A Remarkable Property of Plane Waves in General Relativity*

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The complete general-relativistic plane-wave space–times (gravitational, electromagnetic, or both) are examined in relation to the focusing effect they exert on null cones. The following remarkable property is then obtained. No spacelike hypersurface exists in the space–time which is adequate for the global specification of Cauchy data. As a consequence, it is not possible to imbed a plane wave globally in any hyperbolic normal pseudo-Euclidean space.

1. INTRODUCTION

Recent successes in the application of symmetry groups to strong interaction physics[1] have prompted some authors to suggest[2] that these and other new symmetries might possibly arise in some way out of space–time geometry, or from some extension of the space–time concept. Since they commute with the rotational symmetries which generate spin, the new symmetries would, according to this view, be associated with some space "orthogonal" to the four-dimensional continuum of which we are normally aware. Thus, the idea has been revived, of an *imbedding space*[3] which could possibly house, in addition to the space–time manifold itself, the extra perpendicular dimensions required for the new symmetries. The number (and signature) of these new dimensions is to be determined, in this approach, by the condition that the imbedding space be minimal and pseudo-Euclidean. It is thus of some relevance to try to determine the number and nature of the extra dimensions required, for the isometric imbedding of a general, physically interesting, general-relativistic space–time.

The problem divides naturally into two parts, that of local imbedding and that of global imbedding. It is known,[4] however, that ten dimensions—with any signature[5] -8, -6, $\cdots$, $+4$—are adequate for the *local* isometric imbedding of analytic space–time (signature -2). Furthermore, it would seem that ten dimensions are also locally *necessary* for most general-relativistic manifolds. For even if empty space–times can be always imbedded locally in fewer than ten dimensions (as seems unlikely!), it should be recalled that, in the presence of matter, Einstein's field equations amount, effectively, merely to *inequalities* on the curvature of space–time. That is, unless detailed equations for all matter fields present are assumed (and these would be exceedingly complicated when written solely in terms of the energy tensor), we are left with inequalities on the Ricci tensor which state (presumably) the positive definiteness of energy and, say, the nonnegativeness of the trace of the energy tensor. Since ten dimensions are minimal for the local isometric imbedding of a *general* 4-manifold, it seems fair to assume that ten (with a variety of possible signatures) is also correct for the local isometric imbedding of relativistic manifolds in general. The exceptions which can be locally imbedded in *fewer* than ten dimensions would from a class "of measure zero" for which special equations or symmetries, etc. might be present—although, of course, these exceptions include a large number of the known relativistic models.[6]

The problem of global imbedding[6] is much more difficult in general. It is not even known, at present, whether for a given indefinite Riemannian manifold, a global isometric imbedding into *some* pseudo-Euclidean space is always possible.[7] Furthermore, unlike the case of local imbeddings the choice of *signature* for the imbedding space seems to be important in the global case.[8] In fact, it turns out that there are *physically interesting space-times*—e.g., the plane-wave metrics—which cannot be globally isometrically imbedded in any pseudo-Euclidean n space of signature $2-n$ (hyperbolic normal signature). There are also many "unphysical" space–time models with the above property, e.g., any hyperbolic normal 4-space which contains a

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[1] See, for example, A. Salam, Rapporteur's Report on Symmetry Theories, 1964 Dubna International Conference on High Energy Physics (to be published).

[2] Y. Ne'eman, Rev. Mod. Phys. **37**, 227 (1965), this issue.

[3] Earlier suggestions in the use of such imbeddings for a similar purpose were made by C. Fronsdal, Nuovo Cimento **13**, 988 (1959); D. W. Joseph, Phys. Rev. **126**, 319 (1962).

[4] A. Friedman, J. Math. Mech. **10**, 625 (1961).

[5] The number of positive eigenvalues of the metric tensor minus the number of negative eigenvalues.

[6] See, for example, C. Fronsdal, Phys. Rev. **116**, 778 (1959); J. Rosen, Rev. Mod. Phys. **37**, 204 (1965), preceding article.

[7] The case of positive definite Riemannian manifolds was settled in the affirmative by J. Nash, Ann. Math. **63**, 20 (1956).

[8] For example, while it is a classical theorem of D. Hilbert, Trans. Am. Math. Soc. **2**, 87 (1901), that the Lobatchevsky plane cannot be globally imbedded in ordinary Euclidean 3-space, it is easy to achieve a global imbedding as a suitable pseudosphere in pseudo-Euclidean 3-space. A result of this kind, when the space to be imbedded has an indefinite metric, has also been obtained by A. M. Garsia (private communication to Y. Ne'eman).

closed timelike smooth curve. (The Gödel cosmological model would, in this sense, be classed as "unphysical.")

Remarkably enough, the above result for plane waves is not a topological one in the most obvious sense, the plane waves having a *Euclidean* topology and the timelike and null lines being all *open*. Instead, the result rests essentially on the surprising fact, obtained here, that a plane wave admits no spacelike hypersurface which would be adequate for the global specification of Cauchy data.

2. HYPERBOLIC NORMAL IMBEDDINGS

Before discussing the plane-wave metrics in detail, let us consider the general question of isometrically imbedding a hyperbolic normal r-manifold $\mathfrak{M}_r$ (i.e., of signature $2-r$) in a hyperbolic normal pseudo-Euclidean $\mathcal{E}_n$. The $\mathcal{E}_n$ may be given by coordinates $x_0, x_1, \cdots, x_{n-1}$ and the metric

$$ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - \cdots - dx_{n-1}^2. \tag{2.1}$$

The manifold $\mathfrak{M}_r$ is to be a smooth r-dimensional subset of $\mathcal{E}_n$ whose induced metric given by (2.1) has signature $2-r$. This amounts to saying that the tangent r space to $\mathfrak{M}_r$ at any one of its points contains a timelike vector of $\mathcal{E}_n$. ["Timelike" means that the "squared length" of the vector, according to the metric (2.1), is positive. Similarly, "null" would mean that this "squared length" is zero, etc. The terms "past" and "future" will refer to the ordering with respect to x_0, which is thus thought of as a "time" coordinate.] At each point of $\mathfrak{M}_r$, the tangent r space contains two half-cones of nonzero null vectors—the *past* and *future* half-cones. The systems of past and future half-cones are thus disconnected from each other.

Consider, now, the section of $\mathcal{E}_n$ by some spacelike hyperplane $\mathcal{H}_{n-1}$ meeting $\mathfrak{M}_r$ given, say, by $x_0 = c$ (where c is constant). The induced metric on $\mathcal{H}_{n-1}$ is negative definite, so the induced metric on the intersection $\mathcal{S}_{r-1}$, of $\mathcal{H}_{n-1}$ with $\mathfrak{M}_r$, is also negative definite. That is to say, $\mathcal{S}_{r-1}$ is a *spacelike hypersurface* in $\mathfrak{M}_r$. It must, in fact, be a *nonsingular* (i.e., smooth) hypersurface, since $\mathcal{H}_{n-1}$ can be nowhere tangent to $\mathfrak{M}_r$. Denote the parts of $\mathcal{E}_n$ and $\mathfrak{M}_r$, for which $x_0 < c$ by $\mathcal{E}_n^-$ and $\mathfrak{M}_r^-$, respectively. Similarly, denote those parts for which $x_0 > c$ by $\mathcal{E}_n^+$ and $\mathfrak{M}_r^+$. Note that $\mathcal{S}_{r-1}$ is the boundary between the two disconnected portions $\mathfrak{M}_r^-$ and $\mathfrak{M}_r^+$ of the manifold $\mathfrak{M}_r$.

Observe that any smooth connected timelike or null curve in $\mathcal{E}_n$ meets $\mathcal{H}_n$ in at most one point, since x_0 may be used as a parameter on the curve. Choose any point P in $\mathcal{E}_n^+$, with coordinates $(x_0, x_1, \cdots, x_{n-1})$. Let $\mathcal{C}_1$ be any smooth connected timelike curve in $\mathcal{E}_n$ whose future end point is P. Let $\mathcal{L}_1$ be the straight line segment joining P to the point $(c, x_1, \cdots, x_{n-1})$. The length $(dx_0^2 - dx_1^2 - \cdots - dx_{n-1}^2)^{\frac{1}{2}}$ of any element $d\mathcal{C}_1$ of $\mathcal{C}_1$ is clearly not more than the length dx_0 of the orthogonal projection $d\mathcal{L}_1$ of $d\mathcal{C}_1$ on $\mathcal{L}_1$. Hence the total length of $\mathcal{C}_1$ is at most $x_0 - c$. Thus, the timelike curves in $\mathcal{E}_n$ which extend into the past from P have bounded total length. A corresponding statement holds if P is in $\mathcal{E}_n^-$.

We may specialize these statements to the case when P and $\mathcal{C}_1$ lie on $\mathfrak{M}_r$ and collect together the following necessary conditions for the imbeddability of $\mathfrak{M}_r$ in $\mathcal{E}_n$.

(2.2) *The null half-cones of $\mathfrak{M}_r$ form two systems disconnected from one another—we may call these the systems of "past" and "future" half-cones of $\mathfrak{M}_r$. To each point of $\mathfrak{M}_r$ corresponds one past and one future half-cone.*

(2.3) *There exists a nonsingular spacelike hypersurface $\mathcal{S}_{r-1}$ in $\mathfrak{M}_r$, the removal of which separates $\mathfrak{M}_r$ into two disconnected portions: $\mathfrak{M}_r^+$, which lies on the future side of $\mathcal{S}_{r-1}$, and $\mathfrak{M}_r^-$, which lies on the past side of $\mathcal{S}_{r-1}$, and furthermore:*

(2.4) *every smooth connected timelike or null curve in $\mathfrak{M}_r$ meets $\mathcal{S}_{r-1}$ in at most one point;*

(2.5) *the smooth connected timelike curves in $\mathfrak{M}_r^+$ [resp. $\mathfrak{M}_r^-$], whose future [resp. past] end points are a given fixed point P in $\mathfrak{M}_r^+$ [resp. $\mathfrak{M}_r^-$], have a fixed upper bound to their lengths.*

The fact, physically desirable for a space–time, that $\mathfrak{M}_r$ cannot contain a *closed* smooth timelike curve is ensured by (2.3) and (2.5). For if such a curve met $\mathcal{S}_{r-1}$, this would contradict (2.3), whereas if it did not meet $\mathcal{S}_{r-1}$, neighboring open curves could be found which contradict (2.5). [However, closed smooth *null geodesics* in $\mathfrak{M}_r$ would not apparently contradict (2.2), $\cdots$, (2.5), although such curves would clearly preclude the imbedding of $\mathfrak{M}_r$ in $\mathcal{E}_n$. The condition that $\mathfrak{M}_r$ contain no closed smooth null geodesics could be added to the list (2.2), $\cdots$, (2.5) but it will not be used here.] In fact, (2.4) is an easy consequence of (2.3).

3. THE PLANE WAVES

The metric for a general gravitational–electromagnetic plane wave $\mathcal{W}_4$ can be put in the form[9]

$$ds^2 = du dv + h_{ij}(u) x_i x_j du^2 - dx_i\, dx_i, \tag{3.1}$$

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[9] This metric (without electromagnetism) was given originally by H. W. Brinkman, Proc. Natl. Acad. Sci. (U.S.) **9**, 1 (1923); Math. Ann. **94**, 119 (1925) in the more general form ("plane fronted" wave) that h_{ij} is allowed to be a suitable function of the x_k as well as of u. N. Rosen [Phys. Z. Sowjetunion **12**, 366 (1937)] obtained the metric $ds^2 = A(dt^2 - dz^2) + B_{ij} dy_i dy_j$ for plane waves (but he only considered the case when B_{ij} is diagonal). Here A and B_{ij} are to be appropriate functions of $t-z$. I. Robinson (Report to the Eddington Group, Cambridge, 1956) showed that this latter metric (with electromagnetism included) can be transformed into the form (2.1) by a coordinate transformation and that the singularities that Rosen found to be necessary with his form of metric were, in fact, spurious coordinate singularities. See also Ref. 10.

The existence of Rosen's coordinate singularities is closely related to the phenomenon that is discussed here. The time development of the hypersurface $t=0$ in the Rosen metric gives only that part of $\mathcal{W}_4$ covered nonsingularly by his coordinates, i.e., $t=0$ is not a global Cauchy hypersurface for $\mathcal{W}_4$.

where

$$h_{ii}(u)\geqslant 0. \quad (3.2)$$

The indices $i, j, \cdots$, take the two values 1, 2 and the summation convention is used throughout. All variables u, v, x_1, x_2 range from $-\infty$ to $+\infty$ and the entire manifold $\mathcal{W}_4$ is covered in a $(1-1)$ fashion by this one nonsingular coordinate patch. Without loss of generality, we may take $h_{ij}(u)$ as symmetric:

$$h_{ij}=h_{ji}. \quad (3.3)$$

The condition for a purely *gravitational* wave (vanishing Ricci tensor) is that the equality in (3.2) should hold:

$$h_{ii}(u)=0. \quad (3.4)$$

For a purely *electromagnetic* wave (vanishing Weyl tensor) we have

$$h_{ij}(u)=h(u)\ \delta_{ij}, \quad (3.5)$$

with $h(u)\geqslant 0$. The amplitude and polarization of the gravitational part of the wave is given, in the general case, by the trace-free part of h_{ij}. This may be chosen quite arbitrarily as a function of u. The amplitude of the electromagnetic part of the wave is measured by the square root of the trace of h_{ij}, which again may be chosen arbitraily as a function of u. The polarization of the electromagnetic part of the wave may also be chosen arbitraily as a function of u but it does not contribute in any way to the curvature. (The electromagnetic field is null everywhere and independent duality rotations can be applied in each hyperplane $u=$const. The gravitational field is also null.)

There is, thus, exactly as much freedom in the construction of the full general-relativistic plane waves as is the case for their linear approximations (spin 1 and 2 zero-mass waves in Minkowski space). There is also exactly as much *symmetry*, namely a five parameter group of motions[10] in general. This group acts transitively on the ∞^5 system of null geodesics of $\mathcal{W}_4$, excepting those null geodesics parallel to the propagation world-direction:

$$u=\text{const.}, \qquad x_i=\text{const.} \quad (3.6)$$

Thus, the null geodesics, which are *not* the propagation lines (3.6), are all equivalent to each other in $\mathcal{W}_4$.

It should be emphasized also that the space–time $\mathcal{W}_4$ is *physically satisfactory from the point of view of causality*. The fact that it contains no smooth closed timelike or null curves is evident from the form of the metric. On any timelike or null curve not parallel to the propagation world-direction (3.6), we can use u as a parameter continuously increasing with time. On a null curve (geodesic) parallel to this direction, v serves as such a parameter.

A particular type of plane wave whose properties are

[10] H. Bondi, F. A. E. Pirani, and I. Robinson, Proc. Roy. Soc. (London) **A251**, 519 (1959).

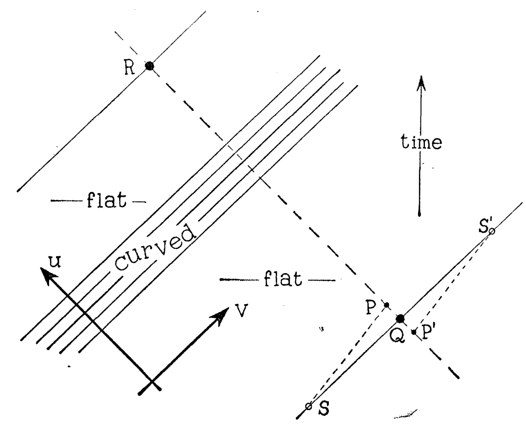

FIG. 1. The sandwich wave $\mathcal{W}_4$ and the related configuration of points mentioned in the test. The diagram essentially represents the two-dimensional section of $\mathcal{W}_4$ by $x_i=0$.

easily studied is the "sandwich" wave.[10] In this case we suppose that the gravitational and electromagnetic amplitudes vanish outside a certain range (a, b) of values of u, but that they do not both vanish everywhere. Thus

$$h_{ij}(u)=0 \text{ unless } a<u<b; \qquad \text{some } h_{ij}(u)\neq 0 \quad (3.7)$$

so that the space–time is flat for $u<a$ and again flat for $u>b$, but it is curved in between (see Fig. 1). For simplicity, the arguments will be given here for the case of sandwich waves only, which are "sufficiently weak" (in a sense to be defined shortly). In fact, general plane waves can also be treated by essentially the same method.

Let Q be a point in $\mathcal{W}_4$. The *complete null cone* $\mathcal{K}_3$ of Q is the set of points lying on all the null geodesics through Q. Now choose Q with coordinates

$$x_i=0, \qquad v=v_0, \qquad u=u_0<a, \quad (3.8)$$

so that Q lies in one of the *flat* regions of $\mathcal{W}_4$. Near Q, the equation of $\mathcal{K}_3$ is therefore

$$(u-u_0)(v-v_0)-x_ix_i=0 \quad (3.9)$$

which can be written

$$v=f_{ij}(u)x_ix_j+v_0 \quad (3.10)$$

where, near Q,

$$f_{ij}(u)=(u-u_0)^{-1}\delta_{ij}. \quad (3.11)$$

If we choose f_{ij} as a function of u so that the hypersurface (3.10) remains null (i.e., with null normal vector) also throughout the *curved* region of $\mathcal{W}_4$, then (3.10) will be the equation of $\mathcal{K}_3$. We may choose f_{ij} symmetric:

$$f_{ij}=f_{ji}. \quad (3.12)$$

The only condition to be satisfied by f_{ij} then turns out to be

$$f_{ij}'+f_{ik}f_{kl}+h_{ij}=0, \quad (3.13)$$

where the prime denotes differentiation with respect to u. Given the starting value (3.11), the equation (3.13) therefore extends the definition of f_{ij} uniquely into the curved region and, with (3.10), defines $\mathcal{K}_3$, until eventually f_{ij} may be come infinite.

To examine this last possibility, note that the trace of (3.13) gives

$$f_{ii}'+\tfrac{1}{2}f_{ii}f_{jj}=-\tfrac{1}{2}(f_{ik}f_{ik}\delta_{jl}\delta_{jl}-f_{ik}\delta_{ik}f_{jl}\,\delta_{jl})-h_{ii}\leqslant 0$$

by Schwarz' inequality and (3.2). Thus

$$\left\{\exp\left(\frac{1}{2}\int f_{ii}\,du\right)\right\}''\leqslant 0, \qquad (3.14)$$

with strict inequality holding for at least *some* values of u. If we choose Q so that u_0 is very large and negative, then in the limiting case $u_0=-\infty$ we get, from (3.11), $f_{ij}=0$ for $u<a$. That is to say, $\mathcal{K}_3$ becomes the null hyperplane $v=v_0$ in the initially flat region [see (3.10)]. Thus, initially $\{\exp(\frac{1}{2}\int f_{ii}du)\}'=0$, so from (3.14) we see that $\exp(\frac{1}{2}\int f_{ii}du)$ eventually becomes zero for some finite value of u. Hence, some component of f_{ij} must become infinite[11,12] for some least value u_1 of u, where $u_1>a$ (since $f_{ij}=0$, for $u_1\leqslant a$).

Let us suppose that the amplitudes h_{ij} become zero *before* this value is reached, that is to say

$$u_1>b. \qquad (3.15)$$

This is the meaning of the notion that the wave $\mathcal{W}_4$ be "sufficiently weak," as alluded to earlier. Now if we replace the value $-\infty$, for u_0, by a finite value which is very large and negative, then for values of u near a, (3.10) will be altered only slightly. Thus u_1 will change only slightly—it will remain finite and satisfy (3.15). Thus the complete null cone $\mathcal{K}_3$ of Q (for Q sufficiently far off) necessarily encounters *singularities* at the other side of the sandwich wave—where $\mathcal{W}_4$ has again become flat.

In order to examine these singularities, we consider (3.13) in the flat space region $u>b$. The equation (3.13) may then be written

$$p_{ij}'=\delta_{ij}, \qquad (3.16)$$

where p_{ij} is the matrix inverse to f_{ij}:

$$p_{ij}f_{jk}=\delta_{ij}. \qquad (3.17)$$

[11] The necessary existence of singularities in the f_{ij}'s is essentially the same phenomenon as that discovered by A. Raychaudhuri, Phys. Rev. **98,** 1123 (1955); and A. Komar, Phys. Rev. **104,** 544 (1956). Here the result is applied to null geodesics rather than timelike ones. A case of this particular phenomenon was also found by I. Ozsváth and E. Schücking in *Recent Developments in General Relativity* (Pergamon Press, Ltd., London–Warsaw, 1962).

[12] This fact has relevance to the question of two colliding weak plane sandwich waves. Each wave warps the other until singularities in the wave fronts ultimately appear. This, in fact, causes the space-time to acquire genuine physical singularities in this case. The warping also produces a scattering of each wave after collision so that they cease to be sandwich waves when they separate (and they are no longer plane—although they have a two-parameter symmetry group).

The solution of (3.16) is

$$p_{ij}(u)=u\,\delta_{ij}-q_{ij}, \qquad (3.18)$$

where q_{ij} is constant and symmetric [by (3.12)]. This gives infinite values for f_{ij} whenever u is an eigenvalue of q_{ij}. Two essentially different cases can occur. Either the eigenvalues of q_{ij} are distinct or else $q_{ij}=u_1\delta_{ij}$ (q_{ij} can be diagonalized since it is symmetric). In this latter case, (3.18) gives the same form as (3.11), that is to say, the null cone $\mathcal{K}_3$ has *two* vertices, namely, Q and also the point R with coordinates

$$x_i=0, \qquad v=v_0, \qquad u=u_1. \qquad (3.19)$$

The null cone of Q is thus *focused*[13] by the wave to the *single point* R (see Fig. 2). This is the case of pure *anastygmatic* focusing and it occurs with a purely electromagnetic wave. In the gravitational case, we expect to get some resultant astygmatism[14] and the eigenvalues of q_{ij} will generally be distinct. In this case the null cone $\mathcal{K}_3$ is focused to a spacelike *line* through R, as will be seen in a moment.

The null geodesics which generate $\mathcal{K}_3$ are the curves on $\mathcal{K}_3$ with null tangent vectors. Hence by (3.1), (3.13), and the derivative of (3.10), using u as a parameter, we get for each particular geodesic,

$$x_i'=f_{ij}x_j \qquad (3.20)$$

with v then given by (3.10). [Note that (3.20) and (3.13) imply the relation $x_i''=-h_{ij}x_j$, which shows that h_{ij} measures the geodesic deviation of the null rays.[14]] There is an exceptional null geodesic $\mathcal{Q}_1$ through Q which is *not* parametrized by u, however, namely the one parallel to the propagation world-direction. Thus,

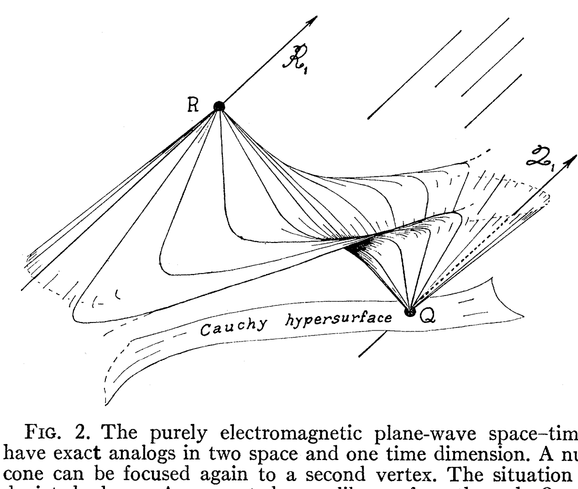

Fig. 2. The purely electromagnetic plane-wave space–times have exact analogs in two space and one time dimension. A null cone can be focused again to a second vertex. The situation is depicted above. A connected spacelike surface through Q can never meet the null line $\mathcal{R}_1$ (if the surface has no boundary).

[13] This focusing effect and its possible relation to energy flux is discussed in greater generality in R. Penrose, Hlavaty Festschr. (to be published).

[14] R. K. Sachs, Proc. Roy. Soc. (London) **A264,** 309 (1961), F. A. E. Pirani and A. Schild, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Phys. **9,** 543 (1961).

$\mathcal{Q}_1$ is given [see (3.6)] by

$$u=u_0, \qquad x_i=0. \tag{3.21}$$

When $\mathcal{K}_3$ enters the flat region $u>b$, we can solve (3.20) using (3.18) [see (3.17)] and obtain

$$x_i=(u\,\delta_{ij}-q_{ij})m_j, \tag{3.22}$$

where the m_j are constant for each geodesic. When u_i is a repeated eigenvalue of q_{ij} (the anastygmatic case) we see that $x_i=0$ for $u=u_1$ so that *all the null geodesics through* Q, *with the single exception of* $\mathcal{Q}_1$, *pass through the point* R [given by (3.19)]. When u_1 is a simple eigenvalue of q_{ij} (the general astygmatic case), we may, for convenience, choose the original x_1 and x_2 axes in (3.1) so that q_{ij} is diagonal with $q_{11}=u_1$, $q_{12}=0$, $q_{22}=u_2(u_2\neq u_1)$. Thus, $x_1=0$ when $u=u_1$, so that *all the null geodesics through* Q, *with the single exception of* $\mathcal{Q}_1$, *pass through some point of the spacelike line*: $x_1=0$, $v=v_0$, $u=u_1$. In fact, through each point of this line there will pass the ∞^1 null geodesics given when m_2 takes a fixed value. In particular, the null geodesics through both Q and R are those given when the linear relation $m_2=0$ holds. The linearity of (3.20) then implies that the values of x_i at points near Q, for these geodesics, are also determined by a linear relation: $x_ic_i=0$ (c_i constant). That is, near Q, the null geodesics on $\mathcal{K}_3$ which eventually pass through R are the ones—except for $\mathcal{Q}_1$—lying on the intersection of $\mathcal{K}_3$ with the timelike hyperplane $x_ic_i=0$. This is a *connected* ∞^1 system of curves (the generators of an ordinary three-dimensional cone). Thus in *both* the above cases we have the property that there are null geodesics through Q, arbitrarily close to $\mathcal{Q}_1$ which pass through a *fixed* point R *not* on $\mathcal{Q}_1$.

The roles of Q and R may be interchanged. If $\mathcal{R}_1$ is the null geodesic through R parallel to the propagation direction (i.e., $\mathcal{R}_1$ has equation $u=u_1$, $v=v_0$, $x_i=0$), then $\mathcal{R}_1$ does *not* pass through Q, but neighboring null geodesics to $\mathcal{R}_1$ through R *do* pass through Q—these being the same null geodesics as in the previous case.[15] Thus, the null geodesics through *both* Q and R form either a two-dimensional or a one-dimensional system, a limiting position of the geodesics of this system giving the *pair* of null geodesics $\mathcal{Q}_1$, $\mathcal{R}_1$. Hence $\mathcal{W}_4$ has a characteristic property:

> $\mathcal{W}_4$ *contains a sequence of null geodesics which converges on a pair of nonintersecting null geodesics.* (3.23)

The convergence in (3.23) is to be taken in the sense that any neighborhood of a point on either of the limit lines contains a point of every geodesic sufficiently far on in the sequence. There is clearly a sense in which this convergence is nonuniform. We can also allow that more than two null geodesics may emerge in the limit, although this will not in fact happen with the "sufficiently weak" waves considered here.

One consequence of (3.23) is that $\mathcal{W}_4$ contains no global Cauchy hypersurface. For a hypersurface to be completely satisfactory globally, for the specification of Cauchy data,[16] we would require that every null geodesic should intersect it exactly once. However, if each null geodesic of the sequence in (3.23) were to meet the hypersurface just once, it is clear that *at most one* of the limiting geodesics could meet it—the other would have to miss the hypersurface altogether. The situation can be made more graphic if we examine the diagram of Fig. 2. A connected spacelike surface through Q must initially lie entirely "below" (i.e., to the past of) the future null cone of Q. As this cone folds down in order to focus again at the point R, the hypersurface gets trapped beneath it (it cannot cross the null cone and remain spacelike everywhere). Thus the hypersurface can never intersect $\mathcal{R}_1$ nor any of the propagation lines beyond $\mathcal{R}_1$. Cauchy data on such a hypersurface could thus give no information for specifying amplitudes for a parallel wave which might lie beyond $\mathcal{R}_1$.

The situation of (3.23) also serves to prove the non-imbeddability of $\mathcal{W}_4$ in a hyperbolic normal $\mathcal{E}_n$, by means of a similar argument to the above. For suppose that $\mathcal{W}_4$ can be isometrically imbedded in $\mathcal{E}_n$. Then there exists a nonsingular spacelike hypersurface $\mathcal{S}_3$ in $\mathcal{W}_4$ which is in accordance with (2.3), (2.4), and (2.5) (with $\mathcal{W}_4$ for $\mathfrak{M}_r$). Thus, by (2.4), every null geodesic of the sequence in (3.23) must meet $\mathcal{S}_3$ in at most one point. Hence, $\mathcal{S}_3$ cannot meet *both* of $\mathcal{Q}_1$, $\mathcal{R}_1$. Suppose, first, that $\mathcal{S}_3$ does not meet $\mathcal{Q}_1$. Then, by (2.3) either $\mathcal{Q}_1$ lies entirely in $\mathcal{W}_4^+$ or entirely in $\mathcal{W}_4^-$. Suppose $\mathcal{Q}_1$ lies in $\mathcal{W}_4^+$. Choose a fixed point P, with $x_i=0$, $v=v_0$, $u=u_0+\epsilon$, where $a-u_0>\epsilon>0$. Then since P and $\mathcal{Q}_1$ both lie in the flat region $u<a$, there will be a straight timelike segment joining P to any point S on the half of $\mathcal{Q}_1$ which lies to the past of Q (see Fig. 1) We may specify S by $x_i=0$, $v=v_0-s$, $u=u_0$ with $s>0$. Then the Minkowski length of the segment SP will be $s\epsilon$ [see (3.1)]. This is clearly unbounded for $s\rightarrow\infty$, whereas SP remains within $\mathcal{W}_4^+$. Thus, (2.5) is contradicted. Similarly, if $\mathcal{Q}_1$ lies in $\mathcal{W}_4^-$, we choose P' with coordinates $x_i=0$, $v=v_0$, $u=u_0-\epsilon$ and S' with $x_i=0$, $v=v_0+s$, $u=u_0$ and let $s\rightarrow\infty$. Again (2.5) is contradicted. The argument is essentially the same if $\mathcal{S}_3$ does not meet $\mathcal{R}_1$.

4. CONCLUDING REMARKS

In order to assess the relevance of this result to the question of imbedding "physically sensible" space-times in a hyperbolic normal $\mathcal{E}_n$, we must examine how "physically sensible" the general relativistic plane waves really are. It seems fair to assume that they are

[15] Note that, somewhat remarkably, $\mathcal{K}_3$ is not a closed set. The null geodesic $\mathcal{R}_1$ is not part of $\mathcal{K}_3$ but it consists of limit points of $\mathcal{K}_3$.

[16] See, for example, Y. Bruhat in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1963), and the references cited therein.

no *less* physical (as idealizations) than their flat space counterparts. But there are, of course, difficulties encountered with plane waves even in flat space. The total energy, for example, is infinite. Whereas the local energy flux remains meaningful for an electromagnetic wave, this is not so in the spin-2 zero-mass case. Also, the angular momentum due to the photon spin in a circularly polarized plane electromagnetic wave cannot be exhibited locally in a gauge invariant way; etc. These concepts all become meaningful only for waves (e.g., wave packets) whose amplitudes fall off suitable in *all* spacial directions so that the *total* energy–momentum and angular momentum, etc., can be defined.[17]

In the general relativistic case, the equivalent condition is usually interpreted as some form of asymptotic flatness condition for the space–time manifold—although some other assumptions are presumably appropriate on the cosmological scale. A question of some interest, therefore, is whether the somewhat strange properties of plane waves encountered here will still be present for waves which locally approximate plane waves, but for which the space–time is asymptotically flat, or asymptotically cosmological in some appropriate sense.

A geometrical approach to the global question of asymptotic flatness has been recently proposed by the author[18] which has relevance to this problem. Moreover, various asymptotically cosmological situations can also be treated by essentially the same method. A space–time is termed *asymptotically simple* if, from the point of view of its *conformal* structure, a sufficiently extensive boundary hypersurface $\mathscr{I}_3$ can be introduced "at infinity" which satisfies certain regularity conditions. For an asymptotically *flat* space–time, it turns out that $\mathscr{I}_3$ is *null*. The cases when $\mathscr{I}_3$ is *spacelike* are also of some interest cosmologically, since these cases are, in an appropriate sense, asymptotically *de Sitter* with a positive cosmological constant. The asymptotic simplicity condition then implies that in these cases the Riemann tensor falls off suitably (the "peeling off" property) for Einstein–Maxwell fields. In particular, the Weyl and Maxwell tensors must fall off as r^{-1} along every null geodesic, a property clearly not shared by the plane waves (r being a linear parameter on the null geodesic).

It is a simple matter to show, in fact, that the key property (3.23) *cannot occur* in any such asymptotically simple space–time. [If $\mathscr{I}_3$ is timelike—the asymptotically de Sitter cases with negative cosmological constant—then there appears to be no reason against (3.23) occuring. However, these cases appear to be of lesser interest cosmologically.] On the other hand, the focusing effect of a wave on null cones encountered here is a quite general phenomenon[13]—although it is peculiarly exact in the case of plane waves. In the case of a wave packet, the effect would be quite similar to that depicted in Fig. 2 for null geodesics fairly close to QR, but the extreme deflections of null geodesics through Q near $\mathfrak{Q}_1$ would not take place. It is still an open question, therefore, whether global Cauchy hypersurfaces or global imbeddings in some hyperbolic normal $\mathcal{E}_n$ will always exist for wave packets.

[17] The question of including a source for the wave should also be considered to be completely physically realistic. However this seems to be not essential for the case of wave packets.

[18] R. Penrose, Phys. Rev. Letters **10**, 66 (1963); in *Relativity Groups and Topology* (the 1963 Les Houches lectures), edited by C. DeWitt and B. DeWitt (Gordon & Breach Publishers, Inc., New York, 1964). A more complete account is due to appear shortly in Proc. Roy. Soc. (London).

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