Embedding of Various Relativistic Riemannian Spaces in Pseudo-Euclidean Spaces

JOE ROSEN*
Department of Physics, Tel-Aviv University, Tel-Aviv, Israel

The embedding of various relativistic Riemannian spaces is performed. In each case the original Riemannian line element, the embedding pseudo-Euclidean line element, and the embedding transformation are shown.

INTRODUCTION

A recent geometric theorem[1] states that an n-dimensional Riemannian space,

$$\text{line element: } ds^2=\sum_{i,j=1}^{n} g_{ij}\, dx^i\, dx^j,$$

can always be locally and isometrically embedded in an m-dimensional pseudo-Euclidean space [line element: $ds^2=(dz^1)^2+\cdots+(dz^p)^2-(dz^{p+1})^2-\cdots-(dz^{p+q})^2, m=p+q$], where $m=\frac{1}{2}n(n+1)$ and neither p nor q is less than the number of positive or negative eigenvalues of g_{ij}, respectively. The embedding is performed by finding the smallest number of z's, as functions of the x's, for which the Riemannian line element is transformed into the pseudo-Euclidean. In general, this involves solving systems of partial differential equations.

In the case of Riemannian spaces of general relativity ($n=4$ and signature of g_{ij} is $+---$), the embedding pseudo-Euclidean space need not be of more than ten dimensions, and for certain examples is of less. The problem of finding the *minimal* embedding space for a given Riemannian space has been reviewed by A. Friedman[2] both for local and global[3] embedding. The only available useful theorem,[4] it appears, is that a vacuum solution cannot be embedded in five dimensions. It seems therefore to be impossible at present to state in general more than the following for the embedding of general relativistic solutions: (1) A local embedding in more than ten dimensions is not minimal; (2) An embedding of a nonflat vacuum solution in six dimensions is minimal; (3) An imbedding of a nonvacuum solution in five dimensions is minimal.

In the following the local (and sometimes also global) embedding of various more or less familiar relativistic Riemannian spaces, whose embedding can be performed without solving systems of partial differential equations, is shown. The Riemannian line elements are listed according to their embedding pseudo-Euclidean line elements. The special cases (e.g., when a Riemannian space is embeddable in more than one pseudo-Euclidean space or is embeddable in different pseudo-Euclidean spaces according to the values of parameters and the ranges of variables) are treated in Sec. J. For the sake of brevity the m-dimensional pseudo-Euclidean line element $ds^2=(dz^1)^2+\cdots+(dz^p)^2-(dz^{p+1})^2-\cdots-(dz^{p+q})^2$, where $m=p+q$, is designated by $(m,\ p+q-)$. "+" is used for temporal coordinates and "−" for spatial.

An appendix is included containing a table of embedding transformations for expressions appearing in Riemannian line elements.

EMBEDDED AND EMBEDDING SPACES AND EMBEDDING TRANSFORMATIONS

A. (4, 1+3−)

1. Minkowskian space of special relativity.

$$(dx^0)^2-(dx^1)^2-(dx^2)^2-(dx^3)^2.$$

$$z^1=x^0$$
$$z^2=x^1$$
$$z^3=x^2$$
$$z^4=x^3.$$

B. (5, 1+4−)

1. Einstein model.[5]

$$dt^2-(1-r^2/R^2)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2),\ R=\text{const.}$$

$$z^1=t$$
$$z^2=R(1-r^2/R^2)^{\frac{1}{2}}$$
$$z^3=r\cos\theta$$
$$z^4=r\sin\theta\cos\phi$$
$$z^5=r\sin\theta\sin\phi.$$

* This work is intended for inclusion in a Ph.D. thesis to be submitted to the Hebrew University of Jerusalem.

[1] A. Friedman, J. Math. Mech. **10,** 625 (1961). It appears that previous to Friedman, this had been proven rigorously for positive definite metrics only.

[2] A. Friedman, Symposium on Particle Symmetries and the Embedding Problem, Southwest Center for Advanced Studies, Dallas, 1964 [Rev. Mod. Phys. **37,** 201 (1965), preceding paper].

[3] J. Nash, Ann. Math. **63,** 20 (1955).

[4] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, New Jersey, 1926), p. 197 ff.

[5] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University Press, London, 1934), p. 333 ff.

2. de Sitter model.[5]

$(1-r^2/R^2)dt^2-(1-r^2/R^2)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2)$, $R=$ const.

$z^1=(R^2-r^2)^{\frac{1}{2}}\sinh(t/R)$

$z^2=(R^2-r^2)^{\frac{1}{2}}\cosh(t/R)$

$z^3=r\cos\theta$

$z^4=r\sin\theta\cos\phi$

$z^5=r\sin\theta\sin\phi$.

3. Einstein–de Sitter model.[6]

$dt^2-\exp[g(t)][dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2)]$.

$$z^1=\exp(\tfrac{1}{2}g)(\tfrac{1}{4}r^2+1)+\int[dt/2\dot{g}\exp(\tfrac{1}{2}g)]$$

$$z^2=\exp(\tfrac{1}{2}g)(\tfrac{1}{4}r^2-1)+\int[dt/2\dot{g}\exp(\tfrac{1}{2}g)]$$

$z^3=\exp(\tfrac{1}{2}g)r\cos\theta$

$z^4=\exp(\tfrac{1}{2}g)r\sin\theta\cos\phi$

$z^5=\exp(\tfrac{1}{2}g)r\sin\theta\sin\phi$.

4. Homogeneous model.[7]

$dt^2-\exp[g(t)](1+r^2/4R^2)^{-2}[dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2)]$,

$R=$ const.

$$z^1=\int[1+\tfrac{1}{4}R^2\dot{g}^2\exp(g)]^{\frac{1}{2}}dt$$

$z^2=R\exp(\tfrac{1}{2}g)(1-r^2/4R^2)/(1+r^2/4R^2)$

$z^3=\exp(\tfrac{1}{2}g)r\cos\theta/(1+r^2/4R^2)$

$z^4=\exp(\tfrac{1}{2}g)r\sin\theta\cos\phi/(1+r^2/4R^2)$

$z^5=\exp(\tfrac{1}{2}g)r\sin\theta\sin\phi/(1+r^2/4R^2)$.

5. Friedman model.[8]

$dt^2-R^2(t)[d\chi^2+S^2(\chi)(d\theta^2+\sin^2\theta\, d\phi^2)]$;

where

$$S(\chi)=\begin{cases}\sinh\chi\\ \chi\\ \sin\chi\end{cases}\quad\text{for}\quad\epsilon=\begin{cases}-1\\ 0;\\ +1\end{cases}\quad \dot{R}^2=kM/4\pi R+\Lambda R^2/3-\epsilon;$$

$k, M, \Lambda=$ const.

$\epsilon=-1$	0	$+1$
$z^1=R\cosh\chi$	$\frac{1}{2}R(\chi^2+1)+\int(dt/2\dot{R})$	$\int(\dot{R}^2+1)^{\frac{1}{2}}dt$
$z^2=R\sinh\chi\cos\theta$	$\frac{1}{2}R(\chi^2-1)+\int(dt/2\dot{R})$	$R\cos\chi$
$z^3=R\sinh\chi\sin\theta\cos\phi$	$R\chi\cos\theta$	$R\sin\chi\cos\theta$
$z^4=R\sinh\chi\sin\theta\sin\phi$	$R\chi\sin\theta\cos\phi$	$R\sin\chi\sin\theta\cos\phi$
$z^5=\int(\dot{R}^2-1)^{\frac{1}{2}}dt$	$R\chi\sin\theta\sin\phi$	$R\sin\chi\sin\theta\sin\phi$.

C. (6, 2+4−)

1. Exterior Schwarzschild solution.[9]

$(1-2m/r-\Lambda r^2/3)dt^2-(1-2m/r-\Lambda r^2/3)^{-1}dr^2$

$-r^2(d\theta^2+\sin^2\theta\, d\phi^2)$; $m, \Lambda=$ const.

$z^1=(1-2m/r-\Lambda r^2/3)^{\frac{1}{2}}\cos t$

$z^2=(1-2m/r-\Lambda r^2/3)^{\frac{1}{2}}\sin t$

$$z^3=\int\left[\frac{(m/r^2-\Lambda r/3)^2+1}{1-2m/r-\Lambda r^2/3}-1\right]^{\frac{1}{2}}dr$$

$z^4=r\cos\theta$

$z^5=r\sin\theta\cos\phi$

$z^6=r\sin\theta\sin\phi$.

[6] See Ref. 5, p. 415 ff.
[7] See Ref. 5, p. 361 ff.
[8] O. Heckmann and E. Schücking, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1963), p. 438.
[9] See Ref. 5, p. 245 ff.

2. Exterior Schwarzschild solution in conformal space.[10]

$(r-a)dt^2/(r+a)-(r+a)$

$\times[dr^2+(r^2-a^2)(d\theta^2+\sin^2\theta\, d\phi^2)]/(r-a)$, $a=$const.

$$z^1=[(r-a)/(r+a)]^{\frac{1}{2}}\cos t$$

$$z^2=[(r-a)/(r+a)]^{\frac{1}{2}}\sin t$$

$$z^3=\int\{[a^2+2a(r+a)^3]/(r-a)(r+a)^3\}^{\frac{1}{2}}\, dr$$

$$z^4=(r+a)\cos\theta$$

$$z^5=(r+a)\sin\theta\cos\phi$$

$$z^6=(r+a)\sin\theta\sin\phi.$$

3. Interior Schwarzschild solution.[9]

$\frac{1}{4}[3(1-r_1^2/R^2)^{\frac{1}{2}}-(1-r^2/R^2)^{\frac{1}{2}}]^2dt^2-(1-r^2/R^2)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2)$; r_1, $R=$const.

$$z^1=\tfrac{1}{2}[3(1-r_1^2/R^2)^{\frac{1}{2}}-(1-r^2/R^2)^{\frac{1}{2}}]\cos t$$

$$z^2=\tfrac{1}{2}[3(1-r_1^2/R^2)^{\frac{1}{2}}-(1-r^2/R^2)^{\frac{1}{2}}]\sin t$$

$$z^3=[(\tfrac{1}{4}+R^2)(1-r^2/R^2)]^{\frac{1}{2}}$$

$$z^4=r\cos\theta$$

$$z^5=r\sin\theta\cos\phi$$

$$z^6=r\sin\theta\sin\phi.$$

4. Reissner–Weyl (charged-particle) solution.[11]

$(1-2m/r+4\pi e^2/r^2)dt^2-(1-2m/r+4\pi e^2/r^2)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2)$; m, $e=$const.

$$z^1=(1-2m/r+4\pi e^2/r^2)^{\frac{1}{2}}\cos t$$

$$z^2=(1-2m/r+4\pi e^2/r^2)^{\frac{1}{2}}\sin t$$

$$z^3=\int\left[\frac{\frac{1}{4}(2m/r^2-8\pi e^2/r^3)^2+1}{1-2m/r+4\pi e^2/r^2}-1\right]^{\frac{1}{2}} dr$$

$$z^4=r\cos\theta$$

$$z^5=r\sin\theta\cos\phi$$

$$z^6=r\sin\theta\sin\phi.$$

5. Degenerate static vacuum field, class $A1$.[12]

$(1-b/r)dt^2-(1-b/r)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2)$, $0\leq\theta\leq\pi$, ϕ mod 2π, $b=$const.

a. $0<b<r<\infty$.

$$z^1=(1-b/r)^{\frac{1}{2}}\cos t$$

$$z^2=(1-b/r)^{\frac{1}{2}}\sin t$$

$$z^3=\int[b(b/4r^3+1)/(r-b)]^{\frac{1}{2}}\, dr$$

$$z^4=r\cos\theta$$

$$z^5=r\sin\theta\cos\phi$$

$$z^6=r\sin\theta\sin\phi.$$

b. $b<0<r<\infty$.

$$z^1=\int[b(b/4r^3-1)/(r-b)]^{\frac{1}{2}}\, dr$$

$$z^2=(1-b/r)^{\frac{1}{2}}\sinh t$$

$$z^3=(1-b/r)^{\frac{1}{2}}\cosh t$$

$$z^4=r\cos\theta$$

$$z^5=r\sin\theta\cos\phi$$

$$z^6=r\sin\theta\sin\phi.$$

6. Spherically symmetric maximal analytic extension of above.[12,13]

$-b^2[4\exp(-r)(du^2-dv^2)/r+r^2(d\theta^2+\sin^2\theta\, d\phi^2)]$, $r=$ transcendental function of (u^2-v^2) defined by $u^2-v^2=(r-1)\exp(r)$, $0\leq\theta\leq\pi$, ϕ mod 2π, $-1<u^2-v^2<\infty$, $b=$const. >0.

$$z^1=b(1-1/r)^{\frac{1}{2}}\cos[2\,\text{argtanh}\,(v/u)]$$

$$z^2=b(1-1/r)^{\frac{1}{2}}\sin[2\,\text{argtanh}\,(v/u)]$$

$$z^3=b\int\{[(r-1)(4r^3+1)/r^5]^{\frac{1}{2}}\, d(u^2-v^2)/2(u^2-v^2)\}$$

$$z^4=br\cos\theta$$

$$z^5=br\sin\theta\cos\phi$$

$$z^6=br\sin\theta\sin\phi.$$

7. Degenerate static vacuum field, class $B1$.[12]

$-(1-b/r)d\phi^2-(1-b/r)^{-1}dr^2-r^2(d\theta^2-\sin^2\theta\, dt^2)$, $0<\theta<\pi$, $b=$const.

a. $0<b<r<\infty$.

$$z^1=(1-b/r)^{\frac{1}{2}}\cosh\phi$$

$$z^2=r\sin\theta\sinh t$$

$$z^3=r\sin\theta\cosh t$$

$$z^4=r\cos\theta$$

$$z^5=\int[b(b/4r^3+1)/(r-b)]^{\frac{1}{2}}\, dr$$

$$z^6=(1-b/r)^{\frac{1}{2}}\sinh\phi.$$

[10] V. Fock in *Recent Developments in General Relativity* (Pergamon Press, Ltd., London, 1962), p. 209.
[11] See Ref. 5, p. 265 ff.
[12] J. Ehlers and W. Kundt in Ref. 8, p. 49.
[13] M. D. Kruskal, Phys. Rev. **119**, 1743 (1960).

b. $b<0<r<\infty$.

$$z^1=\int[b(b/4r^3-1)/(r-b)]^{\frac{1}{2}}\,dr$$

$$z^2=r\sin\theta\sinh t$$

$$z^3=r\sin\theta\cosh t$$

$$z^4=r\cos\theta$$

$$z^5=(1-b/r)^{\frac{1}{2}}\cos\phi$$

$$z^6=(1-b/r)^{\frac{1}{2}}\sin\phi.$$

8. Analytic locally static extension of above.[12]

$-b^2[d\rho^2/(1-\rho^2/4)^4+\rho^2\,d\psi^2+(\cosh^2\tau\,dz^2-d\tau^2)/(1-\rho^2/4)^2]$; $0\leq\rho<2$; ψ mod 2π; $-\infty<z,\tau<\infty$; $b=$const. >0.

$$z^1=b\rho\cosh\psi$$

$$z^2=b\sinh\tau/(1-\rho^2/4)$$

$$z^3=b\cosh\tau\sin z/(1-\rho^2/4)$$

$$z^4=b\cosh\tau\cos z/(1-\rho^2/4)$$

$$z^5=b\int[1+1/(1-\rho^2/4)^3]^{\frac{1}{2}}\,d\rho$$

$$z^6=b\rho\sinh\psi.$$

D. (6, 3+3−)

1. Degenerate static vacuum field, class $A2$.[12]

$(b/z-1)dt^2-(b/z-1)^{-1}dz^2-z^2(dr^2+\sinh^2 r\,d\phi^2)$, $0\leq r<\infty$, ϕ mod 2π, $0<z<b=$const.

$$z^1=(b/z-1)^{\frac{1}{2}}\cos t$$

$$z^2=(b/z-1)^{\frac{1}{2}}\sin t$$

$$z^3=z\cosh r$$

$$z^4=z\sinh r\cos\phi$$

$$z^5=z\sinh r\sin\phi$$

$$z^6=\int[b(b/4z^3+1)/(b-z)]^{\frac{1}{2}}\,dz.$$

2. Maximal analytic extension of above.[12]

$-b^2[4\exp(-z)(du^2-dv^2)/z+z^2(dr^2+\sinh^2 r\,d\phi^2)]$, $z=$ transcendental function of (u^2-v^2) defined by $u^2-v^2=(1-z)\exp(z)$, $u^2-v^2<1$, $0\leq r<\infty$, ϕ mod 2π, $b=$ const. >0.

$$z^1=b(1/z-1)^{\frac{1}{2}}\cos[2\,\mathrm{argtanh}\,(v/u)]$$

$$z^2=b(1/z-1)^{\frac{1}{2}}\sin[2\,\mathrm{argtanh}\,(v/u)]$$

$$z^3=bz\cosh r$$

$$z^4=bz\sinh r\cos\phi$$

$$z^5=bz\sinh r\sin\phi$$

$$z^6=b\int\{[(1-z)(4z^3+1)/z^5]^{\frac{1}{2}}\,d(u^2-v^2)/2(u^2-v^2)\}.$$

3. Degenerate static vacuum field, class $B2$.[12]

$-(b/z-1)d\phi^2-(b/z-1)^{-1}dz^2-z^2(dr^2-\sinh^2 r\,dt^2)$, $0<r<\infty$, $0<z<b=$const.

$$z^1=(b/z-1)^{\frac{1}{2}}\cosh\phi$$

$$z^2=z\cosh r$$

$$z^3=z\sinh r\sinh t$$

$$z^4=z\sinh r\cosh t$$

$$z^5=\int[b(b/4z^3+1)/(b-z)]^{\frac{1}{2}}\,dz$$

$$z^6=(b/z-1)^{\frac{1}{2}}\sinh\phi.$$

4. Analytic static extension of above.[12]

$-b^2[d\rho^2/(1+\rho^2/4)^4+\rho^2\,d\psi^2+(dz^2-\cosh^2 z\,d\tau^2)/(1+\rho^2/4)^2]$; $0\leq\rho<\infty$; ψ mod 2π; $-\infty<z,\tau<\infty$; $b=$const. >0.

$$z^1=b\rho\cosh\psi$$

$$z^2=b\cosh z\cos\tau/(1+\rho^2/4)$$

$$z^3=b\cosh z\sin\tau/(1+\rho^2/4)$$

$$z^4=b\sinh z/(1+\rho^2/4)$$

$$z^5=b\int[1+1/(1+\rho^2/4)^3]^{\frac{1}{2}}\,d\rho$$

$$z^6=b\rho\sinh\psi.$$

E. (7, 2+5−)

1. Petrov space T_1, group G_4, metric 4.[14]

$-(kx^4+1)^{\frac{4}{3}}[(dx^1)^2+(dx^2)^2]-(kx^4+1)^{-\frac{2}{3}}(dx^3)^2+(dx^4)^2$, $k=$const.

$$z^1=\frac{1}{3}\int[9+k^2(kx^4+1)^{-8/3}]^{\frac{1}{2}}\,dx^4$$

$$z^2=(kx^4+1)^{\frac{2}{3}}\cosh x^2$$

$$z^3=(kx^4+1)^{\frac{2}{3}}\sinh x^2$$

$$z^4=(kx^4+1)^{\frac{2}{3}}\cos x^1$$

$$z^5=(kx^4+1)^{\frac{2}{3}}\sin x^1$$

$$z^6=(kx^4+1)^{-\frac{1}{3}}\cos x^3$$

$$z^7=(kx^4+1)^{-\frac{1}{3}}\sin x^3.$$

---

[14] A. Z. Petrov in Ref. 10, p. 379. The present metrics 1 and 2 of space T_1, group G_4, are obtained from those given by Petrov by setting $y^2=x^2/c_3$, $y^3=x^3/c_1^2c_3$ for metric 1 and $y^1=x^1/c_3$, $y^4=x^4/c_1^2c_3$ for metric 2 and in both cases setting $g=c_2/c_1c_3$ and dropping the scale factor.

F. (7, 3+4−)

1. Static cylindrically symmetric magnetic or electric geon.[15]

$$(1+r^2/a^2)^2(dt^2-dr^2-dz^2)-r^2\,d\phi^2/(1+r^2/a^2)^2,\ a=\text{const.}$$

$$z^1=(1+r^2/a^2)\cos t$$

$$z^2=(1+r^2/a^2)\sin t$$

$$z^3=r\cosh\phi/(1+r^2/a^2)$$

$$z^4=r\sinh\phi/(1+r^2/a^2)$$

$$z^5=(1+r^2/a^2)\cos z$$

$$z^6=(1+r^2/a^2)\sin z$$

$$z^7=\int\{[(1+r^2/a^2)^6+(1-r^2/a^2)^2]^{\frac{1}{2}}\,dr/(1+r^2/a^2)^2\}.$$

2. Petrov space T_1, group G_4, metric 5.[14]

$$-(kx^3+1)^{\frac{4}{3}}[(dx^1)^2+(dx^2)^2]-(dx^3)^2+(kx^3+1)^{-\frac{2}{3}}(dx^4)^2,$$
$k=\text{const.}$

$$z^1=(kx^3+1)^{-\frac{1}{3}}\cos x^4$$

$$z^2=(kx^3+1)^{-\frac{1}{3}}\sin x^4$$

$$z^3=(kx^3+1)^{\frac{2}{3}}\cosh x^2$$

$$z^4=(kx^3+1)^{\frac{2}{3}}\sinh x^2$$

$$z^5=(kx^3+1)^{\frac{2}{3}}\cos x^1$$

$$z^6=(kx^3+1)^{\frac{2}{3}}\sin x^1$$

$$z^7=\frac{1}{3}\int[9+k^2(kx^3+1)^{-8/3}]^{\frac{1}{2}}\,dx^3.$$

3. Petrov space T_1, group G_4, metric 6.[14]

$$-(kx^3+1)^{\frac{4}{3}}(dx^1)^2-(kx^3+1)^{-\frac{2}{3}}(dx^2)^2-(dx^3)^2+(kx^3+1)^{\frac{4}{3}}(dx^4)^2,\ k=\text{const.}$$

$$z^1=(kx^3+1)^{\frac{2}{3}}\cos x^4$$

$$z^2=(kx^3+1)^{\frac{2}{3}}\sin x^4$$

$$z^3=(kx^3+1)^{-\frac{1}{3}}\cosh x^2$$

$$z^4=(kx^3+1)^{-\frac{1}{3}}\sinh x^2$$

$$z^5=(kx^3+1)^{\frac{2}{3}}\cos x^1$$

$$z^6=(kx^3+1)^{\frac{2}{3}}\sin x^1$$

$$z^7=\frac{1}{3}\int[9+k^2(kx^3+1)^{-8/3}]^{\frac{1}{2}}\,dx^3.$$

[15] M. A. Melvin, Phys. Letters **8**, 65 (1964).

G. (8, 3+5−)

1. Degenerate static vacuum field, class C.[12]

$$[|f(-y)|\,dt^2-dy^2/|f(-y)|-f(x)\,d\phi^2-dx^2/f(x)]/(x+y)^2;\ f(u)\equiv(u^3+au+b);$$
$a, b=\text{const.};\ x+y>0;\ f(-y)<0;\ f(x)>0.$

$$z^1=|f(-y)|^{\frac{1}{2}}\cos t/(x+y)$$

$$z^2=|f(-y)|^{\frac{1}{2}}\sin t/(x+y)$$

$$z^3=[f(x)]^{\frac{1}{2}}\cosh\left\{\int[dx/f(x)]\right\}\Big/(x+y)$$

$$z^4=[f(x)]^{\frac{1}{2}}\sinh\left\{\int[dx/f(x)]\right\}\Big/(x+y)$$

$$z^5=[f(x)]^{\frac{1}{2}}\cos\phi/(x+y)$$

$$z^6=[f(x)]^{\frac{1}{2}}\sin\phi/(x+y)$$

$$z^7=|f(-y)|^{\frac{1}{2}}\cos\left\{\int[dy/f(-y)]\right\}\Big/(x+y)$$

$$z^8=|f(-y)|^{\frac{1}{2}}\sin\left\{\int[dy/f(-y)]\right\}\Big/(x+y).$$

H. (10, 4+6−)

1. Weyl's static rotationally symmetric solution.[15,16]

$$\exp[2\psi(r,z)]dt^2-\exp[-2\psi(r,z)]\{\exp[2\gamma(r,z)]\times(dr^2+dz^2)+r^2\,d\phi^2\}.$$

$$z^1=r\exp(-\psi)$$

$$z^2=\exp(\psi)\cos t$$

$$z^3=\exp(\psi)\sin t$$

$$z^4=\exp(\gamma-\psi)\cosh z$$

$$z^5=\exp(\gamma-\psi)\sinh z$$

$$z^6=\exp(\gamma-\psi)\cos r$$

$$z^7=\exp(\gamma-\psi)\sin r$$

$$z^8=r\exp(-\psi)\cos\phi$$

$$z^9=r\exp(-\psi)\sin\phi$$

$$z^{10}=\exp(\psi).$$

2. Cylindrical gravitational wave line element.[16]

$$\exp[-2\psi(\rho,t)]\{\exp[2\gamma(\rho,t)](dt^2-d\rho^2)-\rho^2\,d\phi^2\}-\exp[2\psi(\rho,t)]\,dz^2.$$

$$z^1=\rho\exp(-\psi)$$

[16] N. Rosen, Bull. Res. Council Israel **3**, 328 (1954).

$z^2 = \exp(\psi)$

$z^3 = \exp(\gamma - \psi) \cos t$

$z^4 = \exp(\gamma - \psi) \sin t$

$z^5 = \exp(\gamma - \psi) \cos \rho$

$z^6 = \exp(\gamma - \psi) \sin \rho$

$z^7 = \rho \exp(-\psi) \cos \phi$

$z^8 = \rho \exp(-\psi) \sin \phi$

$z^9 = \exp(\psi) \cos z$

$z^{10} = \exp(\psi) \sin z.$

3. "Anti Mach" model.[17]

$-(dx^1)^2 + 4x^4 dx^1 dx^3 - 2dx^2\, dx^3 - 2(x^4)^2 (dx^3)^2 - (dx^4)^2.$

$z^1 = x^4$

$z^2 = 2\,\mathrm{Re}\,(\sqrt{x^4}) \cos \tfrac{1}{2}(x^1 + x^3) + 2\,\mathrm{Im}\,(\sqrt{x^4}) \cos \tfrac{1}{2}(x^1 - x^3)$

$z^3 = 2\,\mathrm{Re}\,(\sqrt{x^4}) \sin \tfrac{1}{2}(x^1 + x^3) + 2\,\mathrm{Im}\,(\sqrt{x^4}) \sin \tfrac{1}{2}(x^1 - x^3)$

$z^4 = (x^2 - x^3)/\sqrt{2}$

$z^5 = (x^2 + x^3)/\sqrt{2}$

$z^6 = \sqrt{2} x^4 \cos x^3$

$z^7 = \sqrt{2} x^4 \sin x^3$

$z^8 = 2\,\mathrm{Re}\,(\sqrt{x^4}) \cos \tfrac{1}{2}(x^1 - x^3) + 2\,\mathrm{Im}\,(\sqrt{x^4}) \cos \tfrac{1}{2}(x^1 + x^3)$

$z^9 = 2\,\mathrm{Re}\,(\sqrt{x^4}) \sin \tfrac{1}{2}(x^1 - x^3) + 2\,\mathrm{Im}\,(\sqrt{x^4}) \sin \tfrac{1}{2}(x^1 + x^3)$

$z^{10} = x^1.$

I. (10, 5+5−)

1. Gödel model.[8]

$a^2\{[dx^0 + \exp(x^1) dx^2]^2 - (dx^1)^2 - \tfrac{1}{2}\exp(2x^1)(dx^2)^2 - (dx^3)^2\}$, $a = \text{const.}$

$z^1 = ax^0$

$z^2 = (a/\sqrt{2}) \exp(x^1) \cos x^2$

$z^3 = (a/\sqrt{2}) \exp(x^1) \sin x^2$

$z^4 = a\sqrt{2} \exp(\tfrac{1}{2}x^1) \cos \tfrac{1}{2}(x^0 + x^2)$

$z^5 = a\sqrt{2} \exp(\tfrac{1}{2}x^1) \sin \tfrac{1}{2}(x^0 + x^2)$

$z^6 = ax^1$

$z^7 = ax^3$

$z^8 = (a/\sqrt{2}) \exp(x^1)$

$z^9 = a\sqrt{2} \exp(\tfrac{1}{2}x^1) \cos \tfrac{1}{2}(x^0 - x^2)$

$z^{10} = a\sqrt{2} \exp(\tfrac{1}{2}x^1) \sin \tfrac{1}{2}(x^0 - x^2).$

J. Special Cases

1. Degenerate static vacuum field, class $A3$.[12]

$dt^2/z - z dz^2 - z^2(dr^2 + r^2\, d\phi^2)$, $0 \le r < \infty$, $\phi \bmod 2\pi$, $0 < z < \infty$.

a. (6, 2+4−).

$z^1 = \tfrac{1}{2} r^2 z - \tfrac{1}{4} z^2 - 1/16z^2 + \tfrac{1}{2} z$

$z^2 = \sinh t/\sqrt{z}$

$z^3 = \cosh t/\sqrt{z}$

$z^4 = zr \cos \phi$

$z^5 = zr \sin \phi$

$z^6 = \tfrac{1}{2} r^2 z - \tfrac{1}{4} z^2 - 1/16z^2 - \tfrac{1}{2} z.$

b. (6, 3+3−).

$z^1 = \cos t/\sqrt{z}$

$z^2 = \sin t/\sqrt{z}$

$z^3 = \tfrac{1}{2} r^2 z - \tfrac{1}{4} z^2 + 1/16z^2 + \tfrac{1}{2} z$

$z^4 = \tfrac{1}{2} r^2 z - \tfrac{1}{4} z^2 + 1/16z^2 - \tfrac{1}{2} z$

$z^5 = zr \cos \phi$

$z^6 = zr \sin \phi.$

2. Degenerate static vacuum field, class $B3$.[12]

$-d\phi^2/z - z dz^2 - z^2(dr^2 - r^2\, dt^2)$; $0 < z, r < \infty$.

a. (6, 2+4−).

$z^1 = \tfrac{1}{2} r^2 z - \tfrac{1}{4} z^2 - 1/16z^2 + \tfrac{1}{2} z$

$z^2 = zr \sinh t$

$z^3 = zr \cosh t$

$z^4 = \cos \phi/\sqrt{z}$

$z^5 = \sin \phi/\sqrt{z}$

$z^6 = \tfrac{1}{2} r^2 z - \tfrac{1}{4} z^2 - 1/16z^2 - \tfrac{1}{2} z.$

b. (6, 3+3−)

$z^1 = zr \sinh t$

$z^2 = \tfrac{1}{2} r^2 z - \tfrac{1}{4} z^2 + 1/16z^2 + \tfrac{1}{2} z$

$z^3 = \cosh \phi/\sqrt{z}$

$z^4 = \sinh \phi/\sqrt{z}$

$z^5 = \tfrac{1}{2} r^2 z - \tfrac{1}{4} z^2 + 1/16z^2 - \tfrac{1}{2} z$

$z^6 = zr \cosh t.$

---

[17] I. Ozsváth and E. Schücking in Ref. 10, p. 339.

3. Static extension of above.[12]

$-d\phi^2/z - z dz^2 - z^2(dx^2 - d\tau^2),\ 0<z<\infty.$

a. (6, 2+4−).

$z^1 = z\tau$

$z^2 = \frac{1}{2}(\tau^2 - x^2 + \frac{1}{2}z + 1/8z^3 - 1)z$

$z^3 = \frac{1}{2}(\tau^2 - x^2 + \frac{1}{2}z + 1/8z^3 + 1)z$

$z^4 = \cos\phi/\sqrt{z}$

$z^5 = \sin\phi/\sqrt{z}$

$z^6 = zx.$

b. (6, 3+3−).

$z^1 = z\tau$

$z^2 = \frac{1}{2}(\tau^2 - x^2 + \frac{1}{2}z - 1/8z^3 - 1)z$

$z^3 = \cosh\phi/\sqrt{z}$

$z^4 = \sinh\phi/\sqrt{z}$

$z^5 = \frac{1}{2}(\tau^2 - x^2 + \frac{1}{2}z - 1/8z^3 + 1)z$

$z^6 = zx.$

4. Petrov space T_2, group G_6.[14]

$2dx^1dx^4 - \sin^2 x^4(dx^2)^2 - \sinh^2 x^4(dx^3)^2.$

a. (6, 1+5−).

$z^1 = x^1 + \frac{1}{8}\sin 2x^4 + \frac{1}{8}\sinh 2x^4 + x^4$

$z^2 = \sin x^4 \cos x^2$

$z^3 = \sin x^4 \sin x^2$

$z^4 = \sinh x^4 \cos x^3$

$z^5 = \sinh x^4 \sin x^3$

$z^6 = x^1 + \frac{1}{8}\sin 2x^4 + \frac{1}{8}\sinh 2x^4.$

b. (6, 2+4−).

$z^1 = x^1 - \frac{1}{8}\sin 2x^4 + \frac{1}{8}\sinh 2x^4 + \frac{1}{2}x^4$

$z^2 = \sin x^4 \cosh x^2$

$z^3 = \sin x^4 \sinh x^2$

$z^4 = \sinh x^4 \cos x^3$

$z^5 = \sinh x^4 \sin x^3$

$z^6 = x^1 - \frac{1}{8}\sin 2x^4 + \frac{1}{8}\sinh 2x^4 - \frac{1}{2}x^4.$

c. (6, 3+3−).

$z^1 = x^1 - \frac{1}{8}\sin 2x^4 - \frac{1}{8}\sinh 2x^4$

$z^2 = \sin x^4 \cosh x^2$

$z^3 = \sinh x^4 \cosh x^3$

$z^4 = \sin x^4 \sinh x^2$

$z^5 = \sinh x^4 \sinh x^3$

$z^6 = x^1 - \frac{1}{8}\sin 2x^4 - \frac{1}{8}\sinh 2x^4 - x^4.$

5. Petrov space T_1, group G_4, metric 3.[14]

$-(kx^3+1)^2(dx^1)^2 - (kx^3+1)(dx^2)^2 - (dx^3)^2/(kx^3+1)$
$+(kx^3+1)^2(dx^4)^2,\ k = \text{const}.$

a. (6, 2+4−): $kx^3+1>0$.

$z^1 = (kx^3+1)x^4$

$z^2 = \frac{1}{2}k(kx^3+1)[(x^1)^2 - (x^4)^2] - \frac{1}{2}(1/k - k/4)\ln(kx^3+1) + \frac{1}{2}x^3$

$z^3 = \frac{1}{2}k(kx^3+1)[(x^1)^2 - (x^4)^2] - \frac{1}{2}(1/k - k/4)\ln(kx^3+1) - \frac{1}{2}x^3$

$z^4 = (kx^3+1)^{\frac{1}{2}}\cos x^2$

$z^5 = (kx^3+1)^{\frac{1}{2}}\sin x^2$

$z^6 = (kx^3+1)x^1.$

b. (6, 3+3−): $kx^3+1>0$ or $kx^3+1<0$.

$z^1 = (kx^3+1)x^4$

$z^2 = \frac{1}{2}k(kx^3+1)[(x^1)^2 - (x^4)^2] - \frac{1}{2}(1/k + k/4)\ln|kx^3+1| + \frac{1}{2}x^3$

$z^3 = \mathrm{Re}[(kx^3+1)^{\frac{1}{2}}]\cosh x^2 + \mathrm{Im}[(kx^3+1)^{\frac{1}{2}}]\sinh x^2$

$z^4 = \mathrm{Re}[(kx^3+1)^{\frac{1}{2}}]\sinh x^2 + \mathrm{Im}[(kx^3+1)^{\frac{1}{2}}]\cosh x^2$

$z^5 = \frac{1}{2}k(kx^3+1)[(x^1)^2 - (x^4)^2] - \frac{1}{2}(1/k + k/4)\ln|kx^3+1| - \frac{1}{2}x^3$

$z^6 = (kx^3+1)x^1.$

c. (6, 4+2−): $kx^3+1<0$.

$z^1 = (kx^3+1)x^4$

$z^2 = |kx^3+1|^{\frac{1}{2}}\cos x^2$

$z^3 = |kx^3+1|^{\frac{1}{2}}\sin x^2$

$z^4 = \frac{1}{2}k(kx^3+1)[(x^1)^2 - (x^4)^2] - \frac{1}{2}(1/k - k/4)\ln|kx^3+1| + \frac{1}{2}x^3$

$z^5 = \frac{1}{2}k(kx^3+1)[(x^1)^2 - (x^4)^2] - \frac{1}{2}(1/k - k/4)\ln|kx^3+1| - \frac{1}{2}x^3$

$z^6 = (kx^3+1)x^1.$

6. Petrov space T_1, group G_4, metric 1.[14]

$-(y^2+g)^2(dy^1)^2 - (y^2+g)(dy^2)^2/(y^2+1)$
$-(y^2+1)(dy^3)^2/(y^2+g) + (y^2+g)^2\cos^2 y^1(dy^4)^2,$
$g = \text{const}.$

a. $(4, 1+3-): g=1.$

$$z^1=(y^2+1)\cos y^1 \sinh y^4$$
$$z^2=(y^2+1)\cos y^1 \cosh y^4$$
$$z^3=(y^2+1)\sin y^1$$
$$z^4=y^3.$$

b. $(6, 2+4-): g<1, y^2<-1; g<1, -g<y^2; 1<g, y^2<-g;$ or $1<g, -1<y^2.$

c. $(6, 3+3-): g<1, -1<y^2<-g;$ or $1<g, -g<y^2<-1.$

The transformations for b and c are summarized as follows: $ds^2=(dz^1)^2-(dz^2)^2-(dz^3)^2\pm(dz^4)^2\pm(dz^5)^2 \pm(dz^6)^2.$

$$z^1=(y^2+g)\cos y^1 \sinh y^4$$
$$z^2=(y^2+g)\cos y^1 \cosh y^4$$
$$z^3=(y^2+g)\sin y^1$$
$$z^4=|(y^2+1)/(y^2+g)|^{\frac{1}{2}}\times\begin{cases}\cos y^3\\ \cosh y^3\end{cases}$$
$$z^5=|(y^2+1)/(y^2+g)|^{\frac{1}{2}}\times\begin{cases}\sin y^3\\ \sinh y^3\end{cases}$$
$$z^6=\int\left|\frac{(1-g)[4(y^2+g)^3\pm(1-g)]}{4(y^2+1)(y^2+g)^3}\right|^{\frac{1}{2}}dy^2.$$

If g	and y^2,	use for			with signs of		
		z^4	z^5	z^6	$(dz^4)^2$	$(dz^5)^2$	$(dz^6)^2$
$g<1$	$y^2<-1$	cosh	sinh	$-$	$+$	$-$	$-$
	$-1<y^2<-g$	cosh	sinh	$-$	$-$	$+$	$+$
	$-g<y^2$	cos	sin	$+$	$-$	$-$	$+$
$1<g$	$y^2<-g$	cos	sin	$+$	$-$	$-$	$+$
	$-g<y^2<-1$	cosh	sinh	$-$	$-$	$+$	$+$
	$-1<y^2$	cosh	sinh	$-$	$+$	$-$	$-$

7. Petrov space T_2, group G_4, metric 2.[14]

$$-(y^1+g)(dy^1)^2/(y^1+1)-(y^1+g)^2(dy^2)^2 -(y^1+g)^2\sin^2 y^2(dy^3)^2+(y^1+1)(dy^4)^2/(y^1+g),$$
$g=$const.

a. $(4, 1+3-): g=1.$

$$z^1=y^4$$
$$z^2=(y^1+g)\cos y^2$$
$$z^3=(y^1+g)\sin y^2\cos y^3$$
$$z^4=(y^1+g)\sin y^2\sin y^3.$$

b. $(6, 1+5-): g<1, -1<y^1<-g;$ or $1<g, -g<y^1<-1.$

c. $(6, 2+4-): g<1, y^1<-1; g<1, -g<y^1; 1<g, {}^1<-g;$ or $1<g, -1<y^1.$

The transformations for b and c are summarized as follows: $ds^2=-(dz^1)^2-(dz^2)^2-(dz^3)^2\pm(dz^4)^2\pm(dz^5)^2 \pm(dz^6)^2.$

$$z^1=(y^1+g)\cos y^2$$
$$z^2=(y^1+g)\sin y^2\cos y^3$$
$$z^3=(y^1+g)\sin y^2\sin y^3$$
$$z^4=|(y^1+1)/(y^1+g)|^{\frac{1}{2}}\times\begin{cases}\cos y^4\\ \cosh y^4\end{cases}$$
$$z^5=|(y^1+1)/(y^1+g)|^{\frac{1}{2}}\times\begin{cases}\sin y^4\\ \sinh y^4\end{cases}$$
$$z^6=\int\left|\frac{(1-g)[4(y^1+g)^3\pm(1-g)]}{4(y^1+1)(y^1+g)^3}\right|^{\frac{1}{2}}dy^1.$$

If g	and y^1,	use for			with signs of		
		z^4	z^5	z^6	$(dz^4)^2$	$(dz^5)^2$	$(dz^6)^2$
$g<1$	$y^1<-1$	cos	sin	$-$	$+$	$+$	$-$
	$-1<y^1<-g$	cos	sin	$-$	$-$	$-$	$+$
	$-g<y^1$	cosh	sinh	$+$	$-$	$+$	$+$
$1<g$	$y^1<-g$	cosh	sinh	$+$	$-$	$+$	$+$
	$-g<y^1<-1$	cos	sin	$-$	$-$	$-$	$+$
	$-1<y^1$	cos	sin	$-$	$+$	$+$	$-$

8. Normal hyperbolic space admitting a constant null bivector (plane-fronted gravitational waves).[12,18] $-dx^2-dy^2-2dudv-2H(x, y, u)\, du^2$.

a. $(8, 3+5-)$.

$z^1=\text{Im}\,[(2H)^{\frac{1}{2}}]\sin u$

$z^2=\text{Re}\,[(2H)^{\frac{1}{2}}]+\text{Im}\,[(2H)^{\frac{1}{2}}]\cos u$

$z^3=(u-v)/\sqrt{2}$

$z^4=(u+v)/\sqrt{2}$

$z^5=x$

$z^6=y$

$z^7=\text{Re}\,[(2H)^{\frac{1}{2}}]\cos u+\text{Im}\,[(2H)^{\frac{1}{2}}]$

$z^8=\text{Re}\,[(2H)^{\frac{1}{2}}]\sin u.$

b. $(7, 2+5-): H\geq 0$.

c. $(7, 3+4-): H\leq 0$.

9. Robinson–Trautman metric.[19]

$C(u, v, y, z)\,dv^2+2dudv-u^2(dy^2+dz^2)/Q^2(v, y, z)$.

a. $(10, 4+6-)$.

$z^1=\text{Re}\,(\sqrt{C})\cos v$

$z^2=\text{Re}\,(\sqrt{C})\sin v+\text{Im}\,(\sqrt{C})$

$z^3=(u+v)/\sqrt{2}$

$z^4=u\cosh z/Q$

$z^5=u\sinh z/Q$

$z^6=u\cos y/Q$

$z^7=u\sin y/Q$

$z^8=(u-v)/\sqrt{2}$

$z^9=\text{Re}(\sqrt{C})+\text{Im}\,(\sqrt{C})\cos v$

$z^{10}=\text{Im}\,(\sqrt{C})\sin v.$

b. $(9, 3+6-): C\leq 0$.

c. $(9, 4+5-): C\geq 0$.

10. Petrov space T_3, group G_2.[14]

$-\exp(x^2)[\exp(-2x^4)(dx^1)^2+(dx^2)^2]-2dx^3\,dx^4$
$+x^2[x^3+\exp(x^2)](dx^4)^2$.

a. $(10, 4+6-)$.

$z^1=\text{Re}\,(\sqrt{x})\cos x^4$

$z^2=\text{Re}\,(\sqrt{x})\sin x^4+\text{Im}\,(\sqrt{x})$

$z^3=\exp(\frac{1}{2}x^2-x^4)$

$z^4=(x^3-x^4)/\sqrt{2}$

$z^5=(x^3+x^4)/\sqrt{2}$

$z^6=\exp(\frac{1}{2}x^2-x^4)\cos x^1$

$z^7=\exp(\frac{1}{2}x^2-x^4)\sin x^1$

$z^8=2\exp(\frac{1}{2}x^2)$

$z^9=\text{Re}\,(\sqrt{x})+\text{Im}\,(\sqrt{x})\cos x^4$

$z^{10}=\text{Im}\,(\sqrt{x})\sin x^4$

where $x\equiv x^2[x^3+\exp(x^2)]$.

b. $(9, 3+6-): x\leq 0$.

c. $(9, 4+5-): x\geq 0$.

[18] F. A. E. Pirani in Ref. 10, p. 89.
[19] I. Robinson and A. Trautman, Proc. Roy. Soc. (London) **A265**, 463 (1962).

11. Petrov space T_2, group G_5.[14]

$2dx^1dx^4+\alpha(x^4)(dx^2)^2+2\beta(x^4)dx^2\,dx^3+\gamma(x^4)(dx^3)^2$; where α, β, γ satisfy a certain equation; $\alpha, \beta, \gamma \leq 0$ is assumed.

a. $(10, 3+7-)$.

$$z^1=(-2\beta)^{\frac{1}{2}}\cos\tfrac{1}{2}(x^2-x^3)$$
$$z^2=(-2\beta)^{\frac{1}{2}}\sin\tfrac{1}{2}(x^2-x^3)$$
$$z^3=x^1-\frac{1}{8}\int[(\alpha')^2/\alpha+(\gamma')^2/\gamma]\,dx^4+\tfrac{1}{2}x^4$$
$$z^4=x^1-\frac{1}{8}\int[(\alpha')^2/\alpha+(\gamma')^2/\gamma]\,dx^4-\tfrac{1}{2}x^4$$
$$z^5=(-\alpha)^{\frac{1}{2}}\cos x^2$$
$$z^6=(-\alpha)^{\frac{1}{2}}\sin x^2$$
$$z^7=(-\gamma)^{\frac{1}{2}}\cos x^3$$
$$z^8=(-\gamma)^{\frac{1}{2}}\sin x^3$$
$$z^9=(-2\beta)^{\frac{1}{2}}\cos\tfrac{1}{2}(x^2+x^3)$$
$$z^{10}=(-2\beta)^{\frac{1}{2}}\sin\tfrac{1}{2}(x^2+x^3).$$

b. $(10, 4+6-)$.

$$z^1=(-2\beta)^{\frac{1}{2}}\cos\tfrac{1}{2}(x^2-x^3)$$
$$z^2=(-2\beta)^{\frac{1}{2}}\sin\tfrac{1}{2}(x^2-x^3)$$
$$z^3=x^1+\frac{1}{8}\int[(\alpha')^2/\alpha-(\gamma')^2/\gamma]\,dx^4+\tfrac{1}{2}x^4$$
$$z^4=(-\alpha)^{\frac{1}{2}}\cosh x^2$$
$$z^5=(-\alpha)^{\frac{1}{2}}\sinh x^2$$
$$z^6=x^1+\frac{1}{8}\int[(\alpha')^2/\alpha-(\gamma')^2/\gamma]\,dx^4-\tfrac{1}{2}x^4$$
$$z^7=(-\gamma)^{\frac{1}{2}}\cos x^3$$
$$z^8=(-\gamma)^{\frac{1}{2}}\sin x^3$$
$$z^9=(-2\beta)^{\frac{1}{2}}\cos\tfrac{1}{2}(x^2+x^3)$$
$$z^{10}=(-2\beta)^{\frac{1}{2}}\sin\tfrac{1}{2}(x^2+x^3).$$

c. $(10, 5+5-)$.

$$z^1=(-2\beta)^{\frac{1}{2}}\cos\tfrac{1}{2}(x^2-x^3)$$
$$z^2=(-2\beta)^{\frac{1}{2}}\sin\tfrac{1}{2}(x^2-x^3)$$
$$z^3=x^1+\frac{1}{8}\int[(\alpha')^2/\alpha+(\gamma')^2/\gamma]\,dx^4+\tfrac{1}{2}x^4$$
$$z^4=(-\alpha)^{\frac{1}{2}}\cosh x^2$$
$$z^5=(-\gamma)^{\frac{1}{2}}\cosh x^3$$
$$z^6=(-\alpha)^{\frac{1}{2}}\sinh x^2$$
$$z^7=(-\gamma)^{\frac{1}{2}}\sinh x^3$$
$$z^8=x^1+\frac{1}{8}\int[(\alpha')^2/\alpha+(\gamma')^2/\gamma]\,dx^4-\tfrac{1}{2}x^4$$
$$z^9=(-2\beta)^{\frac{1}{2}}\cos\tfrac{1}{2}(x^2+x^3)$$
$$z^{10}=(-2\beta)^{\frac{1}{2}}\sin\tfrac{1}{2}(x^2+x^3).$$

REMARKS

For many of the Riemannian spaces listed above, more than one transformation gave the same embedding space, the additional transformation or transformations usually being similar to the one shown. In any case the signs of the pseudo-Euclidean coordinates are arbitrary and arbitrary constants may be added to them.

The possibility that the extra dimensions obtained by embedding the physical Riemannian space may be a source of elementary particle symmetries, has been discussed elsewhere.[20–23] The present study was motivated by such ideas, since their verification depends upon the symmetry structure admitted by these dimensions.

ACKNOWLEDGMENTS

The hospitality of the Israel Atomic Energy Commission, the Southwest Center for Advanced Studies, and the University of Texas is gratefully acknowledged.

APPENDIX

The following table should be useful as an aid in embedding Riemannian spaces. It contains embedding transformations for expressions appearing in Riemannian line elements.

[20] C. Fronsdal, Nuovo Cimento **13**, 988 (1959).
[21] D. W. Joseph, Phys. Rev. **126**, 319 (1962).
[22] N. Rosen, J. Rosen, and Y. Ne'eman, *Proceedings of the 1964 Coral Gables Conference on Symmetry Principles at High Energy* (W. H. Freeman and Company, San Francisco, 1964), p. 93.
[23] Y. Ne'eman and J. Rosen, Ann. Phys. (to be published).

Original form	Transformed form	Transformation[a]
$f^2(x)\,dx^2$	dp^2	$p=\int f(x)\,dx$
$dx\,dy$	dp^2-dq^2	$p=\frac{1}{2}(x+y),\ q=\frac{1}{2}(x-y)$
$u^2\,dx^2$	$dp^2+dq^2-du^2$ or $-dr^2+ds^2+du^2$	$p=u\cos x,\ q=u\sin x$; $r=u\cosh x,\ s=u\sinh x$
$u^2(dx^2+dy^2)$	$dp^2+dq^2-dr^2+ds^2$	$p=u\cos x,\ q=u\sin x,\ r=u\cosh y,\ s=u\sinh y$ or $p=u\cos y,\ q=u\sin y,\ r=u\cosh x,\ s=u\sinh x$
$u^2(dx^2-dy^2)$	$dp^2+dq^2-dr^2-ds^2$	$p=u\cos x,\ q=u\sin x,\ r=u\cos y,\ s=u\sin y$ or $p=u\sinh x,\ q=u\cosh y,\ r=u\cosh x,\ s=u\sinh y$
$u^2dx\,dy$	$dp^2+dq^2-dr^2-ds^2$	$p=u\cos\frac{1}{2}(x+y),\ q=u\sin\frac{1}{2}(x+y),\ r=u\cos\frac{1}{2}(x-y),\ s=u\sin\frac{1}{2}(x-y)$ or $p=u\sinh\frac{1}{2}(x+y),\ q=u\cosh\frac{1}{2}(x-y),\ r=u\cosh\frac{1}{2}(x+y),\ s=u\sinh\frac{1}{2}(x-y)$
$dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2)$	$dx^2+dy^2+dz^2$	$x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta$
$dr^2+r^2(d\theta^2-\sin^2\theta\,d\phi^2)$	$dx^2-dy^2+dz^2$	$x=r\sin\theta\cosh\phi,\ y=r\sin\theta\sinh\phi,\ z=r\cos\theta$
$dr^2-r^2(d\theta^2+\sinh^2\theta\,d\phi^2)$	$-dx^2-dy^2+dz^2$	$x=r\sinh\theta\cos\phi,\ y=r\sinh\theta\sin\phi,\ z=r\cosh\theta$
$dr^2-r^2(d\theta^2-\sinh^2\theta\,d\phi^2)$	$-dx^2+dy^2+dz^2$	$x=r\sinh\theta\cosh\phi,\ y=r\sinh\theta\sinh\phi,\ z=r\cosh\theta$
$dr^2+r^2(d\theta^2+\cos^2\theta\,d\phi^2)$	$dx^2+dy^2+dz^2$	$x=r\cos\theta\cos\phi,\ y=r\cos\theta\sin\phi,\ z=r\sin\theta$
$dr^2+r^2(d\theta^2-\cos^2\theta\,d\phi^2)$	$dx^2-dy^2+dz^2$	$x=r\cos\theta\cosh\phi,\ y=r\cos\theta\sinh\phi,\ z=r\sin\theta$
$dr^2-r^2(d\theta^2+\cosh^2\theta\,d\phi^2)$	$dx^2-dy^2-dz^2$	$x=r\cosh\theta\cosh\phi,\ y=r\cosh\theta\sinh\phi,\ z=r\sinh\theta$
$dr^2-r^2(d\theta^2-\cosh^2\theta\,d\phi^2)$	$dx^2+dy^2-dz^2$	$x=r\cosh\theta\cos\phi,\ y=r\cosh\theta\sin\phi,\ z=r\sinh\theta$

[a] The signs of the transformed coordinates are arbitrary, and arbitrary constants may be added.