Seminar on the Embedding Problem*

Introduction

In the 1925 preface to his textbook on "Riemannian Geometry," L. P. Eisenhart was noting that, "The recent physical interpretation of intrinsic differential geometry of spaces has stimulated the study of this subject." In such a context, he devoted a large part of the book to the description of Riemann spaces embedded in flat spaces with added dimensionality. It turned out, however, that general relativity was handy enough when treated in terms of the curvilinear coordinates of the Riemann space itself; the embedding seemed to add extraneous spatial extensions which added little to the understanding of gravitation and the cosmology. From time to time, some interesting results would be derived in this way, but they would also be directly derivable from the Riemannian metric, interest in the embedding thus subsiding again. An example of this sort was provided by C. Fronsdal's study of the complete Schwarzschild solution via its embedding [Phys. Rev. **116**, 778 (1958)]; the remarkable insight into that problem did not give rise to an interest in the embedding method, since the same result was achieved shortly afterward by M. D. Kruskal [Phys. Rev. **119**, 1743 (1960)] without using it.

The theory of elementary-particle physics was developed in terms of a flat space and its symmetry properties—the Lorentz group. It is not surprising, therefore, that attempts to describe properties of particles in curved space–time should lead to a renewed interest in the embedding method. The applications vary, from the development of additional techniques in handling the Lorentz group, to attempts to identify the "internal" symmetries of particle physics with the symmetries of the normal, "extraneous," piece of the embedding space.

We have deemed it worthwhile to convene some physicists and mathematicians interested in this subject to a discussion of the method and its application. The following papers formed the nucleus of a seminar, which, we hope, will prove fruitful to both relativity and particle physicists, and may provide some mathematicians with the motivation to complete the study of those differential geometry aspects which are extremely mysterious at this stage.

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Isometric Embedding of Riemannian Manifolds into Euclidean Spaces

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This is a review of differential geometry results pertaining to the problem of embedding curved space–time in a pseudo-Euclidean space.

I. INTRODUCTION

Recent work in elementary-particle physics ties up internal symmetries of elementary particles, under nonstrong interaction, with symmetries of generalized space–time curvature. The mathematical aspect of the problem is then to obtain information on the embedding class of various four-dimensional relativistic metrics; the embedding considered is isometric and smooth, and is either local or global.

In the mathematical literature there are numerous classical results on local (isometric) embeddings, but only a few which may be of help in determining the embedding class for the present metrics. As for global embedding, there are several fairly recent results, but they only apply to positive definite metrics.

In this talk I shall survey most of the known results concerning global embeddings, as well as some results concerning local embeddings which might be useful for relativistic metrics.

II. GLOBAL ISOMETRIC EMBEDDING

In this section we consider isometric embedding of a Riemannian manifold V_n of dimension n into a Euclidean space E_m of dimension m with metric $ds^2=dx_1^2+\cdots dx_m^2$. We say that V_n is of class $C^p(p\geq 1)$ if the local coordinates of V_n are related to each other by p times continuously differentiable functions. The metric of V_n is said to be of class C^p if, in local coordinates, the metric tensor has $p-1$ times continuously differentiable components. Finally, if the manifold V_n is of class C^p and if its metric is of class C^p, then we say that V_n is a C^p Riemannian manifold. An embedding is said to be of class C^p if the functions defining it are p times continuously differentiable.

For an embedding of class C^p, $p\leq 2$, the Riemann curvature tensor in the image manifold is not defined and, in fact, may not exist (since its definition involves the first two derivatives of the metric tensor). This is why it is natural to consider only embeddings of class C^p with $p\geq 3$. However, I will mention some results concerning C^1 imbedding for the reason that the dimension of the embedding (or enveloping) space is surprisingly small.

THEOREM 1. *Any compact C^1 Riemannian manifold V_n (with or without boundary) has a C^1 isometric embedding in E_{2n}. Any noncompact C^1 Riemannian manifold has a C^1 isometric embedding in E_{2n+1}.*

Actually a more general result is valid. Before stating it, let us introduce two concepts. A *short embedding* is an embedding which, at each point, does not increase the line element. A point P in the embedding space belongs to the *limit set* of the embedding of V_n if and only if there is a divergent sequence in V_n whose image converges to P.

THEOREM 2. *If a compact C^1 Riemannian manifold V_n (with or without a boundary) has a C^1 embedding in E_k where $k\geq n+1$, then it has also a C^1 isometric embedding in E_k. If a noncompact C^1 Riemannian manifold V_n has a C^1 short embedding in E_k, $k\geq n+1$, which does not meet its limit set, then it has also a C^1 isometric embedding in E_k.*

In view of well-known embedding theorems of Whitney, the assumptions of Theorem 2 always hold if $k=2n$, $k=2n+1$ in the compact and noncompact cases respectively. Thus Theorem 1 follows from Theorem 2.

Theorem 1 is due to Nash.[1] Theorem 2 was proved by Nash[1] in the weaker form $k\geq n+2$ and, in its present form, by Kuiper.[2] From Theorem 2 one deduces:

COROLLARY 1. *For any point of a C^1 Riemannian manifold V_n there is a neighborhood which has a C^1 isometric embedding in E_{n+1}.*

COROLLARY 2. *The flat n-torus (i.e., the metric product of n circles) has a C^1 isometric embedding in E_{n+1}.*

COROLLARY 3. *The hyperbolic space H_n [i.e., E_n provided with the metric $ds^2=\frac{4}{3}(d\rho^2+\varphi(\rho)d\sigma^2)$, where $d\sigma$ is the Euclidean surface element on the unit sphere of E_n, ρ is the radial distance and $\varphi(\rho)=$ (sinh $c\rho)^2/c^2$] has a C^1 isometric embedding in E_{n+1}.*

We turn to imbeddings of class C^p, $p\geq 3$.

THEOREM 3. *Any compact C^p Riemannian manifold ($p\geq 3$) has a C^p isometric embedding into E_m, where $m=\frac{1}{2}n(3n+11)$. The same result holds for noncompact manifolds, but with $m=\frac{1}{2}n(n+1)(3n+11)$.*

The theorem is due to Nash.[3] The bound on m seems much too high, but so far there has been no progress in decreasing this bound for the general case. In the special case of the hyperbolic space H_n, Blansula[4] found, by explicit formulas, a C^∞ isometric embedding in E_{6n-5} if $n>2$ and in E_6 if $n=2$.

We now state a result in the converse direction, i.e., in the direction of giving a lower bound on m.

THEOREM 4. *Let a compact C^4 Riemannian manifold V_n have the following property: At each point of V_n there is a q plane such that all the sectional curvatures of all the 2 planes in it are ≤ 0. Then V_n does not have a C^4 isometric embedding into any E_m with $m\leq n+q-1$.*

Thus, in particular, the flat n torus has no C^4 isometric embedding in E_{2n-1} (it has, of course, C^∞ isometric embedding in E_{2n}). Compare this result with Corollary 2.

Theorem 4 was proved by Chern and Kuiper[5] in case $q=2,3$ and by Ōtsuki[6] for general q. The special case where V_n is flat was previously established by Tompkins.[7] Ōtsuki[8,9] obtained other related results. Thus, he gave an example in case $n=q=2$ of a compact surface with everywhere negative Gaussian curvature which can be isometrically embedded in E_m with $m=n+q=4$. He also showed that if the condition on the curvature (in Theorem 4) is assumed only at one point, with $q=n$, then there is no C^4 isometric embedding in E_{2n-2}.

In the proofs of all the previous theorems, the positivity of the metric plays an indispensable role and, in fact, the proofs break down if the metric is indefinite. We shall illustrate this just in the case of Theorem 4. The proof here is based on the following geometric idea:

Suppose V_n is a submanifold of E_m and take 0 to be a fixed point in E_m. As a point P varies on V_n, the Euclidean distance $\overline{OP}$ achieves a maximum at some point P_0. At P_0 the manifold must then be "concave toward 0" and this will impose some "positivity condition" on the Riemann curvature tensor at P_0. In order for this condition not to contradict the assumption of negative sectional curvatures at P_0, the dimension m must be sufficiently large (i.e., $m\geq n+q-1$).

It is clear that in the case of indefinite metric, even if one could find a local maximum for $\overline{OP}$, attained at

[1] J. Nash, Ann. Math. **60**, 383 (1954).
[2] N. H. Kuiper, Ned. Akad. Wetensh. Proc. Ser. A58=Indig. Math. **17**, 546, 683 (1955).
[3] J. Nash, Ann. Math. **63**, 20 (1955).
[4] D. Blansula, Monatsh. Math. **59**, 217 (1955).
[5] S. S. Chern and N. H. Kuiper, Ann. Math. **56**, 422 (1952).
[6] T. Ōtsuki, Proc. Japan Akad. **29**, 99 (1953).
[7] C. Tompkins, Duke Math. J. **5**, 58 (1939).
[8] T. Ōtsuki, Math. J. Okayama Univ. **3**, 95 (1954).
[9] T. Ōtsuki, Math. J. Okayama Univ. **5**, 95 (1956).

some point P_0, the manifold still need not be "concave" at P_0 as before.

III. LOCAL ISOMETRIC EMBEDDING

In this section, all embeddings are local.

THEOREM 5. *Any Riemannian manifold V_n with analytic positive definite metric can be analytically and isometrically embedded in E_m, where $m=\frac{1}{2}n(n+1)$.*

The theorem is due to Janet,[10] Cartan,[11] and Burstin.[12] We next consider V_n with indefinite metric. The notation $V_n(p,q)$ indicates that the tensor metric has p positive and q negative eigenvalues $(p+q=n)$. The Euclidean space with metric $ds^2=dx_1^2+\cdots+dx_p^2-dx_{p+1}^2-\cdots-dx_n^2$ is denoted by $E_n(p,q)$ where $q=n-p$. We finally set $V_n(n,0)=V_n$.

THEOREM 6. *Any Riemannian manifold $V_n(p,q)$ with analytic metric can be analytically and isometrically embedded in $E_m(r,s)$ where $m=\frac{1}{2}n(n+1)$ and where r,s are any prescribed integers satisfying: $r\geq p$, $s\geq q$.*

The theorem is due to Friedman.[13]

From now on all the embeddings are isometric and sufficiently smooth. It is useful to introduce the following concepts:

Let k_0 be the smallest nonnegative integer such that $V_n(p,q)$ can be embedded in $E_{n+k_0}(p,q+k_0)$. For each k, $0\leq k\leq k_0$, we define the kth *embedding class* of $V_n(p,q)$ to be the smallest number N_k such that $V_n(p,q)$ has an embedding in $E_{n+Nk}(p+a_k,\ q+k)$ (where $a_k+k=N_k$). The *embedding class* of $V_n(p,\ q)$ is defined to be $\min_{0\leq k\leq k_0} N_k$.

According to Theorem 6, $N_k\leq\frac{1}{2}n(n-1)$ for all k.

FUNDAMENTAL PROBLEM. *Given $V_n(p,q)$, determine the N_k.*

For many special relativistic metrics $(n=4)$ upper bounds (less than 6) on N_k are known. For a recent treatment and references, see Rosen.[14] Fujitana, Ikeda, and Matsumoto[15] also considered embedding of generalized Schwarzschild fields in $E_6(p,q)$.

To determine whether or not $N_k=0$ one simply checks whether or not the Riemann curvature tensor vanishes identically (see, for instance, Ref. 16).

[10] M. Janet, Ann. Soc. Polon. Math. **5,** 38 (1926).

[11] É. Cartan, Ann. Soc. Polon. Math. **6,** 1 (1927).

[12] C. Burstin, Rec. Math. Moscou (Math. Sbornik) **38,** 74 (1931).

[13] A. Friedman, J. Math. Mech. **10,** 625 (1961).

[14] J. Rosen, Ph.D. thesis, Hebrew University, Jerusalem, 1964 (to be published).

[15] T. Fujitana, M. Ikeda, and M. Matsumoto, J. Math. Kyoto Univ. **1**, 43, 63, 255 (1961/62).

[16] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, New Jersey, 1949).

For definite metrics Thomas[17] and Rosenskon[18] gave an algebraic criterion to determine whether or not the imbedding class is 1. This criterion involves rather lengthy calculations (evaluation of a fairly large number of determinants). Their work extends with minor changes to the case of indefinite metrics. Thus, there is always a sure (though lengthy) way to determine whether or not $N_k=1$. There are however instances where some fairly simple necessary (or sufficient) conditions can be used to check whether or not $N_k=1$. We state a result of Schouten and Struik[19] which involves a necessary condition.

THEOREM 7. *If the Ricci tensor of $V_n(p,q)$ is zero, then $N_k\neq 1$.*

Certain extensions of the work of Thomas[17] to imbedding classes >1 were given by Allendoerfer.[20] He considered only V_n, but his result undoubtedly extends to $V_n(p,q)$. He proved the following theorem:

THEOREM 8. *If at each point in V_n the first normal space is of dimension q and the type is ≥ 3, then V_n can be embedded in E_{n+q}.*

The first normal space is defined as follows: Suppose V_n is already embedded in some $E_{n+p}(p>q)$ [this is always true if $p=\frac{1}{2}n(n-1)$] and is given by $y^i=y^i(x^\alpha)$. Then the first normal space is the vector space generated by the vectors $Y_{\alpha\beta}$ with components

$$Y_{\alpha\beta}{}^i=(\partial^2y^i/\partial x^\alpha\partial x^\beta)-\Gamma_{\alpha\beta}{}^\gamma(\partial y^i/\partial x^\gamma).$$

The definition of type is quite involved. The type is always $\leq[n/q]$. Since, in relativity, $n=4$, Theorem 8 can only be applied in case $q=1$. Thus its usefulness may be only in giving a sufficient condition for the embedding class to be equal to 1.

We shall conclude with an example where $N_k\neq N_h$ for $k\neq h$. First we recall the following fact (see Ref. 16):

If $V_n(p,q)$ has a constant curvature then its embedding class is 1, i.e., there is a space $E_{n+1}(r,s)$ (with $r\geq p$, $s\geq q$) such that $V_n(p,q)$ is (locally) isometric to a portion of a hypersphere in $E_{n+1}(r,s)$. Thus, $N=\min_{0\leq k\leq k_0}N_k=1$.

Take now V_n with constant negative curvature. As proved by Lieber,[21] V_n can be embedded in E_{2n-1} but cannot be imbedded in E_{2n-2}, i.e., $N_0=n-1$ whereas $N=1$.

[17] T. Y. Thomas, Acta Math. **67,** 169 (1936).

[18] N. A. Rosenskon, Izvest. Akad. Nauk SSSR **7,** 253 (1943).

[19] J. A. Schouten and D. J. Struik, Am. J. Math. **43,** 213 (1921).

[20] C. B. Allendoerfer, Am. J. Math. **61,** 633 (1939).

[21] E. Lieber, Doklady Akad. Nauk SSSR **55,** 291 (1947).