fringes is said to be the manifestation of *spatial coherence* between the light reaching P from the two pinholes, since the fringe contrast depends on the spatial separation of the pinholes. Interference fringes will in general be observed near P only if

$$\Delta\theta\Delta l \lesssim \lambda_0, \tag{2.3}$$

where $\Delta\theta$ is the angle which the line P_1P_2 subtends at the source and $\lambda_0=c/\nu_0$ is the effective wavelength of the light. If R denotes the distance between the source σ and the plane $\mathfrak{A}$ of the pinholes, it follows that in order that fringes are observed near P, the pinhole must be situated within an area around Q (see Fig. 2), of size

$$\Delta A \sim (R\Delta\theta)^2 \sim R^2(\lambda_0/\Delta l)^2 = (c^2/\nu_0^2)(R^2/S), \tag{2.4}$$

where $S=(\Delta l)^2$ is the area of the source. This area is said to be the *area of coherence* of the light in the plane $\mathfrak{A}$, around the point Q. It should be noted that according to (2.4), the area of coherence will be large if R is large, i.e., if the plane $\mathfrak{A}$ of the pinholes is sufficiently far away from the source. The solid angle $\Delta\Omega$ which the area of coherence subtends at the source is, in this case (thermal source), given by

$$\Delta\Omega \sim \Delta A/R^2 \sim (c^2/\nu_0^2)(1/S). \tag{2.5}$$

A rough elementary derivation of Eq. (2.3) may be obtained by considering the interference pattern at $\mathfrak{B}$ to arise from the superposition of independent interference patterns formed by light from different elements of the source, and noting the condition for the individual interference patterns to remain approximately in step.

Although the above discussion has been confined to light from a thermal source which directly illuminates a distant plane, it is evident that the concept of the area of coherence applies much more generally. Such an area can be defined in terms of a similar two-pinhole experiment, without reference to the source. The area of coherence depends, of course, on the location of the plane of the pinholes and will not, in general, be given by Eq. (2.4).

2.3. Volume of Coherence and the Degeneracy Parameter

Suppose now that the field is that of a nearly plane, quasimonochromatic wave. The right-angled cylinder, whose base is the area of coherence in a plane normal to the direction of propagation and whose height is the coherence length, may be called the *volume of coherence*. It is also the volume corresponding to one cell of phase space of the photons [see also A. Kastler (1964)]. It occupies a domain of space of volume $\Delta V=c\Delta t\Delta A$.

When Δt is given by (2.1) and ΔA by (2.4),

$$\Delta V \sim c\lambda_0^2R^2/\Delta\nu(\Delta l)^2 = (R/\Delta l)^2(\lambda_0/\Delta\lambda)\lambda_0^3, \tag{2.6}$$

where $\Delta\lambda=\Delta(c/\nu_0)=c\Delta\nu/\nu_0^2$.

It will be seen later that the average number of photons in the same state of polarization, which can be found in a "volume of coherence," i.e., the average number of photons in the same state of polarization which traverse the area of coherence per coherence time, is a very significant parameter. It is known as the *degeneracy parameter* δ of the light [Mandel (1961a)].

If E_ν is the average number of photons emitted per unit area of the source, per unit frequency interval, per unit solid angle around the direction normal to the source, per unit time, then evidently

$$\delta=\tfrac{1}{2}E_\nu S\Delta\nu\Delta\Omega\Delta t. \tag{2.7}$$

The factor $\frac{1}{2}$ on the right-hand side arises from the fact that the light is assumed to be generated by a thermal source and hence is unpolarized; thus it may be regarded as a mixture of photons of two mutually orthogonal polarizations, present in equal amounts. In the usual situations, where the expressions (2.5) and (2.1) for $\Delta\Omega$ and Δt apply, (2.7) evidently reduces to

$$\delta \sim \tfrac{1}{2}(c^2/\nu_0^2)E_\nu. \tag{2.8}$$

This expression is seen to be independent of the geometry.

In particular, for blackbody radiation emerging from an equilibrium enclosure

$$E_\nu = (2\nu^2/c^2)[\exp(h\nu/KT)-1]^{-1}, \tag{2.9}$$

where K is the Boltzman constant, T the absolute temperature of the radiation, h the Planck constant. In this case (2.8) becomes

$$\delta \sim [\exp(h\nu/KT)-1]^{-1}. \tag{2.10}$$

The expression on the right of (2.10) is precisely the expression first derived by Einstein (1912) in his study of radiation in a cavity in thermal equilibrium with the walls of the cavity. He showed that this quantity describes the average number of photons in a cell of phase space, which is what in quantum statistics is known as the *degeneracy* parameter of the radiation. However, the definition of degeneracy given here in terms of the "volume of coherence" applies also to light far away from its source, whether or not the light is of thermal origin.

The concept of volume of coherence may be shown to correspond to a quantum-mechanically defined cell of phase space [cf. Hanbury Brown and Twiss (1957a), p. 321]. To see this consider again a beam of quasi-monochromatic light, propagated in the direction z normal to the plane of a distant thermal source of square shape and sides Δl. In that case, the photons of different momenta and polarization will be effectively independent and the beam is describable in terms of the "one photon" phase space. Now the volume of an elementary cell of phase space is given by

$$\Delta p_x \Delta p_y \Delta p_z \Delta q_x \Delta q_y \Delta q_z = h^3, \tag{2.11}$$

where Δp_x, $\cdots$ and Δq_x, $\cdots$ are the uncertainties in the components of the momentum and position coordinates[6] of a photon in the beam. In the present case one readily finds from elementary geometrical considerations that

$$\Delta p_x = \Delta p_y = (h\nu_0/c)(\Delta l/R). \tag{2.12}$$

Further, if the square of the "angular size" $\Delta l/R$ of the source is assumed to be negligible compared to $\Delta\nu/\nu_0$, the uncertainty in Δp_z arises mainly from the uncertainty in the frequency and is given by

$$\Delta p_z = (h/c)\Delta\nu. \tag{2.13}$$

On substituting from (2.12) and (2.13) into (2.11), one obtains

$$\Delta q_x \Delta q_y \Delta q_z = c\lambda_0^2 R^2/\Delta\nu(\Delta l)^2, \tag{2.14}$$

which is seen to be identical with the expression (2.6) for the volume of coherence.

Finally it may be useful to note typical orders of magnitude of some of the parameters which we have been discussing. The values of these parameters are vastly different for thermal light and for light generated by some optical masers. For example, the bandwidth $\Delta\nu$ of the best "monochromatic" thermal light that can be produced in the laboratory is of the order of 10^8 cps, while for maser light values of 10^2 cps or even smaller may be achieved. The corresponding coherence times are, therefore, of the order of 10^{-8} and 10^{-2} sec, the coherence lengths of the order of 1 and 10^6 m, respectively. The largest value of the degeneracy parameter which can be obtained with thermal light produced in a laboratory is of the order 10^{-3}, while values greater than 10^{14} have now been achieved with maser light [cf. Mandel (1961a); Gabor (1961), pp. 133, 146)]. Thus the usual thermal light is non-degenerate ($\delta \ll 1$), while maser light is normally highly degenerate ($\delta \gg 1$).

3. THE LAWS OF INTERFERENCE AND THE DESCRIPTION OF SECOND-ORDER COHERENCE

3.1. The Classical Description

In the preceding section we have introduced rough criteria which indicate conditions under which simple interference effects may usually be expected to take place. We will now examine some of the interference effects more fully, and introduce a precise quantitative measure of so-called *second-order coherence*, of which these effects may be regarded as manifestation. This measure, which is classical in this context, will later be shown (in Sec. 3.2) to correspond precisely to a measure defined quantum mechanically.

Let $\mathbf{V}^{(r)}(\mathbf{r}, t)$ denote a real classical wave function, characterizing the field at the point $\mathbf{r}$ at time t. This function may represent, for example, the electric field or the vector potential. We purposely do not specify the nature of the wave function $\mathbf{V}^{(r)}$ any more closely at this stage, since the main analysis is independent of the particular choice of the wave function, and different choices may be best suited to describe different experimental situations. In the case of photoelectric detection, it is appropriate, as will be shown in Sec. 6.1, to consider the vector potential (actually in a certain complex representation) as the basic field variable. But detection processes may be envisaged, for which other choices may be more suitable.

For any realistic light beam, $\mathbf{V}^{(r)}$ will be a fluctuating function of time, which may be regarded as a typical member of an ensemble consisting of all possible realizations of the field. There are several reasons why $\mathbf{V}^{(r)}$ fluctuates. When the light is produced by thermal sources for example, fluctuations arise mainly because $\mathbf{V}^{(r)}$ consists of a large number of Fourier components which are independent of each other, so that their superposition gives rise to a fluctuating field which is only describable in statistical terms. But even light from a well-stabilized source, such as an optical

[6] Strictly speaking the position of a photon cannot be defined more closely than to within a region of linear dimensions of the order of a wavelength [cf. Akhiezer and Berestetsky (1953), p. 17; Acharaya and Sudarshan (1960)].

maser (where the Fourier components may be coupled to some extent because of saturation properties of the maser), will exhibit some random fluctuations, since the effect of spontaneous emission is never entirely absent. In addition there will be further contributions to such fluctuations from vibrations of the mirrors as the ends of the resonant cavity.

As already mentioned, coherence effects are essentially a manifestation of the correlation which may exist between the fluctuations at two or more space–time points. Now correlation effects can, of course, best be treated by the techniques of the theory of probability and a general treatment along these lines will be given in Sec. 4. However the description of the averaged light intensity only needs second-order correlations, and in this section we will briefly introduce them in connection with simple experiments. Before doing so, it will be useful to introduce a complex representation of the field, which is a generalization of the representation frequently used to describe the idealized case of a strictly monochromatic field, and has proved to be of considerable importance in the development of coherence theory.

Let us assume that $\mathbf{V}^{(r)}$ may be represented as a Fourier integral with respect to the time variable,[7]

$$\mathbf{V}^{(r)}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \mathbf{v}(\mathbf{r}, \nu) \exp(-2\pi i \nu t)\, d\nu, \quad (3.1)$$

and note that because $\mathbf{V}^{(r)}$ is real, $\mathbf{v}(\mathbf{r}, -\nu) = \mathbf{v}^*(\mathbf{r}, \nu)$, where the asterisk denotes the complex conjugate. Hence the negative frequency components ($\nu<0$) do not provide any information which is not already contained in the positive ones ($\nu>0$) and so may be safely omitted. Thus, in place of $\mathbf{V}^{(r)}$, we may employ the function

$$\mathbf{V}(\mathbf{r}, t) = \int_{0}^{\infty} \mathbf{v}(\mathbf{r}, \nu) \exp(-2\pi i \nu t)\, d\nu. \quad (3.2)$$

$\mathbf{V}(\mathbf{r}, t)$ is known as the *complex analytic signal*[8] or complex amplitude associated with $\mathbf{V}^{(r)}(\mathbf{r}, t)$. This concept was introduced by Gabor (1946) [see also, Born and Wolf (1964), Sec. 10.2; Beran and Parrent (1964), Chap. 2.)]. The term "complex analytic signal" arises from the fact that, by a well-known theorem [Titchmarsh (1948), p. 128], the absence of negative frequency components in (3.2) ensures that each Cartesian component of $\mathbf{V}$, considered as a function of complex t, will be analytic and regular in the lower half of the complex t-plane. Now from (3.2) and (3.1) it follows that the real part of $\mathbf{V}$ is $\frac{1}{2}\mathbf{V}^{(r)}$, while from the analytic property just mentioned it follows that the real and imaginary parts of $\mathbf{V}$ form a Hilbert transform pair:

$$\mathbf{V}(\mathbf{r}, t) = \tfrac{1}{2}[\mathbf{V}^{(r)}(\mathbf{r}, t) + i\mathbf{V}^{(i)}(\mathbf{r}, t)], \quad (3.3)$$

where

$$\mathbf{V}^{(i)}(\mathbf{r}, t) = \pi^{-1} P \int_{-\infty}^{\infty} \frac{\mathbf{V}^{(r)}(\mathbf{r}, t')}{t'-t}\, dt',$$

$$\mathbf{V}^{(r)}(\mathbf{r}, t) = -\pi^{-1} P \int_{-\infty}^{\infty} \frac{\mathbf{V}^{(i)}(\mathbf{r}, t')\, dt'}{t'-t}, \quad (3.4)$$

and P denotes the Cauchy principal value of the integral at $t'=t$.

This transition from the real field $\mathbf{V}^{(r)}$ to the complex field $\mathbf{V}$ has other significant features. The complex field $\mathbf{V}$ will later be shown to appear naturally in the theory of photoelectric detection of light fluctuations. It will be seen to be an eigenvalue of the operator used in the theory of the quantized field to represent the annihilation of a photon at the space–time point $(\mathbf{r}, t)$.

Consider now a beam of quasimonochromatic light. We assume first that it is linearly polarized, so that we may represent it by a (complex) scalar function $V(\mathbf{r}, t)$. Now because of the high frequency of optical vibrations, V cannot be measured as a function of time with any presently available optical detectors. The mean optical periods are of the order of 10^{-15} sec, whereas the fastest optical detectors presently available (best photodetectors) have resolving times of the order of 10^{-10} sec, although special techniques now exist by means of which resolving times down to about 10^{-12} sec may be achieved. However, although one cannot study the rapid time variations of the field experimentally, one can make measurements of the correlations of the field at two or more space–time points. Let us consider the correlation at two space–time points and its determination from a simple interference experiment.

The light vibrations at points $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$ are isolated by placing an opaque screen across the beam, with pinholes at the two points, and we observe the effect at a screen $\mathcal{B}$, some distance beyond $\mathcal{A}$ (Fig. 3). To a first approximation, the instantaneous field at a

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[7] In the case when $\mathbf{V}^{(r)}(\mathbf{r},t)$ represents a typical member of a stationary ensemble of random functions as we will later assume, it will not be square-integrable and hence the Fourier integral representation of $\mathbf{V}^{(r)}(\mathbf{r}, t)$ may not exist. This difficulty can be overcome in the usual way, by considering in place of $\mathbf{V}^{(r)}$ the truncated function

$$\mathbf{V}_T^{(r)}(\mathbf{r}, t) = \mathbf{V}^{(r)}(\mathbf{r}, t), \quad \text{when } |t|<T$$
$$= 0, \quad \text{when } |t|>T$$

and proceeding to the limit as $T\to\infty$ at the end of calculations involving formulas which contain the analytic signal associated with $\mathbf{V}_T^{(r)}$ [cf. Born and Wolf (1964), Chap. X].

[8] The customary definition of the complex analytic signal associated with $V^{(r)}$ differs trivially from (3.2) by having a multiplicative factor 2 in front of the integral. In the context of this article the present definition seems preferable, since, as we shall see later (Secs. 3.2, 4), it leads to a more elegant correspondence with quantities that arise naturally in the quantum treatment of coherence.

point P on the screen $\mathcal{G}$ is given by

$$V(\mathbf{r}, t) = K_1 V(\mathbf{r}_1, t-t_1) + K_2 V(\mathbf{r}_2, t-t_2), \quad (3.5)$$

where $t_1 = s_1/c$ and $t_2 = s_2/c$ are the times needed for the light to travel from P_1 to P and P_2 to P, respectively, c is the velocity of light, and K_1 and K_2 are constant factors which depend on the size of the pinholes and the geometry. From elementary diffraction theory it follows that K_1 and K_2 are pure imaginary numbers.

The instantaneous intensity $I(\mathbf{r}, t)$ at the point $P(\mathbf{r})$ and time t may be defined by the formula[9]

$$I(\mathbf{r}, t) = V^*(\mathbf{r}, t) V(\mathbf{r}, t). \quad (3.6)$$

From (3.5) and (3.6) it follows that

$$I(\mathbf{r}, t) = |K_1|^2 I_1(\mathbf{r}_1, t-t_1) + |K_2|^2 I(\mathbf{r}_2, t-t_2) + 2\mathcal{R}\{K^*_1 K_2 V^*(\mathbf{r}_1, t-t_1) V(\mathbf{r}_2, t-t_2)\}, \quad (3.7)$$

where $\mathcal{R}$ denotes the real part. If we take the average of $I(\mathbf{r}, t)$ over an ensemble of different realizations of the field and denote this ensemble average by $\langle \cdots \rangle_e$, we obtain

$$\langle I(\mathbf{r}, t) \rangle_e = |K_1|^2 \langle I(\mathbf{r}_1, t-t_1) \rangle_e + |K_2|^2 \langle I(\mathbf{r}_2, t-t_2) \rangle_e + 2\mathcal{R}\{K^*_1 K_2 \Gamma(\mathbf{r}_1, \mathbf{r}_2, t-t_1, t-t_2)\}, \quad (3.8)$$

where

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) = \langle V^*(\mathbf{r}_1, t_1) V(\mathbf{r}_2, t_2) \rangle_e \quad (3.9)$$

and

$$\langle I(\mathbf{r}_j, t_j) \rangle_e = \langle V^*(\mathbf{r}_j, t_j) V(\mathbf{r}_j, t_j) \rangle_e = \Gamma(\mathbf{r}_j, \mathbf{r}_j, t_j, t_j), \quad (j=1, 2). \quad (3.10)$$

$\Gamma(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2)$ obviously represents the correlation between the field at $\mathbf{r}_2$ and the complex-conjugate field at $\mathbf{r}_1$, at times t_2 and t_1, respectively, and $\langle I(\mathbf{r}_j, t_j) \rangle_e$ represents the (ensemble) averaged intensity of the light at the pinhole P_j, at time t_j ($j=1, 2$). We shall see shortly [Eq. (3.20)] that under the usual conditions the third term on the right-hand side of Eq. (3.8) gives rise to a sinusoidal modulation of the averaged intensity $\langle I(\mathbf{r}, t) \rangle$ with $\mathbf{r}$.

Usually one is concerned with *stationary fields*, in which case all our ensemble averages are independent of the origin of time; moreover the field is as a rule also *ergodic*. Under these conditions the ensemble averages become time-independent and may be replaced by the corresponding time averages.[10] Let us denote the time average of a stationary process $f(\mathbf{r}, t)$ by $\langle f(\mathbf{r}, t) \rangle_t$, i.e.,

$$\langle f(\mathbf{r}, t) \rangle_t = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(\mathbf{r}, t)\, dt. \quad (3.11)$$

Then the "ensemble correlation function" $\Gamma(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2)$ may be replaced by the corresponding time correlation function and this function depends on the time arguments only through their difference $t_1 - t_2$. Hence if we set

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \langle V^*(\mathbf{r}_1, t) V(\mathbf{r}_2, t+\tau) \rangle_t = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} V^*(\mathbf{r}_1, t) V(\mathbf{r}_2, t+\tau)\, dt, \quad (3.12)$$

the expression (3.8) for the averaged intensity at P becomes, under the assumption of stationarity and ergodicity,

$$\langle I(\mathbf{r}, t) \rangle = |K_1|^2 \langle I(\mathbf{r}_1, t) \rangle + |K_2|^2 \langle I(\mathbf{r}_2, t) \rangle + 2\mathcal{R}\{K^*_1 K_2 \Gamma(\mathbf{r}_1, \mathbf{r}_2, t_1 - t_2)\}, \quad (3.13)$$

where we have omitted the subscripts t or e for the

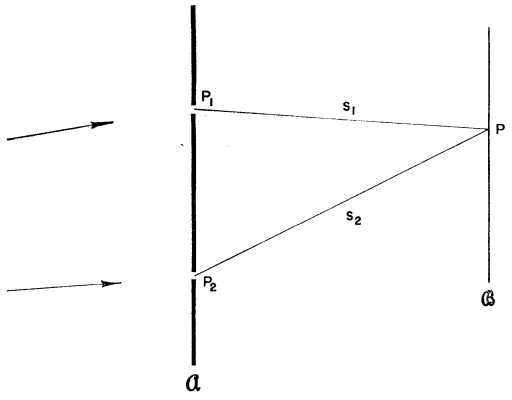


FIG. 3. The meaning of the second-order coherence functions $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ and $\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ illustrated with the help of a two-beam interference experiment.

[9] $I(\mathbf{r}, t)$ is not strictly proportional to the square of the real field variable $\mathbf{V}^{(r)}(\mathbf{r}, t)$. However, it may easily be shown that if the light is quasimonochromatic, $\frac{1}{2}I(\mathbf{r}, t)$ represents a short-time average of $V^{(r)2}$, taken over a time interval of a few mean periods of the light vibrations.

[10] In discussion of actual interference experiments, where the detector integrates over a time interval long compared with the coherence time, the time average may appear to describe the experimental situation more realistically than the ensemble average. However it should be remembered that for a stationary ergodic light beam the long record of the time variations of the field contains within it a large number of typical members of the statistical ensemble. In any case the ensemble average represents the average over a large number of separate measurements.

For discussions of stationarity and ergodicity, see, for example, Davenport and Root (1958), Goldman (1953), or Yaglom (1962).

two types of average, since it is now unnecessary to distinguish between them.

We note that, if the last term on the right-hand side of (3.13) does not vanish, the averaged intensity $\langle I(\mathbf{r}, t)\rangle$ is not equal to the sum of the (averaged) intensities of the two beams which reach the point of observation P from the two pinholes. It differs from this sum by the term $2\Re\{K^*_1 K_2 \Gamma(\mathbf{r}_1, \mathbf{r}_2, t_1-t_2)\}$. Hence if $\Gamma\neq 0$, the superposition of the two beams will give rise to an *interference effect.*

The correlation function $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ is known as the *mutual coherence function,*[11] and is the central quantity in the elementary theory of optical coherence. It is convenient to normalize Γ by setting

$$\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)=\frac{\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)}{[\Gamma(\mathbf{r}_1, \mathbf{r}_1, 0)]^{\frac{1}{2}}[\Gamma(\mathbf{r}_2, \mathbf{r}_2, 0)]^{\frac{1}{2}}}$$

$$=\frac{\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)}{[\langle I(\mathbf{r}_1)\rangle]^{\frac{1}{2}}[\langle I(\mathbf{r}_2)\rangle]^{\frac{1}{2}}}. \quad (3.14)$$

From Schwarz' inequality it then readily follows that this normalization ensures that

$$0\leq |\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)| \leq 1. \quad (3.15)$$

Let us also set

$$\langle I^{(j)}(\mathbf{r}, t)\rangle = |K_j|^2\langle I(\mathbf{r}_j, t)\rangle$$

$$= |K_j|^2\langle V^*(\mathbf{r}_j, t)V(\mathbf{r}_j, t)\rangle = |K_j|^2\Gamma(\mathbf{r}_j, \mathbf{r}_j, 0), \quad (3.16)$$

($j=1, 2$). $\langle I^{(1)}(\mathbf{r}, t)\rangle$ obviously represents the average intensity of the light reaching the point $P(\mathbf{r})$ through the pinhole at P_1 only, and $\langle I^{(2)}(\mathbf{r}, t)\rangle$ has a strictly analogous interpretation.

From (3.13), (3.14), and (3.16), it follows that the averaged intensity of the light at P may be expressed in the form

$$\langle I(\mathbf{r}, t)\rangle = \langle I^{(1)}(\mathbf{r}, t)\rangle + \langle I^{(2)}(\mathbf{r}, t)\rangle$$

$$+2[\langle I^{(1)}(\mathbf{r}, t)\rangle]^{\frac{1}{2}}[\langle I^{(2)}(\mathbf{r}, t)\rangle]^{\frac{1}{2}}\Re[\gamma(\mathbf{r}_1, \mathbf{r}_2, (s_1-s_2)/c)]. \quad (3.17)$$

From (3.17) and (3.14) it is evident that an experimental measurement of the averaged intensities $\langle I(\mathbf{r}, t)\rangle$, $\langle I^{(1)}(\mathbf{r}, t)\rangle$, $\langle I^{(2)}(\mathbf{r}, t)\rangle$, $\langle I(\mathbf{r}_1, t)\rangle$, and $\langle I(\mathbf{r}_2, t)\rangle$ immediately provides information about the real parts of the correlation functions[12] $\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ and $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$. However, it is the absolute value of the normalized *complex* correlation function γ, rather than its real part, that is a true measure of the "sharpness" of the interference effects to which superposition of the two beams may give rise. To see this let us examine (3.17) more closely. We set

$$\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = |\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)| \exp\{i[\alpha(\mathbf{r}_1, \mathbf{r}_2, \tau)-2\pi\nu_0\tau]\}, \quad (3.18)$$

where

$$\alpha(\mathbf{r}_1, \mathbf{r}_2, \tau) = \arg\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)+2\pi\nu_0\tau. \quad (3.19)$$

On substituting from (3.18) into (3.17) we obtain the following expression for the averaged intensity in the plane $\mathcal{B}$ of observation:

$$\langle I(\mathbf{r}, t)\rangle = \langle I^{(1)}(\mathbf{r}, t)\rangle + \langle I^{(2)}(\mathbf{r}, t)\rangle$$

$$+2[\langle I^{(1)}(\mathbf{r}, t)\rangle]^{\frac{1}{2}}[\langle I^{(2)}(\mathbf{r}, t)\rangle]^{\frac{1}{2}}|\gamma(\mathbf{r}_1, \mathbf{r}_2, (s_1-s_2)/c)|$$

$$\times\cos[\alpha(\mathbf{r}_1, \mathbf{r}_2, (s_1-s_2)/c)-\delta], \quad (3.20)$$

where

$$\delta=2\pi\nu_0(s_1-s_2)/c=k_0(s_1-s_2), \quad (3.21)$$

with $k_0=2\pi\nu_0/c=2\pi/\lambda_0$, and λ_0 representing the effective wavelength of the light. Now, in general, the intensities $\langle I^{(1)}\rangle$ and $\langle I^{(2)}\rangle$ of the two beams will change slowly with the position $P(\mathbf{r})$ on the screen $\mathcal{B}$. Moreover, as we shall see later [cf. Eqs. (3.26) and (3.27)] $|\gamma|$ and α will also change slowly over any part of the screen $\mathcal{B}$ over which the change in the path delay (s_1-s_2) is small compared to the coherence length of the light. However, the cosine term in (3.20) will change rapidly, because of the presence of the term δ which is inversly proportional to the very small effec-

[11] The mutual coherence function as customarily employed in the literature, follows the original definition given by Wolf (1955) [see also, Wolf (1954 b) and Blanc-Lapierre and Dumontet (1955)]. It corresponds to the quantity which in our present notation would be denoted by $\frac{1}{4}\Gamma(\mathbf{r}_2, \mathbf{r}_1, \tau)$. The factor $\frac{1}{4}$ arises from the fact that in the present article we have supressed a multiplicative factor 2 in the definition of the analytic signal, for reasons explained in footnote 8. The inversion of the order of the two position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ is here made for similar reasons.

[12] The imaginary parts of these functions could, in principle, be determined from the knowledge of their real parts for all values of τ since, as may readily be shown from (3.2) and (3.12). Γ and consequently also γ are analytic signals, and so their real and imaginary parts are coupled by Hilbert transform relations.

The real and imaginary parts $\Gamma^{(r)}$ and $\Gamma^{(i)}$ of Γ may be expressed in terms of the real field $V^{(r)}$ and its Hilbert conjugate $V^{(i)}$. In fact [cf. Roman and Wolf (1960 a), p. 474–476; Mandel (1963 d), p. 241–242],

$$\Gamma^{(r)}=\tfrac{1}{2}\langle V^{(r)}(\mathbf{r}_1, t)V^{(r)}(\mathbf{r}_2, t+\tau)\rangle=\tfrac{1}{2}\langle V^{(i)}(\mathbf{r}_1, t)V^{(i)}(\mathbf{r}_2, t+\tau)\rangle,$$

$$\Gamma^{(i)}=\tfrac{1}{2}\langle V^{(r)}(\mathbf{r}_1, t)V^{(i)}(\mathbf{r}_2, t+\tau)\rangle=-\tfrac{1}{2}\langle V^{(i)}(\mathbf{r}_1, t)V^{(r)}(\mathbf{r}_2, t+\tau)\rangle.$$

In particular it follows from these relations that

$$\langle V^{(r)2}(\mathbf{r}, t)\rangle=\langle V^{(i)2}(\mathbf{r}, t)\rangle=\tfrac{1}{2}\langle V^*(\mathbf{r}, t)V(\mathbf{r}, t)\rangle,$$

$$\langle V^{(r)}(\mathbf{r}, t)V^{(i)}(\mathbf{r}, t)\rangle=0.$$

tive wavelength λ_0 of the light. Hence over a sufficiently small portion of the screen $\mathfrak{S}$, the averaged intensity $\langle I(\mathbf{r}, t)\rangle$ will vary nearly sinusoidally and periodically with position.

The usual measure of the sharpness of interference fringes is the so-called *visibility*, and is due to Michelson. The visibility $\Omega(\mathbf{r})$ at a point $P(\mathbf{r})$ in an interference pattern is defined as

$$\mathcal{V}(\mathbf{r}) \equiv \frac{\langle I\rangle_{\max} - \langle I\rangle_{\min}}{\langle I\rangle_{\max} + \langle I\rangle_{\min}}, \tag{3.22}$$

where $\langle I\rangle_{\max}$ and $\langle I\rangle_{\min}$ represent the intensity maxima and minima in the immediate neighborhood of P. Now from (3.20) we have, to a good approximation,

$$\begin{aligned}\langle I\rangle_{\max} &= \langle I^{(1)}(\mathbf{r}, t)\rangle + \langle I^{(2)}(\mathbf{r}, t)\rangle \\ &+ 2[\langle I^{(1)}(\mathbf{r}, t)\rangle]^{\frac{1}{2}}[\langle I^{(2)}(\mathbf{r}, t)\rangle]^{\frac{1}{2}} \,|\, \gamma(\mathbf{r}_1, \mathbf{r}_2, (s_1 - s_2)/c) \,|, \\ \langle I\rangle_{\min} &= \langle I^{(1)}(\mathbf{r}, t)\rangle + \langle I^{(2)}(\mathbf{r}, t)\rangle \\ &- 2[\langle I^{(1)}(\mathbf{r}, t)\rangle]^{\frac{1}{2}}[\langle I^{(2)}(\mathbf{r}, t)\rangle]^{\frac{1}{2}} \,|\, \gamma(\mathbf{r}_1, \mathbf{r}_2, (s_1 - s_2)/c) \,|\end{aligned} \tag{3.23}$$

and hence (3.22) becomes

$$\mathcal{V}(\mathbf{r}) = 2(\mu + \mu^{-1})^{-1} \,|\, \gamma(\mathbf{r}_1, \mathbf{r}_2, (s_1 - s_2)/c) \,|, \tag{3.24}$$

where $\mu = [\langle I^{(1)}(\mathbf{r}, t)\rangle / \langle I^{(2)}(\mathbf{r}, t)\rangle]^{\frac{1}{2}}$. In particular, when the averaged intensities of the two beams are equal, as is frequently the case, then $\mu = 1$ and (3.24) reduces to $\mathcal{V}(\mathbf{r}) = |\, \gamma(\mathbf{r}_1, \mathbf{r}_2, (s_1 - s_2)/c) \,|$, i.e., $|\, \gamma \,|$ is then simply equal to the visibility of the fringes and may thus be determined from simple measurements.

The argument (phase) of γ also has a simple operational significance. It follows from (3.19), (3.20), and (3.21) that the positions of the maxima of the averaged intensity in the fringe pattern are, to a very high degree of approximation, given by

$$\begin{aligned}&\arg \gamma(\mathbf{r}_1, \mathbf{r}_2, (s_1 - s_2)/c) \\ &\quad \equiv \alpha(\mathbf{r}_1, \mathbf{r}_2, (s_1 - s_2)/c) - (2\pi/\lambda_0)(s_1 - s_2) \\ &\quad = 2m\pi, \qquad (m = 0, \pm 1, \pm 2, \cdots).\end{aligned} \tag{3.25}$$

The positions of the maxima given by (3.25) coincide with those which would be obtained if the two pinholes were illuminated by strictly monochromatic light of wavelength λ_0, and the phase of the vibrations at P_1 was retarded with respect to that at P_2 by the amount $\Delta = \alpha(\mathbf{r}_1, \mathbf{r}_2, (s_1 - s_2)/c)$. Hence, for the purpose of describing the interference effects near the point P, $\alpha(\mathbf{r}_1, \mathbf{r}_2, (s_1 - s_2)/c)$ may be regarded as representing the "effective retardation" of the light at P_1 with respect to the light at P_2. Equation (3.25) shows that the argument of γ may be determined from measurement of the position of the maxima of the fringe pattern.

We have seen that, on the one hand, γ is a measure of the correlation of the complex field at two points P_1 and P_2, and, on the other hand, it is a measure of the sharpness and location of the fringe maxima obtained by superposing the beams propagated from these points. For this reason it is customary to refer to γ as the *complex degree of coherence* of the field at the points P_1 and P_2. This term is, however, ambiguous, since γ depends not only on the location of the two points P_1 and P_2 but also on the location of the point of observation [characterized by P_1, P_2, *and* $\tau = (s_1 - s_2)/c$]. It may be more appropriate to reserve the term "*complex degree of coherence*" for the quantity $\mathrm{Max}_\tau\, \gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$. However, when the light is quasi-monochromatic, and when τ is restricted to a small enough range of values as is usually the case, this distinction is not very significant. For one may then show [cf. Eq. (3.36) below and (3.14)] that

$$\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau_1) \approx \gamma(\mathbf{r}_1, \mathbf{r}_2, \tau_2) \exp[-2\pi i \nu_0(\tau_1 - \tau_2)], \tag{3.26}$$

(ν_0 = midfrequency of the light) for any two values τ_1 and τ_2 such that

$$|\, \tau_1 - \tau_2 \,| \ll 1/\Delta\nu, \tag{3.27a}$$

i.e., such that their absolute difference is small compared to the coherence time of the light[13] [cf. Eq. (2.1)]. Thus, over a τ range satisfying the condition (3.27a), γ and also Γ are effectively periodic in τ, with period equal to the midperiod $2\pi/\nu_0$ of the light.

In relating the correlation functions γ and Γ to results of measurements, we have, of course, implicitly assumed that the detecting apparatus measures the average of the instantaneous intensity $I(\mathbf{r}, t) = V^*(\mathbf{r}, t) V(\mathbf{r}, t)$. In practice this will almost certainly be the case if V is identified with the appropriate field variable, and if the detector performs a time average over a time interval that is long compared to the characteristic time scales of the fluctuating field—i.e., long compared to the mean period and the coherence time of the light. (Alternatively the ensemble average of the intensity may be found from a succession of measurements, whether the measurement times are long or short.) Under these conditions the time average may be assumed to differ by an inappreciable amount from the average over an infinitely long time span, defined by Eq. (3.11). If these conditions are not satisfied, other types of interference effects (transient interference) may take place; these will be discussed in Sec. 7.

[13] If the light is not cross-spectrally pure (cf. Sec. 5.5), the effective spectral width $\Delta\nu$ must be interpreted with some caution.

Let us now return to (3.20) and restrict ourselves to a region of the pattern such that Eq. (3.17) is satisfied, i.e., such that (see Fig. 3)

$$|s_1-s_2| \ll \Delta s, \tag{3.27}$$

where $\Delta s=c/\Delta\nu$ is the coherence length of the light (cf. Sec. 2.1). Then from (3.24) it follows, in view of (3.26), that when $\gamma=0$, no interference fringes are formed at all in the region of the pattern under consideration ($\mathcal{V}=0$), and this is what one traditionally understands by the statement that the two light beams reaching the point P and its neighborhood are mutually *incoherent*. On the other hand, when $|\gamma|=1$, one obtains, according to (3.24) and (3.26), interference fringes with maximum possible visibility; if, moreover, the average intensities of the two beams are equal ($\mu=1$), as is frequently the case, the visibility $\mathcal{V}$ is unity (zero minima of total average intensity), and this is what is traditionally understood by the statement that the two beams are mutually *completely coherent*. The intermediate cases $0<|\gamma|<1$ characterize *partial coherence*.

We shall refer to the coherence effects just discussed as *coherence effects of the second order*, since they are characterized by a correlation function which depends on *two* space–time points. A general classification of coherence effects and a quantitative definition of coherence of any order will be given in Sec. 4.4.

It is a basic property of the mutual coherence function $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$, in terms of which the degree of coherence was defined, that in vacuo it obeys two wave equations:

$$\nabla^2_j\Gamma=c^{-2}(\partial^2\Gamma/\partial\tau^2), \qquad (j=1, 2) \tag{3.28}$$

where ∇^2_j denotes the Laplacian operator with respect to the coordinates of the point $P_j(\mathbf{r}_j)$. These two wave equations for the mutual coherence function[14] were first derived by Wolf (1955) [see also, Born and Wolf, (1964), Sec. 10.7.1]. With the help of Eqs. (3.28), one may study the distribution of second-order coherence throughout an optical field. Some examples of this type will be given in Sec. 5.1.

It is evident that the phenomena of temporal coherence and spatial coherence, which were briefly mentioned in Secs. 2.1 and 2.2, are characterized by $\Gamma(\mathbf{r}_1, \mathbf{r}_1, \tau)$ and $\Gamma(\mathbf{r}_1, \mathbf{r}_2, 0)$, respectively. In the first case the dependence of the correlation on the parameter τ is exhibited, with the points P_1 and P_2 coincident and fixed; in the second case the dependence on the position of the two points is exhibited, while the time delay τ ($\ll 1/\Delta\nu$) is kept fixed. However, it is now clear that only in the simplest cases can one sharply distinguish between temporal and spatial coherence. In general, these two types of coherence are not independent. This is clear from the fact that Γ obeys the two wave equations (3.28).

An interesting consequence of the two equations is the fact that spatially incoherent light may become partially coherent, or almost completely spatially coherent by the very process of propagation. A familiar example of this situation is the coherence of light which reaches a telescope from a distant star. The star light orginates from very many almost independently radiating atoms and so is initially spatially incoherent at the source (star). However, since on a good observing night, relatively sharp diffraction fringes (with very low minima) are formed by the starlight in the focal plane of an observing telescope, the light has obviously acquired spatial coherence in the process of propagation. This and other consequences of the elementary theory of coherence will be discussed in Secs. 5.1 and 5.2.

In the terminology of the theory of stochastic processes, the mutual coherence function $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ is known as the complex *cross-correlation function* of the two random processes characterized by $V(\mathbf{r}_1, t)$ and $V(\mathbf{r}_2, t)$ and the "self-coherence function" $\Gamma(\mathbf{r}, \mathbf{r}, \tau)$ is known as the *autocorrelation function* of the random process $V(\mathbf{r}, t)$. A basic theorem of the stochastic theory, the so-called *Wiener–Khintchine theorem* [cf. Rice (1944), Secs. 2.1 and 2.2; Wang and Uhlenbeck (1945)], asserts that the *power-spectrum* (*spectral density*) $W(\mathbf{r}, \nu)$ of the random process and the autocorrelation function $\Gamma(\mathbf{r}, \mathbf{r}, \tau)$ form a Fourier transform pair. Moreover, in the present case where V is an analytic signal, Γ itself may be shown to be an analytic signal (cf. footnote 12), so that the Fourier transform of Γ does not contain any components of negative frequencies:

$$\Gamma(\mathbf{r}, \mathbf{r}, \tau)=\int_0^\infty W(\mathbf{r}, \mathbf{r}, \nu)\exp(-2\pi i\nu\tau)\,d\nu, \tag{3.29}$$

$$W(\mathbf{r}, \mathbf{r}, \nu)=\int_{-\infty}^\infty \Gamma(\mathbf{r}, \mathbf{r}, \tau)\exp(2\pi i\nu\tau)\,d\tau. \tag{3.30}$$

If the field variable $V(\mathbf{r}, t)$ is identified with the analytic signal associated with the electric field (still assumed to be linearly polarized), then $W(\mathbf{r}, \mathbf{r}, \nu)$ represents the spectrum of the light at the point $P(\mathbf{r})$, or more precisely the *electric energy spectrum* of the light. It may be worthwhile to stress that the spectrum is *not* the Fourier transform of the field variable V itself, as is often incorrectly assumed. In fact, in the case of a stationary random field, which we are now considering, the random field variable V is, in general, not square integrable [as immediately follows from

[14] The corresponding equations for the "ensemble correlation function" $\Gamma(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2)$ defined by (3.9) are

$$\Delta_j^2\Gamma=c^{-2}(\partial^2\Gamma/\partial t_j^2), \qquad (j=1, 2), \tag{3.28a}$$

and follow at once from the fact that $V(\mathbf{r}, t)$ obeys the wave equation. These equations reduce to (3.28) under assumption of stationarity and ergodicity.

(3.12) on the assumption that the mean intensity $\langle I(\mathbf{r}, t)\rangle = \Gamma(\mathbf{r}, \mathbf{r}, 0)$ is nonzero], so that the Fourier transform of V may not even exist.

More generally, the cross-power spectrum $W(\mathbf{r}_1, \mathbf{r}_2, \nu)$ and the cross-correlation function (the mutual-coherence function) $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ form a Fourier-transform pair and again the frequency integral extends only over the positive frequency range:

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \int_0^\infty W(\mathbf{r}_1, \mathbf{r}_2, \nu) \exp(-2\pi i\nu\tau)\, d\nu, \quad (3.31)$$

$$W(\mathbf{r}_1, \mathbf{r}_2, \nu) = \int_{-\infty}^\infty \Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) \exp(2\pi i\nu\tau)\, d\tau. \quad (3.32)$$

$W(\mathbf{r}_1, \mathbf{r}_2, \nu)$ is sometime known as the *cross-spectral density* or *mutual spectral density* of the light vibrations at the points $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$.

When the light is quasimonochromatic, $|W(\mathbf{r}_1, \mathbf{r}_2, \nu)|$ will be appreciable only for ν values which lie in an interval $\nu_0 - \frac{1}{2}\Delta\nu$, $\nu_0 + \frac{1}{2}\Delta\nu$, where $\Delta\nu/\nu_0 \ll 1$. Let us rewrite (3.31) in the form

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \tilde{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, \tau) \exp(-2\pi i\nu_0\tau) \quad (3.33)$$

where

$$\tilde{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, \tau) = \int_{-\nu_0}^\infty \tilde{W}(\mathbf{r}_1, \mathbf{r}_2, \mu) \exp(-2\pi i\mu\tau)\, d\mu \quad (3.34)$$

and

$$\tilde{W}(\mathbf{r}_1, \mathbf{r}_2, \nu) = W(\mathbf{r}_1, \mathbf{r}_2, \nu_0 + \nu). \quad (3.35)$$

In view of the behavior of $|W|$, the "shifted" mutual spectral density $|\tilde{W}|$, considered as function of μ, will have appreciable values only when μ lies in a small interval of effective width $\Delta\nu$ around $\mu = 0$. Thus the function $\tilde{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, \tau)$ contains only low-frequency components and hence, when considered as function of τ, it will change very slowly compared with variations arising from the periodic term $\exp(-2\pi i\nu_0\tau)$. In fact $\tilde{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, \tau)$ considered as function of τ will remain effectively constant in any τ interval (τ_1, τ_2) for which $|\tau_1 - \tau_2| \ll 1/\Delta\nu$, i.e., one whose duration is small compared to the coherence time of the light. From this result and from (3.33) the following useful formula then follows:

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau_2) = \Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau_1) \exp[-2\pi i\nu_0(\tau_1 - \tau_2)], \quad |\tau_1 - \tau_2| \ll 1/\Delta\nu. \quad (3.36)$$

So far we have considered the light to be linearly polarized, but it is not difficult to extend the considerations to light of any state of polarization. In place of the complex scalar field variable $V(\mathbf{r}, t)$ we must employ appropriate complex vector field variables, for example the complex fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$, which are the analytic signals associated with the real electric and magnetic vectors $\mathbf{E}^{(r)}(\mathbf{r}, t)$ and $\mathbf{H}^{(r)}(\mathbf{r}, t)$ respectively. In place of the mutual coherence function Γ, one must now introduce the second-order coherence tensors

$$\begin{aligned}
\mathcal{E}_{jk}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) &= \langle E^*_j(\mathbf{r}_1, t_1) E_k(\mathbf{r}_2, t_2)\rangle, \\
\mathcal{H}_{jk}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) &= \langle H^*_j(\mathbf{r}_1, t_1) H_k(\mathbf{r}_2, t_2)\rangle, \\
\mathcal{G}_{jk}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) &= \langle E^*_j(\mathbf{r}_1, t_1) H_k(\mathbf{r}_2, t_2)\rangle, \\
\tilde{\mathcal{G}}_{jk}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) &= \langle H^*_j(\mathbf{r}_1, t_1) E_k(\mathbf{r}_2, t_2)\rangle, \quad (3.37)
\end{aligned}$$

where the suffixes j, k refer to Cartesian components, and where the averages are to be regarded as ensemble averages.

When the field is stationary and ergodic each of the correlation tensors (3.37) again depends on t_1 and t_2 only through the difference $t = t_1 - t_2$, and becomes equal to the corresponding tensor defined by means of a time average [cf. Wolf (1954b, 1956)]:

$$\mathcal{E}_{jk}(\mathbf{r}_1, \mathbf{r}_2, \tau) = \langle E^*_j(\mathbf{r}_1, t) E_k(\mathbf{r}_2, t+\tau)\rangle, \quad (3.38)$$

etc.

These coherence tensors provide the mathematical framework for the description of all second-order coherence phenomena and lead to a unified treatment of coherence and polarization effects. Some examples relating to polarization will be given in Sec. 5.6. Other applications of these tensors to problems of interference and diffraction have been described by Germey (1963) and Karczewski (1963a, b).

Since the electric and magnetic fields are coupled by Maxwell's equations, one may show that these coherence tensors are related by two sets of tensorial differential equations. When the field is stationary and ergodic, one set of these equations, relating to a field *in vacuo*, becomes [Wolf (1956); Roman and Wolf (1960a)][15]

$$\begin{aligned}
\epsilon_{jkl}\partial^1_k \mathcal{E}_{lm} - \frac{1}{c}\frac{\partial}{\partial\tau}\tilde{\mathcal{G}}_{jm} &= 0, \\
\epsilon_{jkl}\partial^1_k \mathcal{G}_{lm} - \frac{1}{c}\frac{\partial}{\partial\tau}\mathcal{H}_{jm} &= 0, \\
\epsilon_{jkl}\partial^1_k \tilde{\mathcal{G}}_{lm} + \frac{1}{c}\frac{\partial}{\partial\tau}\mathcal{E}_{jm} &= 0, \\
\epsilon_{jkl}\partial^1_k \mathcal{H}_{lm} + \frac{1}{c}\frac{\partial}{\partial\tau}\mathcal{G}_{jm} &= 0, \quad (3.39) \\
\partial^1_j \mathcal{E}_{jk} = 0, \qquad \partial^1_j \mathcal{G}_{jk} &= 0, \\
\partial^1_j \tilde{\mathcal{G}}_{jk} = 0, \qquad \partial^1_j \mathcal{H}_{jk} &= 0.
\end{aligned}$$

[15] For the same reasons as mentioned in footnote 11, the correlation tensors of the form (3.38) differ trivially from those originally introduced. As a consequence of this slight change, the sign of each term involving the derivative $\partial/\partial\tau$ in (3.39) is opposite to that which can be found in the corresponding equations in the literature.

Generalizations of Eqs. (3.39) to regions of space containing currents and charges have been given by Roman (1961 a) and Beran and Parrent (1962, 1964).

Here ϵ_{jkl} represents the completely antisymmetric unit tensor of Levi-Civita, and $\partial^1{}_k=\partial/\partial x^k{}_1$ are the Cartesian components of the operator ∇ taken with respect to the coordinates of the point $P_1(\mathbf{r}_1)$. There is a similar set of equations involving the second point $P_2(\mathbf{r}_2)$.

Some conservation laws were derived from these equations by Roman and Wolf (1960b) and Roman (1961b).

3.2. The Quantum-Mechanical Description

We shall now give a quantum-mechanical description of the simple interference experiment represented by Fig. 3. As practically all image detectors of light rely on the photoelectric effect (this includes photoelectric detectors, photographic emulsions, the eye, etc.), the operator which most closely corresponds to the 'observable' in the measurement is the positive frequency part of the field operator, or $\hat{\mathbf{A}}^{(+)}(\mathbf{r}, t)$. $\mathbf{A}^{(+)}(\mathbf{r}, t)$ is also the annihilation operator associated with the vector potential of the field at $\mathbf{r}$, t in the Heisenberg picture [cf. Schweber (1961), p. 170]. The operator corresponding to the measurement of the total light intensity over all polarization components at the space–time point $\mathbf{r}$, t will be[16] $\hat{\mathbf{A}}^{(-)}(\mathbf{r},t)\cdot\hat{\mathbf{A}}^{(+)}(\mathbf{r}, t)$ (cf. Sec. 6.1 below). This can be seen at once by an argument due to Glauber (1963b). For the rate of absorption of photons by an ideal detector will be proportional to

$$\sum_j |\langle s_j \mid \hat{\mathbf{A}}^{(+)}(\mathbf{r}, t) \mid s_i\rangle|^2 = \langle s_i \mid \hat{\mathbf{A}}^{(-)}(\mathbf{r}, t)\cdot\hat{\mathbf{A}}^{(+)}(\mathbf{r}, t) \mid s_i\rangle,$$

where $\mid s_i\rangle$ and $\mid s_j\rangle$ are initial and final states of the radiation field and the sum is taken over all final states [see also, Eq. (4.5)]. $\mathbf{A}^{(+)}(\mathbf{r}, t)$ may be expressed in terms of the annihilation operators $\hat{a}_{\mathbf{k},s}$ for photons of momentum $\hbar\mathbf{k}$ and polarization s ($s=1, 2$) in the form [cf. Messiah (1962), p. 1031]

$$\hat{\mathbf{A}}^{(+)}(\mathbf{r}, t) = (hc/L^3)^{\frac{1}{2}}\sum(1/k^{\frac{1}{2}})\hat{a}_{\mathbf{k},s}\boldsymbol{\varepsilon}_{\mathbf{k},s}\exp[i(\mathbf{k}\cdot\mathbf{r}-ckt)], \tag{3.40}$$

where the $\hat{a}_{\mathbf{k},s}$ and $\hat{a}^\dagger{}_{\mathbf{k},s}$ obey the commutation relations

$$[\hat{a}_{\mathbf{k},s}, \hat{a}_{\mathbf{k}',s'}]=[\hat{a}^\dagger{}_{\mathbf{k},s}, \hat{a}^\dagger{}_{\mathbf{k}',s'}]=0,$$
$$[\hat{a}_{\mathbf{k},s}, \hat{a}^\dagger{}_{\mathbf{k}',s'}]=\delta_{\mathbf{k},\mathbf{k}'}\delta_{s,s'}, \tag{3.41}$$

and the $\boldsymbol{\varepsilon}_{\mathbf{k},s}$ form a set of complex orthogonal unit vectors defined up to a unitary transformation [cf. Messiah (1962), p. 1032], with

$$\boldsymbol{\varepsilon}^*{}_{\mathbf{k},s}\cdot\boldsymbol{\varepsilon}_{\mathbf{k},s'}=\delta_{s,s'},$$
$$\mathbf{k}\cdot\boldsymbol{\varepsilon}_{\mathbf{k},s}=0. \tag{3.42}$$

If $\hat{\rho}$ is the density operator [cf. ter Haar (1961)] of the combined field due to both light beams, then, according to the usual rules of quantum mechanics, the expectation value $\langle I(\mathbf{r}, t)\rangle$ of the intensity at $(\mathbf{r}, t)$ summed over all polarizations, will be given by

$$\langle I(\mathbf{r}, t)\rangle = \mathrm{Tr}\,[\hat{\rho}\hat{\mathbf{A}}^{(-)}(\mathbf{r}, t)\cdot\hat{\mathbf{A}}^{(+)}(\mathbf{r}, t)]. \tag{3.43}$$

It is worth noting that the operators $\hat{\mathbf{A}}^{(+)}(\mathbf{r}, t)$ and $\hat{a}_{\mathbf{k},s}$ are closely related to the complex fields $\mathbf{V}(\mathbf{r}, t)$, and their Fourier series complex amplitudes $v_{\mathbf{k},s}$ of the classical description. For the eigenvalues of $\hat{a}_{\mathbf{k},s}$ are all complex numbers $v_{\mathbf{k},s}$. Let $\mid v_{\mathbf{k},s}\rangle$ be the eigenstate belonging to the eigenvalue $v_{\mathbf{k},s}$. Then[17]

$$\hat{a}_{\mathbf{k},s} \mid v_{\mathbf{k},s}\rangle = v_{\mathbf{k},s} \mid v_{\mathbf{k},s}\rangle. \tag{3.44}$$

Now consider the state

$$\mid \{v_{\mathbf{k},s}\}\rangle \equiv \prod_{\mathbf{k},s} \mid v_{\mathbf{k},s}\rangle, \tag{3.45}$$

where $\{v_{\mathbf{k},s}\}$ is to be interpreted as the set of all $v_{\mathbf{k},s}$. From (3.40), (3.44), and (3.45) we find that

$$\begin{aligned}\hat{\mathbf{A}}^{(+)}(\mathbf{r}, t) \mid \{v_{\mathbf{k},s}\}\rangle &= (hc/L^3)^{\frac{1}{2}}\sum_{\mathbf{k},s} k^{-\frac{1}{2}} v_{\mathbf{k},s}\boldsymbol{\varepsilon}_{\mathbf{k},s}\exp[i(\mathbf{k}\cdot\mathbf{r}-ckt)] \mid \{v_{\mathbf{k},s}\}\rangle \\ &= \mathbf{V}(\mathbf{r}, t) \mid \{v_{\mathbf{k},s}\}\rangle,\end{aligned} \tag{3.46}$$

so that the complex analytic wave amplitude $\mathbf{V}(\mathbf{r}, t)$ defined by

$$\mathbf{V}(\mathbf{r}, t) = (hc/L^3)^{\frac{1}{2}}\sum_{\mathbf{k},s} k^{-\frac{1}{2}} v_{\mathbf{k},s}\boldsymbol{\varepsilon}_{\mathbf{k},s}\exp[i(\mathbf{k}\cdot\mathbf{r}-ckt)], \tag{3.47}$$

is also an eigenvalue of the operator $\hat{\mathbf{A}}^{(+)}(\mathbf{r}, t)$. This

[16] Throughout this article operators are denoted by circumflex. From here on angular brackets are to be understood as representing the quantum-mechanical expectation of the operator within the brackets, or the ensemble average of the corresponding statistical variable.

[17] This may be seen at once from the definition of $a_{\mathbf{k},s}$, by allowing $\hat{a}_{\mathbf{k},s}$ to operate on a state of the form

$$\mid v_{\mathbf{k},s}\rangle = \exp(-\mid v_{\mathbf{k},s}\mid^2/2)\sum_{n_{\mathbf{k},s}=0}^{\infty}[v_{\mathbf{k},s}{}^{n_{\mathbf{k},s}}/(n_{\mathbf{k},s}!)^{\frac{1}{2}}] \mid n_{\mathbf{k},s}\rangle,$$

where the $\mid n_{\mathbf{k},s}\rangle$ are Fock states, and $v_{\mathbf{k},s}$ is any complex number. We then arrive at Eq. (3.44). For an introduction to the states $\mid v_{\mathbf{k},s}\rangle$ see Louisell (1964), p. 126; also Schiff (1955), p. 67. The states have recently been studied in detail by Glauber (1963 b, c), who refers to them as coherent states for reasons which will become clear in Sec. 4.4. The states were also used by Schwinger (1953) in his formulation of quantum electrodynamics.

already points to the close connection between the quantum-mechanical description and the classical description of the field in terms of the complex analytic signals, which will be further examined in Sec. 4.3.

It is natural to choose the set of states $|\{v_{k,s}\}\rangle$ as basis for the representation of the density operator $\hat{\rho}$ [cf. Glauber (1963a, b, c)]. Although they are not orthogonal and form an overcomplete set, they satisfy a closure relation of the form [Klauder (1960), p. 125–126]

$$\hat{1}=\prod_{k,s}\pi^{-1}\int |v_{k,s}\rangle\langle v_{k,s}|\, d^2v_{k,s}, \tag{3.48}$$

where $d^2v_{k,s}$ indicates an integral over the complex $v_{k,s}$ plane. A valuable feature of the $|\{v_{k,s}\}\rangle$ representation has recently been found by Sudarshan (1963a, b), and formulated rigorously by Klauder, McKenna, and Currie (1965) and Mehta and Sundershan (1965) (see Sec. 4.3 and Appendix).

Thus, if we represent the state of one beam in the basis $|\{v'_{k,s}\}\rangle$, and that of the other in the basis $|\{v''_{k,s}\}\rangle$, the density operator of the combined field may be expressed in the form

$$\hat{\rho}=\iint\Phi(\{v'_{k,s}\},\{v''_{k,s}\})\times|\{v'_{k,s}\},\{v''_{k,s}\}\rangle\langle\{v''_{k,s}\},\{v'_{k,s}\}|\, d^2\{v'_{k,s}\}\, d^2\{v''_{k,s}\}. \tag{3.49}$$

In order to avoid the problem of symmetrizing the state of the combined field, we assume that the two beams do not share any common $\mathbf{k},s$ modes. This would be the situation if two plane beams, inclined at a small angle, were superposed. The functional $\Phi(\{v'_{k,s}\},\{v''_{k,s}\})$ here plays the role of a generalized weighting or 'probability' functional.

On substituting from (3.49) into (3.43), we find that

$$\langle I(\mathbf{r},t)\rangle=\mathrm{Tr}\iint\Phi(\{v'_{k,s}\},\{v''_{k,s}\})\times|\{v'_{k,s}\},\{v''_{k,s}\}\rangle\langle\{v''_{k,s}\},\{v'_{k,s}\}|\times\hat{\mathbf{A}}^{(-)}(\mathbf{r},t)\cdot\hat{\mathbf{A}}^{(+)}(\mathbf{r},t)\, d^2\{v'_{k,s}\}\, d^2\{v''_{k,s}\}. \tag{3.50}$$

Now $\hat{\mathbf{A}}^{(-)}(\mathbf{r},t)$ is the total operator for the combined field so that, in view of (3.46),

$$\langle\{v''_{k,s}\},\{v'_{k,s}\}|\,\hat{\mathbf{A}}^{(-)}(\mathbf{r},t)=\{\mathbf{V}'^{*}(\mathbf{r},t)+\mathbf{V}''^{*}(\mathbf{r},t)\}\langle\{v''_{k,s}\},\{v'_{k,s}\}|. \tag{3.51}$$

On making use of (3.51) and its conjugate in (3.50), and remembering that the trace is invariant under cyclic permutation of operators, we obtain

$$\begin{aligned}\langle I(\mathbf{r},t)\rangle&=\mathrm{Tr}\iint\Phi(\{v'_{k,s}\},\{v''_{k,s}\})[\mathbf{V}'^{*}(\mathbf{r},t)+\mathbf{V}''^{*}(\mathbf{r},t)]\cdot[\mathbf{V}'(\mathbf{r},t)+\mathbf{V}''(\mathbf{r},t)]\\&\quad\times|\{v''_{k,s}\},\{v'_{k,s}\}\rangle\langle\{v'_{k,s}\},\{v''_{k,s}\}|\, d^2\{v'_{k,s}\}\, d^2\{v''_{k,s}\}\\&=\iint\Phi(\{v'_{k,s}\},\{v''_{k,s}\})[\mathbf{V}'^{*}(\mathbf{r},t)+\mathbf{V}''^{*}(\mathbf{r},t)]\cdot[\mathbf{V}'(\mathbf{r},t)+\mathbf{V}''(\mathbf{r},t)]\, d^2\{v'_{k,s}\}\, d^2\{v''_{k,s}\}.\end{aligned} \tag{3.52}$$

Since $\Phi(\{v'_{k,s}\},\{v''_{k,s}\})$ is a weighting function for the distribution of different complex amplitudes $\mathbf{V}'(\mathbf{r},t)$ and $\mathbf{V}''(\mathbf{r},t)$ it seems natural to interpret the integral in (3.52) as defining an ensemble average. The implication of such an interpretation will be considered more carefully in Sec. 4.3 below. With this understanding we may write

$$\langle I(\mathbf{r},t)\rangle=\langle[\mathbf{V}'^{*}(\mathbf{r},t)+\mathbf{V}''^{*}(\mathbf{r},t)]\cdot[\mathbf{V}'(\mathbf{r},t)+\mathbf{V}''(\mathbf{r},t)]\rangle, \tag{3.53}$$

and Eq. (3.53) is identical with the ensemble average of the classical Eq. (3.6) when $\mathbf{V}'(\mathbf{r},t)+\mathbf{V}''(\mathbf{r},t)$ is replaced by $\mathbf{V}(\mathbf{r},t)$, in the special case of linearly polarized light.

If, as in (3.5) and (3.6), we restrict ourselves for the sake of simplicity to linearly polarized light and express $\mathbf{V}'(\mathbf{r},t)$ and $\mathbf{V}''(\mathbf{r},t)$ in terms of the values at $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, i.e., if we set

$$V'(\mathbf{r},t)=K_1V'(\mathbf{r}_1,t-t_1)$$

$$V''(\mathbf{r},t)=K_2V''(\mathbf{r}_2,t-t_2),$$

we would, of course, on expanding (3.53), reproduce all the steps of the argument leading to Eq. (3.8) or Eq. (3.13).

Two conclusions of this analysis are worth noting. First of all, since on expansion Eq. (3.53) becomes

$$\langle I(\mathbf{r},t)\rangle=\langle|\,\mathbf{V}'(\mathbf{r},t)\,|^2\rangle+\langle|\,\mathbf{V}''(\mathbf{r},t)\,|^2\rangle+2\Re\langle[\mathbf{V}'^{*}(\mathbf{r},t)\cdot\mathbf{V}''(\mathbf{r},t)]\rangle, \tag{3.54}$$

we see that no cross or interference terms appear if $\mathbf{V}'(\mathbf{r},t)$ and $\mathbf{V}''(\mathbf{r},t)$ are orthogonal to each other. Thus, two orthogonally polarized light beams will not give rise to interference effects even though they may be strongly correlated. Secondly, although

$$\langle\mathbf{V}'(\mathbf{r},t)\cdot\mathbf{V}''(\mathbf{r},t)\rangle=\langle\mathbf{V}'(\mathbf{r},t)\rangle\cdot\langle\mathbf{V}''(\mathbf{r},t)\rangle=0 \tag{3.55}$$

if the two beams are statistically independent, the converse is not necessarily true. Absence of second-order correlation does not imply statistical independence, except for "Gaussian light" (generated by thermal sources, cf. Sec. 4.5). It is in principle possible to observe higher order correlation effects with light beams which do not give rise to interference fringes.

But even when the two beams are statistically quite independent, as reflected in the factorization of the diagonal density matrix $\Phi(\{v'_{k,s}\}, \{v''_{k,s}\})$ into the product of separate density matrices

$$\Phi(\{v'_{k,s}\}, \{v''_{k,s}\}) = \Phi_1(\{v'_{k,s}\})\Phi_2(\{v''_{k,s}\}), \quad (3.56)$$

we shall see that certain transient, not entirely predictable, interference effects may be observed. These will be discussed in Sec. 7.

4. GENERAL STATISTICAL DESCRIPTION OF THE FIELD

We will now generalize the foregoing discussions of correlations in optical fields in a systematic way. We shall see that such a generalization is equally feasible in classical terms and in terms of the quantized field, and that, moreover, there is an interesting correspondence between the two descriptions when Sudarshan's phase-space representation is employed.

4.1. The Classical Description

Consider an optical field which is represented by the complex analytic wave amplitude $\mathbf{V}(\mathbf{r}_i, t_i)$ at the space–time point $\mathbf{r}_i, t_i$ (we shall sometimes abbreviate this to $\mathbf{V}_i$). It would seem clear that the most complete "classical" description of the field possible will be given by the joint N-fold probability distribution, for all N [cf. Wolf (1963)]

$$p_N(\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_N) d^2\mathbf{V}_1 d^2\mathbf{V}_2 \cdots d^2\mathbf{V}_N,$$

where

$$d^2\mathbf{V}_k = d\mathbf{V}^{(r)}{}_k d\mathbf{V}^{(i)}{}_k$$

$$(\mathbf{V}_k = \mathbf{V}^{(r)}{}_k + i\mathbf{V}^{(i)}{}_k; \quad k = 1, 2, \cdots, N).$$

We may imagine that this joint distribution of the complex field has been derived from the known probability functional of the real field $\mathbf{V}^{(r)}(\mathbf{r}, t)$, when the appropriate transformation to the new variates $\mathbf{V}^{(i)}(\mathbf{r}, t)$ [Eq. (3.4)] is applied. However we will leave open questions regarding the construction of p_N which have not been examined in general. In principle this distribution allows us to calculate the ensemble average of any function $F(\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_N)$ of the field at N space–time points, from the rule

$$\langle F(\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_N) \rangle = \int \cdots \int F(\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_N) \times p_N(\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_N)\, d^2\mathbf{V}_1\, d^2\mathbf{V}_2 \cdots d^2\mathbf{V}_N. \quad (4.1)$$

It should be noted that, since the $\mathbf{V}$ are complex vectors, p_N is a distribution of $6N$ real variates, and the integral in (4.1) extends over $3N$ complex planes. We shall see in Sec. 6.1 that the complex representation is particularly suitable for the quantum-mechanical discussion of photoelectric measurements of the field.

As the specification of the complex $\mathbf{V}$ is equivalent to the simultaneous specifications of the amplitude and phase of the classical wave, and for quasimonochromatic light of $\mathbf{V}^{(r)}$ and its time derivative, it might appear that $p_N(\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_N)$ will have no analog in the description of the quantized field. We shall see however that a very close correspondence exists nevertheless.

We have already encountered the second-order moments of the $\mathbf{V}$ in the discussion of a simple interference experiment. More complicated correlation measurements in general involve the higher order moments—or correlation tensors— of the $\mathbf{V}$, defined by

$$\Gamma^{(N,M)}{}_{j_1,\cdots,j_N,j_{N+1},\cdots,j_{N+M}}(x_1, \cdots, x_N; x_{N+1}, \cdots, x_{N+M}) = \langle V^*_{j_1}(x_1) \cdots V^*_{j_N}(x_N)\, V_{j_{N+1}}(x_{N+1}) \cdots V_{j_{N+M}}(x_{N+M}) \rangle, \quad (4.2)$$

where x_i stands for the space and time coordinates $\mathbf{r}_i, t_i$ and the j_i are now polarization indices referring to the Cartesian components of the complex vector $\mathbf{V}(\mathbf{r}_i, t_i)$. Such general moments of order $N+M$ were introduced [Wolf (1963, 1964); Mandel 1964a)], for the discussion of correlation experiments with light from nonthermal sources. Moreover, by analogy with the second-order cross-spectral densities defined in Sec. 3.1 [Eq. (3.32)] one may also define a general $(N+M)$th-order cross-spectral density, although we shall not need to use it here. For the usual case of stationary ergodic optical fields the ensemble averages may be replaced by time averages and the $\Gamma^{(N,M)}$ functions do not depend on the origin of time. In the following it will be taken for granted that an $(N+M)$th order correlation, which is written as an explicit function of only $(N+M-1)$ time variables, refers to a stationary field.

We might mention that several experiments in which the fourth-order correlation is effectively measured have been performed, and proposals have been made for measuring sixth-order correlations. In addition the fourth-order correlations are occasionally valuable for describing interference effects with random features. These questions will be discussed in more detail in Secs. 6 and 7.

4.2. The Description in Terms of Quantized Fields

In the quantum-mechanical treatment of the radiation field, the transverse[18] field at each space–time point $\mathbf{r}$, t in the Heisenberg picture is represented by the Hermitian operator $\hat{\mathbf{A}}(\mathbf{r}, t)$, which may be given a Fourier series expansion in terms of plane waves

$$\hat{\mathbf{A}}(\mathbf{r}, t) = (\hbar c/L^3)^{\frac{1}{2}} \sum_{\mathbf{k},s} (1/k^{\frac{1}{2}}) [\hat{a}_{\mathbf{k},s} \boldsymbol{\varepsilon}_{\mathbf{k},s} \exp[i(\mathbf{k}\cdot\mathbf{r} - ckt)] + \hat{a}^{\dagger}_{\mathbf{k},s} \boldsymbol{\varepsilon}^*_{\mathbf{k},s} \exp[-i(\mathbf{k}\cdot\mathbf{r} - ckt)]]$$
$$= \hat{\mathbf{A}}^{(+)}(\mathbf{r}, t) + \hat{\mathbf{A}}^{(-)}(\mathbf{r}, t). \tag{4.3}$$

$\hat{\mathbf{A}}^{(+)}(\mathbf{r}, t)$ and $\hat{\mathbf{A}}^{(-)}(\mathbf{r}, t)$ are the annihilation and creation operators, which were already encountered in Sec. 3.2. These operators are Hermitian conjugates of each other and obey commutation rules which are derivable from (3.41). We have already noted that $\hat{\mathbf{A}}^{(+)}(\mathbf{r}, t)$ has right eigenstates of the form $|\{v_{\mathbf{k},s}\}\rangle$ and corresponding eigenvalues which are the complex wave amplitudes $\mathbf{V}(\mathbf{r}, t)$ given by (3.47). Similarly $\hat{\mathbf{A}}^{(-)}(\mathbf{r}, t)$ has left eigenstates $\langle\{v_{\mathbf{k},s}\}|$ and corresponding eigenvalues $\mathbf{V}^*(\mathbf{r}, t)$.

The coherence properties of the field can now be described by correlations of the field operators in a number of possible ways. Thus Senitzky (1962b) defined a correlation operator as a symmetrized product of real operators at two points. However a different operator appears to be more useful in the present context. Corresponding to the tensor equation (4.2), Glauber (1963a, b, c, 1964) has introduced correlation tensors defined in terms of ordered products of the complex field operators of the form[19]

$$G^{(N,M)}{}_{j_1\cdots j_N, j_{N+1}\cdots j_{N+M}}(x_1\cdots x_N, x_{N+1}\cdots x_{N+M}) = \mathrm{Tr}\,[\hat{\rho}\hat{A}^{(-)}{}_{j_1}(x_1)\cdots A^{(-)}{}_{j_N}(x_N) \times \hat{A}^{(+)}{}_{j_{N+1}}(x_{N+1})\cdots\hat{A}^{(+)}{}_{j_{N+M}}(x_{N+M})], \tag{4.4}$$

where x_i stands for the space–time point $\mathbf{r}_i$, t_i and where $\hat{\rho}$ is the density operator representing the (generally mixed) state of the field. An operator of the type

$$\hat{A}^{(-)}{}_{j_1}(x_1)\cdots\hat{A}^{(-)}{}_{j_N}(x_N)\hat{A}^{(+)}{}_{j_1}(x_1)\cdots\hat{A}_{j_N}(x_N)$$

is needed to describe N-fold photoelectric delayed coincidence measurements of the field, at the space–time points $x_1\cdots x_N$, with detectors sensitive to polarizations $j_1\cdots j_N$, respectively [cf., however, Holliday and Sage (1964), who suggest a different operator]. For, if $|S_1\rangle$ is the state of the field before the measurement and $|S_2\rangle$ after the measurement, the matrix element corresponding to the absorption of photons of the specified polarizations at each detector at the given times is [cf. Glauber (1963b), p. 2531]

$$\langle S_2 | \hat{A}^{(+)}{}_{j_1}(x_1)\cdots\hat{A}^{(+)}{}_{j_N}(x_N) | S_1\rangle.$$

The rate at which such absorptions occur, summed over all final states, is therefore proportional to

$$\sum_{s_2} |\langle s_2 | \hat{A}^{(+)}{}_{j_1}(x_1)\cdots\hat{A}^{(+)}{}_{j_N}(x_N) | s_1\rangle|^2 = \sum_{s_2}\langle s_1 | \hat{A}^{(-)}{}_{j_1}(x_1)\cdots\hat{A}^{(-)}{}_{j_N}(x_N) | s_2\rangle\langle s_2 | \hat{A}^{(+)}{}_{j_1}(x_1)\cdots\hat{A}^{(+)}{}_{j_N}(x_N) | s_1\rangle$$
$$= \langle s_1 | \hat{A}^{(-)}{}_{j_1}(x_1)\cdots\hat{A}^{(-)}{}_{j_N}(x_N)\hat{A}^{(+)}{}_{j_1}(x_1)\cdots\hat{A}^{(+)}{}_{j_N}(x_N) | s_1\rangle, \tag{4.5}$$

and, when the right-hand side of this equation is averaged over the ensemble of initial states of the system, it becomes identical with the right-hand side of (4.4) for $x_{N+r} = x_r$, $j_{N+r} = j_r$, $(r = 1, 2, \cdots, N)$ and $M = N$.

A number of properties of the $G^{(N,M)}$ follow from the definition. Thus [cf. Glauber (1963b), Sec. 3],

$$G^{(N,M)*}{}_{j_1,\cdots,j_N,j_{N+1},\cdots,j_{N+M}}(x_1, \cdots, x_N; x_{N+1}, \cdots, x_{N+M}) = G^{(M,N)}{}_{j_{N+1},\cdots,j_{N+M},j_1,\cdots,j_N}(x_{N+1}, \cdots, x_{N+M}; x_1, \cdots, x_N) \tag{4.6}$$

and

$$G^{(N,N)}{}_{j_1,\cdots,j_N,j_1,\cdots,j_N}(x_1, \cdots, x_N; x_1, \cdots, x_N) \geq 0, \tag{4.7}$$

and it may also be shown that

$$|G^{(N,M)}{}_{j_1,\cdots,j_N,j_{N+1},\cdots,j_{N+M}}(x_1, \cdots, x_N, x_{N+1}, \cdots, x_{N+M})|^2 \leq G^{(N,N)}{}_{j_1,\cdots,j_N,j_1,\cdots,j_N}(x_1, \cdots, x_N, x_1, \cdots, x_N) G^{(M,M)}{}_{j_{N+1},\cdots,j_{N+M},j_{N+1},\cdots,j_{N+M}}(x_{N+1}, \cdots, x_{N+M}, x_{N+1}, \cdots, x_{N+M}). \tag{4.8}$$

The similarities between the quantum-mechanical correlations $G^{(N,M)}$ and the classically defined correlation $\Gamma^{(N,M)}$ suggest that there is a close correspondence between them. We shall see that this correspondence is brought out particularly clearly by the phase space representation. We should mention that, while the normally ordered products of creation and annihilation operators have played a preferred role in the theory of optical coherence, other products and their physical significance have also been discussed recently [Jordan

[18] As the transverse part of the field is the only part leading to observable effects, we shall deal only with the transverse part.

[19] Actually in Glauber's correlations, the operator $\hat{\mathbf{A}}$ is replaced by the electric field operator $\hat{\mathbf{E}}$ [cf. also Eq. (5.61a) and (5.79) below]. As we shall see later, the present definitions are somewhat more convenient for applications to questions relating to photoelectric detection of light fluctuations.

Glauber discusses only $(N+N)$ order products of the $2N$ field operators, to which he refers as "Nth"-order correlations. However the N, M type correlations may be useful in the treatment of the coherence properties of harmonically generated light. [See, for example, Ducuing and Bloembergen (1964); Bloembergen (1964)].

(1964); Kano (1964b, 1965); Mandel (1964d); Mehta and Sudarshan (1965)].

4.3. A Phase-Space Representation

Shortly after the introduction of the higher-order correlations it was shown by Sudarshan (1963a, b) that there exists an interesting phase-space (or "diagonal" $|\, v_{k,s}\rangle$-space) representation of the density operator $\hat{\rho}$ of the electromagnetic field, which brings out a correspondence between the classical and quantum-mechanical descriptions. The possibility of such a representation, in some special cases has also been noted by Glauber (1963c), who called it the P representation, but its universal validity was first recognized and established by Sudarshan.

Sudarshan indicated that every density operator $\hat{\rho}$ of a free electromagnetic field may be represented in the "diagonal" form in the basis formed by the eigenstates $|\, \{v_{k,s}\}\rangle$ of the annihilation operator $\hat{\mathbf{A}}^{(+)}(x)$ [cf. Eq. (3.46)]:

$$\hat{\rho}=\int \Phi(\{v_{k,s}\})\; |\; \{v_{k,s}\}\rangle\langle\{v_{k,s}\}\; |\; d^2\{v_{k,s}\}. \quad (4.9)$$

Sudarshan's original formulation of this representation was somewhat heuristic. It was based on a formal series (see Appendix below) involving the Dirac delta function and its derivatives, for the functional $\Phi(\{v_{k,s}\})$ in terms of the elements $(\{n_{k,s}\}, \{n'_{k,s}\})$ of $\hat{\rho}$ in the Fock representation. Rigorous formulations of Sudarshan's theorem were given later by Klauder, McKenna, and Currie (1965)[19a] and Mehta and Sudarshan (1965) (cf. end of Appendix below). Although in general $\Phi(\{v_{k,s}\})$ in Eq. (4.9) is to be regarded as a generalized functional, it is found that in many important cases $\Phi(\{v_{k,s}\})$ may be taken to be an ordinary, nonnegative well-behaved functional.

We may look on $\Phi(\{v_{k,s}\})$ as a phase-space distribution describing the mixed state of the field in the multidimensional space of the complex $\{v_{k,s}\}$. This is further emphasized by the fact that the $\{v_{k,s}\}$ are eigenvalues of the $\{\hat{a}_{k,s}\}$, whose real and imaginary (Hermitian and anti-Hermitian) parts $\Re(\hat{a}_{k,s})$, $\Im(\hat{a}_{k,s})$ behave like canonically conjugate operators [Mandel (1963a)]. Thus, in view of (3.41),

$$[\Re(\hat{a}_{k,s}),\, \Im(\hat{a}_{k,s})]=(i/2)[\hat{a}_{k,s},\, \hat{a}^{\dagger}{}_{k,s}] =i/2. \quad (4.10)$$

Let us now evaluate the correlation tensors $G^{(N,M)}$ given by (4.4), when the density operator $\hat{\rho}$ is expressed in the "diagonal" form (4.9). We have

$$G^{(N,M)}{}_{j_1,\cdots,j_N,j_{N+1},\cdots,j_{N+M}}(x_1,\, \cdots,\, x_N,\, x_{N+1},\, \cdots,\, x_{N+M}) = \mathrm{Tr}\int \Phi(\{v_{k,s}\})|\; \{v_{k,s}\}\rangle\langle\{v_{k,s}\}| \times \hat{A}^{(-)}{}_{j_1}(x_1)\cdots\hat{A}^{(-)}{}_{j_N}(x_N)\,\hat{A}^{(+)}{}_{j_{N+1}}(x_{N+1})\cdots \times \hat{A}^{(+)}{}_{j_{N+M}}(x_{N+M})\; d^2\{v_{k,s}\}.$$

By repeatedly applying the conjugate of the eigenvalue relation (3.46),

$$\langle\{v_{k,s}\}\; |\; \hat{A}^{(-)}{}_{j_r}(x_r)=V^*{}_{j_r}(x_r)\,\langle\{v_{k,s}\}\; |, \quad (r=1,\, 2,\, \cdots,\, N)$$

in this integral, and then inverting the product of $|\, \{v_{k,s}\}\rangle\langle\{v_{k,s}\}\; |$ with the remaining operator and applying (3.46) again, we arrive at

$$G^{(N,M)}{}_{j_1,\cdots,j_N,j_{N+1},\cdots j_{N+M}}(x_1,\, \cdots,\, x_N,\, x_{N+1},\, \cdots,\, x_{N+M}) = \mathrm{Tr}\int \Phi(\{v_{k,s}\})\, V^*{}_{j_1}(x_1)\cdots V^*{}_{j_N}(x_N) \times V_{j_{N+1}}(x_{N+1})\cdots V_{j_{N+M}}(x_{N+M})\; |\; \{v_{k,s}\}\rangle\langle\{v_{k,s}\}\; |\; d^2\{v_{k,s}\}$$

$$=\int \Phi(\{v_{k,s}\})\, V^*{}_{j_1}(x_1)\cdots V^*{}_{j_N}(x_N) \times V_{j_{N+1}}(x_{N+1})\cdots V_{j_{N+M}}(x_{N+M})\; d^2\{v_{k,s}\}$$

$$=\langle V^*{}_{j_1}(x_1)\cdots V^*{}_{j_N}(x_N)\, V_{j_{N+1}}(x_{N+1})\cdots V_{j_{N+M}}(x_{N+M})\rangle. \quad (4.11)$$

The angular brackets in Eq. (4.11) may be interpreted as defining an expectation value of the product of the $V_j(x)$, when the phase-space distribution $\Phi(\{v_{k,s}\})$ is used as weighting or "probability" function. Although $\Phi(\{v_{k,s}\})$ is a real function of the complex $\{v_{k,s}\}$, it is not necessarily positive everywhere like a classical probability. Functions of this type were first discussed

[19a] In a recent publication [*Quantum Optics and Electronics, Les Houches, 1964*, edited by C. DeWitt, A. Blandon, and C. Cohen-Tannoudji (Gordon and Breach Publishers, New York, 1965), p. 144], Glauber refers to this representation and states that:

"In a recent preprint, Klauder, McKenna, and Currie confirm the conclusion that no useful weight function P need exist for arbitrary density operators. To minimize this difficulty they express matrix elements of the density operator through a limiting procedure involving an infinite sequence of operators expressed as P representations. This procedure, however, does not preserve the most useful property of the P representation, the reduction of statistical averages to simple integrals over the complex α plane."

To us, this interpretation of the main result of Klauder, McKenna, and Currie appears to be misleading, since these authors (and also Mehta and Sudarshan, 1965, see also end of Appendix below) show precisely in what sense the convergence has to be interpreted to make Sudarshan's theorem rigorous. In this connection, it may not be inappropriate to recall that when a wave function $\psi(q)$ is expanded in terms of a complete orthonormal set $\phi_n(q)$, $(n=1, 2, 3, \cdots)$,

$$\psi(q)=\sum_{n=1}^{\infty} C_n\phi_n(q), \quad (1)$$

what is generally meant is not ordinary pointwise convergence in a strict mathematical sense, but convergence in the mean-square sense [see, for example, E. C. Kemble, *The Fundamental Principles of Quantum Mechanics with Elementary Applications* (Dover Publications, Inc., New York, 1958), p. 138], viz.,

$$\lim_{N\to\infty}\int_{-\infty}^{\infty} |\, \psi(q)-\sum_{n=1}^{N} C_n\phi_n(q)\; |^2\, dq=0. \quad (2)$$

Nevertheless, the limiting procedure implied in (2) does not prevent a wide use of the symbolic expansion (1) in the calculation of "averages as simple integrals" over the q plane.

by Wigner (1932) [cf. also, Moyal (1949); Baker (1958); Sudarshan (1962), Sec. 4.1; Mehta (1964)], who pointed out that they are usable as weighting or "probability" distributions for the calculation of expectation values of physical observables. The name "quasiprobability" for such functions was suggested by Baker (1958). The choice of a phase-space distribution is not unique in general, in that there exist different distributions leading to the same expectation value of any function $F(\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_N)$ of the classical field variables [cf. Mehta (1964)]. The nonuniqueness is in part due to the many alternative ways in which quantum-mechanical operators may be associated with functions of the classical field variables. [For a review of the various ways of associations see Shewell (1959).]

The right-hand side of (4.11) is of exactly the same form as the expression (4.2), for the classically defined correlation functions $\Gamma^{(N,M)}$. The analogy may be made even closer by introducing "configuration space probability densities"[20] $p_N(\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_N)$ which correspond to $\Phi(\{v_{\mathbf{k},s}\})$. It is only necessary to integrate $\Phi(\{v_{\mathbf{k},s}\})$ over the $\{v_{\mathbf{k},s}\}$ variables, subject to N constraints of the type [cf. Mandel (1963a)]

$$\mathbf{V}(\mathbf{r}_i, t_i) = \left(\frac{hc}{L^3}\right)^{\frac{1}{2}} \sum_{\mathbf{k},s} \frac{1}{k^{\frac{1}{2}}} v_{\mathbf{k},s} \boldsymbol{\varepsilon}_{\mathbf{k},s} \exp[i(\mathbf{k}\cdot\mathbf{r}_i - ckt_i)] \qquad (i=1, 2, \cdots, N). \quad (4.12)$$

Thus

$$p_N\{\mathbf{V}(\mathbf{r}_1, t_1), \cdots, \mathbf{V}(\mathbf{r}_N, t_N)\} = \int \Phi(\{v_{\mathbf{k},s}\}) \prod_{i=1}^{N} \delta\{\mathbf{V}(\mathbf{r}_i, t_i) - (hc/L^3)^{\frac{1}{2}} \sum_{\mathbf{k},s} k^{-\frac{1}{2}} v_{\mathbf{k},s} \boldsymbol{\varepsilon}_{\mathbf{k},s} \exp[i(\mathbf{k}\cdot\mathbf{r}_i - ckt_i)]\}\, d^2\{v_{\mathbf{k},s}\}. \quad (4.13)$$

With this choice of the probability densities, the correlations $\Gamma^{(N,M)}$ are identical with the corresponding correlations $G^{(N,M)}$. A description of the field in terms of Φ or the associated probability densities p_N is sometimes called *semiclassical* [cf. Sudarshan (1963a, b)], although it is in fact a representation of a strictly quantum-mechanical description.[20a]

Whether one uses as weighting functions the probability densities $\{p_N\}$ defined classically in Sec. 4.1 or quantum mechanically by Eq. (4.13), it is clear that the resulting formulas have a strict correspondence. Moreover this correspondence persists throughout the whole theory of coherence. For this reason we will from now on denote all the field correlations by Γ, leaving the choice of the weighting functions open, unless we are specifically concerned with the distinction between correlations defined classically and quantum mechanically. In any case we shall see that a strict *equality* between the "probability distributions" and therefore between the corresponding correlations exists at least in the case of thermal light.

It has sometimes been stated [Glauber (1963b, p. 2533)] that, for a stationary, ergodic field, the correlation function $G^{(1,1)}$ defined quantum mechanically by Eq. (4.4), would, in the limit of a strong, low frequency field, reduce to the mutual coherence function $\Gamma^{(1,1)}$ defined by Eq. (3.12) in terms of the unquantized field. It should be evident from the preceding remarks (and is supported by explicit calculations relating to blackbody radiation [Mehta and Wolf (1964a, b)] that exact equality may exist between the correlation functions G and Γ, irrespective of the strength and the frequency of the field.

The phase-space representation also leads to an interesting expression for the probability distribution $p(n)$ of the number of photons n within a given volume V of space at time t (see footnote 6) for an arbitrary state of the field [cf. Ghielmetti (1964)]. We choose V as the volume of normalization, and observe that the probability for an arbitrary distribution $\{n_{\mathbf{k},s}\}$ among the different modes is given by the expectation value of the corresponding projection operator. Thus

$$p(\{n_{\mathbf{k},s}\}) = \mathrm{Tr}\,[\hat{\rho}\,|\,\{n_{\mathbf{k},s}\}\rangle\langle\{n_{\mathbf{k},s}\}\,|]$$

and from (4.9),

$$= \mathrm{Tr} \int \Phi(\{v_{\mathbf{k},s}\})\,|\,\{v_{\mathbf{k},s}\}\rangle\langle\{v_{\mathbf{k},s}\}\,| \times |\,\{n_{\mathbf{k},s}\}\rangle\langle\{n_{\mathbf{k},s}\}\,|\,d^2\{v_{\mathbf{k},s}\}$$

$$= \int \Phi(\{v_{\mathbf{k},s}\})\,|\,\langle\{v_{\mathbf{k},s}\}\,|\,\{n_{\mathbf{k},s}\}\rangle\,|^2\,d^2\{v_{\mathbf{k},s}\}$$

$$= \int \Phi(\{v_{\mathbf{k},s}\}) \prod_{\mathbf{k},s} \left[\frac{(v^*_{\mathbf{k},s} v_{\mathbf{k},s})^{n_{\mathbf{k},s}}}{n_{\mathbf{k},s}!} \exp(-v^*_{\mathbf{k},s} v_{\mathbf{k},s})\right] d^2\{v_{\mathbf{k},s}\}, \quad (4.14)$$

when we make use of the known scalar product of the $|\,\{n_{\mathbf{k},s}\}\rangle$ and $|\,\{v_{\mathbf{k},s}\}\rangle$ states [cf. Glauber (1963c)]. Now

$$p(n) = \sum_{\{n_{\mathbf{k},s}\}} p(\{n_{\mathbf{k},s}\})\,\delta_{n,m}$$

where

$$m = \sum_{\mathbf{k},s} n_{\mathbf{k},s}$$

and from (4.14), with the help of the multinomial

[20] Like Φ, the p_N's so defined are not necessarily nonnegative. The use of negative probability functions for the calculation of expectation values has been discussed, for example, by Bartlett (1944).

[20a] Throughout the present article we use the adjective "semiclassical" in its more customary sense, namely, to refer to a treatment of problems involving the interaction of the electromagnetic field with matter in which the field is treated classically and the interaction quantum mechanically.

theorem, we arrive at

$$p(n)=\int\Phi(\{v_{\mathbf{k},s}\})\frac{U^n}{n!}e^{-U}\,d^2\{v_{\mathbf{k},s}\},\qquad(4.15a)$$

where

$$U=\sum_{\mathbf{k},s}|\,v_{\mathbf{k},s}\,|^2.$$

Equation (4.15a) may also be written in the alternative form

$$p(n)=\int_0^\infty P(U')\frac{U'^n}{n!}e^{-U'}\,dU',\qquad(4.15b)$$

where

$$P(U')=\int\Phi(\{v_{\mathbf{k},s}\})\,\delta(U'-U)\,d^2\{v_{\mathbf{k},s}\}.$$

This is the required distribution. We note that it is in the form of an average over Poisson distributions, weighted with the phase-space distribution Φ. It is interesting to observe that somewhat similar results were already obtained by Bothe (1927). Expressions identical to (4.15b) were obtained more recently by Mandel (1958) and Mandel, Sudarshan, and Wolf (1964) from semiclassical considerations of the photoelectric measurement process [see also Kelley and Kleiner (1964)]. These results will be described in Sec. 6.2. We shall see that Eqs. (4.15) lead to a useful characterization of differences between some optical fields, such as those produced by lasers and by thermal sources[20b] [cf. Mandel (1964a)].

As we shall see, for many purposes the discussion of coherence is more readily based on the correlations than on the distributions themselves, and the distinction between the quantum mechanical and classical descriptions then disappears formally, although it will still be implicit in the manner in which the expectation values are calculated (i.e., in the choice of the probability distributions p_N).

4.4. Order and Degree of Coherence

The relation between the second-order correlation function $\Gamma^{(1,1)}{}_{j,j}(\mathbf{r}_1,\mathbf{r}_2,t_1,t_2)$ at two space–time points and the visibility of the interference fringes formed if the radiation at $\mathbf{r}_1$ and $\mathbf{r}_2$ (linearly polarized in direction j) is allowed to interfere has already been described in Sec. 3.1. For a stationary, linearly polarized field, the modulus of the normalized correlation function $|\,\gamma^{(1,1)}{}_{j,j}(\mathbf{r}_1,\mathbf{r}_2,t_1-t_2)\,|$, which lies between 0 and 1, was seen to be a useful measure of the degree of coherence. More generally we may take $|\,\gamma^{(1,1)}{}_{j_1,j_2}(x_1,x_2)\,|$ to be a measure of the degree of second-order coherence at the space–time points x_1, x_2 between the j_1 and j_2 components of the field. The condition for full second-order coherence between these components at x_1 and x_2 is then given by [cf. Mandel and Wolf (1961a); Parrent (1959a, b); Streifer (1964); Mehta, Wolf, and Balachandram (1965)]

$$|\,\gamma^{(1,1)}{}_{j_1,j_2}(x_1,x_2)\,|=1.\qquad(4.16)$$

The relation (4.16) has been generalized by Glauber (1963b, 1964) to define a necessary condition for $2N$th-order coherence.[21] He introduces a $2N$th-order normalized correlation by

$$\gamma^{(N,N)}{}_{j_1,\cdots,j_{2N}}(x_1,\cdots,x_{2N})=\Gamma^{(N,N)}{}_{j_1,\cdots,j_{2N}}(x_1,\cdots,x_{2N})\Big/\prod_{r=1}^{2N}\{\Gamma^{(1,1)}{}_{j_r,j_r}(x_r;x_r)\}^{\frac{1}{2}},\qquad(4.17)$$

which reduces to the usual form for $N=1$. $\Gamma^{(1,1)}{}_{j_r,j_r}(x_r,x_r)$ is of course the mean light intensity associated with the polarization component j_r at the space–time point x_r.

By analogy with the condition (4.16) for second-order coherence, the relation

$$|\,\gamma^{(n,n)}{}_{j_1,\cdots,j_{2n}}(x_1,\cdots,x_{2n})\,|=1\qquad\text{for all } n\leq N\qquad(4.18)$$

has been proposed as condition for $2N$th-order coherence [Glauber (1963b, 1964)]. As $\Gamma^{(N,N)}{}_{j_1,\cdots,j_N,j_1,\cdots,j_N}(x_1,\cdots,x_N;x_1,\cdots,x_N)$ is a real positive quantity proportional to the joint N-fold photoelectric counting rate for polarization components $j_1,\cdots,j_N$ at the space–time points $x_1,\cdots,x_N$, it follows that $\gamma^{(N,N)}{}_{j_1,\cdots,j_N,j_1,\cdots,j_N}(x_1,\cdots,x_N;x_1,\cdots,x_N)$ is real and positive also. Thus the condition (4.18) for $2N$th-order coherence implies that the joint n-fold counting rate reduces to the products of the counting rates of the n separate detectors for $n\leq N$. The condition for $2N$th-order coherence therefore rules out the possibility of intensity correlations up to the Nth order. On this definition the intensity correlation effects, first observed by Hanbury Brown and Twiss (1956a, 1957a, b) and discussed in Sec. 6.4 below, do not exist with light having coherence of order higher than 2. Moreover, in view of the foregoing interpretation, a radiation field having a limited number—say N—of photons cannot be coherent to an order greater than $2N$.

Again by analogy with $|\,\gamma^{(1,1)}{}_{j,j}(x_1,x_2)\,|$ it is tempting to regard $|\,\gamma^{(N,N)}{}_{j_1,\cdots,j_{2N}}(x_1,\cdots,x_{2N})\,|$ as the degree of $2N$th-order coherence. However this quantity does not appear to be bounded by 0 and 1, in general, and such an interpretation therefore seems to be unjustified.

[20b] These considerations have been criticized by Glauber (*Quantum Electronics III*, p. 108, 1964) in highly acerbic terms. The criticism has not however prevented Glauber from making similar deductions from a similar equation in a recent publication (Glauber, *Quantum Optics and Electronics, Les Houches 1964*, edited by C. deWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach Publishers, New York, 1965)], pp. 65, 181).

[21] Note that Glauber refers to it as Nth-order coherence.

It has been pointed out [Glauber (1963b, 1964)] that the condition (4.18) of $2N$th-order coherence is not rotationally invariant and ought to be replaced by a stronger condition. Now it was shown [Mandel and Wolf (1961a); Parrent (1959a, b)] that the relation (4.16) for some space–time points x_1 and x_2, with $j_1=j_2$ implies factorization of the form

$$\Gamma^{(1,1)}{}_{j,j}(x_1, x_2) = V^*{}_j(x_1)\, V_j(x_2). \tag{4.19}$$

By analogy Glauber (1963b, 1964) has proposed the factorization property

$$\Gamma^{(n,n)}{}_{j_1,\cdots,j_{2n}}(x_1, \cdots, x_{2n}) = \prod_{r=1}^{n} V^*{}_{j_r}(x_r) \prod_{r=n+1}^{2n} V_{j_r}(x_r)$$

$$\text{for all } n \leq N \tag{4.20}$$

as the condition for $2N$th-order coherence. It is evident that (4.20) implies (4.18) and that, in general, such a strong condition would not be expected to hold except over a limited region of space–time.

Nevertheless it is easy to show that, at least in principle, there exist radiation fields which are coherent to all orders at all space–time points [cf. Glauber (1963b, 1964)]. Consider a field which is in an eigenstate $|\{v'_{\mathbf{k},s}\}\rangle$ of the annihilation operator $\hat{A}^{(+)}(\mathbf{r}, t)$ satisfying Eq. (3.46). Then the density matrix in the Sudarshan representation (4.9) will be of the form

$$\Phi(\{v_{\mathbf{k},s}\}) = \delta(\{v_{\mathbf{k},s}\} - \{v'_{\mathbf{k},s}\}), \tag{4.21}$$

and when this is substituted in the Eq. (4.11) the usual argument leads to

$$\Gamma^{(N,M)}{}_{j_1,\cdots,j_N,j_{N+1},\cdots j_{N+M}}(x_1, \cdots, x_N; x_{N+1}, \cdots, x_{N+M})$$

$$= \prod_{r=1}^{N} V^*{}_{j_r}(x_r) \prod_{r=N+1}^{N+M} V_{j_r}(x_r). \tag{4.22}$$

Thus a field in a state $|\{v'_{\mathbf{k},s}\}\rangle$ is coherent to all orders at all space–time points. However, it may be shown that the only state of this kind which represents a stationary field is the vacuum state.

Although the foregoing definitions all refer to coherence of even order, it is clear that the factorization property (4.22), and a normalized correlation function of the type $\gamma^{(N,M)}{}_{j_1,\cdots,j_{N+M}}(x_1, \cdots, x_{N+M})$, can just as readily be used to define coherence of odd order, or more generally of order $N+M$, where N and M are arbitrary nonnegative integers. Such a description may appear natural for the discussion of coherence properties of optical harmonics generated in nonlinear media [cf. Franken and Ward (1963); Bloembergen (1963)].

The correlation functions for $N>2$, $M>2$, and the corresponding measures of coherence, have as yet played a negligible role in practice. Indeed the usefulness of the whole concept of higher order coherence still remains to be shown. As we shall see, for optical fields generated by thermal sources in equilibrium (e.g., a hot star, an incandescent filament, a gas discharge, etc.) the (1+1)-order correlations contain all the required information about the field, and no full coherence of order greater than 2 is possible.

4.5. Thermal Light

The radiation field generated by a thermal source is of a particularly simple kind, in which the ensemble distribution of the complex field amplitude $\mathbf{V}(\mathbf{r}, t)$ is Gaussian. Until fairly recently such optical fields were of course the only ones of any significance.

The Gaussian nature of the field fluctuations has usually been proved by an appeal to the central limit theorem of statistics, for the radiation field at each point in space generally consists of contributions from many independent atomic radiators. An interesting discussion of the question of the representation of the probability distribution took place between Von Laue and Einstein in 1915 [Von Laue (1915a, b); Einstein (1915); see also Einstein and Hopf (1910)], and a thorough investigation of the problem in classical terms was made more recently [Janossy (1957, 1959); see also, Van Cittert (1934, 1939); Blanc-Lapierre and Dumontet (1955); and Dumontet (1956a), p. 173].

The quantum-mechanical derivation of the statistical properties of the field is usually based on Bloch's theorem [Bloch (1932)] for oscillators in thermodynamic equilibrium. In the following we shall make use of the Sudarshan phase-space representation to show simply that, when the quantized radiation field is in equilibrium at temperature T and is therefore describable by a canonical ensemble, the joint probability distribution for all the Fourier amplitudes $v_{\mathbf{k},s}$ is a product of Gaussian distributions [cf. Mandel (1963a)]. The joint probability distribution of the complex field $\mathbf{V}(\mathbf{r}, t)$ at any number of space time points is therefore Gaussian also, since $\mathbf{V}(\mathbf{r}, t)$ is a linear combination of the $v_{\mathbf{k},s}$.

The density operator for a quantum-mechanical system in equilibrium at temperature T is given by [cf. Messiah (1961), p. 448]

$$\hat{\rho} = \frac{\exp(-\hat{H}/KT)}{\mathrm{Tr}\,[\exp(-\hat{H}/KT)]}, \tag{4.23}$$

where $\hat{H}$ is the Hamiltonian of the system and K is Boltzmann's constant. Now for the quantized free electromagnetic field we have [cf. Messiah (1961), p. 439]

$$\hat{H} = \sum_{\mathbf{k},s} \hat{a}^{\dagger}{}_{\mathbf{k},s} \hat{a}_{\mathbf{k},s} c\hbar\boldsymbol{k}, \tag{4.24}$$

if the zero-point energy is neglected. In order to arrive at a "diagonal" representation of $\hat{\rho}$ in the basis of the states $|\{v_{\mathbf{k},s}\}\rangle$, we substitute (4.24) in (4.23) and make use of the closure relation (3.48). This gives

$$\hat{\rho} = \hat{F}/\mathrm{Tr}\,(\hat{F}), \tag{4.25}$$

where

$$\hat{F}=\prod_{k,s}\pi^{-1}\int \exp\left(-\hat{a}^{\dagger}_{k,s}\hat{a}_{k,s}c\hbar k/KT\right) \mid v_{k,s}\rangle\langle v_{k,s}\mid d^2v_{k,s}. \tag{4.26}$$

Now the $\mid v_{k,s}\rangle$ states have a simple expansion in terms of the eigenstates $\mid n_{k,s}\rangle$ of the number operator $\hat{a}^{\dagger}_{k,s}\hat{a}_{k,s}$,

$$\hat{a}^{\dagger}_{k,s}\hat{a}_{k,s}\mid n_{k,s}\rangle=n_{k,s}\mid n_{k,s}\rangle, \tag{4.27}$$

in the form [cf. Klauder (1960), p. 125]

$$\mid v_{k,s}\rangle=\sum_{n_{k,s}}\exp\left(-\mid v_{k,s}\mid^2/2\right)\{v_{k,s}{}^{n_{k,s}}/[(n_{k,s})!]^{\frac{1}{2}}\}\mid n_{k,s}\rangle, \tag{4.28}$$

and, on introducing this into (4.26) and making use of the eigenvalue equation (4.27), we obtain

$$\hat{F}=\prod_{k,s}\frac{1}{\pi}\int\sum_{n_{k,s}}\sum_{m_{k,s}}\exp\left(-n_{k,s}c\hbar k/KT\right)\exp\left(-\mid v_{k,s}\mid^2\right)\times\frac{v_{k,s}{}^{n_{k,s}}v_{k,s}{}^{*m_{k,s}}}{(n_{k,s}!m_{k,s}!)^{\frac{1}{2}}}\mid n_{k,s}\rangle\langle m_{k,s}\mid d^2v_{k,s}. \tag{4.29}$$

By making the substitution $v_{k,s}=r_{k,s}\exp(i\theta_{k,s})$ and integrating over $\theta_{k,s}$ we see at once that contributions to the integral in (4.29) vanish unless $n_{k,s}=m_{k,s}$. With the help of the substitution $v_{k,s}\exp(-\frac{1}{2}c\hbar k/KT)=u_{ks}$, (4.29) and (4.25) allow us to express $\hat{\rho}$ in the form

$$\hat{\rho}=\frac{\prod_{k,s}\pi^{-1}\int\exp\{-\mid u_{k,s}\mid^2[\exp\left(c\hbar k/KT\right)-1]\}\mid u_{k,s}\rangle\langle u_{k,s}\mid d^2u_{k,s}}{\prod_{k,s}\pi^{-1}\int\exp[-\mid u_{k,s}\mid^2(\exp\left(c\hbar k/KT\right)-1)]\,d^2u_{k,s}}$$

$$=\prod_{k,s}\frac{\exp\left(c\hbar k/KT\right)-1}{\pi}\int\exp\{-\mid u_{k,s}\mid^2[\exp\left(c\hbar k/KT\right)-1]\}\mid u_{k,s}\rangle\langle u_{k,s}\mid d^2u_{k,s}, \tag{4.30}$$

which has the form (4.9). It follows that the joint probability density for the set $\{v_{k,s}\}$ may be taken to be

$$\Phi(\{v_{k,s}\})=\prod_{k,s}(1/\pi\langle n_{k,s}\rangle)\exp\left[-\mid v_{k,s}\mid^2/\langle n_{k,s}\rangle\right], \tag{4.31}$$

where $\langle n_{k,s}\rangle$ is the well-known expectation value of the number operator $\hat{a}^{\dagger}_{k,s}\hat{a}_{k,s}$ in thermal equilibrium [Huang (1963), p. 255; cf also Kano (1964b)]

$$\langle n_{k,s}\rangle=[\exp\left(c\hbar k/KT\right)-1]^{-1}. \tag{4.32}$$

Since the probability distributions of all the $v_{k,s}$ are independent Gaussian distributions with zero means, it is clear from (3.47) that the resultant complex wave amplitude $\mathbf{V}(\mathbf{r},t)$ will also obey a Gaussian distribution with zero mean and variance [cf. Kenney and Keeping (1954), p. 169]

$$\langle\mathbf{V}^*(\mathbf{r},t)\cdot\mathbf{V}(\mathbf{r},t)\rangle=\langle I(\mathbf{r},t)\rangle=(hc/L^3)\sum_{k,s}(\langle n_k\rangle/k). \tag{4.33}$$

Thus, if X and Y denote the real and imaginary parts of one component $V_j(\mathbf{r},t)$, the joint probability distribution will be

$$p(X,Y)=(1/\pi\langle I_j\rangle)\exp\left[-(X^2+Y^2)/\langle I_j\rangle\right], \tag{4.34}$$

where use has been made of the fact that $\langle XY\rangle=0$, $\langle X^2\rangle=\langle Y^2\rangle=\frac{1}{2}\langle I_j\rangle$ (cf. footnote 12).

Although all the foregoing applies strictly only to blackbody radiation, it can be seen at once that the conclusions remain valid if the spectrum is modified in an arbitrary way by the introduction of any linear filters. For a Gaussian random process remains Gaussian under linear transformation [cf. Davenport and Root (1958), p. 155]. It follows incidentally that thermal light will, in general, be distinguishable from light from a nonthermal source, such as an optical maser, by its fluctuation properties, even if the corresponding spectral and angular distributions coincide.

Now it is an important property of Gaussian processes that they are completely determined by their first and second moments. Thus, for radiation fields produced by thermal sources, the higher-order correlations of type (4.2) all reduce to products of second-order correlations [Reed (1962); Mehta (1965b), p. 398]:

$$\Gamma^{(N,N)}{}_{j_1,\cdots j_N,j_{N+1},\cdots,j_{2N}}(x_1,\cdots,x_N;x_{N+1},\cdots,x_{2N})=\sum_{\pi}\Gamma^{(1,1)}{}_{j_1,j_{N+1}}(x_1,x_{N+1})\cdots\Gamma^{(1,1)}{}_{j_N,j_{2N}}(x_N,x_{2N}),$$

$$\Gamma^{(N,M)}=0\quad\text{if}\quad N\neq M, \tag{4.35}$$

where $\sum_{\pi}$ stands for the sum over all $N!$ possible permutations of the indices 1 to N. This no doubt accounts for the emphasis that has been given to second-order correlations in the past. In particular the correlations for a blackbody radiation field have been calculated and will be discussed in Sec. 5.7.

From (4.35) we note also that, when all the coordi-

nates x_i and polarization indices j_i coincide,

$$\Gamma^{(N,N)}{}_{j,\cdots,j}(x,\cdots,x)$$
$$=N![\Gamma^{(1,1)}{}_{j,j}(x,x)]^N=N!\langle I_j(x)\rangle^N, \quad (4.36)$$

where $\langle I_j(x)\rangle$ is the mean intensity of the particular polarization component. It follows therefore that the conditions (4.18) and (4.20) for full coherence can only be satisfied for $N=1$ [cf. Glauber (1963b, 1964)], so that the concept of complete higher-order coherence is not relevant to thermal light.

While the phase-space description of a thermal radiation field in terms of $\Phi(\{v_{\mathbf{k},s}\})$ is the one most generally useful, it is simply related to the alternative description in terms of occupation numbers $n_{\mathbf{k},s}$ of the different $\mathbf{k}$, s modes. This can be seen by transforming the density matrix $\Phi(\{v_{\mathbf{k},s}\})$ given by (4.31) to the basis of the energy eigenstates $|\{n_{\mathbf{k},s}\}\rangle$ with the help of the general transformation [Sudarshan (1963a, b)]

$$\rho(\{n_{\mathbf{k},s}\},\{n'_{\mathbf{k},s}\})$$
$$=\prod_{\mathbf{k},s}\int\Phi(\{v_{\mathbf{k},s}\})\exp(-|v_{\mathbf{k},s}|^2)\cdot\frac{v^{n_{\mathbf{k},s}}{}_{\mathbf{k},s}v^{*n'_{\mathbf{k},s}}{}_{\mathbf{k},s}}{(n_{\mathbf{k},s}!n'_{\mathbf{k},s}!)^{\frac{1}{2}}}d^2v_{\mathbf{k},s}, \quad (4.37)$$

when we find

$$\rho(\{n_{\mathbf{k},s}\},\{n'_{\mathbf{k},s}\})$$
$$=\prod_{\mathbf{k},s}[(1+\langle n_{\mathbf{k},s}\rangle)(1+1/\langle n_{\mathbf{k},s}\rangle)^{n_{\mathbf{k},s}}]^{-1}\delta_{n_{\mathbf{k},s},n'_{\mathbf{k},s}}. \quad (4.38)$$

The density matrix is diagonal and corresponds to a set of statistically independent Bose–Einstein distributions [cf. Morse (1962), p. 218] for the occupation numbers of different modes [see also Holliday (1964); Kano (1964b)].

In practice neither the distributions of $\mathbf{V}(\mathbf{r},t)$ nor of $n_{\mathbf{k},s}$ for an optical field are measurable directly. However, the total instantaneous intensity $\mathbf{V}^*(\mathbf{r},t)\cdot\mathbf{V}(\mathbf{r},t)=I(\mathbf{r},t)$, or the intensity $V^*_j(\mathbf{r},t)V_j(\mathbf{r},t)=I_j(\mathbf{r},t)$ associated with a particular polarization component j, may be accessible to measurement with the help of a photoelectric detector. The detection process will be considered more specifically in Sec. 6.1. Here we merely wish to indicate the forms of the distributions $p(I_j)$ and $p(I)$ for a Gaussian variate $\mathbf{V}(\mathbf{r},t)$.

According to (4.34) X and Y are statistically independent Gaussian variates with zero means and variances $\frac{1}{2}\langle I_j\rangle$. Then it follows from the well-known properties of the chi-square distribution in statistics [Kenney and Keeping (1954), p. 98] that

$$I_j=X^2+Y^2$$

obeys the exponential distribution

$$p(I_j)=(1/\langle I_j\rangle)\exp(-I_j/\langle I_j\rangle). \quad (4.39)$$

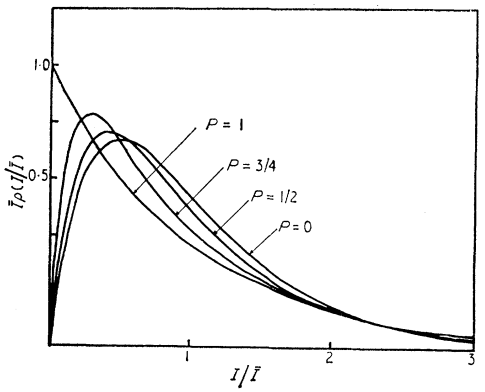


FIG. 4. Some examples of the probability density $p(I)$ [Eq. (4.41)] for different values of the degree of polarization P. [After L. Mandel (1963c).]

The distribution of the total instantaneous intensity I in general depends on the degree of polarization $P(0\leq P\leq 1)$ of the light, and this quantity will be discussed more fully in Sec. 5.6. In the special case in which the two orthogonal polarization components $V_1(\mathbf{r},t)$ and $V_2(\mathbf{r},t)$ obey the cross-spectral purity condition (see Sec. 5.5), the derivation of $p(I)$ becomes very simple, for the mean intensities $\langle I_1\rangle$, and $\langle I_2\rangle$ of the two components may be expressed as [cf. Eq. (5.58)]

$$\langle I_1\rangle=\tfrac{1}{2}(1+P)\langle I\rangle,$$
$$\langle I_2\rangle=\tfrac{1}{2}(1-P)\langle I\rangle, \quad (4.40)$$

and both $p_1(I_1)$ and $p_2(I_2)$ are of the form (4.39). The resultant intensity I therefore obeys the distribution [cf. Mandel (1963c)]

$$p(I)=\int_0^\infty p_1(I_1)p_2(I-I_1)\,dI_1$$
$$=\frac{1}{P\langle I\rangle}\left\{\exp\left[-\frac{2I}{\langle I\rangle(1+P)}\right]-\exp\left[-\frac{2I}{\langle I\rangle(1-P)}\right]\right\}. \quad (4.41)$$

Some examples of the distribution $p(I)$ for different values of P are given in Fig. 4. It is seen that all the curves start at the origin, except the one for $P=1$ corresponding to fully polarized light. The reason is that in such a beam the ensemble of spin states degenerates to a single member. An expression closely related to (4.41) has also been given [Blanc-Lapierre (1956)] for the fluctuations of the envelope of a Gaussian random process [see also Hurwitz, (1945)]. Finally we find from (4.41) that the fluctuations of intensity

obey the relation

$$\langle(\Delta I)^2\rangle=\tfrac{1}{2}(1+P^2)\langle I\rangle, \qquad (4.42)$$

which leads to upper and lower limits for $\langle(\Delta I)^2\rangle$.

4.6. Nonthermal Light

The most important example of nonthermal or non-Gaussian light is that produced by an optical maser, which also differs in several other important respects from typical thermal light. Thus the degeneracy parameter δ(cf. Sec. 2.3)—or the average occupation number per unit cell of phase space—may be very great ($\delta\gg1$) for maser light, while it is usually very small for thermal light (typical figures were quoted at the end of Sec. 2). The nonthermal nature of the radiation from a laser was recognized early on by several workers [Golay (1961); Smith and Williams (1962); Mandel (1962a, b); Glauber (1963a); Mandel and Wolf (1963a, b); but see also Forrester (1961a, b); Corcoran and Pao (1962); Bolwijn, Alkemade, and Boschloo (1963)] and has now been demonstrated experimentally [Bellisio, Freed, and Haus. (1964); Bailey and Sanders (1964); Armstrong and Smith (1964, 1965); Freed and Haus (1965)].

A laser differs from a thermal source most significantly in the strong coupling that exists between the source and the field, leading to one (or possibly a few) strongly preferred atomic transition(s). The radiation is produced mainly by stimulated emission. It may be in thermal equilibrium with the source, but the strong coupling rules out the substitution of the "free" Hamiltonian (4.24) in the density operator (4.23). The description of the combined system of source and field in a maser has been discussed by many authors [Weber (1957, 1959); Senitzky (1958, 1959, 1960, 1961, 1961b, 1962); Louisell, Yariv, and Siegman (1961); Schwinger (1961); McCumber (1963); Gordon, Walker, and Louisell (1963); Gordon, Louisell, and Walker (1963); Lamb (1964); Schwabl and Thirring (1964); Wagner and Hellwarth (1964)]. We shall not go into the rather lengthy analyses which involve solving the equations of motion of the coupled system, but merely note that for the case of an ideal laser oscillating in a single $\mathbf{k}'$, s' mode, the radiation field is very close to being in a "classical" state of well-defined complex amplitude $v'_{\mathbf{k}',s'}$. The case for considering this the proper state of a laser field has recently been argued by Pauli Brunner, and Richter (1963), Jordan and Ghielmett, (1964), Picard and Willis (1965), and Haken (1964) [but see also, Mandel (1964c) and Kano (1964a)]. As the phase of $v'_{\mathbf{k}',s'}$ will in general be random, we can write an approximation to the "diagonal density matrix" of the field in the form:

$$\Phi(\{v_{\mathbf{k},s}\})=(2\pi\,|\,v'_{\mathbf{k}',s'}\,|)^{-1}\delta(|\,v_{\mathbf{k}',s}\,|-|\,v'_{\mathbf{k}',s'}\,|)\prod_{\mathbf{k},s\neq\mathbf{k}',s'}(2\pi\,|\,v_{\mathbf{k},s}\,|)^{-1}\delta(|\,v_{\mathbf{k},s}\,|). \qquad (4.43)$$

Although (4.43) does not represent a pure state, it is easy to show that a field in such a state is nevertheless coherent to all even orders of the type $(N+N)$ in the sense of the coherence condition (4.20) [cf. also Paul (1963)].

The state represented by (4.43) leads to a particularly simple expression for the probability distribution $p(n)$ of the number of photons counted in a volume δV at a given time (or falling on a surface in a given time interval). It is merely necessary to substitute (4.43) into (4.15). We then find that

$$p(n)=\frac{|\,v'_{\mathbf{k}',s'}\,|^{2n}}{n!}\exp[-|\,v'_{\mathbf{k}',s'}\,|^2], \qquad (4.43a)$$

which is a Poisson distribution in n. Recent photoelectric measurements of the fluctuations of a single-mode laser beam, appear to be in agreement with this distribution [Bellisio, Freed, and Haus (1964); Bailey and Sanders (1964); Armstrong and Smith (1964, 1965); but compare also the results of Freed and Haus (1965)], although as yet they provide a test only of the second moment of n.

However, if the distribution (4.43a) is regarded as empirically correct, it can be shown to have the following implication for the correlations of a single-mode laser field [Mandel (1965)]:

$$\langle\hat{A}^{(-)m}{}_j(\mathbf{r},t)\hat{A}^{(+)m}{}_j(\mathbf{r},t)\rangle=\langle\hat{A}^{(-)}{}_j(\mathbf{r},t)\hat{A}^{(+)}{}_j(\mathbf{r},t)\rangle^m, \qquad (4.43b)$$

where the angular brackets denote the quantum-mechanical expectation value. The density operator of the field would then be an operator satisfying (4.43b), and a form very different from Eq. (4.43) is possible.

Of course in practice the light from an optical maser contains contributions from spontaneous emission and from several modes which, moreover, may be coupled [cf. Paananen, Tang, and Statz (1963); Lamb (1964); Haken and Sauermann (1963); DeLang and Bouwhuis (1963); Lipsett and Mandel (1963, 1964a); Haus and Mullen (1962, 1963); Bellisio, Freed, and Haus (1964)]. It is clear that in practice the density matrix may be much more complicated than that of Eq. (4.43). Indeed when a large number of independent modes is present, it may again approach the Gaussian form (4.31).[21a]

Nevertheless it is of some interest to see the consequences of Eq. (4.43) for an ideal single-mode laser beam. If $V_j(\mathbf{r},t)$ is some particular polarization component of $\mathbf{V}(\mathbf{r},t)$ and

$$\mathcal{R}[V_j(\mathbf{r},t)]=X, \qquad \mathcal{I}[V_j(\mathbf{r},t)]=Y,$$

[21a] A more detailed discussion of the properties of a laser field, and particularly of the spectral linewidth, has recently been given by Lamb [*Quantum Optics and Electronics, Les Houches, 1964*, edited by C. DeWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach Publishers, New York, 1965), p. 377]. (See also Glauber, *ibid.*, p. 165.)

where $\mathcal{R}$ and $\mathcal{I}$ denote the real and imaginary parts, then, corresponding to Eq. (4.34), we have from (4.43)

$$p(X, Y)=\frac{1}{2\pi(X^2+Y^2)^{\frac{1}{2}}}\delta\{(X^2+Y^2)^{\frac{1}{2}}-\langle I_j\rangle^{\frac{1}{2}}\}, \quad (4.44)$$

where

$$\langle I_j\rangle^{\frac{1}{2}}=|\,v_{k',s'}\,||\,(\epsilon_{k',s'})_j\,|\,(hc/k'L^3)^{\frac{1}{2}},$$

and from the properties of the Dirac δ function, it then follows that

$$p(X, Y)=(1/\pi)\,\delta(X^2+Y^2-\langle I_j\rangle). \quad (4.44a)$$

It is seen that X and Y are no longer statistically independent, although it is easy to show that $\langle XY\rangle$ still vanishes (cf. footnote 12). By integrating over one of the variates we find

$$\begin{aligned} p(X)&=\frac{2}{\pi}\int_0^\infty \delta(X^2+Y^2-\langle I_j\rangle)\,dY \\ &=[\pi(\langle I_j\rangle-X^2)^{\frac{1}{2}}]^{-1} \quad \text{for} \quad |\,X\,|<\langle I_j\rangle^{\frac{1}{2}}, \\ &=0 \quad \text{for} \quad |\,X\,|>\langle I_j\rangle^{\frac{1}{2}}. \end{aligned} \quad (4.45)$$

Unlike the Gaussian distribution, this probability distribution has a *minimum* at $X=0$. The two distributions are compared in Fig. 5 and are seen to correspond to well-known positional distributions of the harmonic oscillator in the ground state and in the limit of very high excitation. This suggests that the field of a laser may be pictured as a set of highly excited quantum oscillators. The quantity $\langle I_j\rangle$ is, of course, the expectation value of the light intensity associated with the j component. In the ideal case represented by (4.43), I_j never departs from $\langle I_j\rangle$ and

$$p(I_j)=\delta(I_j-\langle I_j\rangle). \quad (4.46)$$

It is easy to see that these simple distributions for the ideal single-mode laser field become very different when two or more modes are present. Thus suppose that we are dealing with a field consisting of a superposition of n independent laser modes having the same direction of propagation and the same state of polarization. If $p(X)$ describes the distribution of the real part of the resultant complex amplitude and $p_1(X_1)$, $p_2(X_2)$, $\cdots$, $p_n(X_n)$ describe the distributions of the separate components, then evidently

$$\begin{aligned} p(X)=&\int_{-\langle I_1\rangle^{\frac{1}{2}}}^{\langle I_1\rangle^{\frac{1}{2}}}\cdots\int_{-\langle I_n\rangle^{\frac{1}{2}}}^{\langle I_n\rangle^{\frac{1}{2}}} p_1(X_1)\,p_2(X_2)\cdots p(X_n) \\ &\times\delta(X-X_1-X_2-\cdots-X_n)\,dX_1\,dX_2\cdots dX_n. \end{aligned} \quad (4.46a)$$

In the special case of a two-mode field with equal mean intensities $\frac{1}{2}\langle I\rangle$, this becomes, with the help of Eq. (4.45)

$$\begin{aligned} p(X)&=\frac{1}{\pi^2}\int_{X-(\frac{1}{2}\langle I\rangle)^{\frac{1}{2}}}^{(\frac{1}{2}\langle I\rangle)^{\frac{1}{2}}}\frac{dx}{[\frac{1}{2}\langle I\rangle-x^2]^{\frac{1}{2}}[\frac{1}{2}\langle I\rangle-(X-x)^2]^{\frac{1}{2}}}, \\ &\qquad \text{for} \quad |\,X\,|<(2\langle I\rangle)^{\frac{1}{2}} \\ &=0 \quad \text{for} \quad |\,X\,|>(2\langle I\rangle)^{\frac{1}{2}} \end{aligned}$$

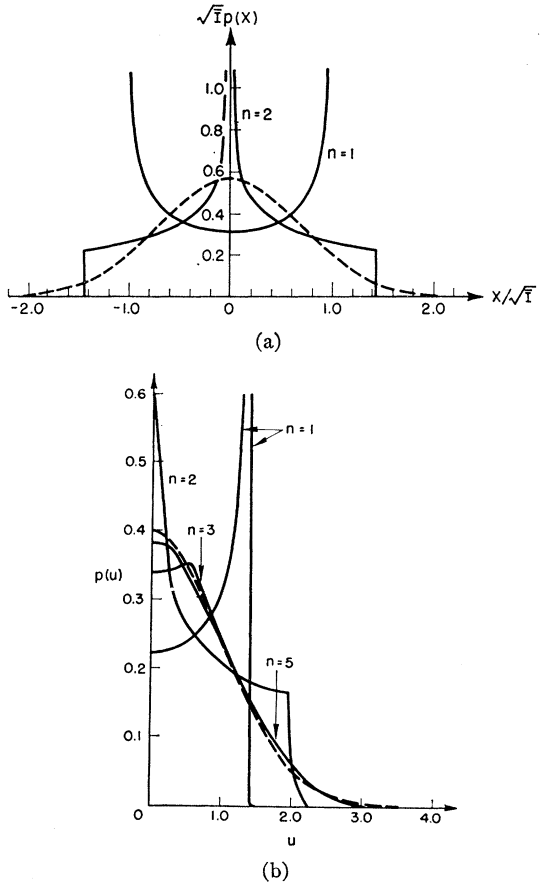


FIG. 5. (a) A comparison of the probability densities $p(X)$ [$X=\mathcal{R}V(r, t)$] for thermal light (broken lines) and light from an ideal maser with one mode ($n=1$) and two independent modes ($n=2$). [After L. Mandel (1964a, 1965).] (b) Probability densities for the real part of the instantaneous complex amplitude resulting from the superposition of n sinusoidal waves together with Gaussian noise. The broken curve shows the distribution for Gaussian noise alone [After H. Hodara (1964).]

which can be shown to be expressible as a complete elliptic integral of the first kind [see, for example, Grobner and Hofreiter (1950), part II, p. 47], i.e.,

$$p(X)=\frac{1}{\pi^2}\left(\frac{2}{\langle I\rangle}\right)^{\frac{1}{2}}K\left[\left(1-\frac{X^2}{2\langle I\rangle}\right)^{\frac{1}{2}}\right]. \quad (4.46b)$$

A similar result has also been obtained by Hodara (1964) for a simple model. This distribution is also illustrated in Fig. 5(a). It will be seen to differ very appreciably from the corresponding single-mode distribution. With the addition of further independent modes $p(X)$ tends rapidly towards the Gaussian form. Hodara (1965) has recently calculated curves for the distribution of the real part of the instantaneous complex amplitude resulting from the superposition of several sine waves plus Gaussian noise. Some of these curves are illustrated in Fig. 5(b). In the absence of Gaussian noise these curves correspond to a particular situation described by the general distribution (4.46a).

As has already been pointed out, in practice the distributions may depart appreciably from the above simple forms and the higher-order correlations will in general be required for a full description of the statistical properties. Among other examples of nonthermal light are the Cerenkov radiation produced by a relativistic electron moving through a medium [cf. Sommerfeld (1954), Sec. 47], and the dipole radiation from an electron beam moving over a periodic structure as in the Smith–Purcell effect [Smith and Purcell (1953); Kastler (1955); Toraldo di Francia (1960); Ishiguro and Tako (1961)]. Little appears to be known about the statistical properties of such radiation.

4.7. Entropy of an Optical Field

For any physical system described by an ensemble of quantum states, it is, in principle, possible to define an entropy according to the rules of Boltzmann, Gibbs, and von Neumann [see, for example, Landau and Lifshitz (1958), Chap. V]. Not surprisingly, the earliest discussions of the entropy for a quantized field were concerned with blackbody radiation in thermal equilibrium at a given temperature. Indeed it was the study of entropy of the electromagnetic field that led Planck to the introduction of quanta [Planck (1901a, b, 1959); see also Klein (1962)]. The problem was taken up also by Einstein (1905).

Later von Laue and other workers applied thermodynamical considerations to problems of interference [von Laue (1906, 1907, 1915c, pp. 405–410] and diffraction [von Laue (1909, 1910); Toraldo di Francia (1948)]. Some related questions were considered by Kahan (1952) and Jones (1953).

Of particular relevance to the subject under review in this article is the early investigation of von Laue (1907). Von Laue studied how the entropy of a light beam which is obtained by the superposition of two or three beams that are partially coherent with respect to each other depends on the intensities of the individual beams and on their degree of coherence. Very recently Gamo (1964a, Sec. 7.2; 1964b) found some interesting generalizations of Laue's results, and showed how one may calculate the entropy of an optical field that is produced by a thermal source and is propagated through an optical system. Gamo's analysis is based on a representation of the partially coherent field as the sum of certain mutually incoherent elementary fields, defined with the help of the sampling theorem of information theory.[22] Such a representation of a partially coherent field was first introduced by Gabor (1956a, b) and by Gamo (1956, 1957a, b, 1958a, b, 1960).

Since Gamo's results were already reviewed elsewhere [Gamo (1964a)] we will restrict ourselves to indicating how an expression for the statistical entropy of an optical field may be derived by means of the Sudarshan representation of the density matrix. For this purpose we recall the conclusion of Sec. 4.3 that we may regard the space spanned by the eigenvalues $\{v_{\mathbf{k},s}\}$ of the annihilation operators $\hat{a}_{\mathbf{k},s}$, for all $\mathbf{k}$, s, as the phase space of the field. It follows that the "diagonal elements" $\Phi(\{v_{\mathbf{k},s}\})$ of the density operator $\hat{\rho}$ of the field, when expressed in the Sudarshan representation in which the $|\{v_{\mathbf{k},s}\}\rangle$ are the basis states, constitute the phase-space distribution of the field.

In view of the commutation rule (4.10) obeyed by the real and imaginary parts of $\hat{a}_{\mathbf{k},s}$, we first quantize the phase space in cells of volume $(\frac{1}{2})^n$, where n is the number of possible $\mathbf{k}$, s modes of the field. Let there be N cells which we label successively by $j=1, 2, \cdots, N$. Then the probability of finding the system in a state corresponding to the jth cell is

$$p_j=\int_{\text{cell } j}\Phi(\{v_{\mathbf{k},s}\})\, d^2\{v_{\mathbf{k},s}\}, \tag{4.47}$$

where the integral extends over the volume of the cell j. Unlike $\Phi(\{v_{\mathbf{k},s}\})d^2\{v_{\mathbf{k},s}\}$, the probability p_j has physical significance in quantum mechanics and is expected to be nonnegative. The statistical or thermodynamic entropy of the field in the mixed state represented by $\hat{\rho}$ is then given by [see, for example, Landau and Lifshitz (1958), pp. 25 and 34]

$$S=-K\sum_{j=1}^{N}p_j\log p_j, \tag{4.48}$$

where K is the Boltzmann constant. The entropy is a measure of the disorder in the field and obviously depends on the correlation properties. As is well known, S is a maximum when the p_j's are all equal. Broadly speaking, this implies that the values of $v_{\mathbf{k},s}$ are then uniformly (and independently for different $\mathbf{k}$, s) distributed over the range allowed by the external constraints.

It is not difficult to give a rough, intuitive description of the significance of maximum entropy for the coherence properties of the field. For, when the range of possible $\mathbf{k}$, s modes and their amplitudes $v_{\mathbf{k},s}$ is as wide as possible, it follows from Eq. (3.47), and the well-known properties of Fourier transforms, that the range of the correlations between the $\mathbf{V}(\mathbf{r}, t)$'s in the conjugate domain of configuration space will be as short as possible. This will hold for correlations both in space and time, and we may roughly speak of the optical field as being in its most incoherent state. It is worth noting that the range of correlations in black-

[22] With the development of statistical theory of communication by Shannon (1948) attempts were made to treat problems of optical transmission and image formation in terms of Shannon's communication "channel" [see, for example, Fellgett and Linfoot (1955); Linfoot (1955); O'Neill and Asakura (1961)]. These treatments, in which the optical image is regarded as a "signal" in the sense of Shannon, lead to expressions for the entropy of the optical image, but this "signal" entropy is entirely distinct from the measure of entropy of the field as understood in statistical mechanics or thermodynamics [cf. Gabor (1950, 1961, p. 148–152); O'Neill and Asakura (1961), p. 301; Gordon (1962)].

body radiation is only of the order of the mean wavelength $\bar{\lambda}$ in space and of the order of the mean period $1/\bar{\nu}$ in time (see Sec. 5.7).

On the other hand, it is easy to show that the field which is completely coherent in the sense of the definition of Sec. 4.4, has minimum entropy. Let the state of the field be an eigenstate $|\{v'_{k,s}\}\rangle$ of the annihilation operator $\hat{\mathbf{A}}^{(+)}(\mathbf{r}, t)$. Then $\Phi(\{v_{k,s}\}\rangle$ reduces to the form (4.21), viz., $\Phi(\{v_{k,s}\}) = \delta(\{v_{k,s}\} - \{v'_{k,s}\})$, and the probabilities p_j given by (4.47) vanish for all j except for the one cell, say $j=l$, containing the point $\{v'_{k,s}\}$, for which $p_l = 1$. It follows that the entropy S vanishes for a coherent field, as might be expected.

It seems plausible also to expect that a field which is coherent to the order $2N$ will have a lower entropy than one which is coherent only to the $2M$th order, when $M<N$, and that for a given order of coherence, the field with the greater correlation range in space–time will have the lower entropy. However the behavior of the entropy under different conditions of coherence (of order higher than 2) has so far not been investigated.

Although the provision of a single measure, such as entropy, for the totality of the coherence relationships of the field may be attractive in some circumstances, the general value of the measure is rather doubtful. It is probably for this reason that the measure of entropy has so far played no part in the development of coherence theory.

5. SOME SECOND-ORDER COHERENCE EFFECTS

5.1. Propagation Laws

In Sec. 3 it was noted that *in vacuo* the mutual coherence function obeys two wave equations. If the field is stationary the equations take the form[23] [Eq. (3.28)]

$$\nabla_j^2 \Gamma = c^{-2}(\partial^2 \Gamma/\partial \tau^2) \qquad (j=1, 2). \tag{5.1}$$

These two wave equations, subject to prescribed boundary and initial conditions, may be solved by standard techniques. The propagation of mutual coherence from a finite plane region σ was considered by Parrent (1959a, b) [see also, Beran and Parrent (1964), p. 40] with the help of Green's function techniques. If S_1 and S_2 are two typical points on σ, P_1 and P_2 are two points in the wave field, $R_1 = S_1P_1$, $R_2 = S_2P_2$, and θ_1 and θ_2 are angles which the lines S_1P_1 and S_2P_2 make with the normal to the source (see Fig. 6), then the

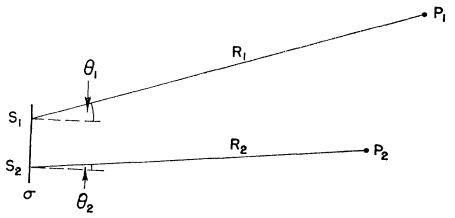


FIG. 6. Illustrating the notation relating to Eq. (5.2), which describes the propagation of mutual coherence Γ from a finite plane region σ *in vacuo*.

solution may be written in the form[24]

$$\Gamma(P_1, P_2, \tau) = \frac{1}{(2\pi)^2}\int\int_\sigma \frac{\cos\theta_1 \cos\theta_2}{R_1^2 R_2^2}\mathfrak{D}\Gamma\left(S_1, S_2, \tau - \frac{R_1 - R_2}{c}\right) dS_1\, dS_2, \tag{5.2}$$

where $\mathfrak{D}$ denotes the differential operator

$$\mathfrak{D} = 1 + \frac{R_1 - R_2}{c}\frac{\partial}{\partial \tau} - \frac{R_1 R_2}{c^2}\frac{\partial^2}{\partial \tau^2}. \tag{5.3}$$

In (5.2) dS_1 and dS_2 denote elements of the surface of the source centered on the points S_1, S_2 and the double integral indicates two-folded integration over σ, as dS_1 and dS_2 explore σ independently. In deriving (5.2) it was assumed that the field obeys the Sommerfeld radiation condition at infinity in the half-space containing the points P_1 and P_2.

If σ is an extended thermal source, then one may assume that the radiation emitted from different elements of σ is effectively mutually incoherent. If, moreover, the radiation is quasimonochromatic and of midfrequency ν_0 then[25] [cf. (3.36)]

$$\Gamma(S_1, S_2, \tau) \sim I(S_1)\delta(S_1 - S_2)\exp(-2\pi i \nu_0 \tau) \qquad (|\tau| < 1/\Delta\nu), \tag{5.4}$$

where $I(S)$ represents the (averaged) intensity at S per unit area of the source. If, moreover $R_1 \gg \lambda_0$, $R_2 \gg \lambda_0$, $(\lambda_0 = c\nu_0)$, the angles θ_1 and θ_2 are sufficiently small and

$$|\tau - (R_1 - R_2)/c| \ll 1/\Delta\nu,$$

(5.2) reduces to

$$\Gamma(P_1, P_2, \tau) \sim \left(\frac{k_0}{2\pi}\right)^2 \exp(-2\pi i \nu_0 \tau)\int_\sigma \frac{I(S)}{R_1 R_2}\exp[-ik_0(R_1 - R_2)]\, dS. \tag{5.5}$$

[23] Since throughout Sec. 5 we shall be concerned with second-order coherence only, we will use the shortened notation Γ for $\Gamma^{(1,1)}$ and γ for $\gamma^{(1,1)}$, as in Sec. 3.

[24] An error in Parrent's formula is corrected here.

[25] The representation of an incoherent source by means of a Dirac delta function is, of course, an idealization not strictly realized in practice, but it is satisfactory for most practical purposes. Some refinements have been discussed by Beran and Parrent (1963).

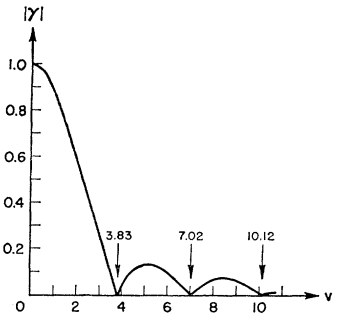


FIG. 7. The behavior of $|\gamma(P_1, P_2, \tau)|$ as function of the distance $P_1P_2=[(x_2-x_1)^2+(y_2-y_1)^2]^{\frac{1}{2}}$ for light from a quasimonochromatic uniform incoherent circular source of radius ρ.

$$v=k_0\rho[(x_2-x_1)^2+(y_2-y_1)^2]^{\frac{1}{2}}/R, \qquad |\tau|\ll 1/\Delta\nu.$$

Equation (5.5) is essentially a mathematical formulation of the so-called *van Cittert–Zernike theorem* [van Cittert (1934); Zernike (1938)]. This formula, which is of the same form as one frequently encounters in the elementary theory of diffraction based on the Huygens–Fresnel principle, is very useful in the solution of coherence problems of instrumental optics.

The van Cittert–Zernike formula (5.5) may be simplified if, as is often the case, the points P_1 and P_2 are situated in a plane parallel to that of the source, and the separation between these two points is small compared with the distance R between the plane containing P_1 and P_2 and the plane of the source σ. Under these conditions (5.5) reduces to

$$\Gamma(P_1, P_2, \tau)\sim\left(\frac{k_0}{2\pi}\right)^2\frac{\exp[i(\psi-2\pi\nu_0\tau)]}{R^2}\int_\sigma I(\xi,\eta)$$

$$\times\exp[ik_0(p\xi+q\eta)]\,d\xi\,d\eta, \qquad (|\tau|\ll 1/\Delta\nu), \quad (5.6)$$

where

$$p=(x_2-x_1)/R, \qquad q=(y_2-y_1)/R, \qquad (5.7a)$$

$$\psi=k_0[(x^2_2+y^2_2)-(x^2_1+y^2_1)]/2R. \qquad (5.7b)$$

Here (x_1, y_1), (x_2, y_2) are coordinates of P_1 and P_2, respectively, and (ξ, η) are coordinates of the source point S referred to Cartesian axes with directions OX, OY in the plane of the source. Equation (5.6) shows that, apart from a simple proportionality factor, the mutual coherence function is now the Fourier transform of the (averaged) intensity distribution across the source.

If, in particular, the quasimonochromatic and spatially incoherent source σ is uniform and circular, of radius ρ, (5.6) and (3.14) then give the following expression for the complex degree of coherence:

$$\gamma(P_1, P_2, \tau)=[2J_1(v)/v]\exp[i(\psi-2\pi\nu_0\tau)]$$

$$(|\tau|\ll 1/\Delta\nu), \quad (5.8)$$

where

$$v=(k_0\rho/R)[(x_2-x_1)^2+(y_2-y_1)^2]^{\frac{1}{2}}, \qquad (5.9)$$

and J_1 is the Bessel function of the first kind and first order. The behavior of $|\gamma|$ for this case is shown in Fig. 7. It is seen that, as the points P_1 and P_2 are separated, the absolute value of $|\gamma|$ decreases from unity and becomes zero (complete second-order incoherence) when $v=3.83$, i.e., when $P_1P_2\equiv[(x_2-x_1)^2+(y_2-y_1)^2]^{\frac{1}{2}}=0.61R\lambda_0/\rho$. As the points are separated still further $|\gamma|$ again acquires nonzero values and becomes zero for the second time when $v=7.02$, i.e., when $P_1P_2=1.12R\lambda_0/\rho$. In instrumental optics the somewhat arbitrary condition $|\gamma|>0.88$ is regarded as representing "almost coherent" light. Now from (5.8) one readily finds that this condition is satisfied when $v<1$, i.e. when $P_1P_2<0.16R\lambda_0/\rho$. Hence one may say that the diameter of the circular area, which a uniform circular quasimonochromatic incoherent source of radius ρ illuminates almost coherently on a plane far away from the source and parallel to it, is $0.16R\lambda_0/\rho$.

We have now obtained a more accurate way of estimating the "area of coherence" than the one given by the order of magnitude relation (2.4). For example, in the case just considered, the area of coherence ΔA, defined by the criterion $|\gamma|>0.88$, is seen to be $\Delta A=\pi(0.16R\lambda_0/\rho)^2=0.25c^2R^2/\nu_0^2S$, where $S=\pi\rho^2$ is the area of the source. This result should be compared with (2.4).

So far we have only considered the distribution of (second order) coherence for propagation in free space from a *plane* surface. If the surface is curved, it is of course in principle still possible to obtain the appropriate solution of the two wave equations (5.1), but there are practical difficulties caused by the requirement of determining the appropriate Green's function. However, various approximate formulas relating to propagation from curved surfaces both in free space and in material media have been found [Zernike (1938); Hopkins (1951, 1953); Wolf (1954a, 1955); Dumontet (1955, 1956b); Born and Wolf (1964), pp. 516, 534]. Some of these formulas bear a somewhat similar relationship to the exact solution of the wave equations for Γ as the Huygens–Fresnel principle bears to the exact solution of the wave equation for the complex disturbance V.

5.2. Stellar Interferometry

Beautiful illustrations of some of the concepts and results of the second-order coherence theory are provided by two methods introduced by Michelson very many years ago. One of these methods concerns the determination of angular diameters of stars, the other concerns the determination of the energy distribution in spectral lines. Originally the principles of these methods were explained in different but equivalent

ways. However, coherence theory provides a deeper understanding of the physical principles involved, and as we see in Sec. 6, it also suggests useful modifications of the experimental techniques. In this section we will consider Michelson's method for the determination of angular diameters of stars. His method for determination of energy distribution in spectral lines will be considered in Sec. 5.3.

As is well known, the angular diameters which stars subtend at the surface of the earth are exceedingly small, so that no available telescopes can resolve a star. In the focal plane of a telescope the star light effectively gives rise to a diffraction pattern appropriate to light from a point source, diffracted at the aperture of the telescope.

Michelson (1890, 1920) [see also, Pease (1931)] showed that information about the angular diameter of the star (and, in principle, also about the intensity distribution across the star) may be obtained with the help of an interferometer, shown schematically in Fig. 8. Light from the star is incident on the outer mirrors M_1 and M_2 of the interferometer, is then reflected at two inner mirrors M_3 and M_4 and is brought to the back focal plane $\mathfrak{F}$ of a telescope to which the interferometer is attached, after passing through two fixed diaphragms S_1 and S_2. The two inner mirrors M_3 and M_4 are fixed while the two outer mirrors M_1 and M_2 can be separated symmetrically in the direction joining S_1 and S_2. In the focal plane $\mathfrak{F}$ of the telescope one then observes the diffraction image of the star on which the telescope is directed, crossed by interference fringes formed by the two beams which pass through the diaphragms.

The visibility of the fringes in the focal plane $\mathfrak{F}$ depends on the separation of the mirrors M_1 and M_2. Michelson showed by an elementary argument that, from measurement of the variation of the visibility with the separation of the two mirrors, one may obtain information about the intensity distribution across the star, at least in cases where one may assume that the distribution is rotationally symmetric. In particular Michelson showed that, if the stellar disk is circular and uniform, the visibility curve, as function of the separation d of the two outer mirrors, will have zeros for certain values of d, and that the smallest of these d-values for which a zero occurs is given by $d_0=0.61\lambda_0/\alpha$, where α is the angular radius of the star. Thus from measurement of d_0 the angular radius of the star may be obtained. Angular diameters of several stars, down to about 0.02 seconds of arc were determined in this way. [Michelson and Pease (1921); Pease (1931).]

From the standpoint of second-order coherence theory the principle of the method is very clear. At the two outer mirrors M_1 and M_2 the incident light is, in general, partially coherent and, according to the simplified form (5.6) of the van Cittert–Zernike theorem, with the appropriate normalization [Eq. (3.14)], the complex degree of coherence is given by (for $|\tau| \ll 1/\Delta\nu$; $\psi \ll 1$)

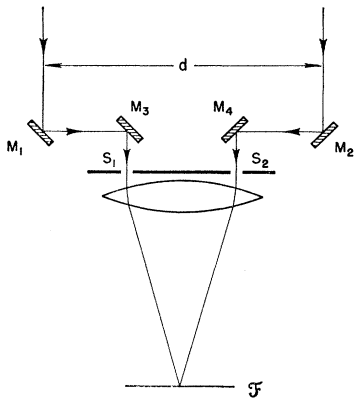


FIG. 8. The Michelson stellar interferometer.

$$\gamma(x_1, y_1; x_2, y_2; \tau) \sim \exp(-2\pi i \nu_0 \tau) \times \frac{\int_\sigma I(u, v) \exp\{-ik_0[(x_1-x_2)u+(y_1-y_2)v]\}\, du\, dv}{\int_\sigma I(u, v)\, du\, dv}. \tag{5.10}$$

Here $I(u, v)$ represents the (averaged) intensity distribution across the star disk, as function of the angular coordinates

$$u=\xi/R, \qquad v=\eta/R,$$

and (x_1, y_1), (x_2, y_2) are the coordinates of the outer mirrors M_1 and M_2. Now according to (3.24) (with $\mu=1$ as appropriate here), $|\gamma|$ represents the visibility of interference fringes which are found when light from the two mirrors M_1 and M_2 is allowed to interfere. Hence the visibility in the focal plane of the telescope is proportional to the absolute value of the Fourier transform of the (averaged) intensity distribution across the star. In particular if the star disk is circular and uniform, (5.10) reduces to (5.8) (with $\psi \approx 0$), and the smallest separation of the mirrors for which γ, and hence the visibility, vanishes is given by $v=3.83$, i.e., $d_0 \equiv [(x_2-x_1)^2+(y_2-y_1)^2]^{\frac{1}{2}}=0.61\lambda_0/\alpha$, in agreement with Michelson's result.

5.3. Interference Spectroscopy

Let us now briefly consider the other method, also due to Michelson, referred to at the beginning of Sec. 5.2, of determining the energy distribution in spectral

lines. Here a beam of quasimonochromatic light is divided into two beams in a Michelson interferometer (Fig. 1) and the beams are superposed after some path difference $c\tau$ has been introduced between them. In the region of superposition interference fringes are formed, the visibility of which depends on the path difference $c\tau$. Michelson showed that, from the measured visibility curve (i.e., the curve which represents the visibility $\mathcal{V}$ as function of τ), one may obtain information about the energy spectrum of the light.

The principle of the method may again be readily understood from the standpoint of coherence theory. If we assume the two beams to be of the same intensity, then, by an argument similar to that leading to (3.24), the visibility of the fringes is given by

$$\mathcal{V}(\tau) = |\gamma(\tau)|. \tag{5.11}$$

Here, for brevity, $\gamma(\tau)$ is written in place of $\gamma(\mathbf{r}, \mathbf{r}, t)$, where $\mathbf{r}$ specifies the position of the (fixed) dividing mirror M_1. Let us represent $\gamma(\tau)$ as a Fourier integral,

$$\gamma(\tau) = \int_0^\infty w(\nu) \exp(-2\pi i\nu\tau)\, d\nu, \tag{5.12}$$

where, according to (3.29) and (3.14), $w(\nu)$ denotes the normalized spectral density of the light. [No negative frequency components are present in the Fourier integral (5.12), since $\gamma(\tau)$ is an analytic signal.]

It will be useful to express $\gamma(\tau)$ in the form

$$\gamma(\tau) = \tilde{\gamma}(\tau) \exp(-2\pi i\nu_0\tau), \tag{5.13}$$

where ν_0 is the midfrequency of the light. Then from (5.12) and (5.13),

$$\tilde{\gamma}(\tau) = \int_{-\infty}^\infty \tilde{w}(\mu) \exp(-2\pi i\mu\tau)\, d\mu, \tag{5.14}$$

with

$$\begin{aligned} \tilde{w}(\mu) &= w(\nu_0+\mu) \quad \text{when} \quad \mu \geq \nu_0, \\ &= 0 \qquad\quad \text{when} \quad \mu < -\nu_0. \end{aligned} \tag{5.15}$$

From (5.11), (5.13), and (5.14) it follows that

$$\mathcal{V}(\tau) = \left| \int_{-\infty}^\infty \tilde{w}(\mu) \exp(-2\pi i\mu\tau)\, d\mu \right|. \tag{5.16}$$

Suppose first that the spectrum $w(\nu)$ is symmetric about the mid frequency ν_0. Then $\tilde{w}$ will be an even function and hence the integral appearing in (5.16) will be real. It follows that in this case one may write in place of (5.16)

$$\mathcal{V}(\tau) = \pm \int_{-\infty}^\infty \tilde{w}(\mu) \exp(-2\pi i\mu\tau)\, d\mu, \tag{5.17}$$

or, taking the Fourier inverse and using the fact that $\mathcal{V}$ is an even function of τ [which follows from (5.11), (3.14), and (3.12)],

$$\tilde{w}(\mu) \equiv w(\nu_0+\mu) = \pm 2 \int_0^\infty \mathcal{V}(\tau) \cos 2\pi\mu\tau\, d\tau. \tag{5.18}$$

Equation (5.18) shows that, under the assumption of *symmetry*, the spectral energy distribution, referred to the midfrequency ν_0 as origin, may be calculated from the knowledge of the visibility curve, provided the ambiguity in the sign of the integral may be removed; this may usually be done by an appeal to physical plausibility.

If, on the other hand, the spectrum is not symmetric, the Fourier transform of the "shifted" spectral density $\tilde{w}(\mu)$ is no longer everywhere real and so Eq. (5.16) no longer leads to (5.17). In order to determine the spectrum, according to (5.11), (5.13), and (5.14) it is now necessary to know, in addition to the visibility curve, the phase $\alpha(\tau)$ of the Fourier transform $\tilde{\gamma}(\tau)$ of $\tilde{w}(\mu)$, or alternatively the phase $\phi(\tau)$ of the complex degree of coherence. The two are of course, related by the equation

$$\phi(\tau) = \alpha(\tau) - 2\pi\nu_0\tau, \tag{5.19}$$

which follows from (5.13). According to (3.25), $\phi(\tau)$ may be determined from measurement of the position of the maxima of the fringe pattern. Unfortunately, such positional measurements are exceedingly difficult to perform and, in fact, apart from those of Pérard (1928, 1935), none appear to have been reported in the literature.

It has been commonly assumed, following a discussion of this question by Rayleigh (1892), that measurements of the visibility and the position of the fringes provide two *independent* sets of information, so that both have to be performed in order to obtain the data necessary for determining asymmetric spectral profiles. However, it has recently been pointed out by Wolf (1962) that this is not necessarily so, since the analytic properties of the complex degree of coherence and the physical constraint that its spectrum be nonnegative impose severe restrictions on the phase $\phi(\tau)$ of $\gamma(\tau)$ [see also, Marathay (1963); Roman and Marathay (1963)].

To indicate the nature of the restriction arising from the analytic properties of γ we note that, since $\gamma(\tau) = |\gamma(\tau)| \exp[i\phi(\tau)]$ is analytic and regular in the lower half-plane II of the complex τ-plane [this is so since $\gamma(\tau)$ is an analytic signal—cf. footnote 12], the function

$$\log\gamma(\tau) = \log|\gamma(\tau)| + i\phi(\tau) \tag{5.20}$$

will evidently also be analytic in II, but it will have logarithmic branch points at points (if any) where

$\gamma(\tau)$ has zeros. Assume first that $\gamma(\tau)$ has no zeros at all in II. In that case $\log\gamma(\tau)$ will not only be analytic but will also be regular in II and hence its real and imaginary parts, just like the real and imaginary parts of γ, will be coupled by Hilbert transform relations. One of these two relations is

$$\phi(\tau)=\pi^{-1}P\int_{-\infty}^{\infty}\frac{\log|\gamma(\tau')|}{\tau'-\tau}\,d\tau'. \tag{5.21}$$

Hence in this case, the phase $\phi(\tau)$ is uniquely specified by $|\gamma(\tau)|$ itself. If, however, $\gamma(\tau)$ has zeros in II at points $\tau=\tau_i$ $(i=1, 2, \cdots)$, (5.21) no longer applies, but the following modified relation may be shown to hold instead [cf. Toll (1956), p. 1767]:

$$\phi(\tau)=\pi^{-1}P\int_{-\infty}^{\infty}\frac{\log|\gamma(\tau')|}{\tau'-\tau}\,d\tau'+\sum_i\arg\frac{\tau-\tau_i}{\tau-\tau_i^*}, \tag{5.22}$$

where the summation is extended over all the zeros τ_i of γ in the lower half-plane II.

The relation (5.22) together with (5.11) shows that the phase $\phi(\tau)$ of $\gamma(\tau)$, and hence the spectral profile, may be determined from the knowledge of the visibility function $\mathcal{V}(\tau)$ and the location of the zeros of γ in the half-plane II. Unfortunately nothing appears to be known at present about the physical significance of the zeros, nor about their location in particular cases. However, the location of the zeros cannot be quite arbitrary, since, as already noted, one has the additional constraint that the Fourier transform must be nonnegative. It may be readily shown that each of the additive factors $\arg[(\tau-\tau_i)/(\tau-\tau^*_i)]$ in (5.22) is nonnegative, so that the first term on the right-hand side of (5.22) represents a *lower bound* on the phase function that may be associated with any given $|\gamma(\tau)|$.

It has been shown by Kano and Wolf (1962) that the complex degree of temporal coherence $\gamma(\tau)$ of blackbody radiation has no zeros at all in the lower half-plane II, so that its phase, and hence the spectrum, can be determined from its absolute value $|\gamma(\tau)|$, in spite of the fact that the spectrum of this radiation is not symmetric.

Michelson's method, which, as we just saw, still poses some unsolved problems, is to some extent being superseded by another interferemetric technique, the so-called method of the *interferogram* [see, for example, Fellgett (1958a, b); Jacquinot (1958, 1960); Strong and Vanasse (1959)]. This method, which is proving particularly useful in the infrared region of the spectrum, allows direct determination of the real part $\gamma^{(r)}$ rather than determination of $|\gamma|$. From the knowledge of $\gamma^{(r)}$ as function of τ the spectrum may, in principle, be determined unambiguously.

Two alternative experimental approaches to the recovery of the phase $\phi(\tau)$ from $|\gamma(\tau)|$ have also been suggested recently. One due to Gamo (1963a, b) is based on a triple intensity correlation measurement, and is briefly described in Sec. 6.4. The other due to Mehta (1965a) is based on two separate sets of measurements of $|\gamma(\tau)|$ and $|\gamma_f(\tau)|$, where $\gamma_f(\tau)$ refers to the beam after it has been passed through a particular kind of linear filter.

Finally it may be mentioned, that the problem of complete determination of the spectral distribution of energy from the knowledge of the absolute value of the complex degree of coherence is strictly analogous to problems arising in other branches of physics and engineering, e.g., in the quantum theory of decay [Khalfin (1957, 1960, 1961)], in scattering theory, particularly in the x-ray region [Goldberger, Lewis, and Watson (1963)], in the diffraction theory of image formation [O'Neill and Walther (1963); Walther (1963)] in the theory of linear electric filters [Lee (1932)], and in other fields.

5.4. Coherence Time and Bandwidth

With the help of the second-order complex degree of coherence one may define the coherence time of light somewhat more precisely than by the order of magnitude relation (2.1).

Consider again the simple interference experiment illustrated in Fig. 1, where a beam of light is divided in a Michelson interferometer into two beams, which are then brought to interference after a time delay τ is introduced between them. Since the visibility of the interference fringes observed in the plane $\mathcal{O}$ is proportional to the absolute value of the degree of coherence $\gamma(\mathbf{r}, \mathbf{r}, \tau)$ ($\mathbf{r}$ specifies the position of the dividing mirror M_1), it is evident that one may define a coherence time Δt in terms of $|\gamma(\mathbf{r}, \mathbf{r}, \tau)|$. One possible definition is provided by the formula [Wolf (1958)]

$$(\Delta_1 t)^2=N^{-1}\int_{-\infty}^{\infty}\tau^2|\gamma(\mathbf{r}, \mathbf{r}, \tau)|^2\,d\tau, \tag{5.23}$$

where

$$N=\int_{-\infty}^{\infty}|\gamma(\mathbf{r}, \mathbf{r}, \tau)|^2\,d\tau=\int_0^{\infty}w^2(\mathbf{r}, \mathbf{r}, \nu)\,d\nu, \tag{5.24}$$

and $w(\mathbf{r}, \nu)$ again denotes the normalized spectral density of the light. The equality of the two integrals in (5.24) follows from Parceval's theorem on Fourier transforms [Sneddon (1951), p. 25].

If the effective bandwidth of the light is defined by the equation

$$(\Delta_1\nu)^2=N^{-1}\int_0^{\infty}(\nu-\bar{\nu})^2w^2(\mathbf{r}, \mathbf{r}, \nu)\,d\nu, \tag{5.25}$$

with

$$\bar{\nu}=N^{-1}\int_0^\infty \nu w^2(\mathbf{r},\mathbf{r},\nu)\,d\nu, \tag{5.26}$$

then one may show, by a similar argument as that given by Weyl and Pauli [Weyl (1931), pp. 77, 393] in proving the uncertainty relation, that the following reciprocity inequality always holds [Wolf (1958); Born and Wolf (1964), pp. 541, 542; see also Gabor (1946)]:

$$(\Delta_1 t)(\Delta_1 \nu)\geq 1/4\pi. \tag{5.27}$$

For quasimonochromatic light whose profile approximates to the Gaussian distribution, the inequality sign may be replaced by approximate equality. However, in view of the fact that the lower limits in the frequency integrals in Eqs. (5.24)–(5.26) are zero rather than minus infinity, the equality sign in (5.27) can never be attained[26] [cf. Mayer and Leontovich (1934); Kay and Silverman (1957, 1959); Kharkevich (1960)].

Another measure of coherence time has been suggested by Mandel (1959) [see also, Eq. 6.14) below], from considerations of the extent of the unit cell of photon phase space. It is also defined in terms of γ, as

$$(\Delta_2 t)=\int_{-\infty}^{\infty} |\,\gamma(\mathbf{r},\mathbf{r},\tau)\,|^2\,d\tau. \tag{5.28}$$

If the bandwidth is then defined as

$$(\Delta_2 \nu)=\left[\int_0^\infty w^2(\mathbf{r},\mathbf{r},\nu)\,d\nu\right]^{-1}, \tag{5.29}$$

then, in view of (5.24), one always has

$$(\Delta_2 t)(\Delta_2 \nu)=1. \tag{5.30}$$

For simple types of spectral profile one finds that $(\Delta_1 t)\sim(\Delta_2 t)$, $(\Delta_1 \nu)\sim(\Delta_2 \nu)$ [cf. Mandel and Wolf (1962); Mehta (1963)]. However, if the spectral density distributions are of a more complicated form, as for example in the case of an optical maser (multiply peaked spectral distributions), the two sets of definitions may lead to results of quite different order of magnitude [Mandel and Wolf (1962)]. Hence caution must be exercised in applying a particular definition in certain cases. It is of interest to note that a somewhat similar situation arises in connection with measures of uncertainties of radar signals and the corresponding measures of bandwidth [cf. Woodward (1953), Chap. 7]. The definitions which we discussed in this section are appropriate for experiments where a beam of light is divided into two beams in a region, around a point $P(\mathbf{r})$ in the field. For experiments involving division at two or more widely separated points (e.g., as in the case of Young's interference experiment) more general definitions may be needed [cf. Wolf (1958), Sec. 5; Born and Wolf (1964), pp. 543–544]. However, when the normalized spectral density is the same at every point in the field, when the spectra are of relatively simple form, and when, in addition, the light is cross-spectrally pure (cf. next section), then the definitions given above seem to be quite adequate for most practical purposes.

[26] Since $\gamma(\tau)$ is an analytic signal, only those complex correlation functions are admissible in the calculation of $(\Delta_1 t)$ whose Fourier transforms $w(\nu)$ vanish identically for $\nu<0$.

5.5. Cross-Spectral Purity

We have already mentioned (p. 240) that in general spatial and temporal (second-order) coherence cannot be sharply separated. These two types of coherence are related because the mutual coherence function Γ obeys the two wave equations (3.28), which relate the variation of Γ with respect to its space and time variables. Nevertheless, it may be shown that in many cases of practical interest the second-order complex degree of coherence may be expressed, at least to a good approximation, as the product of two functions, one of which represents spatial coherence, the other temporal coherence. The possibility of such a factorization has been studied by Mandel (1961c), who also showed that it is intimately connected with certain spectral properties of the light.

To begin with let us restrict ourselves to linearly polarized light. The field may then be represented by a complex stochastic scalar wave function $V(\mathbf{r},t)$. Suppose that $V_1(t)\equiv V(\mathbf{r}_1,t)$, $V_2(t)\equiv V(\mathbf{r}_2,t)$ represent the field at two points $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$, respectively, and let $w_1(\nu)$ and $w_2(\nu)$ be their normalized spectral density functions, i.e., the Fourier transforms of their respective temporal normalized correlations $\gamma(\mathbf{r}_1,\mathbf{r}_1,\tau)$ and $\gamma(\mathbf{r}_2,\mathbf{r}_2,\tau)$. Suppose that the light from P_1 and P_2 is superposed at a point $P_3(\mathbf{r}_3)$, and let $w_3(\nu)$ be the corresponding normalized spectral density of the light at this point. In general $w_3(\nu)$ will not be simply related to $w_1(\nu)$ and $w_2(\nu)$, even if $w_1(\nu)$ and $w_2(\nu)$ are identical.

If $w_2(\nu)\equiv w_1(\nu)$ and a region of neighboring points P_3 exists such that $w_3(\nu)\equiv w_1(\nu)$, then $V_1(t)$ and $V(t)$ are said to be *cross-spectrally pure*. It was shown by Mandel (1961c) that in this case $\gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)$ may be expressed as a product of two factors, according to the following *reduction formula*, valid for all τ:

$$\gamma(\mathbf{r}_1,\mathbf{r}_2,\tau)=\gamma(\mathbf{r}_1,\mathbf{r}_2,\tau_0)\gamma(\mathbf{r}_1,\mathbf{r}_1,\tau-\tau_0). \tag{5.31}$$

Here τ_0 is a constant representing the time delay $([P_1P_3]-[P_2P_3])/v$, where $[\cdots]$ denotes optical path length in the medium, assumed to be *nondispersive*, in which the light is propagated and v is the velocity of

light in the medium. The first factor on the right of (5.31) represents spatial coherence, the other temporal coherence. A converse of the result which we just mentioned is evidently also true: If (5.31) holds, the light at P_1 and P_2 is cross-spectrally pure.

It has also been shown with the help of a propagation law for the mutual coherence function that, if the reduction formula (5.31) holds for all pairs of points on a (fictitious) surface intercepting a light beam and if, moreover, the normalized spectral density is constant on this surface, then (5.31) will also hold throughout the field propagated from the surface, provided the dispersion properties of the medium and the path differences involved are sufficiently small. In particular, if the light originates in a spatially incoherent source, then for any two points on the source $\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ may be approximately expressed in the form[25] $\delta(\mathbf{r}_2-\mathbf{r}_1)\, f(\mathbf{r}_1, \tau)$, where f is some function of $\mathbf{r}_1$ and τ; hence the light is cross-spectrally pure at the source. If, moreover, the normalized spectral density is the same at every element of the source, then, according to the result just quoted, the field produced by the source is also cross-spectrally pure, subject, of course, to the restrictions on dispersive properties of the medium and the path differences involved.

An interesting behavior of cross-spectrally pure beams is described by the spectral modulation law (Mandel (1962a)], expressed by Eq. (5.33) below. Suppose that the beams are superposed at the point P, such that the optical path length $[P_1P]=v\tau_1$, $[P_2, P]=v\tau_2$. Now whereas for cross-spectrally pure beams the normalized spectral density remains unchanged when the path delay $v\,|\,(\tau_1-\tau_2)-\tau_0\,|$ is small compared to the coherence length ($c\tau_c$ say), it becomes cosine modulated when $v\,|\,(\tau_1-\tau_2)-\tau_0\,|$ becomes large, and the amplitude of this modulation depends on the degree of spatial coherence of the light. More precisely, for cross-spectrally pure beams one has

(1) If $|\,(\tau_1-\tau_2)-\tau_0\,| \ll \tau_c$:

$$w(\nu) \equiv w_1(\nu)\,[\equiv w_2(\nu)]. \tag{5.32}$$

(2) If $|\,(\tau_1-\tau_2)-\tau_0\,| \gg \tau_c$:

$$w(\nu) = w_1(\nu)\,\{1+\sigma\,|\,\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau_0)\,| \times \cos\,[\arg\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau_0) - 2\pi\nu_0(\tau_1-\tau_2-\tau_0)]\}, \tag{5.33}$$

where

$$\sigma = 2\Big/\left[\left(\frac{\langle I_1\rangle}{\langle I_2\rangle}\right)^{\frac{1}{2}}+\left(\frac{\langle I_2\rangle}{\langle I_1\rangle}\right)^{\frac{1}{2}}\right]. \tag{5.33a}$$

It is of interest to compare these results with the corresponding relations relating to the (time averaged) *intensity* distribution arising from the superposition of two beams:

(3) If $|\,(\tau_1-\tau_2)-\tau_0\,| \ll \tau_c$:

$$\langle I\rangle = [\langle I_1\rangle+\langle I_2\rangle]\{1+\sigma\,|\,\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau_0)\,| \times\cos\,[\arg\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau_0) - 2\pi\nu_0(\tau_1-\tau_2-\tau_0)]\}. \tag{5.34}$$

(4) If $|\,(\tau_1-\tau_2)-\tau_0\,| \gg \tau_c$:

$$\langle I\rangle = \langle I_1\rangle+\langle I_2\rangle. \tag{5.35}$$

Equation (5.34) follows from the interference law (3.20), when (3.26) is used. Equation (5.35) follows from (3.17) when one uses the fact that $\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)\sim 0$ for $\tau \gg \tau_c$. Equation (5.34) shows that when $|\,\tau_1-\tau_2-\tau_0\,|$ is small compared to the coherence time, the degree of coherence $|\,\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau_0)\,|$ may be determined from measurement of the intensity variation in the pattern. When $|\,\tau_1-\tau_2-\tau_0\,|$ greatly exceeds the coherence time, determination of the degree of coherence by this method becomes impossible according to (5.35). However, the spectral modulation law (5.33) shows that in this case the determination is, in principle, possible from the analysis of the spectrum of the combined beam in the region of superposition, provided the light is cross-spectrally pure. It is remarkable that the modulation of the spectrum persists for arbitrarily large path difference $v(\tau_1-\tau_2)$, although there is a lower limit on this quantity. This result has a bearing on investigations which have their origin in an experiment relating to the measurement of the velocity of light, carried out by Alford and Gold (1958) [cf. the end of Sec. 7.2).

So far we have considered only the case of linearly polarized light, but the concept of cross-spectral purity and the associated reduction formulas are, of course, also relevant for light of other states of polarization. In general we must then consider in place of the (second-order) scalar correlation function $\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ the second-order correlation tensor $\gamma_{jk}(\mathbf{r}_1, \mathbf{r}_2, \tau)$, where j and k are polarization indices, and we must then examine the possibility of expressing each component of the tensor as product of simpler terms. Relatively little is known about such questions but some problems of this type have been briefly considered by Mandel and Wolf (1961b, Sec. 3) and Mandel (1963c, p. 1107).

5.6. Partial Polarization

In the last part of Sec. 3.1, second-order coherence tensors of the electromagnetic field were defined and it was mentioned that these tensors lead to a unified treatment of coherence and polarization effects. In the present section we will show how the degree of polar-

ization of a quasimonochromatic plane wave can be expressed in terms of the rotational invariants of the associated coherence matrix, and we will also discuss some related questions.

Consider a stationary and ergodic ensemble of quasimonochromatic plane waves, of midfrequency ν_0 and effective bandwidth $\Delta\nu \ll \nu_0$, propagated in the direction of the positive z axis. Consider a typical wave of this ensemble and let $E_x(\mathbf{r}, t)$, $E_y(\mathbf{r}, t)$ be the complex analytic signals associated with the components of the (real) electric vector in two mutually orthogonal directions perpendicular to the z direction, with x, y, z forming a right-handed triad. Because of transversality of plane waves ($E_z=0$), the electric coherence tensor (3.38) when referred to the x y z directions will have (at most) four nonvanishing components, namely those with suffixes xx, xy, yx, and yy. Let us consider the associated 2×2 matrix

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = [\langle E^*_j(\mathbf{r}_1, t) E_k(\mathbf{r}_2, t+\tau)\rangle], \qquad (j, k = x, y), \quad (5.36)$$

where on the right of (5.36) either an ensemble average or a time average of a typical member of the ensemble may be taken, because of our assumption of stationarity and ergodicity.

We will only consider correlations between the complex field components at the same point ($\mathbf{r}_1=\mathbf{r}_2=\mathbf{r}$) and for small values of $|\tau|$ ($|\tau| \ll 1/\Delta\nu$). Under the last assumption, each element of (5.36) depends on τ only through a multiplicative factor $\exp(-2\pi i\nu_0\tau)$ [cf. (3.36)] so that we then have[27]

$$\Gamma(\mathbf{r}, \mathbf{r}, \tau) = \mathcal{J}^*(\mathbf{r}) \exp(-2\pi i\nu_0\tau), \qquad |\tau| \ll 1/\Delta\nu, \quad (5.37)$$

where $\mathcal{J}^*$ is the matrix which is the complex conjugate of the matrix

$$\mathcal{J} = [\langle E_j(\mathbf{r}, t) E^*_k(\mathbf{r}, t)\rangle] \qquad (j, k = x, y). \quad (5.38)$$

$\mathcal{J}$ is called the *coherence matrix* [Wolf (1954b, 1959); Parrent and Roman (1960)]. Since the waves are assumed to be plane, the matrix $\mathcal{J}=[\mathcal{J}_{jk}]$ will be the same for all points $\mathbf{r}$ which are situated in any one plane perpendicular to the direction of propagation of the wave.

[27] We write $\mathcal{J}^*$ rather than $\mathcal{J}$ on the right-hand side of (5.37), so that the coherence matrix $\mathcal{J}$ is of the customary form and is then formally closely analogous to the density matrix. However, the analogy between the coherence matrix and the density matrix is only partial. Moreover the trace of $\mathcal{J}$ is not unity.

It will be useful to introduce the column vector

$$\mathcal{E}(\mathbf{r}, t) = \begin{pmatrix} E_x(\mathbf{r}, t) \\ E_y(\mathbf{r}, t) \end{pmatrix} \quad (5.39)$$

and its Hermitian conjugate, the row vector

$$\mathcal{E}^\dagger(\mathbf{r}, t) = (E^*_x(\mathbf{r}, t) \quad E^*_y(\mathbf{r}, t)). \quad (5.40)$$

The coherence matrix $\mathcal{J}$ may then be written in the form

$$\mathcal{J}(\mathbf{r}) = \langle \mathcal{E}(\mathbf{r}, t)\mathcal{E}^\dagger(\mathbf{r}, t)\rangle. \quad (5.41)$$

This coherence matrix is clearly Hermitian: $\mathcal{J}^\dagger=\mathcal{J}$. The averaged intensity at the point $\mathbf{r}$ is, in suitable units, given by

$$\langle I(\mathbf{r}, t)\rangle = \langle E^*_x(\mathbf{r}, t) E_x(\mathbf{r}, t)\rangle + \langle E^*_y(\mathbf{r}, t) E_y(\mathbf{r}, t)\rangle = \langle \mathcal{E}^\dagger\mathcal{E}\rangle = \mathrm{Tr}\, \mathcal{J}. \quad (5.42)$$

Let us now consider how the coherence matrix changes when the light passes through some simple (non-image forming) optical device such as a compensator or a polarizer. The vector $\mathcal{E}'$ representing the emerging field is related to the vector $\mathcal{E}$ representing the incident field by a relation of the form

$$\mathcal{E}' = \mathfrak{L}\mathcal{E}, \quad (5.43)$$

where $\mathfrak{L}$ is a 2×2 matrix which characterizes the optical device. It follows that the coherence matrix $\mathcal{J}'$ of the emergent field is given by

$$\mathcal{J}' = \langle \mathcal{E}'\mathcal{E}'^\dagger\rangle = \langle \mathfrak{L}\mathcal{E}\mathcal{E}^\dagger\mathfrak{L}^\dagger\rangle = \mathfrak{L}\langle \mathcal{E}\mathcal{E}^\dagger\rangle\mathfrak{L}^\dagger = \mathfrak{L}\mathcal{J}\mathfrak{L}^\dagger. \quad (5.44)$$

From (5.42) and (5.44) it follows that the averaged intensity of the light emerging from the device is given by

$$\langle I'\rangle = \mathrm{Tr}\,(\mathfrak{L}\mathcal{J}\mathfrak{L}^\dagger). \quad (5.45)$$

The form of the matrix $\mathfrak{L}$ for some simple optical devices has been discussed by Parrent and Roman (1960). [See also, Jones (1941a, b, 1942, 1948); Hsu, Richartz, and Liang (1947); Marathay (1963); O'Neill (1963), Chap. IX]. For a *compensator* which introduces small retardations $\epsilon_1 \ll \nu_0/\Delta\nu$ and $\epsilon_2 \ll \nu_0/\Delta\nu$ between the x and y components, respectively, $\mathfrak{L}=\mathfrak{L}_c$, where

$$\mathfrak{L}_c = \begin{pmatrix} e^{i\epsilon_1} & 0 \\ 0 & e^{i\epsilon_2} \end{pmatrix}. \quad (5.46)$$

For a *polarizer* which only transmits a component of the electric field making an angle θ with the x direction in a plane perpendicular to the direction of propagation,

$\mathfrak{L}=\mathfrak{L}_p$, where

$$\mathfrak{L}_p=\begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}. \tag{5.47}$$

For a transmission through a succession of several devices, the formulas (5.43)–(5.45) still apply, with $\mathfrak{L}$ being given by the product of the $\mathfrak{L}$ matrices which characterize transmission through the individual elements of the device. Thus for a transmission through a compensator, followed by a polarizer,

$$\mathfrak{L}=\mathfrak{L}_p\mathfrak{L}_c=\begin{pmatrix} e^{i\epsilon_1}\cos^2\theta & e^{i\epsilon_2}\cos\theta\sin\theta \\ e^{i\epsilon_1}\sin\theta\cos\theta & e^{i\epsilon_2}\sin^2\theta \end{pmatrix}. \tag{5.48}$$

From (5.48) and (5.45) one readily finds that the averaged intensity of the emerging light wave is given by

$$\langle I'(\delta,\theta,t)\rangle=\mathcal{J}_{xx}\cos^2\theta+\mathcal{J}_{yy}\sin^2\theta + (\mathcal{J}_{xy}e^{-i\delta}+\mathcal{J}_{yx}e^{i\delta})\sin\theta\cos\theta, \tag{5.49}$$

where

$$\delta=\epsilon_2-\epsilon_1, \tag{5.50}$$

and $\mathcal{J}_{xx}$, etc., are the elements, given by (5.38), of the coherence matrix of the light incident on the device. Equation (5.49) shows that the elements of the coherence matrix of a plane, quasimonochromatic light wave may be determined from measurements of the averaged intensity $\langle I'(\delta,\theta,t)\rangle$ for a number of selected values of the parameters δ and θ; i.e., for a number of selected values of the phase delay and the angle of transmission introduced by passing the wave through a compensator and a polarizer [cf. Wolf (1959), Eq. (3.6); Born and Wolf (1964), p. 546]. This determination of the elements of the coherence matrix is analogous to the determination of the mutual coherence function $\Gamma(r_1, r_2, \tau)$ from intensity measurements in a Young's interference experiment (cf. Sec. 3.1). In fact, the analogy between the two situations is closer still, as will now be shown.

If we set

$$\langle I^{(1)}\rangle=\mathcal{J}_{xx}\cos^2\theta, \qquad \langle I^{(2)}\rangle=\mathcal{J}_{yy}\sin^2\theta \tag{5.51}$$

and[28]

$$\gamma_{xy}(0)=|\gamma_{xy}(0)|\exp[i\beta_{xy}(0)]=\mathcal{J}_{xy}/(\mathcal{J}_{xx})^{\frac{1}{2}}(\mathcal{J}_{yy})^{\frac{1}{2}}, \tag{5.52}$$

Eq. (5.49) reduces to

$$\langle I'(\delta,\theta,t)\rangle=\langle I^{(1)}\rangle+\langle I^{(2)}\rangle + 2\langle I^{(1)}\rangle^{\frac{1}{2}}\langle I^{(2)}\rangle^{\frac{1}{2}}|\gamma_{xy}(0)|\cos[\beta_{xy}(0)-\delta]. \tag{5.53}$$

This formula is very similar to Eq. (3.20), which refers to interference between two partially coherent beams. The coefficient $\gamma_{xy}(0)$ now plays the role of the second-order complex degree of coherence $\gamma(\mathbf{r}_1, \mathbf{r}_2, 0)$ of the scalar wave field. It is seen from (5.52) and (5.38) that $\gamma_{xy}(0)$ represents the correlation between the components of the complex electric field at the point $\mathbf{r}$.

Light which is most frequently encountered in nature is *completely unpolarized* (or natural) light. Such light is characterized by the fact that the averaged intensity $\langle I'(\delta,\theta,t)\rangle$ is independent of δ and θ; i.e., it is independent of any phase delay which may be introduced between the x and y components of its electric field and is the same for all its azimuthal components. One may readily show from (5.49) that, for this to be the case, the coherence matrix must be given by

$$\mathcal{J}=\tfrac{1}{2}\langle I\rangle\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.54}$$

where $\langle I\rangle=\mathcal{J}_{xx}+\mathcal{J}_{yy}$ is the total averaged intensity of the light. Thus the coherence matrix of unpolarized light is proportional to the unit matrix.

According to (5.54), the off-diagonal elements of the coherence matrix of unpolarized light are zero. Hence any two mutually orthogonal components of the complex electric field of an unpolarized wave, which are perpendicular to the direction of propagation, are completely uncorrelated [$\gamma_{xy}(0)=0$]. The other extreme case, where the components are *completely correlated*, represents what is traditionally described as completely polarized light [cf. Born and Wolf (1964), pp. 549–550]. In this case $|\gamma_{xy}(0)|=1$, and according to (5.52) this condition implies that $\det\mathcal{J}=\mathcal{J}_{xx}\mathcal{J}_{yy}-\mathcal{J}_{xy}\mathcal{J}_{yx}=0$.

It has been shown by Wolf (1959) that the coherence matrix $\mathcal{J}$ of any quasimonochromatic plane light wave may be uniquely decomposed into the sum of two coherence matrices $\mathcal{J}^{(u)}$ and $\mathcal{J}^{(p)}$, the first of which represents a completely unpolarized wave and the other a completely polarized one.[29] Since the averaged intensity of a wave is, according to (5.42), given by the trace of its coherence matrix, it follows that the ratio

$$P=\operatorname{Tr}\mathcal{J}^{(p)}/(\operatorname{Tr}\mathcal{J}^{(p)}+\operatorname{Tr}\mathcal{J}^{(u)}) \tag{5.55}$$

represents the *degree of polarization* of the wave. Explicit calculations give [Wolf (1959); Born and Wolf (1964), pp. 550–552]

$$P=\{1-[4\det\mathcal{J}/(\operatorname{Tr}\mathcal{J})^2]\}^{\frac{1}{2}}, \tag{5.56}$$

[28] The argument zero in γ_{xy}, $|\gamma_{xy}|$, and β_{xy} is written here to stress that the quantities refer to correlation for zero τ delay between E_x and E_y.

[29] A somewhat different decomposition into two fully polarized components is more useful for discussing intensity fluctuations [Mandel (1963 c)].

and this formula expresses the degree of polarization of the wave in terms of the rotational invariants of its coherence matrix.

There are a number of equivalent alternative expressions for the degree of polarization. Thus it may be shown [see, for example, Wolf (1959); Parrent and Roman (1960); Born and Wolf (1964), p. 552], that in terms of the eigenvalues $\langle I_1\rangle$ and $\langle I_2\rangle$ ($\langle I_2\rangle < \langle I_1\rangle$) of the coherence matrix,

$$P=(\langle I_1\rangle-\langle I_2\rangle)/(\langle I_1\rangle+\langle I_2\rangle). \tag{5.57}$$

Since the total averaged intensity of the wave is given by $\langle I\rangle = \mathrm{Tr}\,\mathcal{J}=\langle I_1\rangle+\langle I_2\rangle$, it follows from (5.57) that the eigenvalues may be expressed in terms of the degree of polarization in the form

$$\langle I_1\rangle=\tfrac{1}{2}(1+P)\langle I\rangle, \qquad \langle I_2\rangle=\tfrac{1}{2}(1-P)\langle I\rangle, \tag{5.58}$$

and this result was used in the derivation of (4.41).

It is to be noted that the correlation coefficient depends on the particular choice of the xy axes, so that $\gamma_{xy}(0)$ will, in general, change as the axes are rotated about the z direction. However, it may be shown [Wolf (1959); Born and Wolf (1964), p. 553] that *the maximum value which* $|\gamma_{xy}(0)|$ *attains as the axes are rotated, is just the degree of polarization P of the wave.* Moreover, the special choice of axes which maximizes $|\gamma_{xy}(0)|$ is such that the intensities $\langle I_x\rangle$ and $\langle I_y\rangle$ are then equal.

The coherence matrix representation which we discussed in this section is intimately related to a representation in terms of so-called Stokes' parameters [Stokes (1852)]. Connections between the coherence matrix, the density matrix, and the Stokes parameters are discussed in many publications, in particular in the following ones: Falkoff and MacDonald (1951); Fano (1954, 1957); McMaster (1954); Tolhoek (1956); Wolf (1954b, 1959); Parrent and Roman (1960); ter Haar (1961); Marathay (1963); O'Neill (1963); Born and Wolf (1964, Sec. 10.8.3). Some applications of the coherence-matrix techniques were described by Kuscer and Ribaric (1959), Ko (1961, 1962) Karczewski and Wolf (1963, 1965a, b), and by Jacobson (1964).

The coherence matrices considered in this section are appropriate for the description of polarization phenomena associated with plane, quasimonochromatic waves, under conditions where only short time delays ($|\tau| \ll 1/\Delta\nu$) are introduced between the components of the electric field. For other wave fields, and under conditions when either long time delays are introduced or when the wave is propagated through more complicated (dispersive) media, more general coherence matrices must be used—namely those associated with the τ-dependent coherence tensors such as (3.38). Alternatively one may employ their frequency dependent Fourier transforms. Historically, coherence matrices essentially of this latter type were introduced (for plane-wave fields) by Wiener (1928, 1929, 1930) [see also, Barakat (1963), and Pancharatnam (1963a, b)].

5.7. Coherence Properties of Blackbody Radiation

In Sec. 4.5 the main statistical properties of blackbody radiation were outlined. In this section we shall evaluate some of the coherence tensors of this radiation.

Expressions for the second-order electric correlation tensor of the real field [essentially the real part of the tensor $\mathcal{E}_{jk}$ defined by (3.38)] of blackbody radiation were derived by Bourret (1960), by means of techniques analogous to those employed in the theory of isotropic turbulence of an incompressible fluid. Bourret employed only concepts of classical theory, but quantum features of the radiation were taken into account in assuming the spectrum to be given by Planck's law. The main results were later rederived by Sarfatt (1963) by explicit quantum-mechanical calculations. Some related questions were briefly considered by Hamm and Harris (1963).

Correlation functions of the associated complex field were studied by Kano and Wolf (1962), by Mehta (1963), and by Mehta and Wolf (1964a, b). The first two papers deal with temporal coherence. The other two deal with temporal as well as spatial coherence, and include a detailed discussion of the behavior of the electric, the magnetic and also of the mixed correlation tensors of the second order, as well as general expressions for the electromagnetic correlation tensors of arbitrary order. The complex electric correlation tensors of arbitrary order were also considered by Glauber (1963b, p. 2787) [see also Mandel (1963a)]. Some of these investigations were based on the classical wave theory, others on the theory of the quantized field, and, as might be expected (in view of the discussion in Sec. 4.5 above), the two approaches have lead to equivalent results. We will briefly outline the derivation of the main formulas based on the quantum field-theoretical approach, and discuss some of their consequences.

The electric field operator $\hat{\mathbf{E}}$ and the magnetic field operator $\hat{\mathbf{H}}$ at the space time point $x\equiv\mathbf{r}$, ct of a radiation field confined to a rectangular box whose sides are of length L, may be expanded in Fourier series of the form [cf. (4.3)]

$$\begin{aligned}\hat{\mathbf{E}}(x)&=\hat{\mathbf{E}}^{(+)}(x)+\hat{\mathbf{E}}^{(-)}(x),\\ \hat{\mathbf{H}}(x)&=\hat{\mathbf{H}}^{(+)}(x)+\hat{\mathbf{H}}^{(-)}(x),\end{aligned} \tag{5.59}$$

where

$$\begin{aligned}\hat{\mathbf{E}}^{(+)}(x)&=\{\hat{\mathbf{E}}^{(-)}(x)\}^{\dagger}=i\left(\frac{\hbar c}{L^3}\right)^{\frac{1}{2}}\sum_{\mathrm{k},s}k^{\frac{1}{2}}\hat{a}_{\mathrm{k},s}\boldsymbol{\varepsilon}_{\mathrm{k},s}e^{ikx},\\ \hat{\mathbf{H}}^{(+)}(x)&=\{\hat{\mathbf{H}}^{(-)}(x)\}^{\dagger}=i\left(\frac{\hbar c}{L^3}\right)^{\frac{1}{2}}\sum_{\mathrm{k},s}k^{\frac{1}{2}}\hat{a}_{\mathrm{k},s}\frac{\mathbf{k}\times\boldsymbol{\varepsilon}_{\mathrm{k},s}}{k}e^{ikx}.\end{aligned} \tag{5.60}$$

Here $kx=\mathbf{k}\cdot\mathbf{r}-kct$ ($k=|\mathbf{k}|$). $\hat{a}_{\mathbf{k},s}$ and $\hat{a}^{\dagger}_{\mathbf{k},s}$ are the annihilation and creation operators for photons of momentum $\mathbf{k}$ and polarization $s(s=1, 2)$ which obey the commutation relations (3.41), and $\boldsymbol{\varepsilon}_{\mathbf{k},s}$ are the unit vectors obeying the relation (3.42).

Now the second-order correlation tensors (3.37) defined in terms of the operators (5.60) are

$$\mathcal{E}_{ij}(x_1, x_2) = \mathrm{Tr}\,\{\hat{\rho}\hat{E}^{(-)}{}_i(x_1)\hat{E}^{(+)}{}_j(x_2)\}, \quad (5.61a)$$

$$\mathcal{H}_{ij}(x_1, x_2) = \mathrm{Tr}\,\{\hat{\rho}\hat{H}^{(-)}{}_i(x_1)\hat{H}^{(+)}{}_j(x_2)\}, \quad (5.61b)$$

$$\mathcal{G}_{ij}(x_1, x_2) = \mathrm{Tr}\,\{\hat{\rho}\hat{E}^{(-)}{}_i(x_1)\hat{H}^{(+)}{}_j(x_2)\}, \quad (5.61c)$$

$$\tilde{\mathcal{G}}_{ij}(x_1, x_2) = \mathrm{Tr}\,\{\hat{\rho}\hat{H}^{(-)}{}_i(x_1)\hat{E}^{(+)}{}_j(x_2)\}, \quad (5.61d)$$

where $\hat{\rho}$ is the density operator of the field.

Next we express $\hat{\rho}$ in the "diagonal form" (4.9) and substitute for the "phase-space distribution function" Φ the expression (4.31) appropriate to blackbody radiation. This gives

$$\hat{\rho}=\int \prod_{\mathbf{k},s} \frac{1}{\pi\langle n_{\mathbf{k},s}\rangle} \times \exp\left[-|v_{\mathbf{k},s}|^2/\langle n_{\mathbf{k},s}\rangle\right] |v_{\mathbf{k},s}\rangle\langle v_{\mathbf{k},s}|\, d^2v_{\mathbf{k},s}. \quad (5.62)$$

As before, the $v_{\mathbf{k},s}$ are the eigenvalues of the annihilation operator $\hat{a}_{\mathbf{k},s}$ [cf. (3.44)], and $\langle n_{\mathbf{k},s}\rangle$ is the expectation value (4.32) of the number operator,

$$\langle n_{\mathbf{k},s}\rangle=(e^{\alpha k}-1)^{-1}, \qquad \alpha=\hbar c/KT. \quad (5.63)$$

From (5.61), (5.62), (5.60), and (3.44), one obtains after a straightforward calculation [Mehta and Wolf (1964b)], on the assumption that the linear dimensions L of the enclosure are large compared with the mean wavelength of the radiation,

$$\mathcal{E}_{ij}(\mathbf{r}, \tau)=\mathcal{H}_{ij}(\mathbf{r}, \tau)=\frac{\hbar c}{4\pi^2}\int \frac{k^2\delta_{ij}-k_ik_j}{k(e^{\alpha k}-1)} \times \exp\{i(\mathbf{k}\cdot\mathbf{r}-kc\tau)\}\, d^3k, \quad (5.64)$$

$$\mathcal{G}_{ij}(\mathbf{r}, \tau)=-\tilde{\mathcal{G}}_{ij}(\mathbf{r}, \tau)=\frac{\hbar c}{4\pi^2}\epsilon_{ijk}\int \frac{k_k}{e^{\alpha k}-1} \times \exp\{i(\mathbf{k}\cdot\mathbf{r}-kc\tau)\}\, d^3\mathbf{k}. \quad (5.65)$$

Here $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$, $\tau=t_2-t_1$ and ϵ_{ijk} is the completely antisymmetric unit tensor of Levi-Civita, i.e., ϵ_{ijk} is $+1$ or -1 according as the subscripts (i, j, k) are even or odd permutations of (x, y, z) and $\epsilon_{ijk}=0$ when two suffices are equal. We see that the arguments (x_1, x_2) of the four tensors only enter in the combinations $\mathbf{r}_2-\mathbf{r}_1$ and t_2-t_1, as might have been expected from considerations of isotropy and stationarity of the radiation, and this has been indicated explicitly by writing $\mathcal{E}_{ij}(\mathbf{r}, \tau)$ in place of $\mathcal{E}_{ij}(x_1, x_2)$, etc.

It will be useful to normalize $\mathcal{E}_{ij}$ in the usual way and to normalize the other tensors similarly. We therefore set

$$\gamma_{ij}(\mathbf{r}, \tau)=(1/N)\mathcal{E}_{ij}(\mathbf{r}, \tau)=(1/N)\mathcal{H}_{ij}(\mathbf{r}, \tau), \quad (5.66)$$

$$\sigma_{ij}(\mathbf{r}, \tau)=(1/N)\mathcal{G}_{ij}(\mathbf{r}, \tau)=-(1/N)\tilde{\mathcal{G}}_{ij}(\mathbf{r}, \tau), \quad (5.67)$$

where

$$N=[\mathcal{E}_{ii}(0, 0)]^{\frac{1}{2}}[\mathcal{E}_{jj}(0, 0)]^{\frac{1}{2}}=\mathcal{E}_{ii}(0, 0)=\mathcal{H}_{ii}(0, 0) \quad \text{(no summation)}$$

$$=\tfrac{16}{45}[\pi^6K^4T^4/(hc)^3]. \quad (5.68)$$

The integrals in (5.64) and (5.65) may be developed into series and one then obtains the following expressions for the normalized correlation functions [Mehta and Wolf (1964a, b)]:

$$\gamma_{ij}(\mathbf{r}, \tau)=\frac{90\alpha^4}{\pi^4}\sum_{n=1}^{\infty}\left\{\frac{\delta_{ij}}{\{(n\alpha+ic\tau)^2+r^2\}^2} +2\frac{r_ir_j-r^2\delta_{ij}}{\{(n\alpha+ic\tau)^2+r^2\}^3}\right\}, \quad (5.69)$$

$$\sigma_{ij}(\mathbf{r}, \tau)=i\frac{180\alpha^4}{\pi^4}r_k\epsilon_{ijk}\sum_{n=1}^{\infty}\frac{n\alpha+ic\tau}{\{(n\alpha+ic\tau)^2+r^2\}^3}. \quad (5.70)$$

Several conclusions may readily be drawn from the formulas (5.69) and (5.70). Let us consider first the normalized correlation tensor γ_{ij}. If one sets $\mathbf{r}=0$ in (5.69) one obtains the following expression for the normalized tensor which characterizes *temporal coherence* of the electric (and also of the magnetic) field:

$$\gamma_{ij}(0, \tau)=(90/\pi^4)\zeta[4, 1+(ic\tau/\alpha)]\delta_{ij}, \quad (5.71)$$

where $\zeta(s, a)$ is the generalized Riemann zeta function [Whittaker and Watson (1940), p. 266]:

$$\zeta(s, a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}. \quad (5.72)$$

We see that the tensor $\gamma_{ij}(0, \tau)$ is *diagonal*. Hence any two orthogonal components of the complex fields $E_i(\mathbf{r}, t)$, $E^*_j(\mathbf{r}, t+\tau)$ [or $H_i(\mathbf{r}, t)$, $H^*_j(\mathbf{r}, t+\tau)$] $(i\neq j)$, considered at the same point $\mathbf{r}$, at times t and $t+\tau$, respectively, are completely uncorrelated, irrespective of the value of τ. The diagonal components are all equal to each other. A curve showing the behavior of the modulus of a diagonal element is shown in Fig. 9. It is to be noticed that the temporal coherence of the diagonal elements only extends over a τ range which is of the order of $5\alpha/c$, this being the order of magnitude of the mean period $1/\bar{\nu}\sim 1.9\alpha/c$ of the radiation [see Mehta (1963)].

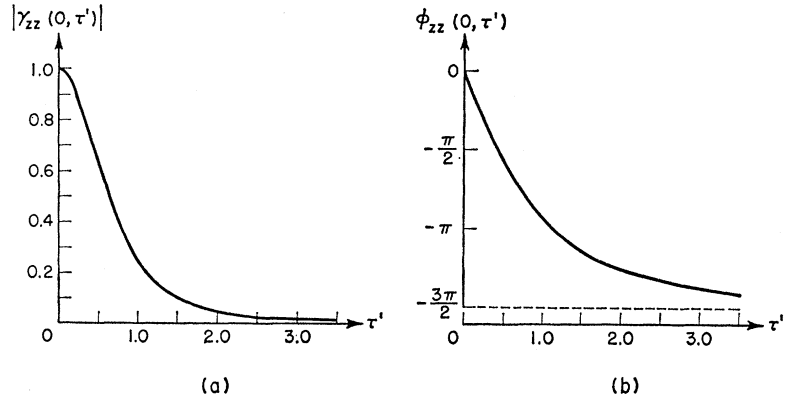


FIG. 9. Temporal coherence of blackbody radiation. (a) The modulus $|\gamma_{zz}(0,\tau)|$ and (b) the argument $\phi_{zz}(0,\tau)$ of a typical diagonal component of the normalized second-order complex electric coherence tensor.

$$|\gamma_{zz}(0,-\tau)|=|\gamma_{zz}(0,\tau)|, \qquad \phi_{zz}(0,-\tau)=-\phi_{zz}(0,\tau),$$

$\tau'=(KT/\hbar)\tau$. [After Y. Kano and E. Wolf (1962). An error in the labeling of the τ' axis in the original figures is corrected here.]

Next consider *spatial coherence* of the field, which is characterized by $\gamma_{ij}(\mathbf{r},0)$. Setting $\tau=0$ in (5.69), we obtain

$$\gamma_{ij}(\mathbf{r},0)=\frac{90}{\pi^4}\sum_{n=1}^{\infty}\left[\frac{\delta_{ij}}{(n^2+r^2/\alpha^2)^2}+\frac{2}{\alpha^2}\frac{r_ir_j-r^2\delta_{ij}}{(n^2+r^2/\alpha^2)^3}\right]. \tag{5.73}$$

The series occurring in (5.73) may be summed and yields the expression

$$\gamma_{ij}(\mathbf{r},0)=(45/4r'^4)\cdot[A(r')\,\delta_{ij}+B(r')\,(r'_ir'_j/r'^2)], \tag{5.74}$$

where

$$A(r')=-r'\coth r'-r'^2\operatorname{cosech}^2 r' -2r'^3\operatorname{cosech}^2 r'\coth r'+4,$$

$$B(r')=3r'\coth r'+3r'^2\operatorname{cosech}^2 r' +2r'^3\operatorname{cosech}^2 r'\coth r'-8, \tag{5.75}$$

with

$$r'=(\pi/\alpha)r. \tag{5.76}$$

Equation (5.74) is valid for all $\mathbf{r}$ but is not suitable for computing $\gamma_{ij}(\mathbf{r},0)$ when r' is small. For r' small one however obtains from (5.73) directly

$$\gamma_{ij}(\mathbf{r},0)=\delta_{ij}[1-\tfrac{8}{21}r'^2+\tfrac{3}{35}r'^4+\cdots] +(r'_ir'_j/r'^2)[\tfrac{4}{21}r'^2-\tfrac{2}{35}r'^4+\cdots]. \tag{5.77}$$

It is seen from (5.73) that the "spatial" coherence tensor $\gamma_{ij}(\mathbf{r},0)$ is *real*. Figure 10 illustrates *longitudinal spatial coherence*, i.e., the variation of $\gamma_{zz}(\mathbf{r},0)$ with r, when $\mathbf{r}$ is along the z axis and *lateral spatia coherence*, i.e., the variation of $\gamma_{zz}(\mathbf{r},0)$ with r, when $\mathbf{r}$ is perpendicular to the z axis. By symmetry, the behavior of each of the other diagonal components of the tensor $\gamma_{ij}(\mathbf{r},0)$ is, of course, the same as that of $\gamma_{zz}(\mathbf{r},0)$. Figure 11 illustrates the behavior of the diagonal as well as off-diagonal elements of $\gamma_{ij}(\mathbf{r},0)$ in various planes.

Let us now consider the normalized "mixed" correlation tensor σ_{ij}. We see from (5.70) that this tensor is completely antisymmetric, so that its diagonal components are zero. Also $\sigma_{ij}(0,\tau)\equiv 0$, so that there is *no temporal coherence* between the complex electric and magnetic fields at the same point. As regards *spatial coherence* we see from (5.70) that a typical off-diagonal

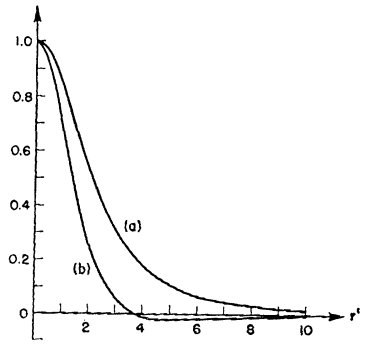


FIG. 10. (a) Longitudinal spatial coherence of blackbody radiation. Variation of a typical diagonal component $\gamma_{zz}(\mathbf{r}',0)$ with $\mathbf{r}'$, when $\mathbf{r}'$ is along the z axis. (b) Lateral spatial coherence of blackbody radiation. Variation of a typical diagonal component $\gamma_{zz}(\mathbf{r}',0)$ with $\mathbf{r}'$, when $\mathbf{r}'$ is perpendicular to the z axis. $\mathbf{r}'=(\pi/\alpha)\mathbf{r}$; $\alpha=\hbar c/KT$. [After R. C. Bourret (1960); J. Sarfatt (1963); and C. L. Mehta and E. Wolf (1964a).]

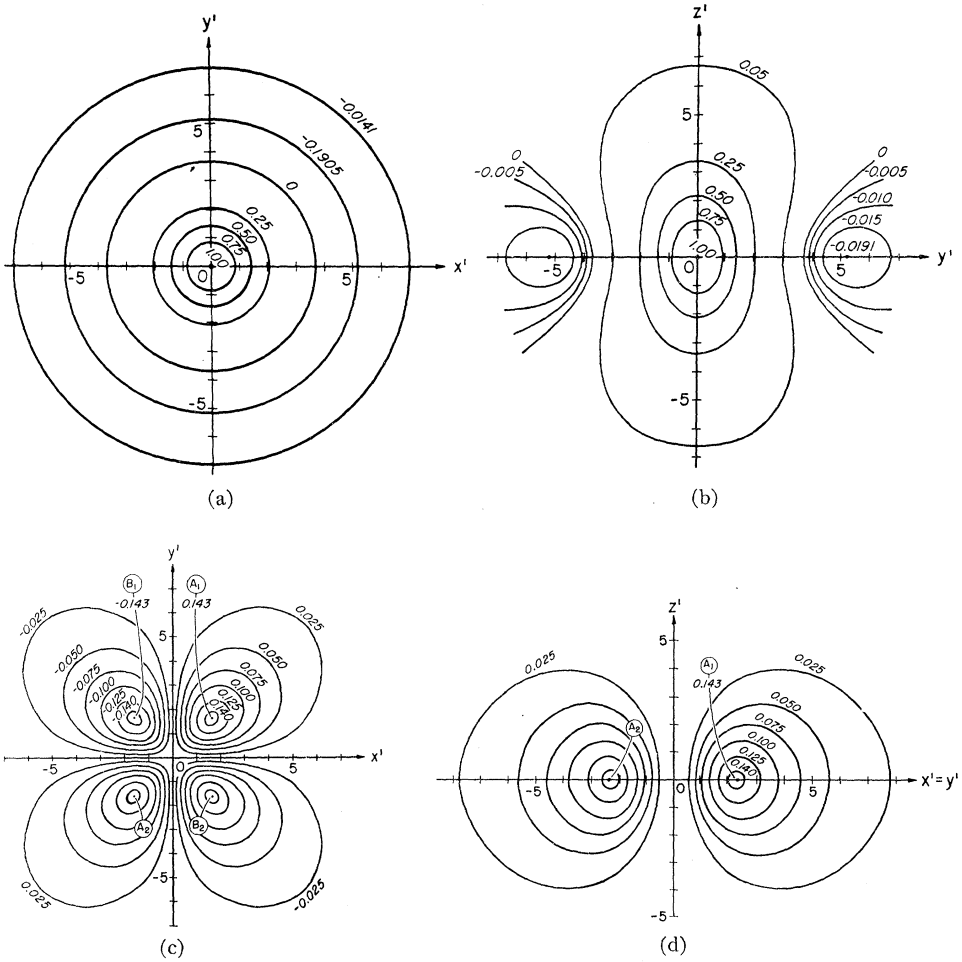


FIG. 11. Spatial coherence of blackbody radiation. The behavior of typical diagonal and off-diagonal components of the normalized second-order complex electric coherence tensor γ_{ij}. [After C. L. Mehta and E. Wolf (1963a).] (a) Contours of $\gamma_{zz}(\mathbf{r}, 0)$ in the xy plane. (b) Contours of $\gamma_{zz}(\mathbf{r}, 0)$ in the yz plane. (c) Contours of $\gamma_{xy}(\mathbf{r}, 0)$ in the xy plane. (d) Contours of $\gamma_{xy}(\mathbf{r}, 0)$ in the plane $x=y$.

component of σ, for $\tau=0$, is given by

$$\sigma_{xy}(\mathbf{r}, 0)=i\frac{180}{\alpha\pi^4}z\sum_{n=1}^{\infty}\frac{n}{(n^2+r^2/\alpha^2)^3}. \qquad (5.78)$$

From (5.78) it follows that $\sigma_{xy}(\mathbf{r}, 0)=0$ for $z=0$. Figures showing the behavior of $\sigma_{xy}(\mathbf{r}, 0)$ have been given by Mehta and Wolf (1964a).

It is seen from Figs. 10 and 11 that spatial coherence extends only over a region whose linear dimension r is of the order of $5\alpha/\pi\sim1.6\alpha$, and this is of the order of magnitude of the mean wavelength $\bar{\lambda}\sim1.9\alpha$ of the radiation [see Mehta (1963)].

For the sake of completeness let us also briefly consider the higher order correlation tensors of blackbody radiation. The correlation tensor of the electric field of an arbitrary order is defined by an expression analogous to (4.4) with the operators $\hat{\mathbf{A}}^{(-)}$ and $\hat{\mathbf{A}}^{(+)}$ being replaced by $\hat{\mathbf{E}}^{(-)}$ and $\hat{\mathbf{E}}^{(+)}$, respectively:

$$\mathcal{E}^{(N,M)}_{j_1,j_2,\cdots,j_{N+M}}(x_1, x_2, \cdots, x_{N+M})$$
$$=\mathrm{Tr}\,\{\hat{\rho}\hat{E}^{(-)}_{j_1}(x_1)\cdots\hat{E}^{(-)}_{j_N}(x_N)$$
$$\times\hat{E}^{(+)}_{j_{N+1}}(x_{N+1})\cdots\hat{E}^{(+)}_{j_{N+M}}(x_{N+M})\}. \qquad (5.79)$$

Here $\hat{\mathbf{E}}^{(-)}_{j_p}(x_p)$ and $\hat{\mathbf{E}}^{(+)}_{j_q}(x_q)$ are Cartesian components of the operators $\hat{\mathbf{E}}^{(-)}(x_p)$ and $\hat{\mathbf{E}}^{(+)}(x_q)$ at the space–time points x_p and x_q, respectively. Now with the help of (4.9), (5.60) and (3.44), these correlation tensors may be expressed as phase-space averages of the product of the components of the complex classical electric field [cf. (4.12)], viz.,

$$\mathcal{E}^{(N,M)}_{j_1,\cdots,j_{N+M}}(x_1, x_2, \cdots, x_{N+M})=\int\Phi(\{v_{\mathbf{k},s}\})$$
$$\times E^*_{j_1}(x_1)\cdots E^*_{j_N}(x_N)E_{j_{N+1}}(x_{N+1})\cdots E_{j_{N+M}}(x_{N+M})$$
$$\times d^2\{v_{\mathbf{k},s}\}. \qquad (5.80)$$

In evaluating the integral (5.80), the classical field $\mathbf{E}$ must be regarded as expressed in the form [cf. (3.47)]

$$\mathbf{E}(x)=i\left(\frac{hc}{L^3}\right)^{\frac{1}{2}}\sum_{\mathbf{k},s}k^{\frac{1}{2}}v_{\mathbf{k},s}\boldsymbol{\varepsilon}_{\mathbf{k},s}\exp(ikx). \qquad (5.81)$$

Now for blackbody radiation, the phase-space distribution $\Phi(\{v_{\mathbf{k},s}\})$ is the multivariate Gaussian dis-

tribution (4.31) in the variables $v_{\mathbf{k},s}$. Hence according to (4.35), all the higher-order electric correlation tensors may be expressed in terms of the second-order ones as follows:

$$\mathcal{E}^{(N,N)}{}_{j_1,j_2,\cdots,j_{2N}}(x_1,\cdots,x_N;x_{N+1},\cdots,x_{2N}) = \sum_{\pi}\mathcal{E}^{(1,1)}{}_{j_1,j_{N+1}}(x_1;x_{N+1})\mathcal{E}^{(1,1)}{}_{j_2,j_{N+2}}(x_2;x_{N+2})\cdots \times\mathcal{E}^{(1,1)}{}_{j_N,j_{2N}}(x_N;x_{2N}); \quad (5.82a)$$

$$\mathcal{E}^{(N,M)}=0, \quad \text{if} \quad N\neq M. \quad (5.82b)$$

On the right-hand side of (5.82a), $\mathcal{E}^{(1,1)}{}_{j_1,j_{N+1}}(x_1,x_{N+1})$, etc., denote the second-order complex electric correlation tensor, which, in the notation of the previous part of this section (5.7) would be denoted by the same symbols, but with the superscript (1, 1) suppressed. The symbol $\sum_\pi$ denotes sum over all $N!$ possible permutations of the indices 1 to N. Similar expressions hold also for the magnetic and the mixed correlation tensors of an arbitrary order.

6. FOURTH- AND HIGHER-ORDER COHERENCE EFFECTS

While the second-order correlation of the complex-field amplitude has the dimension of intensity (i.e., of $|\mathbf{V}(\mathbf{r},t)|^2$) and is useful for describing the intensity distribution of the field, the fourth- and-higher order moments are necessary for describing the correlations of the light intensity fluctuations at two or more space–time points. The first demonstration that such correlations exist in some optical fields was given by Hanbury Brown and Twiss (1956a), who used fast photoelectric detectors to measure the correlations of intensity fluctuations. In the following years the effect was investigated more thoroughly, both by correlation and photoelectric coincidence counting techniques [Hanbury Brown and Twiss (1957a, b); Twiss, Little, and Hanbury Brown (1957); Twiss and Little (1959); Rebka and Pound (1957); Brannen, Ferguson, and Wehlau (1958); Harwit (1960); Janossy, Naray, and Varga (1961)], and it became the basis of a new method—correlation interferometry—for the determination of angular diameters of stars [Hanbury Brown and Twiss (1956b, 1958a, b); Ratcliffe (1956), p. 233; Hanbury Brown (1960, 1964); Gamo (1961, 1963b); Hanbury Brown, Hazard, Davis, and Allen (1964); see also, Bracewell (1958)]. It was later shown that correlation measurements also carry information about the spectral distribution [Forrester (1961a, b); Gamo (1961, 1963a, b); Givens (1961b, 1962); Wolf (1962, (1965); Mandel (1962b, 1963b, 1965)] and about the state of polarization of the light [Wolf (1960); Mandel and Wolf (1961b); Mandel (1963c)].

Most theoretical discussions of intensity correlation effects have been in classical or semiclassical terms [Purcell (1956); Hanbury Brown and Twiss (1957a; Wolf (1957); Janossy (1957, 1959); Mandel (1958, 1959); Kahn (1958); Mandel and Wolf (1961b); Mandel, Sudarshan, and Wolf (1964); Wolf and Mehta (1964)], often with the explicit or implicit assumption that the radiation field of a thermal source may be regarded as a Gaussian random process. The first quantum-mechanical treatment of the origin of correlations appears to have been given by Dicke (1954) [see also, an outline of the method given by Dicke (1964)], and this was later greatly extended [Senitzky (1958), 1962a)]. In particular Fano (1961) showed that two independent excited atoms (corresponding to an incoherent source) would lead to correlation in the excitation of two nearby independent atoms (corresponding to two detectors). More recently, some of these effects have also been discussed in terms of ensembles of quantized fields [Glauber (1963a, b, 1964); Sudarshan (1963b); Goldberger and Watson (1964, 1965); Holliday and Sage (1964); cf. also Mandel (1965)].

As the photoelectric effect plays an essential role in all measurements of intensity correlations, we begin by considering the photoelectric detection process. Following the method of Mandel, Sudarshan, and Wolf (1964), we shall show that the output of an illuminated photodetector carries information about the fluctuations of the radiation field, in the sense that the instantaneous probability of photoemission is proportional to the "instantaneous classical intensity" $\mathbf{V}^*(\mathbf{r},t)\cdot\mathbf{V}(\mathbf{r},t)$ of the light at the photocathode. This conclusion holds as long as the intensity of the light is not too great, so that the photoelectric emission is describable by first-order perturbation theory. It is also related to the fact that $\mathbf{V}(\mathbf{r},t)$ is an eigenvalue of the photon annihilation operator at $\mathbf{r},t$. We shall then examine the form of the fluctuations and their correlations. A detailed analysis of the photoelectric detection process has recently been given by Kelley and Kleiner (1964). The main results are in substantial agreement with those given below for an unquantized field.

6.1. The Photoelectric Detection Process

Consider a plane wave of quasimonochromatic light falling normally on a photoelectric surface containing many electrons in bound states $|\psi_b\rangle$. Under the influence of the light some electrons will make transitions to a continuum of unbound states $|\psi_k\rangle$ and will then be free to be counted by the photodetector with some average probability η. Since the $|\psi_k\rangle$ are eigenstates of the unperturbed Hamiltonian $\hat{H}_0$ of the electron,

$$\hat{H}_0|\psi_k\rangle=E_k|\psi_k\rangle. \quad (6.1)$$

Let the influence of the incident light be described by an interaction Hamiltonian

$$\hat{H}_1(\mathbf{r},t)=(e/mc)\mathbf{A}(\mathbf{r},t)\cdot\hat{\mathbf{p}}, \quad (6.2)$$

where $\mathbf{A}(\mathbf{r}, t)$ is the real vector potential representing the classical field, which is assumed to be in the form of a plane, quasimonochromatic wave, and $\hat{\mathbf{p}}$ is the momentum of an electron. Then it may be shown by the usual method of time-dependent perturbation theory [cf. Mandel, Sudarshan, and Wolf (1964)] that the probability of photoelectric detection $P(t)\Delta t$ at time t within the time interval t, $t+\Delta t$ is given by

$$P(t)\Delta t=(2\pi e^2\eta/\hbar^2m^2c^2)\sum_b\sum_{\text{pol}}W^*(t)W(t)\Delta t. \tag{6.3}$$

$\sum_b$ stands for the sum over all initial bound states of the electrons and $\sum_{\text{pol}}$ for the sum over all polarizations of the final states. $W(t)$ is a complex scalar function whose Fourier spectrum contains only positive frequencies, which is obtained from the field vector $\mathbf{A}(r, t)$ by a linear transformation. In the typical case where the matrix element $\langle\psi_\omega \mid \hat{\mathbf{p}} \mid \psi_b\rangle$ connecting the initial state of energy E_b with the final state of energy $\hbar\omega+E_b$ is independent of ω over the narrow frequency range of the incident light, and the same is true for the density of states $\rho(\omega)$, $P(t)\Delta t$ reduces to

$$P(t)\Delta t=\alpha\mathbf{V}^*(\mathbf{r}, t)\cdot\mathbf{V}(\mathbf{r}, t)\Delta t. \tag{6.4}$$

$\mathbf{V}(\mathbf{r}, t)$ is the vector "analytic signal" [cf. (3.2)] obtained from $\mathbf{A}(\mathbf{r}, t)$ by suppressing the negative frequencies in the Fourier integral expansion, and can be identified with the complex field amplitude of equations (3.2) and (3.47). α is a constant representing the quantum efficiency of the detector and, for the case of one electron, is given by [Mandel, Sudarshan, and Wolf (1964)]

$$\alpha=\frac{2\pi e^2\eta}{\hbar^2m^2c^2}\rho(\omega_0)\sum_b\sum_{\text{pol}}|\,\boldsymbol{\varepsilon}\cdot\langle\psi_{\omega_0}\mid\hat{\mathbf{p}}\mid\psi_b\rangle\,|^2, \tag{6.5}$$

where $\boldsymbol{\varepsilon}$ is the complex unit vector (defined up to a unitary transformation) satisfying the relation $\mathbf{V}(\mathbf{r}, t)=\boldsymbol{\varepsilon}V(\mathbf{r}, t)$ and $\omega_0/2\pi$ is the midfrequency of the light. Thus the instantaneous probability of photodetection is according to (6.4) proportional to the instantaneous classical intensity $I(\mathbf{r}, t)$ of the light. This conclusion was of course to be expected from the fact that $\mathbf{V}(\mathbf{r}, t)$ is an eigenvalue of the annihilation operator $\hat{\mathbf{A}}^{(+)}(\mathbf{r}, t)$, and that the probability of photoemission is proportional to the square modulus of the matrix element $\langle S_2 \mid \hat{\mathbf{A}}^{(+)}(\mathbf{r}, t) \mid S_1\rangle$ connecting the initial state $\mid S_1\rangle$ and the final state $\mid S_2\rangle$ of the field [cf. Eq. (4.5)]. The result emphasizes once again the close connection between the semiclassical and quantum descriptions of the field as applied to photoelectric detection. A similar conclusion was recently drawn by Jaynes and Cummings (1963) from a discussion of the interactions of fields and atomic systems.

6.2. The Probability Distribution of Photoelectric Counts

With the help of the differential probability $P(t)$ it can be shown [cf. Mandel (1958, 1963d); Corcoran and Pao (1962); Kelley and Kleiner (1964)] that the probability $p(n; T, t)$ of counting n photoelectrons in the interval t to $t+T$ is given by the ensemble average over the Poisson distribution

$$p(n; T, t)=(1/n!)\,\langle[\alpha U(T, t)]^n\exp[-\alpha U(T, t)]\rangle, \tag{6.6}$$

where

$$U(T, t)=\int_t^{t+T}I(t')\,dt', \tag{6.7}$$

and the average is to be taken with respect to the ensemble of $U(T, t)$. The resulting expression [see also Eq. (6.12) below] is the analog for photoelectrons of the quantum-mechanical distribution (4.15) for the number of photons of the quantized field. If, as is usually the case, the radiation field is stationary and ergodic, the probability $p(n; T, t)$ will be independent of t and may be written as $p(n, T)$. The operation of ensemble averaging will in general cause $p(n, T)$ to depart from the Poisson distribution. But consider the special case in which the intensity of the incident light does not fluctuate significantly in time. As we have seen in Sec. 4.6, the output of an optical maser oscillating in one mode approximates to this situation. Then, according to (6.6), $p(n, T)$ remains a Poisson distribution with parameter $\langle n\rangle=\alpha\langle I\rangle T$, and the counts obey the statistics of classical particles [Mandel (1964a); Mandel, Sudarshan, and Wolf (1964); see also Mehta and Wolf (1964)].

Consider now the other important case of thermal light falling on the detector, when $\mathbf{V}(\mathbf{r}, t)$ is a Gaussian random process. For light that is cross-spectrally pure with respect to polarization, we have shown in Sec. 4.5 that the corresponding probability distribution $p(I)$ of I depends only on the degree of polarization P and is given by (4.41). Let us assume first of all that T is much shorter than the coherence time of the light, so that $U(T, t)$ reduces to $I(t)\,T$. Then the distribution $p(n, T)$ follows directly from (6.6) and (4.41) [cf. Mandel (1963c); Helstrom (1964)]:

$$\begin{aligned}p(n, T\ll 1/\Delta\nu)&=\int_0^\infty p(I)\frac{(\alpha IT)^n}{n!}\exp(-\alpha IT)\,dI\\&=\frac{1}{P\langle n\rangle}\left\{\left[\frac{1}{1+2/(1+P)\langle n\rangle}\right]^{n+1}\right.\\&\qquad\left.-\left[\frac{1}{1+2/(1-P)\langle n\rangle}\right]^{n+1}\right\},\end{aligned} \tag{6.8}$$

where $\langle n\rangle=\alpha(I)T$ is the expectation value of the number of counts. The expression (6.8) has the form of a distribution of n bosons over $2/(1+P)$ cells of phase space, with $1\leq 2/(1+P)\leq 2$. Polarized light ($P=1$) corresponds to one cell with

$$p(n,\ T\ll 1/\Delta\nu)=1/(1+\langle n\rangle)[1+(1/\langle n\rangle)]^n, \quad (6.9)$$

which is the familar Bose–Einstein distribution formula [cf. Fürth (1928a); Morse (1962), p. 218]. Unpolarized light ($P=0$) corresponds to the two-cell distribution

$$p(n,\ T\ll 1/\Delta\nu)=(1+n)/(1+\langle n\rangle/2)^2(1+2/\langle n\rangle)^n. \quad (6.10)$$

Thus, according to (6.8), the fluctuations of the number of photoelectric counts carry information about the state of polarization of the radiation field, and their measurement allows the degree of polarization P to be determined [cf. Wolf (1960)]. Since we have shown in Sec. 5.6 that the eigenvalues of the coherence matrix are determined by P, it follows that the whole matrix is obtainable from $p(n,\ T\ll 1/\Delta\nu)$. We note that, although the distribution $p(n,\ T)$ refers to the photoelectrons, it is naturally interpreted in terms of fluctuations of the numbers of photons of the quantized field, as is borne out by comparison with Eq. (4.15).

At first sight it may seem strange that the number of cells of the single-photon phase space is determined entirely by the state of polarization, and not also by the distribution of momenta of the photons. The reason is that we have chosen to treat a plane beam—traveling in the 0z direction, say—and a counting interval $T\ll 1/\Delta\nu$ which does not span more than one cell along the z axis of phase space [see also, Gabor (1950)]. From (6.8) it may readily be shown that the variance $\langle(\Delta n)^2\rangle$ of the number of counts registered is given by

$$\langle(\Delta n)^2\rangle=\langle n\rangle[1+\tfrac{1}{2}(1+P^2)\langle n\rangle], \quad (6.11)$$

which is a result that has also been derived more directly [Wolf (1960)]. The formula has recently been generalized to situations where the accessible volume of phase space is limited [Decomps and Kastler (1963); Kastler (1964)].

When T is not necessarily short compared with $1/\Delta\nu$, $p(n;\ T,\ t)$ has to be evaluated by averaging over the ensemble of the integrated intensity U defined by Eq. (6.7). Thus, if $p(U)$ is the distribution of U,

$$p(n;\ T,\ t)=\int_0^\infty p(U)\frac{(\alpha U)^n}{n!}\exp(-\alpha U)\,dU, \quad (6.12)$$

which is analogous to Eq. (4.15) describing the fluctuations of photon numbers of the quantized field. The similarity of (6.12) and (4.15b) suggests that $p(n,\ T)$ reflects the properties of the distribution of the number operator. The fact that Eq. (6.12) can be derived from semiclassical considerations is partly responsible for the success of the semiclassical methods in accounting for the results of measurements. It may be of interest to note that Eq. (6.12) may be inverted, i.e., that it is in general possible to derive the distribution of U from knowledge of $p(n,\ T)$ [Wolf and Mehta (1964)].

In general, no simple expressions for $p(U)$ are known, even for thermal light. However the limiting form of $p(U)$ as $T\to\infty$ for polarized thermal light has been given by Rice (1945, Sec. 3.9), and when this is inserted in (6.12) we obtain [cf. Mandel (1959); Helstrom (1964); Bolgiano (1964)]

$$p(n,\ T\gg 1/\Delta\nu)=\frac{\Gamma(n+T/\tau_c)}{n!\Gamma(T/\tau_c)(1+\langle n\rangle\tau_c/T)^{T/\tau_c}(1+T/\langle n\rangle\tau_c)^n}, \quad (6.13)$$

which is a reasonable approximation when $T\gg 1/\Delta\nu$. This distribution has the form appropriate for n bosons distributed over T/τ_c equal cells of phase space [cf. Mandel (1959)]. The time parameter τ_c defined by

$$\tau_c=\int_{-\infty}^{\infty}|\gamma^{(1,1)}(\mathbf{r},\ \mathbf{r},\ \tau)|^2\,d\tau \quad (6.14)$$

here appears as a natural measure of the coherence time, in the sense that T/τ_c determines the statistical degrees of freedom of the light (see also Sec. 5.4). If the light is partially polarized the corresponding counting distribution is obtained by a convolution of expressions like (6.13) [cf. also, Mandel (1963d); Helstrom (1964)].

The results embodied in Eqs. (6.8) and (6.13) apply to light of thermal origin. Although explicit expressions for $p(U)$ are generally not available, we can readily use (6.12) to relate the moments and cumulants of the distributions of n and U for any kind of light beam. If $M_k(x)$ and $M:(x)$ denote the corresponding moment generating functions, we have from (6.12) and the well-known properties of the Poisson distribution [cf. Kenney and Keeping (1954), p. 74]

$$M_n(x)=\langle\exp[\alpha U(e^x-1)]\rangle, \quad (6.15)$$

where the $\langle\ \rangle$ now denotes the ensemble average over the random variate U. From the definition of the moment generating function

$$M_n(x)=M_U(\alpha e^x-\alpha), \quad (6.16)$$

and, by making a series expansion of this relation in powers of x and equating coefficients, we obtain relations between the corresponding moments. Alterna-

tively (6.16) allows us to write a similar equation connecting the cumulant generating functions $K_n(x)$ and $K_U(x)$ [Kendall (1952), p. 90],

$$K_n(x)=K_U(\alpha e^x-\alpha), \tag{6.17}$$

and, if $\kappa^{(i)}{}_n$ and $\kappa^{(i)}{}_U$ are the corresponding cumulants, we obtain by a power series expansion

$$\sum_{i=1}^{\infty}\frac{x^i}{i!}\kappa^{(i)}{}_n=\sum_{i=1}^{\infty}\frac{\alpha^i(e^x-1)^i}{i!}\kappa^{(i)}{}_U.$$

On expanding the right-hand side as a power series in x and comparing the coefficients of x^i, we obtain the following relations between the cumulants[30] [Mandel (1959, 1963d)], valid for any kind of light:

$$\begin{aligned}
&\kappa^{(1)}{}_n=\alpha\kappa^{(1)}{}_U\\
&\kappa^{(2)}{}_n=\alpha\kappa^{(1)}{}_U+\alpha^2\kappa^{(2)}{}_U\\
&\kappa^{(3)}{}_n=\alpha\kappa^{(1)}{}_U+3\alpha^2\kappa^{(2)}{}_U+\alpha^3\kappa^{(3)}{}_U\\
&\kappa^{(4)}{}_n=\alpha\kappa^{(1)}{}_U+7\alpha^2\kappa^{(2)}{}_U+6\alpha^3\kappa^{(3)}{}_U+\alpha^4\kappa^{(4)}{}_U,\\
&\qquad\text{etc.}
\end{aligned}\tag{6.18}$$

For some cases, such as polarized light from thermal sources, where the distribution of $I(\mathbf{r}, t)$ is exponential, the cumulants $\kappa^{(i)}{}_U$ have been found [Slepian (1958)] and the $\kappa^{(i)}{}_n$ follow directly. But one or two interesting general conclusions can be drawn from (6.18) without explicit knowledge of the cumulants.

Thus from the second equation (6.18), since $\kappa^{(2)}{}_n$ and $\kappa^{(2)}{}_U$ denote the variances $\langle(\Delta n)^2\rangle$ and $\langle(\Delta U)^2\rangle$, we have

$$\langle(\Delta n)^2\rangle=\langle n\rangle+\alpha^2\langle(\Delta U)^2\rangle, \tag{6.19}$$

which shows that the fluctuations of the photoelectric counts are always expressible as the sum of contributions from the fluctuations of classical particles and the fluctuations of classical wave fields [cf. Mandel, Sudarshan, and Wolf (1964)]. The result is reminiscent of the well-known Einstein formula [Einstein (1909a, b); see also, Bothe (1927)], later generalized by Fürth (1928a, b) for energy fluctuations of blackbody radiation in thermal equilibrium. The above relation (6.19), on the other hand, is valid for stationary beams of any type. In particular, when $\langle(\Delta U)^2\rangle$ is very small as it may be for a laser beam, $\langle(\Delta n)^2\rangle\approx\langle n\rangle$.

In general, when the radiation field is very weak, the first terms on the right-hand side of Eqs. (6.18) tend to become dominant. It follows that all the cumulants $\kappa^{(i)}{}_n$ tend towards $\alpha\kappa^{(1)}{}_U=\alpha\langle U\rangle=\alpha\langle I\rangle T=\langle n\rangle$, which means that the distribution of n becomes Poissonian. Thus in very weak fields the photoelectrons obey the statistics of classical particles. In strong fields, on the other hand, it is the last terms in (6.18) which become dominant, and $\kappa^{(i)}{}_n\to\alpha^i\kappa^{(i)}{}_U$. The distribution of n then tends towards the distribution of αU, which is proportional to the integrated classical intensity.

We should emphasize that expressions such as (6.19) apply to light beams, and not necessarily to an isolated optical field, for example one confined in a box. However both Eqs. (6.16) and (6.19) can be generalized to apply to an arbitrary quantum state of an optical field itself, irrespective of the detector, by a re-interpretation of n and U.

Thus, let $\hat{n}$ be the operator corresponding to the number of photons in the volume δV (assumed to be large compared with the wavelength) at a given time t with polarization j. We can express this operator in terms of creation and annihilation operators $\hat{A}'^{\dagger}_j(\mathbf{r}, t)$ and $\hat{A}'_j(\mathbf{r}, t)$, which are linearly related to the

$$\hat{A}^{(-)}{}_j(\mathbf{r}, t) \qquad \text{and} \qquad \hat{A}^{(+)}{}_j(\mathbf{r}, t)$$

operators used previously, such that [cf. Schweber (1962), p. 172][30a]

$$\hat{n}=\int_{\delta V}\hat{A}'^{\dagger}_j(\mathbf{r}, t)\,A'_j(\mathbf{r}, t)\,d^3\mathbf{r}.$$

Then if the sharp brackets denote the quantum-mechanical expectation, the mth moment of $\hat{n}$ is

$$\langle\hat{n}_m\rangle=\int_{\delta V}\cdots\int\langle\hat{A}'^{\dagger}_j(\mathbf{r}_1, t)\hat{A}'_j(\mathbf{r}_1, t)\cdots \times\hat{A}'^{\dagger}_j(\mathbf{r}_m, t)\hat{A}'_j(\mathbf{r}_m, t)\rangle\,d^3\mathbf{r}_1\cdots d^3\mathbf{r}_m,$$

while, as explained in Sec. 4.2, the mth-order correlation of n is given by

$$\langle\hat{U}_m\rangle\equiv\langle:\hat{n}_m:\rangle=\int_{\delta V}\cdots\int\langle\hat{A}'^{\dagger}_j(\mathbf{r}_1, t)\cdots \times\hat{A}'^{\dagger}_j(\mathbf{r}_m, t)\hat{A}'_j(\mathbf{r}_1, t)\cdots\hat{A}'_j(\mathbf{r}_m, t)\rangle\,d^3\mathbf{r}'\cdots d^3\mathbf{r}_m,$$

where the pair of colons denotes normal ordering of operators.

It may be shown that the generating functions for $\langle\hat{n}_m\rangle$ and $\langle\hat{U}_m\rangle$, i.e.,

$$1+\sum_{m=1}^{\infty}\frac{\langle\hat{n}_m\rangle x^m}{m!} \quad\text{and}\quad 1+\sum_{m=1}^{\infty}\frac{\langle\hat{U}_m\rangle x^m}{m!}$$

are related by [cf. Schwinger (1961); Mandel (1964d);

---

[30] There is an error in Eq. (10) of the paper by Mandel (1959), which is corrected here.

[30a] $\hat{\mathbf{A}}_j'(\mathbf{r}, t)$ is defined by the expansion

$$\hat{\mathbf{A}}_j'(\mathbf{r}, t)=(\hbar c/L^3)^{\frac{1}{2}}\sum_{\mathbf{k},s}\hat{a}_{\mathbf{k},s}(\boldsymbol{\varepsilon}_{\mathbf{k},s})_j\exp[i(\mathbf{k}\cdot\mathbf{r}-ckt)$$

rather than by Eq. (3.40).

Ghielmetti (1964); Louisell (1964) p. 116]

$$\langle \exp[\hat{n}x] \rangle = \langle : \exp[\hat{n}(e^x - 1)] : \rangle, \quad (6.20)$$

which is a quantum-mechanical generalized analog of (6.16). The $\hat{n}_m$ and $\hat{U}_m$ may now be related by equating coefficients of x in the power series expansion of Eq. (6.20). Thus for the expectation value of the variance of $\hat{n}$ we find [cf. Mandel (1964d)]

$$\langle (\Delta n)^2 \rangle = \langle \hat{n}_2 \rangle - \langle \hat{n} \rangle^2 = \langle \hat{n} \rangle + [\langle \hat{U}_2 \rangle - \langle \hat{U}_1 \rangle^2], \quad (6.21)$$

where the sharp brackets denote the quantum-mechanical expectation value for an arbitrary state. This is the quantum-mechanical generalized analog of (6.19) [cf. the somewhat different result of Holliday and Sage (1964)]. While it appears to be formally similar, the term within the square brackets now has the nature of a correlation and is not necessarily positive. Indeed it is evident that the term must be negative for Fock states of the field for which $\langle (\Delta n)^2 \rangle$ vanishes.

6.3. Bunching Effects in Photoelectric Detection

We have already seen that the photoelectric counting distribution will, in general, depart from the Poisson form, and that the variance of the number of counts registered is usually in excess of that given by Poisson statistics. This of course implies that the photoelectrons (or photons) are not arriving at random, but have certain characteristic bunching properties.

The problem of determining correlation effects in the emission of radiation was first examined in detail by Dicke (1954), who showed that the correlation is a consequence of the coupling by the radiation field of the separate radiators of the source. This coupling is very strong in a maser, but is always present to some extent in all sources, and leads to both spatial and temporal correlations of the emitted photons. The correlations and their development in time have been studied in considerable detail by Senitzky (1958, 1961a, 1962a). Here we merely wish to note that bunching properties of the photoelectric counts are implicit in the nature of the distribution $p(n; T, t)$ given by (6.6). The bunching gives rise to correlations of the numbers of counts registered at several detectors.

Consider the conditional probability $p_c(t \mid t+\tau)\Delta\tau$ that a photoelectric count will be registered in the time interval $t+\tau$ to $t+\tau+\Delta\tau$, given that one has been registered at time t and that the field is stationary. From the definition of conditional probability [cf. Kendall (1952), p. 167] it follows with the help of (6.4) that

$$p_c(t \mid t+\tau)\Delta\tau = \langle P(t)P(t+\tau) \rangle \Delta t \Delta\tau / \langle P(t) \rangle \Delta t, = \alpha \langle I(t)I(t+\tau) \rangle \Delta\tau / \langle I \rangle. \quad (6.22)$$

For a stationary process this probability will of course depend only on τ.

We can see immediately that, in the special case where the instantaneous intensity $I(t)$ of the light is substantially constant, $pi(t \mid t+\tau)\Delta\tau$ reduces to $\alpha I \Delta\tau$, which is constant and independent of the previous count. The separate counts are therefore statistically independent and, as shown in Sec. 6.2, the counting distribution is a Poisson distribution. But in all other cases $p_c(t \mid t+\tau)$ depends on τ.

Consider a photoelectric detector illuminated by a plane beam of light from a *thermal* source, for which $\mathbf{V}(\mathbf{r}, t)$ is a Gaussian random process. If $V_1(\mathbf{r}, t)$ and $V_2(\mathbf{r}, t)$ are mutually orthogonal polarization components of $\mathbf{V}(\mathbf{r}, t)$ in a plane normal to the direction of propagation of the light, it can be shown from the moment theorem on the complex Gaussian random process that [cf. Reed (1962); Mehta (1965b), p. 398; the formula also follows immediately from Eq. (A.7) of Mandel and Wolf (1961b)]

$$\langle \mid V_1(\mathbf{r}, t) \mid^2 \mid V_2(\mathbf{r}, t+\tau) \mid^2 \rangle = \langle I_1(\mathbf{r}, t) I_2(\mathbf{r}, t+\tau) \rangle = \langle I_1 \rangle \langle I_2 \rangle [1 + \mid \gamma^{(1,1)}{}_{12}(\mathbf{r}, \mathbf{r}, \tau) \mid^2], \quad (6.23)$$

where $\gamma^{(1,1)}{}_{12}(\mathbf{r}, \mathbf{r}, \tau)$ is the normalized cross-correlation function of $V_1(\mathbf{r}, t)$ and $V_2(\mathbf{r}, t+\tau)$. If the polarization components obey the cross-spectral purity condition[31] (cf. Sec. 5.5)

$$\gamma^{(1,1)}{}_{12}(\tau) = \gamma^{(1,1)}{}_{12}(0)\gamma^{(1,1)}{}_{11}(\tau) = \gamma^{(1,1)}{}_{12}(0)\gamma^{(1,1)}{}_{22}(\tau) \equiv \gamma^{(1,1)}{}_{12}(0)\gamma^{(1,1)}(\tau),$$

(6.23) becomes[31]

$$\langle I_1(\mathbf{r}, t) I_2(\mathbf{r}, t+\tau) \rangle = \langle I_1 \rangle \langle I_2 \rangle [1 + \mid \gamma^{(1,1)}{}_{12}(0) \mid^2 \mid \gamma^{(1,1)}(\tau) \mid^2]. \quad (6.24)$$

By writing

$$I(\mathbf{r}, t) = I_1(\mathbf{r}, t) + I_2(\mathbf{r}, t),$$

and using (6.24), we can express the autocorrelation of the total intensity in the form [cf. Mandel and Wolf (1961b)]

$$\langle I(\mathbf{r}, t) I(\mathbf{r}, t+\tau) \rangle = \sum_{i,j=1,2} \sum \langle I_i(\mathbf{r}, t) I_j(\mathbf{r}, t+\tau) \rangle = (\langle I_1 \rangle^2 + \langle I_2 \rangle^2)[1 + \mid \gamma^{(1,1)}(\tau) \mid^2] + 2\langle I_1 \rangle \langle I_2 \rangle [1 + \mid \gamma^{(1,1)}{}_{12}(0) \mid^2 \mid \gamma^{(1,1)}(\tau) \mid^2]. \quad (6.25)$$

Until now the axes of the orthogonal polarization components $V_1(\mathbf{r}, t)$ and $V_2(\mathbf{r}, t)$ have been arbitrary,

[31] For the sake of brevity we shall sometimes suppress the space coordinate $\mathbf{r}$.

and it is clear from the nature of equation (6.25) that the correlation must be independent of the choice of axes. However, it has been shown (Wolf, 1959) that axes can always be chosen to make $\langle I_1(\mathbf{r}, t)\rangle = \langle I_2(\mathbf{r}, t)\rangle = \frac{1}{2}\langle I(\mathbf{r}, t)\rangle$, and that under these conditions $|\gamma^{(1,1)}{}_{12}(0)|$ is a maximum and equal to the degree of polarization P [cf. also Parrent and Roman (1960); Born and Wolf (1964), p. 553]. With this choice (6.25) immediately reduces to

$$\langle I(\mathbf{r}, t)I(\mathbf{r}, t+\tau)\rangle = \langle I(\mathbf{r}, t)\rangle^2[1+\tfrac{1}{2}(1+P^2)\,|\,\gamma^{(1,1)}(\mathbf{r}, \mathbf{r}, \tau)\,|^2], \quad (6.26)$$

and (6.22) becomes

$$p_c(t\,|\,t+\tau)d\tau = \alpha\langle I(\mathbf{r}, t)\rangle \times[1+\tfrac{1}{2}(1+P^2)\,|\,\gamma^{(1,1)}(\mathbf{r}, \mathbf{r}, \tau)\,|^2]d\tau. \quad (6.27)$$

Since $|\gamma^{(1,1)}(\mathbf{r}, \mathbf{r}, \tau)| \approx 1$ for $\tau \ll \tau_c$, and $\gamma^{(1,1)}(\mathbf{r}, \mathbf{r}, \tau) \approx 0$ for $\tau \gg \tau_c$, we see that there is an enhanced probability for a second photon to be counted in a time less than τ_c after a previous one. This illustrates the well known bunching properties of a photon beam. Although, strictly speaking, $p_c(t\,|\,t+\tau)$ refers to the fluctuation properties of the photoelectrons and not of the photons, we would of course interpret the emission of a photoelectron as a photon count; this is after all the only sense in which photons can be counted in practice.

Figure 12 shows the form of the distribution

$$p_c(t\,|\,t+\tau)$$

for polarized thermal light having a Gaussian spectral profile. We note that there is a maximum clustering factor of 2, as compared with independent events. The origin of the bunching phenomenon is clearly brought out by the analysis of Dicke (1954) of the detailed radiation process in an excited gas, which leads to correlation in the emission of photons and to a twofold enhanced probability for the emission of successive

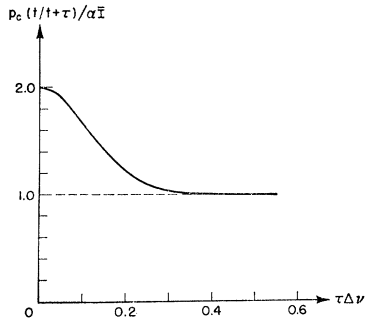


FIG. 12. The conditional probability density $p_c(t\,|\,t+\tau)$ for a Gaussian spectral profile of rms width $\Delta\nu$. [After L. Mandel (1963d).]

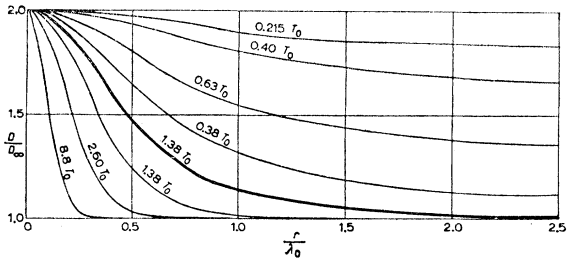


FIG. 13. The variation of density $D(r)$ in a Bose-Einstein gas with distance r from a given molecule, for different temperatures T and for the limit of a weak van der Waals exchange interaction. T_0 is the condensation temperature and λ_0 is the mean de Broglie wavelength at T_0. [After F. London (1943).]

photons with the same momentum $\hbar\mathbf{k}$. It is worth noting that spatial clustering distributions of the form shown in Fig. 12 are well known for a photon gas in thermal equilibrium [Kothari and Auluck (1957)], or indeed for any system of weakly interacting bosons [Uhlenbeck and Gropper (1932); London (1938, 1943)]. This is illustrated in Fig. 13 which shows the mean particle density $D(r)$ at a distance r from another particle [cf. also Landau and Lifshitz (1958, p. 373)].

The bunching effects implicit in Eq. (6.27) lead to an enhancement in the rate of "coincidence" counts recorded by two photodetectors illuminated by two partially coherent thermal light beams. Consider two similar beams of polarized thermal light of instantaneous intensities $I_1(t)$ and $I_2(t)$ incident on two photodetectors. Let the pulses from the two detectors be fed to a coincidence circuit of resolving time T, which gives an output only if the two pulses arrive within a time interval $\frac{1}{2}T$. In view of (6.4) the coincidence rate R of counting is then given by

$$R = \alpha_1\alpha_2\int_{-\frac{1}{2}T}^{\frac{1}{2}T}\langle I_1(t)I_2(t+\tau)\rangle\,dt.$$

By an argument similar to that used in deriving Eq. (6.27) we find [cf. Purcell (1956); Hanbury Brown and Twiss (1957a, 1958a); Mandel (1963d); Sillitto (1963)]

$$R = R_1R_2T[1+|\,\gamma^{(1,1)}(\mathbf{r}_1, \mathbf{r}_2, 0)\,|^2\,\zeta(T)/T], \quad (6.28)$$

where R_1 and R_2 are the separate counting rates of the two detectors and

$$\zeta(T) = \int_{-\frac{1}{2}T}^{\frac{1}{2}T}|\,\gamma^{(1,1)}(\mathbf{r}_1, \mathbf{r}_1, \tau)\,|^2\,d\tau. \quad (6.29)$$

The first term on the right of Eq. (6.28) represents the "accidental coincidence" rate due to random events, while the second is attributable to the bunching properties of the photons. Note that the second term is small for $T \gg \tau_c$ (in which case $\zeta(T) \approx \zeta(\infty) = \tau_c$ [cf. (6.14)]), whereas the two are comparable when $T \ll \tau_c$ and

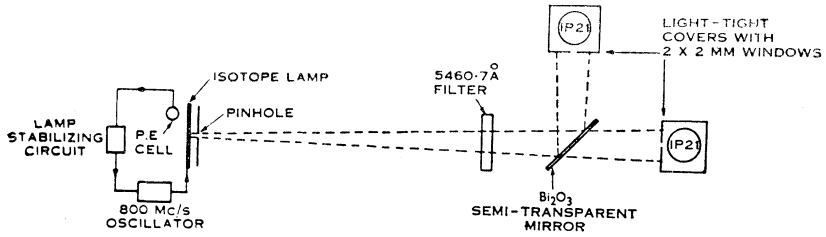


FIG. 14. The apparatus used for the demonstration of excess counting coincidences [After R. Q. Twiss and A. G. Little (1959).]

$|\gamma^{(1,1)}(\mathbf{r}_1, \mathbf{r}_1, \tau)| \approx 1$ (in which case $\zeta(T) \approx T$). The enhanced coincidence effect will therefore be detectable only with circuits having high time resolution and light beams of extremely narrow bandwidth. It is significant that the first attempts to record excess counting coincidences with partially coherent light beams failed through having too small a factor $\zeta(T)/T$ [Adam, Janossy, and Varga (1955a, b); Brannen and Ferguson (1956)], although the first one was admittedly more concerned with the beam splitting action of a half-silvered mirror than with the bunching phenomenon [see also, Hanbury Brown and Twiss (1956c); Z. Bay and P. S. Farago (1963)]. The practical problem of measuring counting fluctuations has been discussed by Alkemade (1959).

The first successful demonstration of the excess counting effect was reported by Twiss, Little, and Hanbury Brown (1957) [see also Twiss and Little (1959)] with the apparatus shown in Fig. 14. By using light from a low pressure Hg^{198} isotope gas discharge having a coherent time of about 0.7×10^{-9} sec and a coincidence circuit with resolving time $T\sim7\times10^{-9}$ sec, they were able to realize a factor

$$|\gamma^{(1,1)}(\mathbf{r}_1, \mathbf{r}_2, 0)|^2 \zeta(T)/T \sim 0.04.$$

The experiment has since been successfully performed by others [Rebka and Pound (1957); Brannen, Ferguson, and Wehlau (1958); but see also, Janossy, Naray, and Varga (1961)].

As the conditional counting probability $p_c(t \mid t+\tau)d\tau$ contains information about the degree of polarization of the light and about the spectral distribution, direct measurements of the interval distribution between counts might be thought to be worthwhile. In particular, when the spectrum of the light becomes too narrow to be readily resolved by the usual spectroscopic techniques, the time resolution called for in measuring $|\gamma^{(1,1)}(\mathbf{r}, \mathbf{r}, \tau)|$ becomes relatively modest. The problem of recovering the spectrum from a knowledge of $|\gamma^{(1,1)}(\mathbf{r}, \mathbf{r}, \tau)|$ has been discussed in Sec. 5.3 and a scheme for measuring $p_c(t \mid t+\tau)$ directly has been proposed by Mandel (1963b) [see also, R. M. Sillitto (1963)] and is shown in Fig. 15. In this arrangement the coincidence circuit counts only when two pulses appear separated by a preselected time delay τ, which can be varied, and $p_c(t \mid t+\tau)$ can then be determined from a series of measurements with different τ. An experiment of this kind has also recently been reported by Martienssen and Spiller (1964), who used an artificially degenerate "pseudothermal" light beam, produced by moving a ground-glass screen in front of a gas laser.

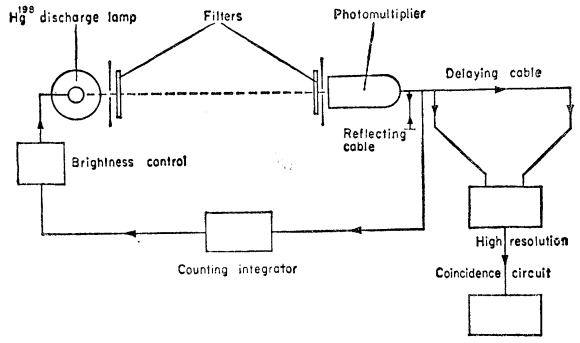


FIG. 15. Outline of the apparatus for measuring the time interval distribution of photons. [After L. Mandel (1963b).]

6.4. Intensity Correlations and Correlation Interferometry

We have already seen that two partially coherent light beams incident on two photodetectors will in general lead to counting "coincidences" in excess of those expected from random events. Historically the correlated emission of photoelectrons was first demonstrated by Hanbury Brown and Twiss (1956a) in the form of a correlation between the two photocurrents

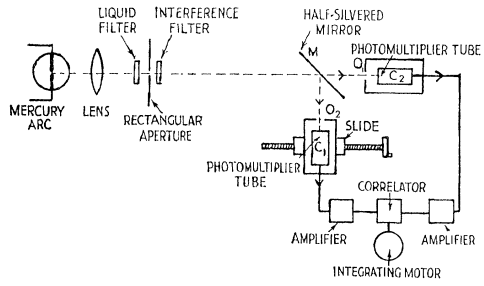


FIG. 16. The apparatus used for the demonstration of correlation between the intensity fluctuations of partially coherent light beams. [After R. Hanbury Brown and R. Q. Twiss (1956a).]

treated as continuous signals. Their apparatus is shown in Fig. 16. Light from a mercury lamp was divided into two beams by a half-silvered mirror and fell on two photocells, whose outputs were sent through band-limited amplifiers to a correlator. The experiment has recently been repeated by Martienssen and Spiller (1964) with the help of their "pseudothermal" source.

Consider a system of N detectors illuminated by beams of partially coherent light. Let suffixes $i=1, \cdots, N$ label the N channels. If n_i is the number of counts registered by the ith detector in a time interval from t to $t+T$, then, from a generalization of the argument leading to Eq. (6.6), we find that

$$\prod_{i=1}^{N} \frac{n_i[\alpha_i U_i(T,t)]^{n_i}}{n_i!} \exp\left[-\alpha_i U_i(T,t)\right],$$

where the angular brackets denote the ensemble average. With the help of (6.6) and a property of the Poisson distribution, this may be shown to reduce to [cf. Mandel, Sudarshan, and Wolf (1964)]

$$\begin{aligned}\langle n_1 n_2 \cdots n_N \rangle &= \alpha_1\alpha_2\cdots\alpha_N \int_t^{t+T}\!\!\cdots\!\int \langle I_1(\mathbf{r}_1,t_1) I_2(\mathbf{r}_2,t_2)\cdots I_N(\mathbf{r}_N,t_N)\rangle\, dt_1\, dt_2\cdots dt_N \\ &= \alpha_1\alpha_2\cdots\alpha_N \int_t^{t+T}\!\!\cdots\!\int \sum_{j_1}\sum_{j_2}\cdots\sum_{j_N} \Gamma^{(N,N)}{}_{j_1 j_2,\cdots,j_N;j_1,j_2,\cdots,j_N}(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N,\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_N, \\ &\qquad t_1,t_2,\cdots,t_N,t_1,t_2,\cdots,t_N)\, dt_1\, dt_2\cdots dt_N, \qquad (6.30)\end{aligned}$$

where α_1, α_2, etc., are constants representing the photoefficiencies of the N photodetectors. Thus the N-fold correlation of the counts is entirely determined by the $2N$th-order correlation function of the field. We can also express this result in another form. Let

$$U_i(T,t) = \int_t^{t+T} I_i(t')\, dt' \qquad (6.31)$$

be the integrated classical intensity at the ith detector. Then we can write (6.30) in the form [cf. Mandel, Sudarshan, and Wolf (1964)]

$$\begin{aligned}&\langle n_1 n_2 \cdots n_N \rangle \\ &= \alpha_1\alpha_2\cdots\alpha_N \langle U_1(T,t)\, U_2(T,t)\cdots U_N(T,t)\rangle, \qquad (6.32)\end{aligned}$$

or, on introducing the deviations $\Delta n_i = n_i - \langle n_i\rangle$, $\Delta U_i = U_i - \langle U_i\rangle$, and making a multinomial expansion, we can write

$$\langle \Delta n_1 \Delta n_2 \cdots \Delta n_N\rangle = \alpha_1\alpha_2\cdots\alpha_N \langle \Delta U_1 \Delta U_2 \cdots \Delta U_N\rangle. \qquad (6.33)$$

Thus the correlations of the photoelectric counts could be described as having their origin in the correlations of the integrated classical intensities. A somewhat different point of view of these correlation effects, related to the approach of Fano (1961), has recently been presented by Goldberger and Watson (1964a, b).

The foregoing relations are quite general and hold for light beams from any source. However for stationary thermal light beams, for which the random process $\mathbf{V}(\mathbf{r},t)$ is (complex) Gaussian, it is well known [cf. Reed (1962); Mehta (1965b), p. 398] that $\Gamma^{(N,N)}$ is in general expressible in terms of products of second-order correlation functions. In particular, when there are just two detectors, and the following strong conditions for cross-spectral purity hold

$$\begin{aligned}&\Gamma^{(1,1)}{}_{j_1 j_2}(\mathbf{r}_1,\mathbf{r}_2,t_1,t_2) = [I_{j_1}(\mathbf{r}_1) I_{j_2}(\mathbf{r}_2)]^{\frac{1}{2}}\, \gamma^{(1,1)}{}_{j_1 j_2}(\mathbf{r}_1,\mathbf{r}_1,0)\gamma^{(1,1)}{}_{j_1 j_1}(\mathbf{r}_1,\mathbf{r}_2,0)\gamma^{(1,1)}{}_{j_1 j_1}(\mathbf{r}_1,\mathbf{r}_1,t_1-t_2), \\ &\gamma^{(1,1)}{}_{j_1,j_2}(\mathbf{r}_1,\mathbf{r}_1,0) = \gamma^{(1,1)}{}_{j_1 j_2}(\mathbf{r}_2,\mathbf{r}_2,0), \qquad (6.34) \\ &\gamma^{(1,1)}{}_{j_1 j_1}(\mathbf{r}_1,\mathbf{r}_2,0) = \gamma^{(1,1)}{}_{j_2 j_2}(\mathbf{r}_1,\mathbf{r}_2,0) \equiv \gamma^{(1,1)}(\mathbf{r}_1,\mathbf{r}_2,0),\end{aligned}$$

we can use a simple generalization of relation (6.26) in (6.30) to yield

$$\langle \Delta n_1 \Delta n_2\rangle = \tfrac{1}{2}(1+P^2)\langle n_1\rangle\langle n_2\rangle\, |\, \gamma^{(1,1)}(\mathbf{r}_1,\mathbf{r}_2,0)\, |^2\, \xi(T)/T. \qquad (6.35)$$

The time

$$\xi(T) = T^{-1} \int_{-T/2}^{T/2}\!\!\int |\, \gamma^{(1,1)}(\mathbf{r}_1,\mathbf{r}_1,t_1-t_2)\, |^2\, dt_1\, dt_2 \qquad (6.36)$$

obeys the inequalities [cf. Mandel (1958, 1959)]

$$\begin{aligned}&\xi(T) \le T \\ &\xi(T) \le \xi(\infty), \qquad (6.37)\end{aligned}$$

and $\xi(\infty)$, which may be shown to be identical with the expression for τ_c given by (6.14), can be interpreted as the coherence time of the light. The relation (6.35) is the equation describing the Hanbury Brown–

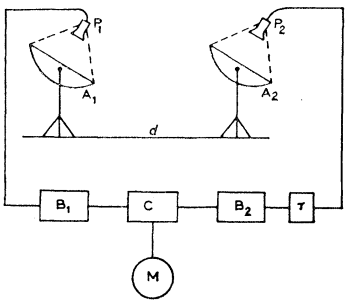


FIG. 17. Schematic diagram of the stellar intensity interferometer: A, mirrors; B, amplifiers; C, multiplier; M, integrator; P, phototubes; τ, delay line. [After R. Hanbury Brown and R. Q. Twiss (1958a).]

Twiss effect with thermal light. In practice the correlation is usually measured as in Fig. 16 by feeding the currents from two photoelectric detectors through two bandlimited amplifiers to an electronic correlator. n_1 and n_2 are then proportional to the signals $S_1(t)$ and $S_2(t)$ in the two correlator channels, and T plays the role of a "resolving time" or reciprocal pass bandwidth [cf. Hanbury Brown and Twiss (1957a); Mandel (1963d)]. When T is much smaller than the coherence time of the light, it is easy to show that the following slightly more general correlation between $S_1(t+\tau)$ and $S_2(t)$ holds:

$$\langle \Delta S_1(t+\tau)\Delta S_2(t)\rangle = \tfrac{1}{2}(1+P^2)\langle S_1\rangle\langle S_2\rangle |\gamma^{(1,1)}(\mathbf{r}_1, \mathbf{r}_2, \tau)|^2. \quad (6.38)$$

Shortly after the successful demonstration of the existence of an intensity-correlation effect, Hanbury Brown and Twiss (1956b, 1958a, b) succeeded in applying the correlation technique to the determination of the angular diameter of the star Sirius. Their apparatus, which has been called a stellar intensity interferometer, is illustrated in Fig. 17. It bears a certain resemblance to the Michelson stellar interferometer (see Fig. 8), but is based on the correlation principle. The light from the star was focused by means of two large reflectors onto two similar photomultipliers, and the correlation of the output currents was examined for different separations of the detectors to yield $|\gamma^{(1,1)}(\mathbf{r}_1, \mathbf{r}_2, \tau)|$. This in turn allowed information about the intensity distribution of the source to be obtained [cf. Eq. (5.10)]. More recently a large stellar interferometer of this type was built at Narrabri, Australia [Hanbury Brown (1964)]. It can operate with a base line up to 600 ft, about 30 times longer than that utilized by Michelson and his collaborators in the original version of their interferometer. With the Narrabri instrument, diameters of certain types of stars (brighter than magnitude +2.5 and with spectral type earlier than *F.0.*) can be determined, down to angular diameters of about 0.0005 sec arc. First preliminary measurements of the diameter of α-Lyrae, were recently reported [Hanbury Brown, Hazard, Davis, and Allen (1964)].

It is an important feature of the technique that the measurements are based on the photoelectric signal $S(t)$, which depends on the slowly varying instantaneous intensity $I(\mathbf{r}, t)$ rather than on the rapidly varying complex wave amplitude $\mathbf{V}(\mathbf{r}, t)$. For this reason small variations of optical path difference, e.g., due to atmospheric fluctuations or mechanical disturbance of the detector, do not affect the measurements, whereas they may be intolerable for the Michelson stellar interferometer. On the other hand, we note from equations (6.35) and (6.38) that the phase of $\gamma^{(1,1)}(\mathbf{r}_1, \mathbf{r}_2, \tau)$ cannot be obtained directly.

A mathematical approach to the derivation of the phase through the analytic properties of $\gamma^{(1,1)}$ has already been described in Sec. 5.3. An alternative approach, in which two coherent light beams of known spectral distribution from an independent source are superposed on the two beams to be correlated, has been suggested by Gamo (1961).[32] Although the correlation of the photoelectric counts can in this case be made to yield information about the phase of $\gamma^{(1,1)}(\mathbf{r}_1, \mathbf{r}_2, \tau)$, the method suffers from experimental difficulties of the kind that surround the Michelson interferometer.

More recently Gamo (1963a, b) has proposed a three-point intensity correlation method for the determination of the phase of $\gamma^{(1,1)}(\mathbf{r}_1, \mathbf{r}_1, \tau)$ for which he suggests the arrangement shown in Fig. 18. A stationary collimated beam of polarized thermal light is split into three beams by partly silvered mirrors. The beams fall on three photodetectors in such a way that the optical path is the same for all three beams, and the photoelectric signals are passed to a triple correlator via delays τ and $\delta\tau$ as shown. If suffixes

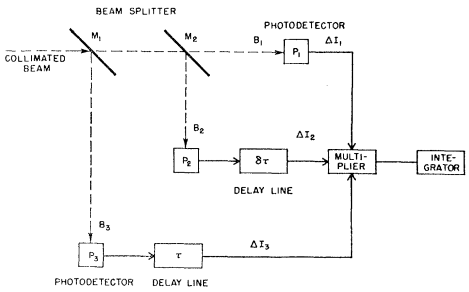


FIG. 18. The apparatus relating to a proposed triple correlation experiment for the determination of the phase of $\gamma^{(1,1)}(\mathbf{r},\mathbf{r},\tau)$. [After H. Gamo (1963a).]

[32] This technique, referred to by Gamo as the *method of the coherent background* must not be confused with another technique, also known by this name, due to Zernike (1948). In Gamo's method the two "background beams," though coherent with respect to each other, are incoherent with respect to the main beams onto which they are superposed. In Zernike's method the background beam and the main beam are coherent with respect to each other. Zernike's method of the coherent background is utilized in his phase contrast technique of observation in a microscope and in Gabor's technique of imaging by reconstructed wavefronts [cf. Born and Wolf (1964), Sec. 8.6 (c) and Sec. 8.10].

1, 2, 3 refer to the three channels, and the resolving time of the correlator is sufficiently short, the triple correlation of the photocurrent fluctuations will be proportional to

$$C=\langle \Delta I_1(t+\tau)\Delta I_2(t+\tau-\delta\tau)\Delta I_3(t)\rangle$$
$$=\Gamma^{(3,3)}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3;\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3;t+\tau,t+\tau-\delta\tau,t;t+\tau,t+\tau-\delta\tau,t)$$
$$-\langle I_1\rangle\langle I_2\rangle\langle I_3\rangle-\langle I_1\rangle\,|\,\Gamma^{(2,2)}(\mathbf{r}_2,\mathbf{r}_3;\mathbf{r}_2,\mathbf{r}_3;t+\tau-\delta\tau,t;t+\tau-\delta\tau,t)\,|^2$$
$$-\langle I_2\rangle\,|\,\Gamma^{(2,2)}(\mathbf{r}_3,\mathbf{r}_1;\mathbf{r}_3,\mathbf{r}_1;t,t+\tau;t,t+\tau)\,|^2$$
$$-\langle I_3\rangle\,|\,\Gamma^{(2,2)}(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_1,\mathbf{r}_2;t+\tau,t+\tau-\delta\tau;t+\tau,t+\tau-\delta\tau)\,|^2. \quad (6.39)$$

With the help of the moment reduction formulas for $\Gamma^{(3,3)}$ and $\Gamma^{(2,2)}$ for a complex Gaussian random process [Reed (1962); Mehta (1965b), p. 398], in the limit of very small $\delta\tau$ this reduces to

$$C=2\langle I_1\rangle\langle I_2\rangle\langle I_3\rangle\,|\,\gamma^{(1,1)}(\tau)\,|^2\cos[Q(\tau)\delta\tau], \quad (6.40)$$

where

$$\pm Q(\tau)\delta\tau=\phi(\tau)-\phi(\tau-\delta\tau)-\phi(\delta\tau), \quad (6.41)$$

and $\phi(\tau)$ is the phase of the normalized autocorrelation function $\gamma^{(1,1)}(\tau)$. Now, for small $\delta\tau$, $\phi(\delta\tau)=2\pi\nu_0\delta\tau$ [cf. Eq. (3.26)], where ν_0 is the midfrequency of the light, so that (6.41) becomes

$$\pm Q(\tau)\approx[d\phi(\tau)/d\tau]-2\pi\nu_0. \quad (6.42)$$

Since $Q(\tau)$ can be obtained experimentally from measurements of C, it follows that $\phi(\tau)$ can be derived. Note however than an ambiguity in the sign of $Q(\tau)$ still leaves a certain ambiguity in $\phi(\tau)$, which can only be resolved by additional measurements.

The method is interesting in that it furnishes one of the very few proposed applications of sixth-order correlation functions of the field.

7. TRANSIENT SUPERPOSITION EFFECTS

All the interference effects treated in Secs. 3 and 5 can be described in terms of the averaged light intensities at different space–time points together with the second order correlations. However, there remain a number of transient phenomena associated with the superposition of light beams, which are observable even with completely independent beams, and are not readily describable in this way. Although the term interference is applied to them, we shall see that in general these phenomena involve the fourth-order correlations of the combined field.

The light beats obtainable with quasimonochromatic beams of slightly different center frequencies are perhaps the best known example of the effects we wish to discuss here, and will be described first. These beat effects must be clearly distinguished from the light beats observed by Brossel and Bitter (1952), Dodd *et al.* (1959) [see also Colegrove *et al.* (1959); Franken (1961); Dodd and Series (1961); Barrat (1959a, b, c, 1961); Cohen-Tannoudji (1961a, b)], under conditions of induced coherence, for example when two, initially independent, atomic states are coupled by an oscillating magnetic field.

7.1. Light Beating Effects between Independent Beams

The first demonstration of beats resulting from the superposition of incoherent beams was given by Forrester, Gudmundsen, and Johnson (1955), who made use of the two spectral components of a Zeemann doublet. Their apparatus is shown in Fig. 19. The photoelectric detector was in the form of a resonant cavity tuned to the Zeeman difference frequency, and they succeeded in obtaining an acceptable signal-to-noise ratio only by severely restricting the response bandwidth of the detector. Indeed it is remarkable that the experiment succeeded at all, for it was carried out with nondegenerate light. The mean number of photons received on a coherence area, in the time for which a steady beat is expected to persist (the coherence time), was much less than 1.

With the development of the laser, which produces light beams for which the degeneracy parameter δ may easily be in excess of 10^{12} [cf. Mandel (1961a)], such beating experiments became much easier to perform. Figures 20 and 21 show the apparatus used by Javan, Ballik, and Bond (1962), and Lipsett and Mandel (1963, 1964a), to determine the coherence times of the He:Ne laser and the ruby laser, respectively, by superposition of beams from two independent sources. Several other workers [Herriott (1962); McMurtry and Siegman (1962)] have used the alternative technique of beating together two or more modes of the same laser, although the statistical independence of the beams is then somewhat questionable (cf. Paananen, Tang, and Statz (1963); Lamb (1964); Haken and Sauermann (1963)]. The different experiments have been discussed by Forrester (1961a, b), McMurtry (1963), Lipsett and Mandel (1963, 1964a, b).

The superposition effects are readily described in the classical terms of Sec. 3.1. If Fig. 3 represents the

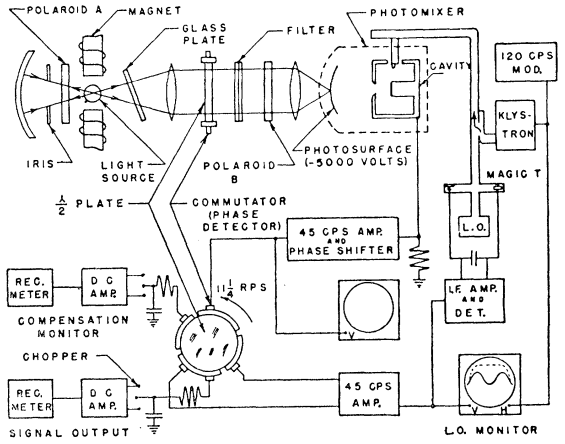


FIG. 19. The apparatus used for the demonstration of beats resulting from the superposition of incoherent beams. [After A. T. Forrester, R. A. Gudmundsen, and P. O. Johnson (1955).]

experimental situation, then the instantaneous intensity at the point of superposition $P(\mathbf{r})$ of the two polarized beams will be represented by Eq. (3.7). However, in this case we cannot expect the ensemble average, given by equation (3.8), to describe the experimental situation in which the detailed behavior of the instantaneous intensity is measured. Indeed the ensemble average of the last "key" term of (3.7) vanishes for incoherent beams. If $I_1(t)$ and $\phi_1(t)$, etc. are the instantaneous intensity and phase of $K_1V(\mathbf{r}_1, t)$, etc. we may write

$$K_1V(\mathbf{r}_1, t)=[I_1(t)]^{\frac{1}{2}}\exp[2\pi i\nu_1 t+i\phi_1(t)]$$
$$K_2V(\mathbf{r}_2, t)=[I_2(t)]^{\frac{1}{2}}\exp[2\pi i\nu_2 t+i\phi_2(t)], \tag{7.1}$$

where ν_1 and ν_2 are the midfrequencies of the two quasimonochromatic beams. With the help of (7.1)

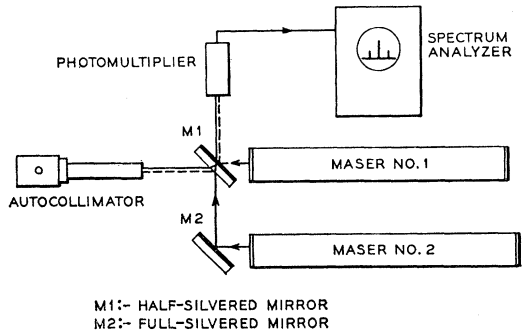


FIG. 20. The apparatus employed for observing beat signals between two optical maser (He:Ne) beams. [After A. Javan, E. A. Ballik, and W. L. Bond (1962).]

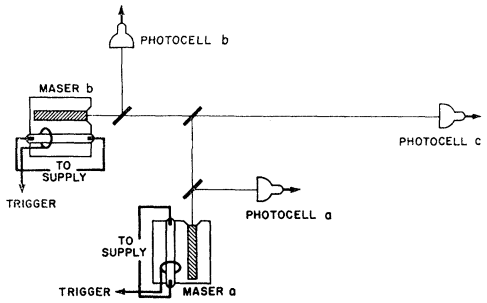
Eq. (3.7) becomes

$$I(\mathbf{r}, t)=I_1(t-t_1)+I_2(t-t_2)+2[I_1(t-t_1)I_2(t-t_2)]^{\frac{1}{2}}\times\cos[2\pi(\nu_2-\nu_1)t+2\pi(\nu_1t_1-\nu_2t_2)+\phi_2(t-t_2)-\phi_1(t-t_1)]. \tag{7.2}$$

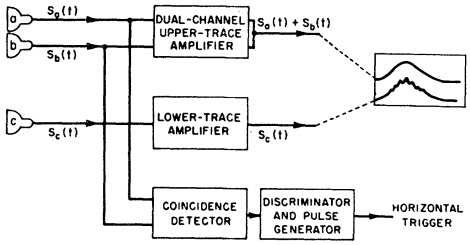
Now the response $S(t)$ of the photoelectric detector in the receiving plane may be expressed in the form (cf. Sec. 6.1)

$$S(t)=\alpha\int_\sigma I(\mathbf{r}, t)\,d\mathbf{r}, \tag{7.3}$$

where α is the quantum efficiency, and the integral



(a)



(b)

FIG. 21. The apparatus employed for observing beat signals between two optical maser (ruby) beams. (a) Over-all arrangement of the system. (b) Block diagram of the beat detection system. [After M. S. Lipsett and L. Mandel (1963).]

is to be taken over the surface σ of the photocathode. In the evaluation of the integral with the help of (7.2) we shall assume that the "sources" P_1 and P_2 are at a great distance from P, so that the wavefronts at the receiver are practically plane and that the intersections of all three planes can be colinear. Let θ_1 and θ_2 be the angles of inclination between the two wavefonts and the receiver, which we take to be of rectangular dimensions $a\times b$ for simplicity. Then we may simplify (7.3) to

$$S(t)=\alpha b\int_0^a I(x, t)\,dx, \tag{7.4}$$

where x is a coordinate defining position across the photocathode in a line normal to the intersection of the wavefronts. Evidently

$$t_1=\tau_1+(x/c)\sin\theta_1,$$
$$t_2=\tau_2+(x/c)\sin\theta_2, \tag{7.5}$$

where τ_1 and τ_2 are constants. Now $I_1(t)$, $I_2(t)$ and $\phi_1(t)$, $\phi_2(t)$ are slowly varying functions [cf. Born and Wolf (1964), p. 496]. If $(a/c)\sin\theta_1$ and $(a/c)\sin\theta_2$ are much shorter than the coherence times of the two beams, both $I_1(t-t_1)$ and $\phi_1(t-t_1)$, etc., will remain constant under the integral (7.4). We then arrive at [cf. Lipsett and Mandel (1963)]

$$S(t)=S_1(t)+S_2(t)+2[S_1(t)\,S_2(t)]^{\frac{1}{2}}(\sin\psi/\psi)$$
$$\times\cos[2\pi(\nu_2-\nu_1)t+2\pi(\nu_1\tau_1-\nu_2\tau_2)$$
$$+\psi+\phi_2(t-\tau_2)-\phi_1(t-\tau_1)], \tag{7.6}$$

where

$$\psi=(\pi a/c)(\nu_1\sin\theta_1-\nu_2\sin\theta_2), \tag{7.7}$$

and the photoelectric signals $S_1(t)$ and $S_2(t)$ are those due to the separate light beams. We see that, for a time short compared with the coherence time, the photoelectric signal will carry a steady sinusoidal modulation at the difference frequency $\nu_2-\nu_1$, of relative modulation amplitude

$$\mathcal{C}=\frac{2}{[S_1(t)/S_2(t)]^{\frac{1}{2}}+[S_2(t)/S_1(t)]^{\frac{1}{2}}}\frac{\sin\psi}{\psi}. \tag{7.8}$$

To ensure an appreciable modulation it is evidently necessary to make ψ small. For small angles θ_1 and θ_2 and nearly equal frequencies ν_1 and ν_2 this implies that

$$(\pi a\nu/c)(\theta_1-\theta_2)\ll 1,$$

or that the two wavefronts have to be aligned to within a fraction of a wavelength within the width a. This suggests that an autocollimator is needed for the alignment of the two laser beams. The beat note will remain steady as long as $S_1(t)$, $S_2(t)$, $\phi_1(t)$, $\phi_2(t)$ remain constant and, in the absence of other effects, the duration of a 'steady' beat note is a measure of the coherence time. Moreover, the spectral excursion of the beat note over a long time is a measure of the spectral width of the light beams.

We stress once again that Eqs. (7.2) and (7.6) refer to instantaneous values, or to single members of the ensembles of fields and photoelectric signals. Because the phase angles $\phi_1(t)$ and $\phi_2(t)$ are randomly distributed over 0 to 2π, the ensemble average of $S(t)$ does not show any sinusoidal dependence on t, and gives no indication of beats. The description of the phenomenon in terms of ensemble averages is not quite so direct, and will be given later for the quantized field. However it is one of the attractive features of the classical description that the operation of averaging can be avoided very easily.

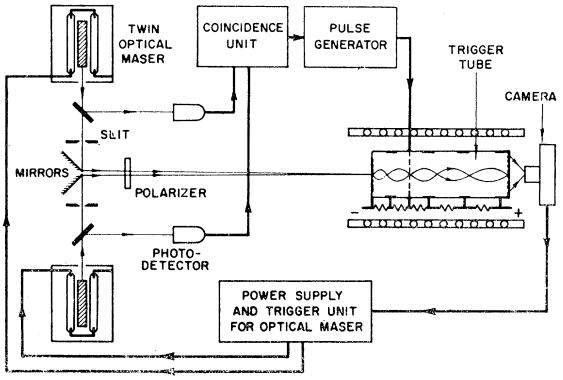


FIG. 22. Outline of the apparatus used for recording transient interference fringes. [After G. Magyar and L. Mandel (1963).]

7.2. Interference Fringes Produced by Independent Beams

It is possible to think of interference fringes as "beats" in the spatial domain, and to describe them by the same mathematical relations, such as equation (7.2), that were used in the previous section. Experimentally the phenomenon is very different however. It is significant that the interference analog of the experiment of Forrester, Gudmundsen, and Johnson (1955) has never been carried out with nondegenerate thermal light, although the possibility has been discussed [Mandel (1960, 1961b, 1962b; Neugebauer (1962)]. The development of the laser made the experiment feasible, and by using laser beams, Magyar and Mandel (1963, 1964) succeeded in recording fringes in the visible.[32a] The outline of their apparatus is shown in Fig. 22. It is not difficult to see why a large value of the degeneracy parameter δ is so important in these experiments. For the number of photons defining the interference pattern in the receiving plane in a time less than the coherence time has an upper limit of δ. When $\delta\ll 1$ it is hard to think of interference fringes at all.

As in the previous section, the instantaneous light intensity at each point in the receiving plane is expressed by equation (7.2), with t_1 and t_2 given by (7.5). The receiver is now in the form of a photoelectric image detector, such as a photographic plate or an image tube, which resolves the pattern in space but integrates over

[32a] An analogous experiment has recently been carried out by Martienssen and Spiller (1964), who used a degenerate "pseudo-thermal" source consisting of a gas laser and a moving ground-glass screen.

time. If T is the exposure time, the effective signal recorded at the point with coordinate x will be

$$S(x, t, T)=\alpha\int_{t}^{t+T} I(x, t')\,dt'. \tag{7.9}$$

In practice T will usually be made small compared with the coherence time τ_c. With the help of (7.2) and (7.5), and with the same restrictions on the size and inclination of the receiver as before, (7.9) becomes [cf. Magyar and Mandel (1963, 1964)]

$$\begin{aligned}S(x, t, T)=S_1(t, T)+S_2(t, T)+2[S_1(t, T)\,S_2(t, T)]^{\frac{1}{2}}\,[\sin\pi(\nu_2-\nu_1)\,T/\pi(\nu_2-\nu_1)\,T]\\ \times\cos[2\pi(\nu_2-\nu_1)\,(t+\tfrac{1}{2}T)+2\pi(x/c)\,(\nu_1\sin\theta_1-\nu_2\sin\theta_2)\\ +2\pi(\nu_1\tau_1-\nu_2\tau_2)+\phi_2(t-\tau_2)-\phi_1(t-\tau_1)].\end{aligned} \tag{7.10}$$

Thus $S(x, t, T)$ is a cosine function of position x, and the recorded signal will show a spatial modulation in the plane of superposition, which we interpret as interference fringes. For small θ_1 and θ_2 and nearly equal ν_1 and ν_2, the spacing of the fringes in the receiving plane will be $c/\nu_1(\theta_1-\theta_2)$, exactly as for coherent beams. The visibility of the ringes is [Eq. (3.22)]

$$\mathcal{V}=\frac{2}{[S_1(t, T)/S_2(t, T)]^{\frac{1}{2}}+[S_2(t, T)/S_1(t_1, T)]^{\frac{1}{2}}}\,\frac{\sin\pi(\nu_2-\nu_1)\,T}{\pi(\nu_2-\nu_1)\,T}, \tag{7.11}$$

and this has its maximum possible value 1 when $S_1(t, T)=S_2(t, T)$ and the exposure $T\ll 1/|\nu_2-\nu_1|$. However if the sources are independent, and there is no close control over the midfrequencies ν_1 and ν_2 of the two light beams, the visibility may never approach its maximum value, even though $T\ll\tau_c$. Because the phase angles $\phi_1(t)$ and $\phi_2(t)$ are randomly distributed over 0 to 2π, the positions of the fringe maxima and minima cannot be predicted beforehand. This feature is one that sharply differentiates the above phenomenon from conventional interference. The simplicity of the foregoing treatment is again due to the avoidance of the operation of ensemble averaging, which would eliminate the modulation term from equation (7.10).

There is another superposition effect, already referred to in Sec. 5.5, which is only distantly related to, and occasionally confused with, the foregoing. It has its origin in an experiment first performed to determine the velocity of light [Alford and Gold (1958)], and has been called the Alford and Gold effect [cf. Givens (1961a, 1962); Mandel (1962a); Paul (1963)]. It occurs with (second-order) coherent or partially coherent light beams, when the optical path difference at the superposition plane greatly exceeds the coherence length. Under these conditions interference fringes are not normally observed, and one may easily be misled into believing that the two beams have no second-order coherence (cf. Sec. 5.5).

However, although the ensemble average of the resultant light intensity is constant over the receiving plane [cf. Eq. (5.35)], the spectral density shows a sinusoidal modulation given by Eq. (5.33). The phase of this modulation depends on the optical path difference, and therefore on the position x in the receiving plane. If the receiver is a photoelectric detector responding to the light intensity in the immediate neighborhood of x, the beats between different spectral components may show up in the photocurrent. It then follows that the output of a sharply tuned filter (of passband $\delta\nu\ll c/$optical-path difference) following the photodetector will vary sinusoidally with x. An observer measuring this output may conclude that the detector is registering interference fringes in the receiving plane, although the ensemble average of the light intensity is constant in this plane.

7.3. The Quantum Theory of Transient Superposition Effects

We have already shown in Sec. 3.2 that a description of ordinary interference effects may readily be given in terms of the quantized field, and that it follows the classical treatment fairly closely. It might therefore be thought that the transient superposition effects discussed in the last two sections can also be described quantum mechanically, in a closely parallel manner. However, here we come up against the basic feature that quantum mechanics is always concerned with expectation values of observables, whereas the calculation of expectation values was deliberately avoided in the simple treatment leading to Eqs. (7.6) and (7.10).

Of course it might be argued that any difficulty can be avoided by considering only pure "coherent" states $|\{v_{\mathbf{k},s}\}\rangle$ of the radiation field, which are eigenstates of the annihilation operator $\hat{\mathbf{A}}^{(+)}(\mathbf{r}, t)$ [cf. Eq. (3.46)] corresponding to the desired eigenvalue $\mathbf{V}(\mathbf{r}, t)$ in the classical description. Such an approach has already been used [Paul, Brunner, and Richter (1963); Paul (1964); Mandel (1964b)]. However, these states do not describe fields encountered in practice and it is easy to show that there are other quantum states of the field for which the approach fails to give any indication of interference effects. The problem has recently

been considered in some detail by Richter, Brunner, and Paul (1964).

Consider two radiation fields in pure states of the kind $|\{v'_{\mathbf{k},s}\}\rangle$ and $|\{v''_{\mathbf{k},s}\}\rangle$, corresponding to the classical complex amplitudes

$$\mathbf{V}'(\mathbf{r},t)=\sum_{\mathbf{k},s}(hc/kL^3)^{\frac{1}{2}}v'_{\mathbf{k},s}\boldsymbol{\varepsilon}_{\mathbf{k},s}\exp[i(\mathbf{k}\cdot\mathbf{r}-ckt)],$$

$$\mathbf{V}''(\mathbf{r},t)=\sum_{\mathbf{k},s}(hc/kL^3)^{\frac{1}{2}}v''_{\mathbf{k},s}\boldsymbol{\varepsilon}_{\mathbf{k},s}\exp[i(\mathbf{k}\cdot\mathbf{r}-ckt)]. \tag{7.12}$$

If we assume that the two fields do not share any common $\mathbf{k}$, s modes, the state of the combined field will be represented by

$$|\{v'_{\mathbf{k},s}\}\rangle\,|\{v''_{\mathbf{k},s}\}\rangle.$$

The expectation value of the total intensity at the space time point $\mathbf{r}$, t in the combined field will then be given by Eqs. (3.50) and (3.52), with the "density matrix" $\Phi(\{v'_{\mathbf{k},s}\},\{v''_{\mathbf{k},s}\})$ diagonal. These equations then reduce to

$$\langle I(\mathbf{r},t)\rangle=[\mathbf{V}'^{*}(\mathbf{r},t)+\mathbf{V}''^{*}(\mathbf{r},t)]\cdot[\mathbf{V}'(\mathbf{r},t)+\mathbf{V}''(\mathbf{r},t)], \tag{7.13}$$

and the subsequent argument becomes identical to the argument of Secs. 7.1 and 7.2. We again find an almost periodic dependence of $\langle I(\mathbf{r},t)\rangle$ on t (beats) and $\mathbf{r}$ (fringes).

However very different states of the quantized field lead to very different results. Consider two energy eigenstates of the field $|\{n'_{\mathbf{k},s}\}\rangle$ and $|\{n''_{\mathbf{k},s}\}\rangle$, corresponding to $\{n'_{\mathbf{k},s}\}$ and $\{n''_{\mathbf{k},s}\}$ numbers of photons in momentum–spin states $\mathbf{k},s$, etc. With the simplifying assumption that no $\mathbf{k}$, s modes are common to both fields, the state of the combined field is represented by $|\{n'_{\mathbf{k},s}\}\rangle\,|\{n''_{\mathbf{k},s}\}\rangle$. The expectation value of the intensity can again be obtained from equation (3.52) with the help of explicit expressions for the density matrix $\Phi(\{v'_{\mathbf{k},s}\},\{v''_{\mathbf{k},s}\})$ [Sudarshan (1963a, b); see also, Appendix to the present article]. However, it is easier to proceed directly from the definition

$$\langle I(\mathbf{r},t)\rangle=\langle\{n''_{\mathbf{k},s}\}\,|\,\langle\{n'_{\mathbf{k},s}\}\,|\,\hat{\mathbf{A}}^{(-)}(\mathbf{r},t)\cdot\langle\mathbf{A}^{(+)}(\mathbf{r},t)\,|\,\{n'_{\mathbf{k},s}\}\rangle\,|\,\{n''_{\mathbf{k},s}\}\rangle, \tag{7.14}$$

and the expansion (3.40), together with the relations [cf. Messiah (1961), p. 436]

$$\begin{aligned}
&\hat{a}_{\mathbf{k}_1,s_1}\,|\,\{n'_{\mathbf{k},s}\}\rangle\,|\,\{n''_{\mathbf{k},s}\}\rangle\\
&\quad=(n'_{\mathbf{k}_1,s_1})^{\frac{1}{2}}\,|\,n'_{\mathbf{k}_1,s_1}-1\rangle\prod_{\mathbf{k},s\neq\mathbf{k}_1,s_1}|\,n'_{\mathbf{k},s}\rangle\,|\,n''_{\mathbf{k},s}\rangle\\
&\qquad\text{if } \mathbf{k}_1,\ s_1 \text{ is a mode of } |\{n'_{\mathbf{k},s}\}\rangle;\\
&\quad=(n''_{\mathbf{k}_1,s_1})^{\frac{1}{2}}\,|\,n''_{\mathbf{k}_1,s_1}-1\rangle\prod_{\mathbf{k},s\neq\mathbf{k}_1,s_1}|\,n'_{\mathbf{k},s}\rangle\,|\,n''_{\mathbf{k},s}\rangle\\
&\qquad\text{if } \mathbf{k}_1,\ s_1 \text{ is a mode of } |\{n''_{\mathbf{k},s}\}\rangle;\\
&\quad=0,\ \text{otherwise.}
\end{aligned} \tag{7.15}$$

As the states $|\,n'_{\mathbf{k},s}\rangle$ and $|\,n'_{\mathbf{k},s}-1\rangle$, etc., are orthogonal, the expansion reduces to

$$\langle I(\mathbf{r},t)\rangle=\sum_{\mathbf{k},s}(hc/kL^3)(n'_{\mathbf{k},s}+n''_{\mathbf{k},s}), \tag{7.16}$$

which does not depend on $\mathbf{r}$ or t. Equation (7.16) therefore gives no indication of any transient interference or beat effects in the superposition of two light beams. However, this conclusion does not imply that the effects are absent, but only that the expectation value of the light intensity is not an appropriate quantity for their description, as we noted earlier. Yet the averaging operation is here unavoidable.

As the detection of a spatial interference pattern, or a beat effect in time, necessarily involves intensity measurements at two or more space–time points, it is natural to examine intensity correlations [cf. Mandel (1964b)]. Consider two space–time points $\mathbf{r}_1$, t_1 and $\mathbf{r}_2$, t_2. If $I(\mathbf{r}_1,t_1)$ and $I(\mathbf{r}_2,t_2)$ are the total intensities, summed over all polarizations, of the combined field at these points, then the ensemble average

$$\langle I(\mathbf{r}_1,t_1)I(\mathbf{r}_2,t_2)\rangle=\mathrm{Tr}[\hat{\rho}\hat{\mathbf{A}}^{(-)}(\mathbf{r}_1,t_1)\hat{\mathbf{A}}^{(-)}(\mathbf{r}_2,t_2):\hat{\mathbf{A}}^{(+)}(\mathbf{r}_1,t_1)\hat{\mathbf{A}}^{(+)}(\mathbf{r}_2,t_2)], \tag{7.17}$$

where the colon signifies the scalar products between the first and third and the second and fourth operators, respectively. If we again make use of the universal "diagonal" representation of the density operator $\hat{\rho}$ [Sudarshan (1963a, b); see also Appendix to the present article] as in Sec. 3.2 and 4.3, together with the assumption of non-overlapping modes, and express the statistical independence of the two superposed light beams in the form

$$\Phi(\{v'_{\mathbf{k},s}\},\{v''_{\mathbf{k},s}\})=\Phi'(\{v'_{\mathbf{k},s}\})\Phi''(\{v''_{\mathbf{k},s}\}), \tag{7.18}$$

we arrive, by the same argument as previously, at

$$\begin{aligned}
&\langle I(\mathbf{r}_1,t_1)I(\mathbf{r}_2,t_2)\rangle\\
&\quad=\langle[\mathbf{V}'^{*}(\mathbf{r}_1,t_1)+\mathbf{V}''^{*}(\mathbf{r}_1,t_1)][\mathbf{V}'^{*}(\mathbf{r}_2,t_2)+\mathbf{V}''^{*}(\mathbf{r}_2,t_2)]:[\mathbf{V}'(\mathbf{r}_1,t_1)+\mathbf{V}''(\mathbf{r}_1,t_1)][\mathbf{V}'(\mathbf{r}_2,t_2)+\mathbf{V}''(\mathbf{r}_2,t_2)]\rangle.
\end{aligned} \tag{7.19}$$

In this relation $\mathbf{V}'(\mathbf{r}_1, t_1)$ and $\mathbf{V}''(\mathbf{r}_1, t_1)$, etc., are the complex classical field amplitudes at $\mathbf{r}_1$, t_1, etc., due to the separate beams, and they are statistically independent. Of the 16 terms resulting from the expansion of (7.19), ten vanish, if we make the usual assumption that the phases of $\mathbf{V}(\mathbf{r}_1, t_1)$, etc., are randomly distributed over 0 to 2π. The remaining terms simplify appreciably if

$$|\mathbf{r}_2-\mathbf{r}_1| \ll 1/\Delta k \quad \text{and} \quad |t_2-t_1| \ll 1/c\Delta k,$$

where $c\Delta k$ is the total frequency spread over both beams, which are assumed to be quasimonochromatic. If, in addition, each beam is polarized, and all the Fourier components of the same beam have the same polarization vector $\boldsymbol{\varepsilon}'$, or $\boldsymbol{\varepsilon}''$, [cf. Messiah (1962), p. 1032] it may be shown that [cf. Mandel (1964b)]

$$\langle I(\mathbf{r}_1, t_1) I(\mathbf{r}_2, t_2)\rangle = \langle I'^2(\mathbf{r}_1, t_1)\rangle + \langle I''^2(\mathbf{r}_1, t_1)\rangle + 2\langle I'(\mathbf{r}_1, t_1)\rangle\langle I''(\mathbf{r}_1, t_1)\rangle \times\{1+|\boldsymbol{\varepsilon}'^*\cdot\boldsymbol{\varepsilon}''|^2\cos[(\mathbf{k}'_0-\mathbf{k}''_0)\cdot(\mathbf{r}_2-\mathbf{r}_1)-c(k'_0-k''_0)(t_2-t_1)]\}. \quad (7.20)$$

$\mathbf{k}'_0$ and $\mathbf{k}''_0$ are here the mid-wave vectors of the two beams. Thus, over a limited space–time region, the intensity correlation shows a sinusoidal dependence on space and time, which can be interpreted both in terms of interference fringes and light beats. The periodicity is found to be the same as in Secs. 7.1 and 7.2. We note however that the positions in space–time of the modulation maxima and minima are not given by (7.20). They are of course randomly distributed for statistically independent light beams, as we have seen already. In so far as the equation (7.20) describes the intensity correlation at two space time points, the above theory is formally somewhat similar to the theory of the Hanbury Brown–Twiss effect in a partially coherent field. (cf. Secs. 6.3 and 6.4). The effects are however very clearly distinguished experimentally.

Discussions of both the experiments and theory on the transient interference effects are sometimes diverted by a remark of Dirac (1947, p. 9) that ". . . each photon interfers only with itself. Interference between different photons never occurs." Although this statement appears to contradict the effect we have been discussing, it can nevertheless be given a meaning. We may think of the detection (of localization in space–time) of a photon as a measurement that forces the photon into a superposition state in which it is partly in each beam. It is the two components of the state of one photon which interfere, rather than two separate separate photons. Of course, the concept of a photon is not helpful in understanding this experiment.

8. CONCLUSIONS

We have shown that the current theory of coherence is capable of describing a very wide class of optical phenomena, ranging from interference patterns to high-speed photoelectric correlations. Moreover we have seen that in most situations considered so far, those coherence properties of the quantized field which are exhibited in experiments with photoelectric detectors may be described by a semiclassical theory. In this theory, the field is described classically, but the photoelectric interaction is treated quantum mechanically. The results appear to be identical with those of the quantized-field theory for first-order processes, as long as the influence of the measuring apparatus on the field may be neglected. The semiclassical theory is, therefore, likely to be adequate for the description of most macroscopic effects. [In this connection see also the articles by Jaynes and Cummings (1963) and Senitzky (1965)]. These remarks do not imply that situations do not exist for which the semiclassical theory is inadequate, but only that such situations are the exception in the optical domain.

The detailed description of the fluctuation properties, implicit in the higher order correlations of the field, has as yet had very little experimental confirmation, as we have indicated. This is an area where further investigation is obviously very desirable and likely to be most fruitful.

APPENDIX

In this Appendix we will briefly consider the possibility of representing any density operator $\hat{\rho}$ in the symbolic "diagonal" form (4.9) of Sudarshan[33] (1963a, b).

Let us consider first a system with only one degree of freedom and let $\rho(n, m)$ be its density matrix in the Fock representation, i.e.,

$$\hat{\rho} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\rho(n, m)\,|n\rangle\langle m|, \quad (A1)$$

where $|n\rangle$ is the eigenstate of the number operator $\hat{a}^\dagger\hat{a}$, $\hat{a}$ is the annihilation operator and $\hat{a}^\dagger$ the creation operator ($[\hat{a}, \hat{a}^\dagger]=1$).

Let $|v\rangle$ and $\langle v|$ be the right and left eigenstates, respectively, of the annihilation and the creation operators,

$$\hat{a}\,|v\rangle = v\,|v\rangle, \qquad \langle v|\,\hat{a}^\dagger = v^*\langle v| \quad (A2)$$

($\langle v|v\rangle=1$). These eigenstates may be expressed in terms of the basic vectors of the Fock representation

[33] A possibility of such a representation in some cases has been noted by Glauber (1963 c, Sec. 7).

in the form [Klauder (1960)]

$$|v\rangle = \sum_{n=0}^{\infty} \exp(-\tfrac{1}{2}|v|^2)[v^n/(n!)^{\frac{1}{2}}]|n\rangle, \quad \text{(A3)}$$

$$\langle v| = \sum_{n=0}^{\infty} \exp(-\tfrac{1}{2}|v|^2)[v^{*n}/(n!)^{\frac{1}{2}}]\langle n|. \quad \text{(A4)}$$

Consider next the operator

$$\iint \Phi(v)\,|v\rangle\langle v|\,d^2v, \quad \text{(A5)}$$

where[34]

$$\Phi(v) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\rho(n,m)(n!m!)^{\frac{1}{2}}}{(n+m)!\pi r} \exp\{r^2+i(m-n)\theta\}\{[-(\partial/\partial r)]^{n+m}\delta(r)\}, \quad \text{(A6)}$$

$v=r\exp(i\theta)$, and the integration in (A5) extends over the whole complex v plane. On substituting from (A3) and (A4) into (A5), one obtains

$$\begin{aligned}\iint \Phi(v)\,|v\rangle\langle v|\,d^2v &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_0^{\infty}\int_0^{2\pi} \frac{\rho(n,m)(n!m!)^{\frac{1}{2}}}{(n+m)!\pi r}\exp[r^2+i(m-n)\theta] \\ &\quad\times\{[-(\partial/\partial r)]^{n+m}\delta(r)\}\exp(-r^2)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{r^{k+l}\exp[i(k-l)\theta]}{(k!)^{\frac{1}{2}}(l!)^{\frac{1}{2}}} r\,dr\,d\theta\,|k\rangle\langle l| \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\rho(n,m)(n!m!)^{\frac{1}{2}}}{(n+m)!}\delta_{n-m,l-k}\frac{(k+l)!}{(k!l!)^{\frac{1}{2}}}\delta_{n+m,k+l}\,|k\rangle\langle l| \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\rho(n,m)\,|n\rangle\langle m| \\ &= \hat{\rho}. \end{aligned} \quad \text{(A7)}$$

Hence the integral (A5) correctly represents the density operator $\hat{\rho}$ in the "diagonal form."

It should be noted that the phase-space distribution function given by (A6) is not an ordinary function; it must be regarded as a generalized (symbolic) function [cf. Friedmann (1956), Chap. 3; Lighthill (1959)].

The sense in which Eq. (A6) is to be understood has recently been discussed by Mehta and Sudarshan (1965). They showed that, if both of the series in (A6) are terminated after N terms, the resulting function $\Phi_N(v)$ can be considered as a tempered distribution with respect to the test function $\langle v|\hat{O}|v\rangle$ corresponding to any operator $\hat{O}$ for which $\langle\hat{O}\rangle$ exists, and that

$$\langle\hat{O}\rangle = \lim_{N\to\infty}\int \Phi_N(v)\langle v|\hat{O}|v\rangle\,d^2v. \quad \text{(A8)}$$

It has also been stated by Sudarshan (1963a, b) that these considerations may be generalized to a system of any finite number of degrees of freedom. The Fock representation basis is then labeled by a sequence $\{n_{k,s}\}$ of non-negative integers and the density matrix by two such sequences,

$$\rho(\{n_{k,s}\},\{m_{ks}\}).$$

In place of (A1) one now has

$$\hat{\rho} = \sum_{\{n_{k,s}\}}\sum_{\{m_{k,s}\}}\rho(\{n_{k,s}\},\{m_{k,s}\})\,|\{n_{k,s}\}\rangle\langle\{m_{k,s}\}|. \quad \text{(A9)}$$

In place of the state $|v\rangle$ one now has a state $|\{v_{k,s}\}\rangle$ labeled by a sequence $\{v_{k,s}\}$ of complex numbers. A "diagonal representation" of $\hat{\rho}$ again exists and is given by

$$\hat{\rho} = \int \Phi(\{v_{k,s}\})\,|\{v_{k,s}\}\rangle\langle\{v_{k,s}\}|\,d^2\{v_{k,s}\}, \quad \text{(A10)}$$

where

$$\begin{aligned}\Phi(\{v_{k,s}\}) &= \sum_{\{n_{k,s}\}}\sum_{\{m_{k,s}\}}\rho(\{n_{k,s}\},\{m_{k,s}\})\prod_{k,s}\frac{(n_{k,s}!m_{k,s}!)^{\frac{1}{2}}}{(n_{k,s}+m_{k,s})!\pi r_{k,s}} \\ &\quad\times\exp\{r_{k,s}^2+i(n_{k,s}-m_{k,s})\theta_{k,s}\}\{(-\partial/\partial r_{k,s})^{n_{k,s}+m_{k,s}}\delta(r_{k,s})\}, \end{aligned}$$

with $v_{k,s}=r_{k,s}\exp(i\theta_{k,s})$. (A11)

Similar comments as above apply to the functional $\Phi(\{v_{k,s}\})$.

ACKNOWLEDGMENT

We are obliged to Dr. C. L. Mehta for helpful discussions and to him and Mr. G. Bedard for technical assistance connected with the preparation of this article.

[34] The delta function with the radial argument r is here assumed to be normalized so that

$$\int_0^{\infty}\delta(r)\,dr = \tfrac{1}{2}.$$

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* *Editor's note.* Attention is called to the resource letter QSL-1 on Quantum and Statistical Aspects of Light, compiled by P. Carruthers, Cornell University, Am. J. Phys. **31,** 5 (1963). This resource letter together with reprints of a number of the important papers in this field are available in the Selected Reprints Series of the American Association of Physics Teachers, which can be ordered from the American Institute of Physics.

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