Generalized Covariance

D. W. JOSEPH
Brace Laboratory of Physics, University of Nebraska, Lincoln, Nebraska

The embedding of space–time in a higher-dimensional pseudo-Euclidean space is considered as a means of studying the representations of coordinate transformations in curved space–time. To the extent that this approach is valid, the existence of particles belonging to various representations of some internal symmetry group is "predicted" in much the same sense that particles of spin 0, $\frac{1}{2}$, 1, etc. are "predicted" by the structure of representations of the Lorentz group.

1. INTRODUCTION

When we know the transformation properties of a certain quantity, we can often guess the laws which govern its behavior merely from the requirement that physical laws must be convariant under transformation from one coordinate system to another; at least, the number of possible such laws is greatly reduced. For example, Einstein in his book on relativity[1] has given numerous examples of the application of this approach to both pre- and post-relativity nonquantum physics. In quantum physics, the significance of transformation properties is perhaps even more frankly admitted, though somewhat distinguished by the language used: when a new particle is discovered the first questions asked are, "What is its spin?—its mass?—its isospin?"—and so forth.

It might be said that there are two levels of subtlety in the uses to which one may put the principle of covariance. On the one hand, when we know the transformation properties of all quantities that enter a certain physical law, then we can make a useful guess about the form of that law. But also, if we are presented with some new quantity (for example, the wave function of a newly discovered particle), we can often guess its transformation properties on the basis of rather meager experimental results. This is, first of all, because any such quantity must have a well-defined behavior under any coordinate transformation—that is, more explicitly, it must belong to some representation of the group of all coordinate transformations. And, secondly, there is the happy empirical fact that simple systems seem to transform according to representations of low dimension. Thus, for instance, it is very useful to know that the representations of the Lorentz group can be characterized by a quantity called spin, which (for a certain class of representations) can take on only integral and half-integral values; then in order to determine that the spin of a new particle is such-and-such, it is only necessary to show that it cannot have any other low value.

2. THE PROBLEM

The Lorentz group was mentioned above, but of course we know that physical space–time is not actually flat, so that Minkowski coordinates are strictly speaking not appropriate, and the Lorentz group is appropriate only in some sort of approximate sense. Certainly the curvature is slight, so one might expect its effect to be small; but is it really legitimate to jump immediately to the Minkowski approximation when questions of symmetry are concerned? One might put it this way: since the curvature is so slight, if the Minkowski approximation is allowable, it should be a very good approximation; but is it even allowable? Experimentally, it seems to be a good approximation in that linear and angular momentum are well conserved; but could the nonprediction of "internal" symmetries be an example of a place where the approximation breaks down completely?

The decision to give up the Lorentz group leads, of course, to difficulties. If we take the traditional approach of using general curvilinear coordinates, the group of coordinate transformations now has an infinite number of parameters; these could, for instance, be taken to be the coefficients in a power series expansion of the new coordinates in terms of the old (by contrast, of course, the Lorentz group has only 10 parameters). It would certainly appear difficult to determine all representations of such a group (as has been done for the Lorentz group).[2] One *particular* class of representations consists of the well-known tensor fields transforming according to

$$\psi'_{\mu\nu\cdots\rho}(x')=\frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x_\beta}{\partial x'_\nu}\cdots\frac{\partial x_\gamma}{\partial x'_\rho}\psi_{\alpha\beta\cdots\gamma}(x), \qquad (1)$$

or the corresponding expression for contravariant fields. Since these fields do not include fields of half-integral spin, we might try to generalize (1) as follows:

$$\psi'(x')=A_{x'}\psi(x), \qquad (2)$$

where the components of ψ have been arranged in a

[1] A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, New Jersey, 1955).

[2] However, Shlomo Sternberg is currently working on the theory of representations of infinite-dimensional Lie groups.

column, and $A_{x'}$ is a matrix depending on the point x', such that the set of all matrices

$$\{A_{x'}\} \tag{3}$$

is a representation of the set

$$\{||\partial x/\partial x'||_{x'}\} \tag{4}$$

of all Jacobian matrices. Note that the form (2) includes all possibilities (1). However, it is not too hard to show[3] that there are no representations (3) which are double-valued, under space rotations, with respect to the matrices (4); hence representations of the form (3) do not include any which can correspond to spin one-half particles. Of course, people have thought of ways to deal with electrons in curved space; what is done is equivalent to introducing a set of pseudo-orthogonal axes ("four legs") at each point of space, and letting the matrices $A_{x'}$ represent the smaller group of (Lorentz) rotations of these axes. If we wish, we can tie the four legs to the coordinate lines by requiring the first to lie along the tangent to the x_1 line, the second to lie in the plane determined by the x_1 and x_2 tangents, and so on. Then indeed we get a set of fields which seem to transform like all the usual representations of the Lorentz group. But this is certainly only a very limited subset of all possible representations [and it may be noted that they are all many-to-one representations of the matrices (4)].

Another approach is to consider the group of all transformations preserving certain boundary conditions in an asymptotically flat space; the result is something akin to the Lorentz group.[4] However, since such spaces need not really correspond to the actual universe, it would seem that we cannot base our search for representations on this approach, for the same reason that the Minkowski approximation is questionable. The idea of restricting the set of all curvilinear coordinate systems to a smaller set is certainly very appealing, and one might hope that an approach such as this will be developed to the point where something more general than asymptotic flatness is allowed.

3. A SOLUTION

The source of our difficulties evidently lies in the use of general curvilinear coordinates. In fact, we would have suffered much the same confusion in looking for representations in flat space had we chosen to use curvilinear coordinates there. However, in that case, there exists a preferred class of coordinate systems, the pseudo-Euclidean ones; and none other need ever be considered.

It turns out that a very similar set of preferred coordinate systems exist for curved space–time; the only difference is that we have to allow ten coordinates, rather than four. Just as we defined pseudo-Euclidean coordinates in flat space by the condition

$$(\Delta s)^2 = (\Delta x_0)^2 - \sum_{i=1}^{3} (\Delta x_i)^2, \tag{5}$$

so we can define pseudo-Euclidean coordinates in curved space by the condition[5–7]

$$(ds)^2 = (dx_0)^2 - \sum_{i=1}^{9} (dx_i)^2. \tag{6}$$

The group of Lorentz transformations, preserving (5), is now replaced by the group of (pseudo) rotations and translations of a ten-dimensional space, preserving (6). Perhaps it hurts one's sense of economy to introduce ten coordinates when four (curvilinear) coordinates would "do"; but the economy in number of coordinate transformations (from an infinite number of parameters to only 55!) is certainly a relief.

Though the introduction of pseudo-Euclidean coordinates is quite straightforward, we must still take account of the fact that the space–time surface R_4 does exist, and find an approximation in which coordinate covariance leads to conservation of momentum and angular momentum. Since all physical observations are restricted to R_4 (or at least to a region very close to it), we would not expect exact symmetry when we apply coordinate transformations to laboratory objects only, unless these transformations preserve R_4. (We would, of course, expect symmetry if we applied the transformations to the whole universe—but this we do not do.) In general, there are no transformations exactly preserving R_4; but there is a set which preserves the tangent hyperplane to R_4 at any given point. On intuitive grounds, we would expect approximate symmetry under such transformations. This effectively amounts to replacing R_4 by its tangent plane T_4, a sort of zeroth-order approximation which we will call the tangent-plane approximation. In this approximation, there are symmetry transformations of two sorts: transformations of points within T_4, analogous to Lorentz transformations; and transformations leaving the points of T_4 fixed, having the appearance of rotations in an "internal" space.

4. DISCUSSION

Higher-dimensional pseudo-Euclidean spaces E_n thus constitute an arena in which we may hope to play the games of quantum mechanics and general relativity

[3] See footnote 6 of D. Joseph, Phys. Rev. **126**, 319 (1962). That paper treats some of the present points in more detail.

[4] R. Sachs, Phys. Rev. **128**, 2851 (1962).

[5] Actually, such a set of coordinates is guaranteed not for all of space but only for some finite region. The possibility of writing (6), locally, with only one positive sign follows from Ref. 6; that the maximum number (ten) of coordinates will be required is suggested by Ref. 7.

[6] Avner Friedman, J. Math. Mech. **10**, 625 (1961).

[7] Joe Rosen, Rev. Mod. Phys. **37**, 204 (1965), this issue.

simultaneously: on the one hand, R_4 can be non-Euclidean, and on the other, the groups of coordinate transformations are sufficiently manageable that we can classify wave functions according to their representations. Other arenas are surely possible, but none seem to be presently available. The E_n arena has the interesting feature that an intuitive approximation strongly suggests the presence of a group of internal symmetries.

Finally, it must be remarked that this tangent-plane approximation is probably not the best "zeroth-order" approximation to R_4. For example, R_4 could have "corrugations" in it, even though it is nearly flat in an intrinsic sense. A solution to this might be to introduce the intrinsically flat space E_4 which best approximates R_4; E_4 then admits a set of pseudo-Euclidean coordinates, so that momentum and angular momentum in the usual sense are conserved to the extent that E_4 fits R_4. To complete the coordinate system, one would then introduce coordinates which are everywhere normal to E_4 and as nearly Euclidean as possible within a range of a few fermis, say, from E_4. These would constitute the "internal" space which would now be not quite Euclidean. It might be interesting to try to estimate the effects which would arise from this non-Euclidean character; they would presumably be quite weak.

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Embedded Space–Time and Particle Symmetries*

YUVAL NE'EMAN†
California Institute of Technology, Pasadena, California

The bootstrap yields the strong-interactions symmetry provided the hadron currents participating in nonstrong interactions are first fed into it; the alternative course, in which the bootstrap is required to generate the entire symmetry uniquely, with no *a priori* information from the nonstrong interactions, seems to lie beyond the present techniques and would also leave unexplained the subsequent emergence of the weaker couplings. We suggest that "internal" symmetries may have a geometrical origin, corresponding to transformations in the global embedding space of the four-dimensional physical Riemann universe. These would be the unitary or orthogonal transformations of the normal subspace, since the latter do transform a small region of curved space–time into itself. This picture fits in with the short ranges of the strong interactions. Cosmological and astrophysical implications are noted.

SO-CALLED "INTERNAL" SYMMETRIES

The conservation laws and the dynamics of particle physics have found their most useful formulation in terms of apparently nonkinematical, i.e., "internal" symmetries. These are generalizations of the concept of "charge," used in electricity for the last 150 years; they appear as a set of charge currents satisfying equations of continuity under certain limiting conditions. In practice, the equations really have "sink" terms due to some symmetry-breaking interactions. We have used the word symmetries since we may relate each conserved neo-charge with the generator of some Lie group, the latter then representing a symmetry of the system. This connection, sometimes known as Noether's theorem, has lately been reformulated appropriately, since what we really want is an inverse theorem leading from the observation of a conserved or quasiconserved quantity to a symmetry group. Okubo[1] had earlier shown in detail why an unrenormalized coupling implies the existence of a conserved current; Horn[2] has now exhibited the emergence of the algebra from the current.

Another starting point for the detection of a conserved current has been shown by Ogievetski and Polubarinov[3] to derive from the observation of particles with unit spin and negative parity, this time in a more exact inverted Yang–Mills technique.[4] Cutkosky[5] has obtained a similar result—i.e., the generation of a symmetry from vector-meson couplings—in terms of an approximate and idealized bootstrap (a theory with no "elementary" strongly interacting particles, in which all hadrons appear as self-bound states of one and the same hadron matter).

The symmetry itself shows up in many different ways—mainly a multiplet structure accounting for the various sets of particle states and relative coupling strengths or amplitudes for strong reactions (this is in fact the "law of force"—like the e^2 in the Coulomb

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[1] S. Okubo, Nuovo Cimento **13**, 292 (1959).

[2] D. Horn (to be published).

[3] V. I. Ogievetski and I. V. Polubarinov, Zh. Eksperim. i Teor. Fiz. **45**, 966 (1963) [English transl.: Soviet Phys.—JETP **18**, 668 (1964)].

[4] C. N. Yang and R. Mills, Phys. Rev. **96**, 191 (1954); also R. Shaw, thesis (unpublished).

[5] R. Cutkosky, Phys. Rev. **131**, 1888 (1963).