Elementary Particles in a Curved Space

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We report a beginning in a project to determine the consequences of the following assumption, "that a physical theory in flat space is obtainable as the limit of a physical theory in a curved space." Because of the absence of groups of motion in general curved spaces, we discuss only the case of constant curvature. Then operators of angular and linear momentum exist, and we show that the interesting irreducible unitary representations of the group of motions reduce very simply to those of the inhomogeneous Lorentz group in the limit of zero curvature.

I. INTRODUCTION

Some of the enigmas of elementrary-particle physics may be characterized as an *embarras du choix.* Some are ancient, as the failure to understand why a relatively small set of states are favored by the existence of elementary particles. Others are more recent, as the ambiguities introduced by divergent integrals in weak interactions. One of the motivations for the research presented here is the hope that the following assumption turn out to be restrictive with respect to some of these ambiguities:

A physical theory that treats space–time as Minkowskian flat must be obtainable as a well-defined limit of a more general physical theory, for which the assumption of flatness is not essential.

It is facile to brush the suggestion aside with arguments approximately as follows: It is known that the important differential equations (Klein–Gordon and Dirac) may easily be generalized to forms that possess general covariance. Or: It is possible to treat gravitation as just another particle field, thus avoiding the idea of a curved space–time altogether. We answer the latter argument by pointing out that *if* such a flat-space, essentially linear and perturbational, treatment of gravitation is adequate, then the smallness of the coupling constant removes this science from relevance to elementary particles; hence our interest in the subject depends on the possibility, however remote, that the nonlinearity is essential—and the only understanding of nonlinear effects has come from Einstein's geometrical interpretation.

With respect to the objection that the Klein–Gordon and Dirac equations may be easily generalized, I say that this is almost totally irrelevant. For the modern theories of elementary particles, both field theory and the phenomenological treatment, are not primarily studies in differential equations. The construction of a physical theory, and the interpretation in particular, rests instead on the concepts of energy, momentum, and spin, whose existence is due to invariance principles—more specifically, the principle of invariance under the inhomogeneous Lorentz group. From this point of view the Klein–Gordon equation is interesting because the solutions form the basis for an irreducible unitary representation of that group.

The inhomogeneous Lorentz group comes to the fore as the group of motions of Minkowski space. The absence of groups of motion in more general Riemannian spaces is a formidable obstacle to the extension of physical models. There is, however, a class of Riemannian spaces in which the road to generalization is well marked. A space of constant curvature has a group of motions that, though it differs from that of a flat space, has the *same number of parameters.* In a space of constant curvature we may define energy and momentum, mass, and spin. We may study the irreducible representations, elementary particles, and their interactions, and we may inquire whether the class of physical theories obtainable as limits as the curvature tends to zero includes all known possibilities—or, as we hope, it is more restrictive. It may be hoped, for example, that divergent integrals may be replaced by regularized ones and that the limit of zero curvature exists if taken after the integrations are carried out. (The principle of gauge invariance supplies a prescription of this kind, but is limited to electromagnetic phenomena.)

In this first report we specialize to the case of constant curvature. We identify the group and make a preliminary study of the irreducible unitary representations, and we show how intimately some of these are related to the representations of the Lorentz group.

II. THE GROUP

A four-dimensional Riemannian space may admit a continuous group of motions with up to ten essential parameters.[1] The maximum number is realized only for a space of constant curvature, that is, the case when the curvature tensor takes the form

$$R_{\mu\nu\lambda\rho}=\rho(g_{\mu\nu}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}). \qquad (1)$$

Here ρ is a constant, which is not an additional assump-

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[1] L. P. Eisenhart, *Continuous Groups of Transformations* (Dover Publications, Inc., New York, 1961), p. 215.

tion but a consequence of the interpretation of $R_{\mu\nu\lambda\rho}$ as a curvature tensor. (Ref. 1, p. 207.)

A space of constant curvature may be realized as a pseudosphere in five-dimensional space, and the group of motions is the set of pseudorotations that take this four-dimensional surface into itself. The algebra of infinitesimal generators is therefore that of O_5 (also known as B_2 or C_2). The commutation relations satisfied by the ten independent rotation operators are

$$[L_{\alpha\beta}, L_{\gamma\delta}]=-i(g_{\alpha\gamma}L_{\beta\delta}+g_{\beta\delta}L_{\alpha\gamma}-g_{\alpha\delta}L_{\beta\gamma}-g_{\beta\gamma}L_{\alpha\delta}),\qquad \alpha,\beta,\gamma,\delta=1,2,3,0,5, \quad (2)$$

where $g_{\alpha\beta}$ is the five-dimensional pseudo-Euclidean metric tensor. The radius of the sphere goes to infinity as the curvature ρ tends to zero; in the limit a flat Minkowski space is obtained and the group becomes the inhomogeneous Lorentz group.

The interpretation of these ten operators is straightforward and corresponds closely to the meaning of the ten operators of the inhomogeneous Lorentz group. We pick a point in four-space and select the coordinates in five-space so that the fifth axis passes through our point. Then $L_{\mu\nu}$, $\mu,\nu=1,2,3,0$, generate ordinary rotations around that point and

$$L_{\mu5}\equiv P_\mu/(|\rho|)^{\frac{1}{2}} \quad (3)$$

generate parallel displacements (covariant translations). The commutation relations between the $L_{\mu\nu}$ and the P_μ are

$$[L_{\mu\nu}, L_{\rho\lambda}]=-i(g_{\mu\rho}L_{\nu\lambda}+g_{\nu\lambda}L_{\mu\rho}-g_{\mu\lambda}L_{\nu\rho}-g_{\nu\rho}L_{\mu\lambda})$$
$$[L_{\mu\nu}, P_\rho]=-i(g_{\mu\rho}P_\nu-g_{\nu\rho}P_\mu)$$
$$[P_\mu, P_\nu]=-i\rho L_{\mu\nu}. \quad (4)$$

These reduce to the commutation relations for the inhomogeneous Lorentz group when $\rho=0$. The signature of the five-space metric is

$$g_{11}=g_{22}=g_{33}=-1,$$
$$g_{00}=+1,$$
$$g_{55}=\rho/|\rho| \qquad (=+1).$$

We shall take $g_{55}=+1$, ρ positive, because in the contrary case the energy operator $P_0\sim L_{05}$ cannot be diagonalized. [This is seen by constructing the raising operator E^+ for L_{05}. It turns out that if $L_{05}\psi=E\psi$ and $\psi'=E^+\psi$, then $L_{05}\psi'=(E+i)\psi'$. But that is impossible if the energy spectrum is real, hence $E^+\psi$ does not exist if ψ is an eigenstate of L_{05}.]

III. REPRESENTATIONS, MASSIVE CASE

The knowledge acquired in the Appendix may be summarized as follows:

There are four classes of unitary irreducible representations of the algebra of infinitesimal motions in a space of constant positive curvature. They are:

m^+: These representations are characterized by the existance of a *lowest* eigenvalue $\bar{m}$ of L_{05}; the states that belong to this eigenvalue form the basis for an irreducible unitary representation of the rotation subalgebra (L_{12}, L_{23}, L_{13}) with spin l, such that $\bar{m}>l$.

m^-: Same as m^+, except that L_{05} has a *highest* eigenvalue $-\bar{m}$.

m^0: These representations have no bound on the energy and, for each energy the spin is unbounded. These representations are expected to be of secondary physical interest and are not discussed further in this report.

I: The identity representation.

We study classes m^+. Let

$$\hat{a}^\mu=0,0,0,1 \quad \text{for} \quad \mu=1,2,3,0.$$

We start with the set of states that belong to the lowest eigenvalue of the energy, denoting them $\psi(\hat{a}, l_z)$ $l_z=-l, -l+1, \cdots, +l$. These states are characterized within the irreducible representation by (suppress the variable l_z)

$$L_{05}\psi(\hat{a})=\bar{m}\psi(\hat{a}), \qquad \bar{m}>l$$
$$(L_{12}^2+L_{23}^2+L_{31}^2)\psi(\hat{a})=l(l+1)\psi, \qquad l=0,\tfrac{1}{2},1,\cdots, \quad (5)$$

where L_{12}, L_{23} and L_{31} are Hermitian matrices. (Both $\bar{m}$ and l take on a unique value.] All the other states may be generated from these $2l+1$ states; we need only devise a system of labels for them.

By a *geodesic Lorentz transformation* on $\hat{a}^\alpha$ we shall mean an interation of a transformation

$$\hat{a}^\mu\rightarrow\hat{a}^\mu+\hat{a}_\nu\Theta^{\nu\mu},$$

where $\Theta^{\nu\mu}$ is infinitesimal, real and antisymmetric. Let a^μ be a four-vector with $a^0>0$ and $a^\mu a_\mu=1$. Let $\alpha(a)$ be the unique geodesic Lorentz transformation that transforms $\hat{a}^\mu$ into a^μ, and $D(a)$ the representative of $\alpha(a)$. Then define $\psi(a, l_z)$, or simply $\psi(a)$, as the transform of $\psi(\hat{a}, l_z)$ by $D(a)$.

Since every Lorentz transformation is a product of two geodesic Lorentz transformations and a rotation, every representative may be expressed in terms of the $D(a)$ and the L_{ij}, $i,j=1,2,3$. The calculation is identical to that which leads to the explicit form of the irreducible unitary representations of the inhomogeneous Lorentz group, and the result for the generators of the homogeneous part is the same, namely,

$$L_{ij}\psi(a)=\left[L_{ij}^{(0)}-i^{-1}\left(a_i\frac{\partial}{\partial a^j}-a_j\frac{\partial}{\partial a^i}\right)\right]\psi(a),$$
$$L_{0i}\psi(a)=\left[\frac{1}{a^0+1}a^jL_{ij}^{(0)}-i^{-1}\left(a_0\frac{\partial}{\partial a^i}-a_i\frac{\partial}{\partial a^0}\right)\right]\psi(a) \quad (6)$$
$$i,j=1,2,3.$$

Here $L_{ij}^{(0)}$ are independent of a^μ and reduce to the L_{ij} when $a^\mu=\hat{a}^\mu$; they are the "spin" part of the L_{ij}.

The statement that $\psi(\hat{a})$ are states with lowest energy means that all the lowering operators for L_{05} must annihilate $\psi(\hat{a})$. Thus

$$(iL_{0i}-L_{5i})\psi(\hat{a})=0,\qquad i=1,2,3.$$

This may be combined with Eq. (5) to read

$$L_{\mu5}\psi(\hat{a})=(\bar{m}\hat{a}_\mu-i\hat{a}^\nu L_{\nu\mu})\psi(\hat{a}).$$

This equation is Lorentz covariant, except for the specialization $a^\mu=\hat{a}^\mu$. Hence, in general,

$$L_{\mu5}\psi(a)=(\bar{m}a_\mu-ia^\nu L_{\nu\mu})\psi(a).\qquad (7)$$

We re-express our results in terms of P_μ [Eq. (3)], and introduce

$$m\equiv\bar{m}\rho^{\frac{1}{2}},\qquad p_\mu\equiv ma_\mu.$$

Summary. Every nontrivial unitary irreducible representation of the group of motions of a space of constant curvature for which the energy is bounded below is of the following form:

$$L_{ij}=L_{ij}^{(0)}-i^{-1}\left(p_i\frac{\partial}{\partial p^j}-p_j\frac{\partial}{\partial p^i}\right),\qquad i=1,2,3$$

$$L_{0i}=\frac{1}{p^0+m}p^jL_{ij}^{(0)}-i^{-1}\left(p_0\frac{\partial}{\partial p^i}-p_i\frac{\partial}{\partial p^0}\right),\qquad i=1,2,3$$

$$P_\mu=p_\mu+\frac{\rho^{\frac{1}{2}}}{im}p^\nu L_{\nu\mu},\qquad \mu=1,2,3,0$$

$$p_\mu^2=m^2,\qquad p_0>0,\qquad m>\rho^{\frac{1}{2}}l,$$

where $L_{ij}^{(0)}$ are the $(2l+1)\times(2l+1)$ dimensional matrices of a unitary irreducible representation of the three-dimensional rotation group.

Remarks. (1) States $\psi(p)$, $\psi(p')$ with different values of p are not orthogonal when $\rho\neq0$. But as ρ tends to zero P_μ tends to p_μ and then $\psi(p)$ and $\psi(p')$ become orthogonal as the operators turn into those of a unitary irreducible representation of the inhomogeneous Lorentz group.

(2) The differential operators P_μ may be interpreted as inducing transformations on p_μ, thus continuing p_μ into the complex domain Re $p_\mu\in v_+$, $p^\mu p_\mu=m^2$. But it is not necessary to enlarge the representation space in this way; the finite translations may equally well be expressed as integration operators over the set of real p_μ. This means that a plane wave is transformed into a wave packet as a result of parallel transfer.

(3) It is possible to define "states of momentum p_μ" in a way that generalizes to the case of constant, nonvanishing curvature. First define the momentum of the states of lowest energy to be $p_\mu=(0,0,0,m)$, where m is the lowest value of the energy. Then a complete set of states is obtained by applying Lorentz transformations to those of lowest energy, and for all of these the momentum is defined by ascribing to p_μ the transformation properties of a four vector. Thus, the set of states $\psi(p)$ may be characterized as those for which the operator $p^\mu P_\mu$ has its lowest value m^2.

APPENDIX

The following discussion is in a form that completely obscures the physics. All the results are summarized at the beginning of Sec. III.

We recast the commutation relations in the Weyl form (almost). If

$$H_1=L_{12},\qquad H_2=L_{05}$$

$$L^\pm=1/\sqrt{2}(L_{13}\pm iL_{23}),\quad M^\pm=1/\sqrt{2}(iL_{03}\pm L_{53})$$

$$E^{+\pm}=\tfrac{1}{2}\sqrt{2}\left(i(L_{01}\pm iL_{02})-(L_{15}\pm iL_{25})\right)$$

$$E^{-\pm}=\tfrac{1}{2}\sqrt{2}\left(i(L_{01}\pm iL_{02})+(L_{15}\pm iL_{25})\right)$$

then

$$[H_1,L^\pm]=\pm L^\pm,\qquad [H_1,M^\pm]=0$$

$$[H_2,L^\pm]=0,\qquad [H_2,M^\pm]=\pm M^\pm$$

$$[H_1,E^{+\pm}]=\pm E^{+\pm},\qquad [H_1,E^{-\pm}]=\pm E^{-\pm}$$

$$[H_2,E^{+\pm}]=+E^{+\pm},\qquad [H_2,E^{-\pm}]=-E^{-\pm}$$

$$[L^+,L^-]=H_1,\qquad [M^+,M^-]=H_2$$

$$[E^{++},E^{--}]=\tfrac{1}{2}(H_1+H_2),\quad [E^{+-},E^{-+}]=\tfrac{1}{2}(-H_1+H_2)$$

$$[L^\pm,M^\pm]=-i\sqrt{2}E^{+\pm},\qquad [L^-,M^\pm]=-i\sqrt{2}E^{-\pm}$$

$$[L^+,E^{\pm-}]=[L^-,E^{\pm+}]=i/\sqrt{2}M^\pm$$

$$[M^+,E^{-\pm}]=[M^-,E^{+\pm}]=-i/\sqrt{2}L^\pm.$$

All other commutators vanish.

We study the four rotation subalgebras:

$$G_1=\{H_1,L^+,L^-\}$$

$$G_2=\{H_2,M^+,M^-\}$$

$$G_3=\{\tfrac{1}{2}(H_1+H_2),E^{++},E^{--}\}$$

$$G_4=\{\tfrac{1}{2}(-H_1+H_2),E^{+-},E^{-+}\}.$$

Let us use the notation $\{H,E^+,E^-\}$ generically. Then the Casimir operator is the same in every case, namely,

$$\alpha=H(H-1)+2E^+E^-.$$

Thus

$$E^+E^-=\tfrac{1}{2}\alpha-\tfrac{1}{2}H(H-1).$$

Now we are interested in unitary representations, that is, $L^\dagger_{\alpha\beta}=+L_{\alpha\beta}$. Then $E^+=(E^-)^\dagger$ in the case of G_1 and $E^+=-(E^-)^\dagger$ in all the other cases. Thus, E^+E^- is positive definite for G_1 but negative definite for G_2, G_3 and G_4. The eigenvalues of H_1 are therefore limited above and below for given α. If the lowest value of $H_1 = -l$ then L^- must annihilate that state and, hence, $\alpha = l(l+1)$. The highest value of H_1 is then $+l$ and that state is annihilated by L^+. For any one of the three other subgroups E^+E^- must be negative for all eigenvalues of H that are realized. The spectrum need have no bounds but then $\alpha \leq H_0(H_0-1)$, where H_0 is the eigenvalue with the smallest absolute magnitude. If a lower bound exists, $H \geq H_{\min}$, then E^- must annihilate that state and hence

$$0 \geq E^-E^+ = [E^-, E^+] = -H_{\min}$$

or

$$H_{\min} \geq 0.$$

The equality is realized for the identy representation only.

A parallel discussion may be carried out for the case of a spectrum bounded above, and we conclude that there are four types of unitary irreducible representations of the groups G_2, G_3, and G_4

$$m^+: \quad H = H_{\min}, H_{\min}+1, H_{\min}+2, \cdots, \qquad H_{\min} > 0$$

$$m^-: \quad H = H_{\max}, H_{\max}-1, H_{\max}-2, \cdots, \qquad H_{\max} < 0$$

$$m^0: \quad H = H_0, H_0 \pm 1, H_0 \pm 2, \cdots$$

$$I: \quad H = 0.$$

Theorem: In a unitary irreducible representation of the whole algebra there appears only one of the four types of irreducible unitary representations of G_2. The proof is based on the

Lemma: If ψ is a basis vector and $(M^+)^n\psi = 0$ for some positive integer n, and if $\mathfrak{L}$ is in the algebra, then $(M^+)^{n+i}\mathfrak{L}\psi = 0$ for $i = 0$, 1 or 2. The proof of the lemma is a trivial consequence of the commutation relations and need not be reproduced here.

Proof of Theorem: According to the lemma, if there exists a set of states tied together by M^+ and M^-, and if this set forms the basis for an irreducible representation of types m^+ or m^- of G_2, then every other state must belong to the same type. Also, if both M^+ and M^- annihilate a state (which would then belong to an identity representation of G_2) then every other state must be annihilated by both $(M^+)^3$ and $(M^-)^3$, which is only true for the identity representation. Hence, the theorem is proved.

In this theorem G_2 may, of course, be replaced by G_3 or G_4. It is also easy to see that the type of representations present must be the same for all three subgroups. Thus we have

Theorem: The algebra of the group of motions of a space of constant curvature has four types (called m^+, m^-, m^0, and I) of unitary irreducible representations characterized by the appearance in each of only one of the four types of unitary irreducible representations of the subalgebras G_2, G_3, and G_4. [In $m^+(m^-)$ the lowest (highest) eigenvalue of H_2 shall be called $\bar{m}(-\bar{m})$.]

Here we limit ourselves to the case m^+ (later we include m^- as well), because only this type offers an energy spectrum that is bounded from below.

Theorem: In an irreducible unitary representation of type m^+ of the whole algebra, the states of lowest energy form an irreducible representation of the rotation subgroup G_1.

Proof: The theorem is analogous to the theorem that highest weights are simple, which is true in the case of compact Lie groups, and the proof is easily carried through in the same way.

Let the highest value of H_1 among the states of lowest value ($\bar{m}$) of H_2 be l, then $\frac{1}{2}(H_2 \pm H_1)$ are bounded below by $\frac{1}{2}(\bar{m}-l)$. Whence the restriction

$$\bar{m} > l.$$