Possible Polarization Experiments in Fermion–Fermion Scattering as a Test of Invariance Principles

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The word "fermion" here means "spin-$\frac{1}{2}$ fermion." Discussion is restricted to processes in which two arbitrary incoming fermions are scattered into two arbitrary fermions (not necessarily the same) in the final state. Invariance under space inversion (P_σ), time-reversal (P_T) invariance, and charge-conjugation (P_C) invariance is not assumed to hold necessarily. Formulas are derived which can be applied to any polarization experiment, including correlation experiments. Necessary conditions are obtained which can be tested experimentally to check whether a certain invariance principle holds in nature. In addition to P_σ, P_T, P_C invariance and their combinations, certain helicity invariance principles (related to γ^5 invariance principles) are discussed. The treatment is relativistic. By a suitable choice of rest frames for each particle, the appearance of some trigonometric functions of the scattering angle is avoided, and σ matrices can be used instead of γ matrices. Experiments in which the two incoming fermions are uncorrelated and one of them is unpolarized, are treated in great detail. It is shown to what extent such experiments can determine the S matrix. A graphical method is introduced which can often be used to simplify discussion of multiple scattering experiments. Fermion–spin-zero-boson scattering is treated as a special case of fermion–fermion scattering. Those consequences of invariance principles which can be checked most easily in general polarization experiments are discussed.

INTRODUCTION

The present paper has three aims. First, we derive some well-known results concerning polarization experiments[1,2] in a form in which their essential simplicity becomes transparent. Most of the unessential complications due to space and velocity transformations do not appear. Second, more general expressions are given, which are valid whether or not some of the usually assumed invariance principles hold. They can be applied to any polarization experiment, including correlation experiments. From these general expressions, the previously mentioned ones follow as special cases. Finally, we derive necessary conditions, which can be checked experimentally to test whether a certain invariance principle holds in a given reaction. Space-parity conservation, time-reversal invariance, and charge-conjugation invariance (if the *CPT* theorem holds) are discussed. In addition, certain helicity invariance principles are defined, and their observable effects described.

In this paper the word "fermion" is used to mean "spin-$\frac{1}{2}$ fermion." We restrict the discussion to processes in which there are two arbitrary incoming fermions which are scattered into two arbitrary fermions (and no other particles) in the final state. The two fermions in the final state may be different from those in the initial state, i.e., the scattering need not be elastic. We call such processes simply "fermion–fermion scattering" and denote them by "f–f scattering." Space-parity conservation, charge-conjugation invariance and time-reversal invariance are not assumed to hold necessarily. Energy–momentum conservation as well as invariance under rotations in three-space are assumed.

In the first section the definition of certain coordinate frames and some questions of notations are discussed. In Sec. 2, formulas are derived which express the results of all polarization experiments in terms of the α matrix, which is simply related to the S matrix. In Sec. 3 the effect of certain invariance principles on the α matrix is studied. A special class of experiments, experiments with polarized beams, is defined and discussed in detail in Sec. 4. In this section formulas are derived which are the generalizations of the well-known ones in the literature.[1,2] These formulas are valid, whether or not any of the usual invariance principles are assumed to hold. The experimentally observable effects of the different invariance principles on experiments with polarized beams are listed systematically. They can be used to check on the validity of these invariance principles. A matrix $\mathbf{A}$ is introduced, and it is shown that performing all possible experiments with polarized beams one can completely determine the $\mathbf{A}$ matrix, but nothing more. The relation between the $\mathbf{A}$ and α matrices is discussed. A graphical method is described which can often be used to simplify the discussion of multiple scattering experiments with polarized beams. Fermion–spin-zero-boson scattering is treated as a special case. Experiments with polarized targets are discussed in Sec. 5. General polarization experiments, including correlation experiments, are treated in Sec. 6.

1. NOTATION AND DEFINITIONS

In any scattering we call one of the incoming particles and one of the outgoing particles "beam particle,"

[1] L. Wolfenstein, Phys. Rev. **96**, 1654 (1954). L. Wolfenstein, Ann. Rev. Nucl. Sci. **6**, 43 (1956).
[2] R. Oehme, Phys. Rev. **98**, 147 (1955).

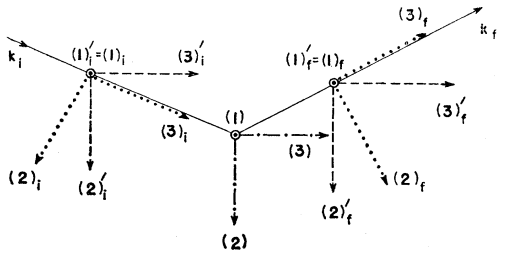

FIG. 1. The figure represents schematically the relation of certain frames discussed in Sec. 1.

the other one the "target particle." The incoming beam (target) particle may be different from the outgoing beam (target) particle if the scattering is inelastic. If one of the incoming particles is at rest in the laboratory, we call that one the incoming target particle, otherwise the choice is arbitrary. The following abbreviations are used: i particle for incoming beam particle, f particle for outgoing ("final") beam particle, r particle for incoming target particle, and s particle for outgoing target particle.

Certain rest frames are defined next. In order to do this we first define a frame K in which the center of mass of the incoming particles is at rest. The three space axes of this frame : (1), (2), (3) are defined by the relations

$$\begin{aligned}(1)&=\hat{\mathbf{k}}_i\times\hat{\mathbf{k}}_f/(\sin\theta),\\(2)&=(\hat{\mathbf{k}}_i-\hat{\mathbf{k}}_f)/(2\sin\tfrac{1}{2}\theta),\\(3)&=(\hat{\mathbf{k}}_i-\hat{\mathbf{k}}_f)/(2\cos\tfrac{1}{2}\theta),\end{aligned}\qquad(1)$$

and $\cos\theta=\hat{\mathbf{k}}_i\cdot\hat{\mathbf{k}}_f$.

Here $\hat{\mathbf{k}}_i$ and $\hat{\mathbf{k}}_f$ are unit three-vectors pointing along the three-momentum of the i particle and f particle, respectively, as measured in any frame in which the center of mass of the incoming particles is at rest. A set of frames K'_t ($t=i, f, r, s$) is obtained from K by parallel translation along the velocity of the tth particle with the velocity of the particle as seen from K. This means that K'_t is a rest frame of the tth particle. Here we have assumed that the rest mass of the particle is different from zero. The three space axes of K'_t are denoted by $(1)'_t$, $(2)'_t$, $(3)'_t$. Finally, the set of frames K_t is obtained from K'_t by a space rotation around the first axis of K'_t in a manner such that the third axis of K_t points opposite to the direction of motion of K as viewed from K'_t. The three space axes of K_t are denoted by $(1)_t$, $(2)_t$, $(3)_t$.

If the rest mass of the t particle is zero, then K'_t is defined to be the frame which is obtained from K by a parallel translation with arbitrary speed along the velocity of the particles as seen from K. The frame K_t is then obtained from K'_t by a rotation around their common first axis in such a manner that the third axis of K_t points in the direction of motion of the particle as seen from K.

The relation of all these frames to each other is represented schematically in Fig. 1.

In the rest of this paper only the frames K_t are used. We call K_t briefly "the" rest frame of the t particle[3] ($t=i, f, r, s$).

The definition of the frames K_t may be somewhat cumbersome, but for our purposes they have distinct advantages. We are interested in polarization phenomena, counting form factors, etc.; momentum vectors of particles are irrelevant from this point of view. By using four distinct rest frames for each scattering we avoid the necessity of relativistically transforming polarizations to a single standard frame. This enables us to use 2×2 Pauli matrices instead of 4×4 Dirac matrices in a relativistic treatment. By using frames whose third axes are always parallel to the velocity of the respective particles we avoid the use of trigonometric functions of scattering angles in many expressions. For example, the third component of polarization in any of the frames K_t is always the longitudinal component of polarization. This would certainly not be true in general. In the frame K, for instance, one would have to use the direction cosines of the velocity of the particle to express the longitudinal component of polarization in terms of the three components each of which is parallel to one of the space axes of K. All the well-known, and from our point of view unessential, parameters of the scattering are contained in the definitions of the K and are thus separated out.[4] They appear only when giving the velocities and orientations of the K_t.

In this paper the polarization vectors of all particles are always given in their respective rest frames, unless explicitly stated otherwise. When we say, for example, that the kth component of polarization of the i particle is P_i^k, we mean the kth component of this polarization in K_i.

To simplify writing we introduce the following notation. The equation

$$\mathbf{P}_t+\mathbf{P}_{t'}=\mathbf{P}_{t''}\qquad(2)$$

means, by definition, that the kth component ($k=1$, 2, 3) of the vector $\mathbf{P}_{t''}$ (as measured in $K_{t''}$) is equal to the kth component of $\mathbf{P}_{t'}$ (as measured in $K_{t'}$) plus the kth component of $\mathbf{P}_t$ (as measured in K_t). Here any of the t, t', t'' can mean i, f, r, or s. Of course, if $K_t=K_{t'}=K_{t''}$, then Eq. (2) simply says that the vector

[3] As the energy of a particle in K_n increases, the transverse component of its polarization measured in K_n approaches zero, but not the transverse component of its polarization as measured in its own rest frame. How this transverse component of polarization will affect the results of experiments is a question of dynamics, and therefore its discussion does not fall within the scope of this work. It is not unreasonable to expect that its detection will become more difficult as the energy increases.

[4] An analogous situation arises in quantum mechanics. There the cumbersome definition of the interaction representation has the advantage of "separating out" the well-known changes caused by the unperturbed Hamiltonian. This information, which is unessential for purposes of studying the perturbing Hamiltonian, is transferred into the definition of the interaction representation.

$\mathbf{P}_{t''}$ is the sum, in the usual sense, of the vectors $\mathbf{P}_t$ and $\mathbf{P}_{t'}$. The last statement is false if $K_t=K_{t'}=K_{t''}$ does not hold. Equation (2) should be regarded as shorthand notation. No deep meaning need be attached to it.

The usual density-matrix formalism[1,2] is employed when describing the polarization states of particles. The polarization state of a collection of i particles, all moving with the same velocity, is characterized by a 2×2 density matrix ρ_i. In general the polarization state of a collection of t particles ($t=i, f, r, s$, as usual), all moving with the same velocity, is characterized by a 2×2 matrix ρ_t. The kth component of polarization of the t particle is defined to be the expectation value per particle of the operator $\sigma_t{}^k$. It is given by the expression

$$P_t{}^k=\mathrm{Tr}\ (\sigma_t{}^k\cdot\rho_t)/N_t, \tag{3a}$$

where

$$N_t=\mathrm{Tr}\ (\rho_t)=\mathrm{Tr}\ (\sigma_t{}^0\cdot\rho_t)\equiv P_t{}^0 \tag{3b}$$

is the expectation value of the operator $\sigma_t{}^0$. We normalize so that it gives the density, the number of t particles per unit volume. The $\sigma_t{}^k$ ($k=1, 2, 3$) are the Pauli matrices. To simplify writing we introduced in Eq. (3b) the notation $\sigma_t{}^0$ for the unit 2×2 matrix (for all t). According to our convention the $P_t{}^k$ is measured in K_t. This implies that $\sigma_t{}^\mu$ ($\mu=0, 1, 2, 3$) and ρ_t are 2×2 matrices operating in the coordinate space given by the axes of K_t. In principle we could choose a different representation of the Pauli matrices in each K_t frame, but we shall agree to choose the same representation for them in all frames. To be specific, we may choose (for any t)

$$\sigma_t{}^1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\equiv\sigma^1,\qquad \sigma_t{}^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}\equiv\sigma^2,$$
$$\sigma_t{}^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\equiv\sigma^3,\qquad \sigma_t{}^0=\begin{pmatrix}1&0\\0&1\end{pmatrix}\equiv\sigma^0. \tag{4}$$

Once the representation of $\sigma_t{}^\mu$ is fixed, the representation of ρ_t is determined.[2]

The advantages of choosing this representation can be illustrated by giving an example. We find, for instance, that a purely forward longitudinal polarization (as viewed from K) is always given in spin space by a spinor $\binom{1}{0}$ whatever the value of t may be.

The polarization state of the two incoming particles at a given energy and c.m. scattering angle (c.m. means center-of-mass system), is described by the 4×4 density matrix ρ_{ir}. This matrix is a sum of terms, each of which is a direct product[5] of two 2×2 matrices. The first 2×2 matrix in each product operates in the spin space of the i particle, the second in the spin space of the r particle. Any such matrix can be expanded in the form

$$\rho_{ir}=\sum_{\mu,\nu=0}^{3}\tfrac{1}{4}P_{ir}{}^{\mu\nu}\sigma_i{}^\mu\otimes\sigma_r{}^\nu. \tag{5}$$

Here the symbol $\otimes$ signifies direct multiplication.[6] The coefficient $P_{ir}{}^{\mu\nu}$ is simply the expectation value of the operator $\sigma_i{}^\mu\otimes\sigma_r{}^\nu$. Similarly the density matrix describing the polarization state of the two outgoing particles ρ_{fs} can be written as

$$\rho_{fs}=\sum_{\xi,\eta=0}^{3}\tfrac{1}{4}P_{fs}{}^{\xi\eta}\sigma_f{}^\xi\otimes\sigma_s{}^\eta. \tag{6}$$

We use the expression "polarization state" of the i particle to mean density as well as the polarization vector, i.e., all four of the numbers $P_i{}^\mu$ ($\mu=0, 1, 2, 3$) as defined in Eq. (3). By "polarization state of two incoming particles" we mean their densities, their polarizations, and correlations between these quantities, i.e., all 16 coefficients in Eq. (5). Similar statements hold for the outgoing particles.

The coefficient $P_{ir}{}^{00}$ is equal to the product of the density of the i particles with the density of the r particles. The equation $P_{ir}{}^{00}=P_i{}^0P_r{}^0$ is true in general. The quantity $P_{ir}{}^{k0}/P_{ir}{}^{00}$ with $k=1, 2, 3$ is equal to $P_i{}^k$, the kth component of polarization of the i particles (we have assumed $P_{ir}{}^{00}\neq0$). If $P_{ir}{}^{00}\neq0$, and $P_{ir}{}^{k0}=0$ for all k, then we say that the i particles are *unpolarized*. Similarly, if $P_{ir}{}^{00}\neq0$ and $P_{ir}{}^{0h}=0$ for all $h=1, 2, 3$, then we say that the r particles are *unpolarized*. The quantity $P_{ir}{}^{kh}/P_{ir}{}^{00}\equiv C_{ir}{}^{kh}$ with $k, h=1, 2, 3$ gives the correlation between the kth component of polarization of the i particle and the hth component of polarization of the r particle. The equation $P_{ir}{}^{kh}/P_{ir}{}^{00}=P_i{}^k\cdot P_r{}^h$ is *not* true in general. If it is true for all k and h, then we say that the i particle and the r particle are *uncorrelated*, because in this case the correlation between $P_i{}^k$ and $P_r{}^h$ is exactly what is expected statistically. If the i particles and the r particles are uncorrelated, then the equation $\rho_{ir}=\rho_i\cdot\rho_r$ holds (but not otherwise). The i particles and the r particles may be correlated even if they are both unpolarized. The coefficients appearing in Eq. (6) can be discussed similarly.

We define the T matrix as

$$T=S-1,$$

where S is the S matrix of the scattering. The T matrix is a sum of terms. Each term is proportional to a direct product of two 2×2 matrices, one of which takes a state of the i particle over into a state of the f particle; the other takes a state of the r particle over

[5] For a discussion of direct multiplication of matrices, see for example, E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* (Academic Press Inc., New York, 1959), p. 17.

[6] It follows from the rules of direct multiplication of two matrices that when taking the transpose, the hermitean conjugate etc. of a direct product of two matrices, the order of the matrices is *not* exchanged. Therefore our ordering convention is not affected by these operations.

into a state of the s particle. Any such matrix can be expanded as

$$T=\sum_{\beta,\gamma=0}^{3}\alpha_{\beta\gamma}\sigma_{fi}{}^{\beta}\otimes\sigma_{sr}{}^{\gamma}, \tag{7}$$

where the rows of $\sigma_{fi}{}^{\mu}$ ($\mu=0$, 1, 2, 3) are labeled by spinor indices describing the spin state of a particle in K_f, and its columns are labeled by spinor indices referring to the frame K_i. Thus the matrices $\sigma_{fi}{}^{\mu}$ take states of the i particle (given in K_i) over into states of the f particle (given in K_f). An analogous statement can be made about the matrices $\sigma_{sr}{}^{\mu}$. We choose the representation of the $\sigma_{fi}{}^{\mu}$ and $\sigma_{sr}{}^{\mu}$, in analogy to Eq. (4), to be

$$\sigma_{fi}{}^{0}=\sigma^{0}, \qquad \sigma_{sr}{}^{0}=\sigma^{0},$$
$$\sigma_{fi}{}^{1}=\sigma^{1}, \qquad \sigma_{sr}{}^{1}=\sigma^{1}, \quad \text{etc.} \tag{8}$$

Invariance under rotations in three-space and energy–momentum conservation requires that the coefficients $\alpha_{\beta\gamma}$ appearing in Eq. (7) depend only on the two independent scalars which can be formed from the three-momenta of the four particles involved in the scattering. These two scalars may be chosen as $\hat{\mathbf{k}}_i\cdot\hat{\mathbf{k}}_f$ and $\mathbf{k}_i\cdot\mathbf{k}_i$. We call that matrix, whose element belonging to the βth row and γth column is $\alpha_{\beta\gamma}$, the α matrix. A knowledge of α is equivalent to knowledge of T, and thus to knowledge of S.

In the rest of this paper we write all direct products of matrices in a manner such that the first matrix operates in the spin spaces of the beam particles, and the second one in the spin spaces of the target particles.[6] Using this convention, together with Eqs. (4) and (8), we may drop all the subscripts on all the σ^{μ} matrices in every expression. All polarization states are then given in the rest frame of the particle, and the T matrix connects states given in such frames. In the following all subscripts on σ^{μ} matrices in Eqs. (3), (5), (6), and (7) accordingly are dropped. With this notation the expressions we derive below assume familiar forms, similar to those encountered in nonrelativistic treatments.

We have chosen to expand the T matrix in the form given by Eqs. (7) and (8) partly because the restrictions imposed on α by the conservation laws we consider turn out to have a very simple form, as shown in the next section. There are of course other ways to decompose 2×2 matrices, by choosing any four linearly independent 2×2 matrices as a basic set. For example, we may choose a basic set so that it should be very simple to take the trace of any product of them. However, it turns out that in this case the restrictions imposed on α by the relevant conservation laws have a more complicated form. For this reason we prefer to use the set of the four σ^{μ} matrices.

Everywhere in this paper summation over repeated indices is *not* implied unless indicated explicitly by a summation sign.

2. GENERAL FORMALISM: THE Ω AND ω MATRICES

Equation (5) shows that the matrix ρ_{ir} contains 16 coefficients $P_{ir}{}^{\mu\nu}$ (μ, $\nu=0$, 1, 2, 3). Once all these are known for a certain c.m. energy and scattering angle, the polarization state of the incoming particles in a scattering at that c.m. energy and scattering angle is determined by them, and vice versa. Similarly, the knowledge of the 16 coefficients $P_{fs}{}^{\xi\eta}$ in Eq. (6) at all c.m. energies and scattering angles is equivalent to the knowledge of the polarization state of the outgoing particles.

Let us assume that ρ_{ir} and T are known. The ρ_{fs} matrix then can be determined from the well-known formula[2]

$$\rho_{fs}=T\rho_{ir}T^{+}. \tag{9}$$

Substituting Eqs. (5), (6), and (7) into Eq. (9), multiplying both sides by $(\sigma^{\xi}\otimes\sigma^{\eta})$ either from the right or from the left, and finally taking the trace in the spin spaces of both the target and the beam particles, we obtain

$$P_{fs}{}^{\xi\eta}=\sum_{\beta,\gamma,\mu,\nu,\epsilon,\kappa=0}^{3}\alpha_{\beta\gamma}\alpha_{\epsilon\kappa}{}^{*}P_{ir}{}^{\mu\nu}\,\mathrm{Tr}\,(\sigma^{\xi}\sigma^{\beta}\sigma^{\mu}\sigma^{\epsilon})\,\mathrm{Tr}\,(\sigma^{\eta}\sigma^{\gamma}\sigma^{\nu}\sigma^{\kappa}). \tag{10}$$

After having taken the trace of both sides in deriving Eq. (10), we have used the relation

$$\mathrm{Tr}\,[(\sigma^{\mu}\otimes\sigma^{\nu})(\sigma^{\mu'}\otimes\sigma^{\nu'})]=(\mathrm{Tr}\,\sigma^{\mu}\sigma^{\mu'})(\mathrm{Tr}\,\sigma^{\nu}\sigma^{\nu'})=4\delta_{\mu\mu'}\delta_{\nu\nu'}.$$

We rewrite Eq. (10) in the form

$$P_{fs}{}^{\xi\eta}=\sum_{\mu,\nu=0}^{3}\Omega_{\xi\eta,\mu\nu}P_{ir}{}^{\mu\nu}, \tag{11}$$

where elements of the Ω matrix are defined by

$$\Omega_{\xi\eta,\mu\nu}\equiv\sum_{\beta,\epsilon,\gamma,\kappa=0}^{3}\alpha_{\beta\gamma}\alpha_{\epsilon\kappa}{}^{*}\,\mathrm{Tr}\,(\sigma^{\xi}\sigma^{\beta}\sigma^{\mu}\sigma^{\epsilon})\,\mathrm{Tr}\,(\sigma^{\eta}\sigma^{\gamma}\sigma^{\nu}\sigma^{\kappa}). \tag{12}$$

All elements of the Ω matrix are real. This can be checked directly, but also follows from the fact that all $P_{ir}{}^{\mu\nu}$ and $P_{fs}{}^{\xi\eta}$ are expectation values of observables, and thus real.

Elements of the $\omega(\xi\eta,\,\mu\nu)$ matrix are defined by the relation

$$\begin{aligned}[\omega(\xi\eta,\,\mu\nu)]_{\beta\gamma,\epsilon\kappa}&\equiv\tfrac{1}{4}\,\mathrm{Tr}\,(\sigma^{\xi}\sigma^{\beta}\sigma^{\mu}\sigma^{\epsilon})\,\mathrm{Tr}\,(\sigma^{\eta}\sigma^{\gamma}\sigma^{\nu}\sigma^{\kappa})\\&=\{-\delta_{\xi\mu}\mathfrak{e}(\xi\,|\,0)\delta_{\beta\epsilon}\mathfrak{e}(\beta\,|\,0)+\delta_{\beta\mu}\delta_{\epsilon\xi}+\delta_{\beta\xi}\delta_{\epsilon\mu}\\&\quad+\sum_{\beta',\epsilon'=0}^{3}i\mathfrak{e}_{\xi\beta'\mu\epsilon'}\delta_{\beta\beta'}\delta_{\epsilon\epsilon'}\mathfrak{e}(0\,|\,\xi,\,\mu)\}\\&\quad\cdot\{-\delta_{\eta\nu}\mathfrak{e}(\eta\,|\,0)\delta_{\gamma\kappa}\mathfrak{e}(\gamma\,|\,0)+\delta_{\gamma\nu}\delta_{\kappa\eta}+\delta_{\gamma\eta}\delta_{\kappa\nu}\\&\quad+\sum_{\gamma',\kappa'=0}^{3}i\mathfrak{e}_{\eta\gamma'\nu\kappa'}\delta_{\gamma\gamma'}\delta_{\kappa\kappa'}\mathfrak{e}(0\,|\,\eta,\,\nu)\}.\end{aligned} \tag{13}$$

In Eq. (16) we have used the notation

$$\varepsilon(a, b, c, \cdots \mid e, f, g, \cdots; h, i, j, \cdots)$$

$$= \begin{cases} +1 \text{ if the set of numbers } \{a, b, c, \cdots\} \text{ is contained in the set of numbers } \{e, f, g, \cdots\}, \text{ or } \{h, i, j, \cdots\}, \\ -1 \text{ otherwise.} \end{cases} \quad (14)$$

The totally antisymmetric four-index tensor $\varepsilon_{\xi\eta\mu\nu}$ (where $\xi, \eta, \mu, \nu = 0, 1, 2, 3$) is defined so that $\varepsilon_{0123} = 1$.

Using the ω matrices we can rewrite Eq. (11) as

$$\Omega_{\xi\eta,\mu\nu} = \sum_{\beta,\gamma,\epsilon,\kappa=0}^{3} 4\alpha_{\beta\gamma}\alpha_{\epsilon\kappa}{}^{*}\omega(\xi\eta, \mu\nu)_{\beta\gamma,\epsilon\kappa}. \quad (15)$$

The Ω matrix depends on the c.m. energy and angle of the experiment to which it refers, because $\alpha_{\beta\gamma}$ which appear in Eq. (12) depend on these quantities.

The ω matrices are all independent of the parameters of the scattering. They form a complete set of 16×16 matrices. We denote the transpose of the $\omega(\xi\eta, \mu\nu)$ matrix by $\tilde{\omega}(\xi\eta, \mu\nu)$, and define it to be that matrix for which $[\tilde{\omega}(\xi\eta, \mu\nu)]_{\beta\gamma,\epsilon\kappa} = [\omega(\xi\eta, \mu\nu)]_{\epsilon\kappa,\beta\gamma}$ holds. Then all ω matrices are hermitean and unitary. In this sense they play a similar role among 16×16 matrices as the sixteen γ matrices play among 4×4 matrices, and the four σ^{μ} matrices among 2×2 matrices.

Generally speaking polarization experiments are those in which some of the $P_{ir}{}^{\mu\nu}$ and $P_{fs}{}^{\xi\eta}$ are measured at certain values of the c.m. energy and scattering angle. The purpose is to deduce from these data via Eq. (11) some information concerning the S matrix at those c.m. energies and scattering angles. Equation (11) shows that the most one can hope to deduce from experiments of this kind is the elements of the Ω matrix. Of course, as shown by Eq. (12), all 256 elements of the Ω matrix are completely determined by the sixteen complex elements of the α matrix, or equivalently, by the T matrix. The converse, however, is not true. The 256 elements of the Ω matrix do not suffice to determine the 16 complex elements of the α matrix uniquely. For example, one may multiply all elements of α by a common phase factor without changing Ω, which means that the α matrix can be determined only up to a phase factor. On the other hand, we know that α contains 16 complex elements, and therefore it depends on 32 real parameters. One of these real parameters, the over-all phase of α, certainly does not influence Ω. Thus all 256 (real) elements of Ω depend at most on 31 real parameters.

3. RESTRICTIONS IMPOSED BY INVARIANCE PRINCIPLES

If all 16 complex elements of the α matrix are independent, then for all values of the c.m. energy and scattering angle one needs 16 complex numbers to describe f–f scattering. The usual way of saying this is that there are 16 independent form factors, each of which is a function of the c.m. energy and angle.[7]

If the S matrix of the scattering process is such that it is invariant under certain operations, then the 16 complex elements of the α matrix are no longer arbitrary. In other words, fewer than 16 independent form factors may be needed to determine f–f scattering. The restrictions imposed upon the α matrix by certain invariance principles are summarized in this section. The detailed derivation of these results is given in Appendix 1.

We make use of a mathematical symbol which is defined in analogy to Eq. (14) as follows:

$$\delta(a, b, c, \cdots \mid e, f, g, \cdots; h, i, j, \cdots; \cdots)$$

$$= \begin{cases} +1 \text{ if the set of numbers } \{a, b, c, \cdots\} \text{ is contained in the set of numbers } \{e, f, g, \cdots\} \text{ or else it is contained in the set of numbers } \{h, i, j, \cdots\} \text{ etc.,} \\ 0 \text{ otherwise.} \end{cases} \quad (16)$$

For example $\delta(1, 2 \mid 1, 2) = 1$, $\delta(1, 2 \mid 2; 3; 4) = 0$, $\delta(1, 2 \mid 1; 2, 3; 2, 4) = 0$, $\delta(a \mid b) = \delta_{ab}$, the usual Kronecker delta.

In Appendix 1 it is shown that if the S matrix is invariant under space inversion (denoted by P_{σ}) and the product of intrinsic parities of the four particles (denoted by η_{σ}) is $+1$, then $\alpha_{\beta\gamma}$ must satisfy

$$\alpha_{\beta\gamma} = \alpha_{\beta\gamma}\delta(\beta, \gamma \mid 0, 1; 2, 3). \quad (17)$$

To satisfy this equation, the elements α_{02}, α_{03}, α_{12}, α_{13}, α_{20}, α_{21}, α_{30}, α_{31} have to be equal to zero. When $\eta_{\sigma} = -1$, then the other eight $\alpha_{\mu\nu}$ must be equal to zero. The nonzero elements can have arbitrary values. One concludes that P_{σ} invariance alone permits eight independent form factors in f–f scattering.

If time-reversal (denoted by P_T) invariance holds, then as shown in Appendix 1, $\alpha_{\beta\gamma}$ must satisfy

$$\alpha_{\beta\gamma} = \alpha_{\beta\gamma}\delta(\beta, \gamma \mid 0, 1, 3; 2), \quad (18)$$

where elements of α^T are the coefficients appearing in $T^T = S^T - 1$. The S^T is the S matrix which describes a scattering in which the f particle is the incoming beam particle with a momentum $-\mathbf{k}_i$ and polarization $-\mathbf{P}_i$ (reversed momentum and spin of the i particle in the original scattering), the i particle is the outgoing beam particle with momentum $-\mathbf{k}_f$, polarization $-\mathbf{P}_f$, and the r and s particles are similarly exchanged with respect to the original scattering. In short, the S^T matrix describes the (Wigner) time-reversed process corresponding to the original scattering, which is described by S.

The particle exchange operation P_{12} is defined as

[7] All form factors referring to a given c.m. energy *and* angle may be considered to be independent. However, unitarity requires that certain integral equations be satisfied by form factors referring to a given c.m. energy, the integral being taken over all angles. In the present paper we are not concerned with these integral equations.

that operation which exchanges the three-momenta and polarizations of the i particle and r particle; at the same time it exchanges the three-momenta and polarizations of the f and s particles. Thus the effect of P_{12} is:

$$\mathbf{k}_i \leftrightarrow \mathbf{k}_r, \qquad \mathbf{k}_f \leftrightarrow \mathbf{k}_s, \tag{19a}$$

$$\mathbf{P}_i \leftrightarrow \mathbf{P}_r, \qquad \mathbf{P}_f \leftrightarrow \mathbf{P}_s, \tag{19b}$$

where $\mathbf{k}_t$ and $\mathbf{P}_t$ are the three momentum and polarization vectors of the t particle ($t=i, f, r, s$). A necessary and sufficient condition for the S matrix to be P_{12} invariant is

$$\alpha_{\beta\gamma}=\alpha_{\gamma\beta}\varepsilon(\beta,\gamma\,|\,0,1;2,3), \tag{20}$$

where $\varepsilon(\beta,\gamma\,|\,0,1;2,3)$ is defined by Eq. (14).

The significance of P_{12} invariance is demonstrated by the following rule: Define $E_{ir}(E_{fs})$ to be that operation which exchanges all the observable physical quantities associated with the i and r particles (f and s particles) with the exception of their energy—momenta and polarizations. If $E_{ir}=E_{fs}\equiv E$ and S is invariant under $\mathcal{E}\cdot E$, where $\mathcal{E}$ is any operation, then S is $\mathcal{E}\cdot P_{12}$ invariant. The simplest special case of this rule is the restriction imposed on the S matrix by the Pauli principle: $E=1$, $\mathcal{E}=1$ and S is P_{12} invariant. Other special cases and a more complete discussion of the relation between P_{12} invariance, the Pauli principle, isotopic-spin invariance, G-parity invariance, etc., is given in Appendix 2.

Next we define certain helicity invariance principles. We say that the beam particle switches its helicity during the scattering if λ_i, the expectation value of the helicity of the i particle, and λ_f, the expectation value of the helicity of the f particle, are related by $\lambda_f=-\lambda_i$. Similarly, we say that the target particle switches sign during the scattering if $\lambda_s=-\lambda_r$.

We consider several distinct helicity invariance principles:

(a) Either both the beam and the target particle switch its helicity during the scattering, or else neither of them does.

(b) One of the scattered particles (either the beam particle or the target particle) switches its helicity during the scattering, but the other does not.

(c) Neither switches its helicity.

(d) Both switch helicity.

(e) The beam particle does not switch helicity, but the target particle does.

(f) The beam particle does switch helicity, but the target particle does not.

In the following we refer to these invariance principles as case (a), case (b), etc. Clearly case (c) and case (d) are subcases of case (a); case (e) and case (f) are subcases of case (b).

The interest in these invariance principles arises from the fact that at high energies they have to hold whenever the S matrix satisfies certain γ^5 invariance principles.[8–10] Cases (a) through (f) impose the following conditions on the α matrix:

$$\text{In case (a)}, \quad \alpha_{\beta\gamma}=\alpha_{\beta\gamma}\delta(\beta,\gamma\,|\,0,3;1,2). \tag{21a}$$

$$\text{In case (b)}, \quad \alpha_{\beta\gamma}=\alpha_{\beta\gamma}(1-\delta(\beta,\gamma\,|\,0,3;1,2)). \tag{21b}$$

$$\text{In case (c)}, \quad \alpha_{\beta\gamma}=\alpha_{\beta\gamma}\delta(\beta,\gamma\,|\,0,3). \tag{21c}$$

$$\text{In case (d)}, \quad \alpha_{\beta\gamma}=\alpha_{\beta\gamma}\delta(\beta,\gamma\,|\,1,2). \tag{21d}$$

$$\text{In case (e)}, \quad \alpha_{\beta\gamma}=\alpha_{\beta\gamma}\delta(\beta\,|\,0,3)\delta(\gamma\,|\,1,2). \tag{21e}$$

$$\text{In case (f)}, \quad \alpha_{\beta\gamma}=\alpha_{\beta\gamma}\delta(\beta\,|\,1,2)\delta(\gamma\,|\,0,3). \tag{21f}$$

All the results presented in this section can be easily visualized if one represents them schematically by pictures in the manner shown in Fig. 2. We refer to such pictures as schemes, and they are to be understood as follows. Rows are imagined to be labeled from zero to three from top to bottom, columns from left to right. A square in the scheme located in the ith row and kth column corresponds to α_{ik}. A shaded (unshaded) square indicates that the corresponding element of α may (may not) be different from zero. A dashed line along the diagonal represents a semisymmetric matrix. A semisymmetric 4×4 matrix is by definition one that satisfies Eq. (20).

We say that the S matrix is $P_a\cdot P_b$ invariant if it is invariant under the combined operation P_b followed by P_a. We say that the S matrix is P_a+P_b invariant if it is invariant under both P_a and P_b separately.

It is easy to find the scheme of α corresponding to an S matrix invariant under $P_a\cdot P_b$, where P_a and P_b are two operations appearing in Fig. 2. One has only to superimpose the scheme corresponding to P_a and that corresponding to P_b and note where a white square of one scheme overlaps with a shaded square of the other scheme. In this manner one finds the locations of all white squares in the scheme corresponding to a $P_a\cdot P_b$ invariant S matrix. For example, let us assume that the S matrix is invariant under P_σ followed by P_T, i.e., it is $P_\sigma\cdot P_T$ invariant. The scheme of the cor-

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[8] Let us go to the limit of infinite particle energies (where the rest mass is negligible). In case (a) the S matrix will then be invariant under the substitution $\Psi_t\to\gamma^5\Psi_t$ carried out simultaneously for all t. The subscript indicates that the field Ψ_t describes the t particle. The S matrix will then also be invariant under the substitution $\Psi_i\to\gamma^5\Psi_i$, $\Psi_f\to-\gamma^5\Psi_f$, $\Psi_r\to\gamma^5\Psi_r$, $\Psi_s\to-\gamma^5\Psi_s$. In case (c) the S matrix will be invariant under the substitution $\Psi_i\to\gamma^5\Psi_i$, $\Psi_f\to\gamma^5\Psi_f$ and also under the substitution $\Psi_r\to\gamma^5\Psi_r$, $\Psi_s\to\gamma^5\Psi_s$. In case (d) the S matrix will be invariant under the substitution $\Psi_i\to\gamma^5\Psi_i$, $\Psi_f\to-\gamma^5\Psi_f$ and also under $\Psi_r\to\gamma^5\Psi_r$, $\Psi_s\to-\gamma^5\Psi_s$. Analogous results hold for cases (b), (e), and (f).

[9] J. Tiomno, Nuovo Cimento **1**, 226 (1955); S. Hori and A. Wakasa, *ibid.* **6**, 304 (1957); J. J. Sakurai, *ibid.* **7**, 649 (1958). Y. Nambu, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics*, edited by J. Prentki (CERN, Geneva, 1962), p. 153.

[10] A. A. Logunov, V. A. Meshcheryakov, and A. N. Tavkhelidze, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics*, edited by J. Prentki (CERN, Geneva, 1962), p. 161.

responding α matrix, for elastic scattering, can be obtained by superimposing the first two schemes in Fig. 2 as described above. The result is shown in Fig. 3(a). Of course, if the CPT theorem holds, then such an α matrix corresponds to an S matrix which is invariant under charge conjugation (denoted by P_C). In this case α has ten independent complex elements. Thus the number of form factors permitted by $P_\sigma \cdot P_T$ invariance in elastic fermion–fermion scattering is ten.

The scheme of an α matrix describing a P_a+P_b invariant scattering can also be obtained by superimposing the scheme of a P_a invariant α matrix and the scheme of a P_b invariant α matrix. A white square appears in the resulting scheme wherever a white square appears in any one or both of the two original schemes. The rest of the squares are shaded. As an example, consider the case when S is $P_\sigma+P_T+P_{12}$ invariant. For elastic scattering the result is shown in Fig. 3(b). In this case the number of nonzero elements of α is six, but due to the semisymmetry of α only five are independent. This result implies that for example in the standard theory of nucleon–nucleon strong elastic scattering, where $P_\sigma+P_T+P_{12}$ invariance is assumed, there are five independent form factors.

4. EXPERIMENTS WITH POLARIZED BEAMS

A. Basic Formulas

We now turn to a special class of scattering experiments which satisfy the following two conditions. First, the target is unpolarized before the scattering and the

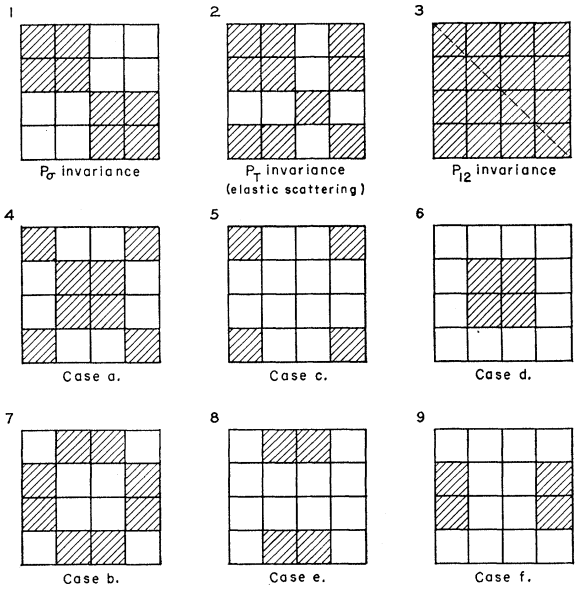

FIG. 2. Schemes representing the restrictions imposed on the α matrix by the invariance principles indicated. Schemes are discussed after Eq. (11). For scheme 1 we have assumed that $\eta_\sigma=1$.

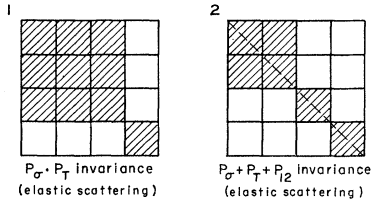

FIG. 3. Schemes representing the restrictions imposed on the α matrix by the invariance principles indicated. These schemes can be constructed by superimposing certain schemes appearing in Fig. 2.

polarizations of the beam and target particles are uncorrelated. This condition means that the coefficients appearing in Eq. (5) satisfy

$$P_{ir}{}^{\mu\nu}=P_{ir}{}^{\mu\nu}\cdot\delta_{\nu 0}. \tag{22}$$

Second, the polarization of the target particles after any scattering is not observed. This means that only those coefficients in Eq. (6) are measured which satisfy

$$P_{fs}{}^{\xi\eta}=P_{fs}{}^{\xi\eta}\cdot\delta_{\eta 0}. \tag{23}$$

Experiments which satisfy these two conditions will be called experiments with polarized beams.

If one performs all possible experiments with polarized beams, then one can determine from Eq. (11) all those elements of the Ω matrix for which the indices η and ν are both equal to zero. There are 16 such elements of the Ω matrix [they are real as explained after Eq. (15)]. The other 240 real elements of Ω can not be determined by such experiments.

How much can we say about the α matrix if we know all $\Omega_{\xi 0,\mu 0}$? To find the answer we first use Eq. (13) to write

$$[\omega(\xi 0,\,\mu 0)]_{\beta\gamma,\epsilon\kappa}=\{-\delta_{\xi\mu}\mathfrak{e}(\xi\mid 0)\,\delta_{\beta\epsilon}\mathfrak{e}(\beta\mid 0)+\delta_{\beta\mu}\delta_{\epsilon\xi}+\delta_{\beta\xi}\delta_{\epsilon\mu}+\sum_{\beta',\gamma'=0}^{3} i\mathfrak{e}_{\xi\beta'\mu\epsilon'}\delta_{\beta\beta'}\delta_{\epsilon\epsilon'}\mathfrak{e}(0\mid\xi,\,\mu)\}\cdot\delta_{\gamma\kappa}. \tag{24}$$

Substituting this into Eq. (15) we find

$$\begin{aligned}\Omega_{\xi 0,\mu 0}&=4\sum_{\gamma=0}^{3}\{-\delta_{\xi\mu}\mathfrak{e}(\xi\mid 0)(\alpha_{0\gamma}\alpha_{0\gamma}{}^{*}-\sum_{\beta=1}^{3}\alpha_{\beta\gamma}\alpha_{\beta\gamma}{}^{*}+\alpha_{\mu\nu}\alpha_{\xi\gamma}{}^{*}\\&\quad+\alpha_{\xi\gamma}\alpha_{\mu\gamma}{}^{*}+\sum_{\beta,\gamma=0}^{3} i\mathfrak{e}_{\xi\beta\mu\epsilon}\alpha_{\beta\gamma}\alpha_{\epsilon\gamma}{}^{*}\mathfrak{e}(0\mid\xi,\,\mu)\}\\&=8\{\delta_{\xi\mu}\mathfrak{e}(\xi\mid 0)\,(\tfrac{1}{2}\sum_{\beta=0}^{3}A_{\beta\beta}-A_{00})+\mathrm{Re}\,A_{\xi\mu}\\&\quad+\tfrac{1}{2}\sum_{\beta,\epsilon=0}^{3}\mathfrak{e}_{\xi\beta\mu\epsilon}\mathfrak{e}(0\mid\xi,\,\mu)\,\mathrm{Im}\,A_{\beta\epsilon},\end{aligned} \tag{25}$$

where the elements of the Hermitian matrix A are defined by

$$A_{\xi\mu}\equiv\sum_{\gamma=0}^{3}\alpha_{\xi\gamma}{}^{*}\alpha_{\mu\gamma}=(\alpha\alpha^{+})\tilde{}\,. \tag{26}$$

Equation (25) shows clearly that all elements $\Omega_{\xi 0,\mu 0}$ are completely determined by the elements of the **A** matrix. The converse is also true. The symmetric part of $\Omega_{\xi 0,\mu 0}$ determines Re (A_{ik}), and the antisymmetric part of $\Omega_{\xi 0,\mu 0}$ determines $(\text{Im } A)_{ik}{}^{D}$, the matrix $(\text{Im } \mathbf{A})^{D}$ being the dual of Im **A**. Thus by performing only experiments with polarized beams one can determine all elements A_{ik} but nothing more.

Once all elements A_{ik} are known, then the α matrix is determined up to an arbitrary unitary transformation. More precisely, we show that once **A** is known, and an α is found which satisfies Eq. (26), then the most general matrix α' which satisfies Eq. (26) is given by

$$\alpha'=\alpha\mathbf{U}, \tag{27}$$

where **U** is unitary.

The fact that the matrix α' given by Eq. (27) does indeed satisfy Eq. (26) is obvious. The fact that all α' matrices satisfying Eq. (26) are of the form given by Eq. (27) is also easily proven in a few lines of algebra. Nevertheless, we prefer to give another proof, which has the advantage of exhibiting a geometrical interpretation of Eq. (27). The 16 complex elements of α may be thought of as specifying the 16 complex components of four four-dimensional vectors (in "index space").

$$\alpha^{\beta}=(\alpha_{\beta 0},\ \alpha_{\beta 1},\ \alpha_{\beta 2},\ \alpha_{\beta 3}) \quad (\beta=0,\ 1,\ 2,\ 3). \tag{28}$$

The diagonal elements $A_{\gamma\gamma}$ $(\gamma=0,\ 1,\ 2,\ 3)$ of the Hermitian matrix **A** determine the "lengths" (i.e., absolute values) of the vectors defined by Eq. (28), while the off-diagonal elements of **A** determine the complex angles between any two of them. That is, they determine the relative orientations of the vectors α^{β} $(\beta=0,\ 1,\ 2,\ 3)$ with respect to each other. No other information is contained in **A**. If we rotate the system of four vectors α^{β} "rigidly," i.e. without changing the lengths or relative orientations of the vectors, then all elements of **A** remain unchanged. All such "rigid" rotations are described in a complex four-dimensional space by unitary matrices. This proves the assertion.

A remark is in order here to qualify the foregoing argument. If Det (**A**) is equal to zero, then by Eq. (26) Det (α) is also equal to zero and thus the characteristic[11] p of α is less than four. In this case a well-known theorem in the theory of matrices states that the four vectors α^{β} $(\beta=0,\ 1,\ 2,\ 3)$ do not span the whole four-dimensional index space, but only a p-dimensional subspace of it. To perform a rigid rotation in this subspace one need not have a transforming matrix which is unitary in the whole four-dimensional index space; one only needs one which transforms the subspace in question "rigidly," and may do anything to vectors perpendicular to this subspace. We may call such matrices "unitary from the p-dimensional subspace spanned by α^{β}" and denote them by $\mathbf{U}(p,\ \alpha^{\beta})$.

[11] See, for example, F. G. Tricomi, *Integral Equations* (Interscience Publishers, Inc., New York, 1957), p. 59.

It is clear that any $\alpha''=\alpha\mathbf{U}(p,\ \alpha^{\beta})$ will satisfy Eq. (26) if α does. However, any such α'' can be written in the form of Eq. (27), thus the original assertion is correct.

If certain invariance principles hold, then the **U** matrix is no longer an arbitrary unitary matrix. In this case **U** has to be such that if α satisfies the restrictions due to the invariance principle in question, then so does α'. It is quite easy to find the restrictions imposed on **U** by the invariance principles discussed in Sec. 3, because of the simple structure of the schemes of α. These restrictions on **U** are so simple that they too can be expressed in terms of schemes. For example, assume P_{σ} invariance and $\eta_{\sigma}=1$, then α has to have the first of the schemes shown in Fig. 2. The matrix α' has the same scheme if and only if the unitary matrix **U** also has this scheme. We found the simple result, that if P_{σ} invariance holds, then the scheme of **U** must be the same as the scheme of α. The same result holds for elastic scattering when P_{σ}, $P_{\sigma}\cdot P_T$, or $P_{\sigma}+P_T$ invariance holds, and for the helicity invariance principles cases (a), (c), and (d). For the helicity invariance principles cases (b), (e), and (f), the scheme of **U** has to be the same as the scheme of **U** (or α) in cases (a), (d), and (c), respectively.

We denote the differential cross section (summed over all polarizations) evaluated at a certain c.m. energy and angle, by $d\sigma$, where $d\sigma$ is a function of the c.m. energy and angle. If the i particles are unpolarized, we denote the differential cross section by $d\sigma(uu)$. The two subscripts remind us that the i particles and the r particles are unpolarized. (At the same time, as everywhere in this section, the incoming beam and target polarizations are uncorrelated.) By definition $d\sigma$ is the expectation value of the operator $\sigma^0\otimes\sigma^0$ after the scattering has taken place. We find from Eqs. (11) and (25) that

$$d\sigma(uu)=P_{fs}{}^{00}/P_{ir}{}^{00}=\Omega_{00,00}$$
$$=A_{00}+A_{11}+A_{22}+A_{33}=\text{Tr } \mathbf{A}. \tag{29}$$

If the i particles are polarized, then $P_{ir}{}^{\mu 0}$ may be nonzero for any value of μ. The differential cross section is denoted in this case by $d\sigma(pu)$ and is given by

$$d\sigma(pu)=P_{fs}{}^{00}/P_{fs}{}^{00}=\sum_{\mu=0}^{3}\Omega_{00,\mu 0}P_{ir}{}^{\mu 0}P_{ir}{}^{00}$$
$$=d\sigma(uu)[1+(\mathbf{U}-\mathbf{V})\mathbf{P}_i], \tag{30}$$

where the three-vector $\mathbf{P}_i$ is defined to be the vector whose kth component $(k=1,\ 2,\ 3)$ is $P_{ir}{}^{k0}/P_{ir}{}^{00}$; it is the polarization of the i particle. The kth component $(k=1,\ 2,\ 3)$ of the three-vectors **U** and **V**, are defined by

$$U^k\equiv(2\Omega_{00,00})^{-1}(\Omega_{00,k0}+\Omega_{k0,00})=(2/\text{Tr } \mathbf{A})\ \text{Re } A_{0k},$$
$$(k=1,\ 2,\ 3) \tag{31a}$$

$$V^k\equiv(2\Omega_{00,00})^{-1}(-\Omega_{00,k0}+\Omega_{k0,00})$$
$$=(2/\text{Tr } \mathbf{A})\tfrac{1}{2}\sum_{\beta,\epsilon=0}^{3}\varepsilon_{0k\beta\epsilon}\ \text{Im } A_{\beta\epsilon}. \tag{31b}$$

We denote the polarization of the f particle by $P_f(uu)$ if the i particles are unpolarized, and by $P_f(pu)$ if the i particles are polarized. The kth ($k=1, 2, 3$) component of this quantity is always equal to $P_{fs}{}^{k0}/P_{fs}{}^{00}$ ($k=1, 2, 3$), and from Eqs. (11), (25), (29), (30), and (31) we find that it can be written as

$$P_f{}^k(uu)=P_{fs}{}^{k0}/[P_{ir}{}^{00}d\sigma(uu)]=\Omega_{k0,00}/\Omega_{00,00}=(U^k+V^k), \tag{32}$$

$$\begin{aligned}P_f{}^k(pu)&=P_{fs}{}^{k0}/[P_{ir}{}^{00}d\sigma(pu)]\\&=(\sum_{\mu=0}^{3}\Omega_{00,\mu0}P_{ir}{}^{\mu0}/P_{ir}{}^{00})^{-1}\sum_{k=0}^{3}\Omega_{k0,\mu0}P_{ir}{}^{\mu0}/P_{ir}{}^{00}\\&=(d\sigma(uu)/d\sigma(pu))(P_f{}^k(uu)+\sum_{h=1}^{3}Z^{kh}P_i{}^h).\end{aligned} \tag{33}$$

The 3×3 matrix appearing in Eq. (33) is defined as

$$Z^{kh}=\frac{\Omega_{k0,h0}}{\Omega_{00,00}}=\frac{2}{\mathrm{Tr}\,\mathbf{A}}[-\delta_{kh}(\tfrac{1}{2}\,\mathrm{Tr}\,\mathbf{A}-A_{00})+\mathrm{Re}\,A_{kh}+\sum_{\epsilon=0}^{3}\varepsilon_{0kh\epsilon}\,\mathrm{Im}\,A_{0\epsilon}]. \tag{34}$$

Equations (29), (30), (32), and (33) are the generalizations of the corresponding equations which were published earlier in the literature.[1,2] They are valid whether or not any of the usually assumed invariance principles hold.[12] In deriving them we have used only invariance under rotations in three-space together with energy–momentum conservation.

In the following discussion we call the vector $(\mathbf{U}-\mathbf{V})$ the "beam-analyzing vector," because it is the scalar product of this vector with $\mathbf{P}_i$ which determines the asymmetry in $d\sigma(pu)$ in what may be called an "analyzing scattering," i.e., a scattering in which the incoming beam particle is polarized and the differential scattering cross section is measured. The vector $(\mathbf{U}+\mathbf{V})$ is called the "beam-polarizing vector" because it determines the polarization of the outgoing beam particles after what may be called a "polarizing scattering," i.e., when the incoming beam particles are unpolarized.[13]

B. Observable Effects of Invariance Principles

In order to test invariance principles, using experiments with polarized beams, we have to determine the observable effects of such principles. More precisely, we are interested in finding the restrictions imposed by invariance principles on the quantities which can be measured by experiments with polarized beams. These quantities are Tr **A**, the vectors U^k, V^k, and the tensor Z^{ik}.

The invariance principles under discussion in this subsection are P_σ, P_T, and $P_\sigma\cdot P_T$ invariance, and the helicity invariance principles cases (a), (c), and (d).[14] [P_{12} invariance will not be discussed here. The helicity invariance principles cases (b), (e), and (f) are mentioned at the end of this subsection.]

We have defined the α matrix so that the restrictions imposed by all the relevant invariance principles (except P_{12}) can be expressed simply by the statement that certain elements of α have to be equal to zero.[14] These restrictions can be summarized in terms of the schemes introduced in Sec. 3. From Eq. (26), which defines the **A** matrix, it follows that the restrictions imposed on **A** by the invariance principles under discussion are also of this simple nature. Certain elements of **A** have to be equal to zero, while the others are arbitrary. In fact, it turns out that the restrictions imposed on **A** by P_σ, P_T, and $P_\sigma\cdot P_T$ invariance, and the helicity invariance principles cases (a), (b), and (c), are exactly the same as the restrictions imposed on α by the same invariance principles. For example, if P_σ invariance holds, then from Eqs. (17) and (26) it follows that $A_{\beta\gamma}$ has to satisfy when $\eta_\sigma=+1$

$$A_{\beta\gamma}=A_{\beta\gamma}\delta(\beta,\gamma\,|\,0,1;2,3), \tag{35}$$

which is exactly the same as Eq. (17) for α. The restrictions on **A** can also be represented by schemes and, as we have just seen, the schemes for **A** are exactly the same as the schemes for α whenever any of the invariance principles mentioned above hold.[15]

To find the restrictions on U^k, V^k, and Z^{ik} due to the invariance principles under discussion, one has only to keep the scheme of **A** (which is the same as the scheme of α) in mind, and put the proper terms equal to zero in Eqs. (29), (31), and (34). For example, if P_σ invariance holds, then from Fig. 1 (1) for $\eta_\sigma=+1$, we find that A_{02}, A_{03}, A_{12}, A_{13} are zero. We know that **A**

[12] Division by $\Omega_{00,00}$ in the definition of **U**, **V**, etc., does not cause difficulties, because $\Omega_{00,00}=0$ if and only if $\alpha=0$, i.e., when there is no interaction whatsoever.

[13] It is shown in part B of this section that when P_σ invariance holds as it does, for example, in the case discussed by Wolfenstein (Ref. 1) then both $(\mathbf{U}+\mathbf{V})$ and $(\mathbf{U}-\mathbf{V})$ can have only one component which is nonzero. These one component quantities are essentially those which were called by Wolfenstein (Ref. 1) the polarizing power and analyzing power, respectively.

[14] We do not list here the restrictions due to P_T, P_σ. P_T and $P_\sigma+P_T$ invariance for inelastic scattering. They correspond exactly to (and are quite easily obtained from) the restrictions obtained for elastic scattering. The difference is that in the inelastic case it is the $(\alpha-\alpha^T)$ matrix (and not α) which has the scheme shown in Fig. 2(2), when $\eta_\sigma=1$. Thus U^k, V^k, $Z^{kh}=-U^{Tk}$, $-V^{Tk}$, $-Z^{Tkh}$, if for elastic scattering U^k, V^k, $Z^{kh}=0$. For the other components $U^k=U^{Tk}$ etc. When $\eta_\sigma=-1$, then the argument is similar, where the superscript T indicates that the corresponding quantity refers to the time reversed reaction. We also do not list the restrictions due to P_C and $P_\sigma\cdot P_C$ invariance. The relations due to P_C invariance are obvious and can be summarized by the statement $\alpha=\alpha^C$, where the superscript C indicates that the quantity refers to the charge-conjugate reaction. The restrictions due to $P_\sigma\cdot P_C$ correspond exactly to the restrictions due to P_σ invariance; in this case it is the $(\alpha-\alpha^C)$ matrix (not α) which has the scheme Fig. 2(1). The observable effects of these restrictions are that if $P_\sigma\cdot P_C$ invariance holds, then $V^k-V^{Ck}=0$, $U^k=U^{Ck}=0$, etc., if P_σ invariance requires that $V^k=0$, $U^k=0$, etc.

[15] This statement would be false for the helicity invariance principles cases (b), (e), and (f). Nevertheless, these cases do not have to be treated separately, as is shown later.

is always Hermitian. Putting this information into Eq. (31) we see that the second and third components of both U^k and V^k are equal to zero. We conclude that both $\mathbf{U}+\mathbf{V}$ (the polarizing vector) and $\mathbf{U}-\mathbf{V}$ (the analyzing vector) can have only their first component different from zero. The same holds when $\eta_\sigma=-1$, the scheme of $\mathbf{A}$ being unchanged. From Eq. (34) we see that $\mathbf{Z}$ does not mix the first component of polarization with the second and third components. In the same way one can immediately see that the results listed below, numbered from 1 to 28, are correct. The number of parameters in parentheses indicates the number of real parameters on which $\mathbf{A}$ depends if the indicated invariance principle holds. It is the number of shaded squares in the scheme of $\mathbf{A}$ (or α), since $\mathbf{A}$ is Hermitian.

P_σ invariance (8 parameters)

1. The beam polarizing vector is perpendicular to the scattering plane, i.e., when scattering unpolarized beam particles on unpolarized target particles the outgoing beam particles can be polarized only perpendicular to the scattering plane.

2. The beam analyzing vector is perpendicular to the scattering plane, i.e., if the incoming beam particles are polarized in the scattering plane, then the differential cross section will be the same as if the incoming beam particles had been unpolarized.

3. Z^{ik} does not mix the first component of polarization with the other two components, i.e., if $\mathbf{P}_i$ is perpendicular to the scattering plane, then so is

$$\sum_{k=1}^{3} Z^{lk} P_i^k$$

[and also $\mathbf{P}_f(pu)$]. If $\mathbf{P}_i$ is parallel to the scattering plane, then so is

$$\sum_{k=1}^{3} Z^{lk} P_i^k$$

[but $\mathbf{P}_f(pu)$ is not, unless $(\mathbf{U}+\mathbf{V})=0$].

P_T invariance, elastic scattering[14] (10 parameters)

4. The first (third) component of the beam polarizing vector is the same as the first (third) component of the beam analyzing vector. The second component of the beam polarizing vector is (-1) times the second component of the beam analyzing vector.

5. The off-diagonal elements of Z^{ik} satisfy the equation $Z^{ik}=(-1)^{i+k}Z^{ki}$.

$P_\sigma P_T$ invariance, elastic scattering (10 parameters)

If the CPT theorem holds, then $P_\sigma \cdot P_T$ invariance is equivalent to charge conjugation invariance, P_C.

6. The first (second) component of the beam polarizing vector is equal to the first (second) component of the beam analyzing vector. The third component of the beam polarizing vector is (-1) times the third component of the beam analyzing vector.

7. The off-diagonal elements of Z^{ik} satisfy the equation $Z^{ik}=[1-2(\delta_{i3}+\delta_{k3})]Z^{ki}$.

$P_\sigma+P_T$ invariance, elastic scattering (6 parameters)

Since any two of the following three operations, P_σ, P_T, and $P_\sigma \cdot P_T$ imply the third one, this case is equivalent to $P_\sigma+P_T+P_\sigma \cdot P_T$ invariance.

8. The beam polarizing vector is perpendicular to the scattering plane.

9. The beam analyzing vector is equal to the beam polarizing vector.

10. The only nonzero off-diagonal elements of $\mathbf{Z}$ are Z^{23} and Z^{32}. They satisfy $Z^{23}=-Z^{32}$.

Helicity invariance case (a) (8 parameters)

11. The beam polarizing vector is longitudinal, i.e., only its third component can be nonzero. Therefore, when unpolarized beam particles are scattered on unpolarized targets, then the outgoing beam particles can have only longitudinal polarization.

12. The beam analyzing vector is longitudinal. Therefore, only the presence of longitudinal polarization in the incoming beam can be detected when measuring the differential cross section of scattering on unpolarized targets.

13. The only nonzero off-diagonal elements of $\mathbf{Z}$ are Z^{12} and Z^{21}. Therefore, $\mathbf{Z}$ does not mix the longitudinal component of polarization with the transverse components of polarization. If the incoming beam polarization is longitudinal, so is the outgoing beam polarization. If the incoming beam polarization is transverse, the longitudinal component of the outgoing beam polarization is the same as if the incoming beam had been unpolarized.

If together with case (a) some other invariance principles hold as well, we may add to statements 11, 12, 13 the following ones:

Case (a)$+P_\sigma$ invariance (4 parameters)

14. Fig. 1 shows that for elastic scattering case (a) $+P_\sigma$ invariance implies P_T invariance and therefore $P_\sigma \cdot P_T$ invariance as well. If the CPT theorem holds, then this implies P_C invariance. (The converse is not true. Case (a) $+P_T$ invariance does *not* imply P_σ invariance, nor does case (a) $+P_\sigma \cdot P_T$ invariance imply P_σ invariance).

15. The beam polarizing vector is zero. No polarization whatsoever can be induced by scattering unpolarized beam particles on unpolarized target particles.

16. The beam analyzing vector is zero. No polarization in the incoming beam can be detected by observing the differential cross section when scattering on an unpolarized target.

There is a simple reason for the fact that both $\mathbf{U}$ and

V are zero in this case. We know that P_σ invariance requires that these vectors be perpendicular to the scattering plane. At the same time in case (a) they have to be longitudinal. Therefore they vanish.

The only parameter which can be determined in experiments with polarized beams in this case is $d\sigma(uu) = d\sigma(pu)$. Multiple scattering experiments with polarized beams can not yield any additional information.

17. **Z** is diagonal.

Case (a) $+P_T$ invariance, elastic scattering (6 parameters)

18. The beam polarizing vector is equal to the beam analyzing vector.

19. The only two off-diagonal elements of **Z** satisfy the relation $Z^{12}=-Z^{21}$.

Case (a) $+P_\sigma\cdot P_T$ invariance, elastic scattering (6 parameters)

20. The beam polarizing vector is (-1) times the beam analyzing vector.

21. The only two off-diagonal elements of **Z** satisfy the relation $Z^{12}=Z^{21}$. Therefore **Z** has three mutually perpendicular eigenvectors, one of which is parallel to the longitudinal direction.[16]

Helicity invariance case (c) (4 parameters)

22. Case (c) for elastic scattering implies P_T invariance, as can be seen from Fig. 1. Therefore, even for nonelastic scattering the conclusions 11, 12, 13, 18, 19 hold now, and in addition we find:

23. **Z** assumes the form:

$$\mathbf{Z}=1-(2/\mathrm{Tr}\,\mathbf{A})\mid R\mid \mathfrak{R}_\perp(\rho) \tag{36}$$

where

$$\mid R\mid =\sqrt{[(\mathrm{Re}\,A_{33})^2+(\mathrm{Im}\,A_{03})^2]},$$

$$\mathrm{Tr}\,\mathbf{A}=\mathrm{Re}\,A_{00}+\mathrm{Re}\,A_{33},\qquad \cos\rho=\mathrm{Re}\,A_{33}/\mid R\mid,\qquad \sin\rho=-\mathrm{Im}\,A_{03}/\mid R\mid,$$

$$\mathfrak{R}_\perp(\rho)=\begin{pmatrix}\cos\rho & \sin\rho & 0\\ -\sin\rho & \cos\rho & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Case (c) $+P_\sigma$ invariance (2 parameters)

This is a special case of case (a) $+P_\sigma$ invariance. Therefore, conclusions 15 and 16 now hold. In addition we find:

24. From conclusion 22 it follows that for elastic scattering case (c) $+P_\sigma$ invariance is equivalent to case (c) $+P_\sigma\cdot P_T$ invariance.

25. **Z** assumes the form

$$\mathbf{Z}=1-(2/\mathrm{Tr}\,\mathbf{A})\,R_{33}\mathfrak{R}_\perp(\rho=0). \tag{37}$$

P_σ invariance requires $\rho=0$. Physically this follows from the observation that if $\mathfrak{R}_\perp(\rho)$ rotates the transverse components of polarization by ρ in the clockwise direction before P_σ has been performed, then it will rotate by $+\rho$ in the counterclockwise direction after a P_σ has been performed. Invariance under P_σ requires $\rho=-\rho$, from which $\rho=0$ follows.

Helicity invariance case (d) (4 parameters)

26. For elastic scattering case (d) implies $P_\sigma\cdot P_T$ invariance. Therefore, even for nonelastic scattering, conclusions 11, 20, and 21 now hold. In addition we find that

27. $Z^{33}=-1$, and $\mathrm{Tr}\,\mathbf{A}=R_{11}+R_{22}$.

Case (d) $+P_\sigma$ invariance (2 parameters)

This is another special case of case (a) $+P_\sigma$ invariance. In addition to conclusions 15 and 16 and 27, we have:

28. For elastic scattering we see from 26 that case (d) $+P_\sigma$ invariance is equivalent to case (d) $+P_\sigma\cdot P_T$ invariance.

All restrictions due to the invariance principles under discussion on all quantities which can be observed in experiments with polarized beams have been given above. In this sense the list of conclusions is complete.[14]

Conclusions 8 and 9 were proven by Wolfenstein.[1] They are called Theorem 1 and Theorem 2 in his work. A result similar to conclusion 15 was proven by Logunov, Meshcheryakov, and Tavkhelidze.[10]

The helicity invariance principles, cases (a), (e), and (f) have not yet been discussed. In fact, it is not necessary to treat them separately. All conclusions obtained above for the cases (a), (c), and (d) are also valid for the cases (b), (e), and (f), respectively. The reason is that the schemes of the **A** matrix in cases (b), (e), and (f) are the same as in the cases (a), (c), and (d) respectively. This is a consequence of Eqs. (21) and (26), and physically it means simply that experiments with polarized beams can not distinguish between cases (a), (c), and (d) or cases (b), (e), and (f), respectively. Indeed, to be able to distinguish between cases (a) and (b), for example, it is necessary to determine not only what happens to the helicity of the beam particle during scattering, but also what happens to the helicity of the target particle. In experiments with polarized beams, however, this can never be done, since the polarization of the target particle after scattering is never observed.

The main importance of these conclusions is that they can be used to test the validity of invariance principles. If it is proven experimentally that any one

[16] We recall again that this does not mean that the direction of a longitudinal incoming beam polarization is left unchanged by **Z**. It means that is is changed by **Z** into a longitudinal outgoing polarization, as discussed before Eq. (2).

of the conclusions concerning $(\mathbf{U}+\mathbf{V})$, $(\mathbf{U}-\mathbf{V})$, or $\mathbf{Z}$, and appearing under the heading of a certain invariance principle, is false, then that invariance principle can *not* hold in nature in that experiment. Several of these conclusions should not be too difficult to check experimentally, since often it is not necessary actually to measure a quantity to test whether a particular invariance holds. It is frequently sufficient to ascertain only that a given term is different from zero to prove that a given invariance does *not* hold.

C. Multiple-Scattering Experiments

This subsection deals with multiple-scattering experiments with polarized beams, that is, multiple-scattering experiments in which every scattering satisfies Eqs. (22) and (23). In addition, we assume that all target particles *are at rest* in the laboratory.

To emphasize that a certain quantity refers to the nth scattering ($n=1, 2, \cdots, N$), we add a subscript n to that quantity. Here N is the total number of scatterings in the multiple-scattering experiment under discussion. We set off the subscript n by a comma from any other subscripts. For example, $\alpha_{\beta\gamma,n}$ is the matrix element $\alpha_{\beta\gamma}$ evaluated at the c.m. energy and angle of the nth scattering, $(U+V)_2{}^k$ is the kth component of the beam polarizing vector evaluated at the c.m. energy and angle of the second scattering, $\mathbf{Z}_3$ is the $\mathbf{Z}$ matrix of the third scattering, $K_{i,2}$ is the rest frame of the i particle of the second scattering, $(\mathbf{1})_{i,2}$ is the first component of this frame, etc.

Since the f particle of the nth scattering is also the i particle of the $(n+1)$st scattering, their polarizations are related. However, $P_{i,n+1}{}^k$, the kth component of the incoming beam polarization in the $(n+1)$st scattering, is *not* equal to $P_{f,n}{}^k$, the kth component of the outgoing beam polarization in the nth scattering. The reason is, that $P_{i,n+1}{}^k$ and $P_{f,n}{}^k$ are the kth component of the same quantity, but measured in two different frames, namely, $K_{i,n+1}$ and $K_{f,n}$, respectively, as discussed in Sec. 1. These two frames are both rest frames of the same particle. Therefore, they are related to each other by an ordinary space rotation at most, not by a general Lorenz transformation. This fact greatly simplifies actual calculations. We define the space-rotation matrix $\mathbf{R}_n$ by the relation

$$\mathbf{P}_{i,n+1}=\mathbf{R}_n\mathbf{P}_{f,n}. \tag{38}$$

The matrix $\mathbf{R}_n$ is completely determined by the "geometry" of the experiment, more precisely, by the velocities of all the particles involved in the nth and $(n+1)$st scattering.

Since all target particles are at rest in the laboratory, the matrix $\mathbf{R}_n$ can always be written as

$$\mathbf{R}_n=\mathbf{R}_n[(\mathbf{3})_{i,n+1},\ \gamma_n]\mathbf{R}_n[(\mathbf{1})_{f,n},\ \delta_n]. \tag{38a}$$

Here the matrix $\mathbf{R}_n[(\mathbf{1})_{f,n},\ \delta_n]$ rotates $K_{f,n}$ around its first axis, $(\mathbf{1})_{f,n}$, by an angle δ_n, so that the third axis of the rotated frame will be $(\mathbf{3})_{i,n+1}$. By definition, $(\mathbf{3})_{f,n}$ points opposite to the direction of motion of K_n as viewed from the rest frame of the f particle of the nth scattering. The $(\mathbf{3})_{i,n+1}$ axis has been defined so that it points opposite to the direction of motion of K_{n+1} as viewed from the same rest frame [since the f particle of the nth scattering and the i particle of the $(n+1)$st scattering are now the same particle]. We see that $\mathbf{R}_n[(\mathbf{1})_{f,n},\ \delta_n]$ is the matrix which takes into account the relative velocities of K_n and K_{n+1}. On the other hand, the matrix $\mathbf{R}_n[(\mathbf{3})_{f,n+1},\ \gamma_n]$ rotates around $(\mathbf{3})_{f,n+1}$ by an angle γ_n so that the first axis of the rotated frame will be $(\mathbf{1})_{f,n+1}$. The angle γ_n is related to the relative orientations of the nth and $(n+1)$st scattering planes, and therefore we conclude that $R_1(\mathbf{3})_{i,n+1},\ \gamma_n]$ is the matrix which takes into account the relative orientation of successive scattering planes.

We define the angle between the nth and $(n+1)$st scattering planes (not a covariant concept) to be γ_n. For example, we say that the nth and $(n+1)$st scattering planes are parallel if $\gamma_n=0$; we say that they are perpendicular if $\gamma_n=\pm\pi/2$.

Those multiple-scattering experiments with polarized beams, in which the i particle of the first scattering is not polarized and the polarization of the f particle of the last scattering is not measured, are called u–u scattering experiments. The symbol signifies originally (before first scattering) unpolarized beam and unpolarized targets.

By performing a single-scattering u–u experiment one can determine one parameter, $d\sigma(uu)$, at each c.m. energy and angle.

In double-scattering u–u experiments the beam particle is generally polarized after the first scattering, and a measurement of the differential cross section in the second scattering, $d\sigma(pu)$, together with Eq. (30) enables us to determine

$$\begin{aligned}(\mathbf{U}-\mathbf{V})_2\mathbf{P}_{i,2}&=(\mathbf{U}-\mathbf{V})_2\mathbf{R}_1\mathbf{P}_f(uu)_1\\&=\mathcal{Q}_{2||}\,\mathcal{P}_{1||}+|\,\mathcal{Q}_{2\perp}\,||\,\mathcal{P}_{1\perp}\,|\cos(\gamma_1+\beta_1),\\&\mathcal{Q}_2\equiv(\mathbf{U}-\mathbf{V})_2,\qquad \mathcal{P}_1\equiv\mathbf{R}_1\mathbf{P}_f(uu)_1.\end{aligned} \tag{39}$$

As usual, the subscripts 1 and 2 indicate that the quantity refers to the first or second scattering. We designate the longitudinal component of a vector by a subscript $||$ and the transverse component by a subscript $\perp$. The angle between $\mathcal{Q}_{2\perp}$ and $\mathcal{P}_{1\perp}$ is $(\gamma_1+\beta_1)$, where γ_1 is the angle between the first and second scattering planes, and consequently β_1 is the angle between $\mathcal{Q}_{2\perp}$ and $\mathcal{P}_{1\perp}$ when $\gamma_1=0$. Choosing γ_1 to be first zero, then π, we can determine both $|\,\mathcal{Q}_{2||}\,|\cdot|\,\mathcal{P}_{1||}\,|$ and $|\,\mathcal{Q}_{2\perp}\,|\cdot|\,\mathcal{P}_{1\perp}\,|\cos\beta_1$. A third triple-scattering experiment in which γ_1 is, for example, $\pi/2$ can serve to determine β_1. We conclude that double-scattering u–u experiments can be used to determine three parameters.

In triple-scattering u–u experiments the differential

cross section of the third scattering is given by Eqs. (32), (33), and (30). One finds

$$d\sigma(pu)_3=d\sigma(uu)_3\left\{1+\frac{d\sigma(uu)_2}{d\sigma(pu)_2}\left[\mathcal{A}_3\mathcal{P}_2+\mathcal{A}_3\mathcal{Z}_2\mathcal{P}_1\right]\right\}, \quad (40)$$

where

$$\mathcal{Z}_n=\mathbf{R}_n\mathbf{Z}_n. \quad (40a)$$

Equation (40) can be used to determine $\mathcal{A}_3\mathcal{Z}_2\mathcal{P}_1$, since all other terms appearing on the right-hand side of that equation are known from single- and double- u–u scattering experiments. To determine $\mathcal{Z}_2$ one has to perform nine different scattering experiments. For example, one may choose the angle between the first and second scattering planes to be 0, π, and $\pi/2$ while keeping the angle between the second and third scattering planes at the values 0, π and $\pi/2$. However, these nine elements do not determine $\mathcal{Z}_2$ completely, because we have seen that $\mathcal{A}_3$ and $\mathcal{P}_1$ can not be determined completely from double u–u scattering experiments. Therefore, only the elements of the matrix

$$\mathbf{H}^{-1}(h_{||}, h_\perp, \chi)\,\mathcal{Z}_2\mathbf{H}(h_{||}, h_\perp, \chi)$$

can be determined, where $h_{||}$, $h_\perp$ and χ are arbitrary. The matrix $\mathbf{H}(h_{||}, h_\perp, \chi)$ is defined to be that matrix which multiplies the longitudinal component of any vector by $h_{||}$, the transverse component by $h_\perp$, and rotates the transverse component by an angle χ.

Higher than triple u–u scattering experiments can yield no new information, since they would only serve to determine terms of the form

$$\mathcal{Z}_1,\ \mathcal{Z}_2,\ \mathcal{Z}_3,\cdots=(\mathbf{H}^{-1}\mathcal{Z}_1\mathbf{H})\cdot(\mathbf{H}^{-1}\mathcal{Z}_2\mathbf{H})\cdot(\mathbf{H}^{-1}\mathcal{Z}_3\mathbf{H})\cdots$$

and each term on the right-hand side of this equation is already known from triple-scattering experiments.

These results are to be contrasted with the well-known ones, that in elastic scattering if $P_\sigma+P_T$ invariance holds, then single-scattering u–u experiments can be used to determine one parameter, $d\sigma(uu)$, but that double-scattering u–u experiments can measure only one additional parameter (not three), that triple-scattering u–u experiments can determine only two more parameters, and finally, that u–u experiments with more than three successive scatterings can also be used to determine new parameters. In fact, choosing the angles between the successive scattering planes properly, two quadruple-scattering u–u experiments can determine two more parameters.

This contrast is easily understood, by remembering that in deriving the previous results we made use of an important implicit assumption. This was that none of the quantities $\mathcal{A}_{||}$, $\mathcal{A}_\perp$, $\mathcal{P}_{||}$, $\mathcal{P}_\perp$ is zero. If this is not a valid assumption, then rotations around the longitudinal axis can not take $\mathcal{A}$ or $\mathcal{P}$ over into three linearly independent vectors in three-space. For example, if $\mathcal{A}$ is purely transverse, which is the case when P_σ invariance holds, a rotation around the longitudinal direction will never remove it from that plane (a two-dimensional subspace of three-space) which is perpendicular to the longitudinal direction.

One of the effects of an invariance principle is to restrict the structure of $\mathbf{Z}$. For example, if P_σ invariance holds, then $Z^{12}=Z^{13}=Z^{21}=Z^{31}=0$. This reduces the number of elements of $\mathbf{Z}$ to be determined, and thus simplifies the task of the experimentalist. Another effect of an invariance principle, however, is to restrict the form of $\mathbf{U}$ and $\mathbf{V}$ and in so doing it affects the very instruments, the "polarizer" and the "analyzer" which are used in a multiple-scattering u–u experiment to determine the elements of $\mathbf{Z}$. For example, if P_σ invariance holds, then the "polarizer" (the first scattering) can not produce a component of polarization parallel to the scattering plane, and the "analyzer" (the last scattering) is "blind" in any direction except the one perpendicular to the scattering plane.

Restrictions on $\mathbf{U}$ and $\mathbf{V}$ may effect our results in another way, namely, they may be used to determine the matrix $\mathbf{H}(h_{||}, h_\perp, \chi)$ to some extent. For example, if $P_\sigma+PI$ invariance holds, then we know that $\mathbf{V}$ and $\mathbf{U}$ are perpendicular to the scattering plane. It follows that in this case $\mathbf{H}=\pm 1$. Therefore, multiple-scattering u–u experiments can determine $\mathcal{Z}$ in this case completely. However, to do this, quadruple or higher order u–u scattering experiments have to be performed.

Summing up, if none of the quantities $\mathcal{A}_{||}$, $\mathcal{A}_\perp$, $\mathcal{P}_{||}$, $\mathcal{P}_\perp$ is zero, then one single-, three double-, and nine triple-scattering u–u experiments have to be performed (in order to learn everything about the S matrix that can be learned by performing u–u scattering experiments). These experiments do not completely determine the elements of the $\mathbf{A}$ matrix. Higher order scattering u–u experiments yield no new information. Therefore, in order to learn more about the S matrix, experiments other than u–u scattering experiments have to be performed. The restrictions imposed by invariance principles decrease the number of parameters to be measured, but at the same time may make it necessary to perform higher than third-order-scattering u–u experiments. In certain cases the effect of the matrix $\mathbf{H}$ can be determined.

Suppose that by some means it is possible to polarize the beam particles along the l direction, and to determine the kth component of polarization of the beam particles (k, $l=1$, 2, 3). Suppose further, that the interaction between the beam and target particles satisfies certain known invariance principles. We ask: In order to determine $\mathbf{Z}$ as closely as possible, how many intermediate scatterings have to be performed, and what should the relative orientation of successive scattering planes be? What combinations of the elements of $\mathbf{Z}$ can we determine by performing a certain multiple-scattering experiment?

To answer these questions, one has to use Eqs. (30), (32), and (33) together with the explicit form of the $\mathbf{R}_n$ rotation matrices which were defined after Eq.

(40). The explicit form of the $\mathbf{R}_n$ depends on the geometry of the experiment, such as the relative orientations of successive scattering planes, velocity of the beam particles, and so on. As the number of successive scatterings increases, the resulting formulas become increasingly more complicated, and the process of obtaining the correct answer increasingly more tedious. It is quite easy to make simple mistakes by failing to visualize correctly the relative orientations of scattering planes, particle velocities in different frames, etc. To illustrate how tedious the answer to questions of the above type can get, we consider two examples.

Suppose that $P_\sigma+P_T$ invariance holds. Suppose further that we perform a quadruple u–u scattering experiment in which each of the scatterings is elastic, the first and second scattering planes are parallel to each other, the second and third scattering planes are perpendicular to each other, and the fourth scattering plane is parallel to the third. We wish to determine which combinations of elements of the $\mathbf{Z}$ matrix we can determine by performing this experiment. As we know, if $P_\sigma+P_T$ invariance holds, then for elastic scattering the polarizing vector and the analyzing vector are both perpendicular to the scattering plane. Therefore, by scattering unpolarized beam particles on an unpolarized target, we can polarize the beam particles parallel to the first axis of their rest frame.[17] When using such a polarizing scattering to induce a polarization in a beam, $l=1$. Similarly an analyzing scattering can determine the first component of beam polarization,[17] and therefore, in this case $k=1$. The intermediate scatterings take place after the polarizing scattering and before the analyzing scattering. The plane of the first scattering (the polarizing scattering) is parallel to the plane of the second scattering (the first intermediate scattering). Therefore, the i particle of the second scattering is polarized along the first axis. (The $\mathbf{R}_1[(\mathbf{1})_{f,1}, \delta_1]$ leaves any direction parallel to $(\mathbf{1})_{f,1}$ unchanged, and since $\gamma_1=0$, the matrix $\mathbf{R}_1[(\mathbf{3})_{i,2}, \gamma_1=0]$ is equal to unity.) The $\mathbf{Z}_2$ matrix of the second scattering will change the magnitude of this polarization, but will leave its direction unchanged, because for elastic scattering $Z_n{}^{21}=Z_n{}^{31}=0$ for all n if $P_\sigma+P_T$ invariance holds. As a result, the polarization of the f particle of the second scattering is parallel to the first axis. The plane of the third scattering (which is the second intermediate scattering) is perpendicular to the second scattering plane. Therefore the i particle of the third scattering will be polarized along the second axis. (The matrix $\mathbf{R}_1[(\mathbf{1})_{f,1}, \delta_1]$ leaves the $(\mathbf{1})_{f,1}$ axis unchanged, and $\mathbf{R}_1[(\mathbf{3})_{1,2}, \gamma_1]$ rotates it by $\gamma_1=\pm\pi/2$, so that it will end up parallel to $(\mathbf{2})_{i,2}$.) The $\mathbf{Z}_3$ matrix will in general have elements which connect the second component to the second and third components, so $\mathbf{P}_{f,3}$ will in general have a second and third component. If there is no third intermediate scattering, then the fourth scattering will be the analyzing scattering. Let the fourth scattering plane be parallel to the third scattering plane. Then $\mathbf{P}_{i,4}$, just as $\mathbf{P}_{f,3}$ will have no first component. Nevertheless $\mathbf{P}_{i,4}\neq\mathbf{P}_{f,3}$, because $K_{i,4}$ is rotated with respect to $K_{f,3}$ around their common first axis by δ_3. This rotation mixes the components of $\mathbf{P}_{f,3}$ to give $\mathbf{P}_{i,4}$. The differential cross section of the analyzing experiment is different from $d\sigma(uu)_4$ only if $P_{i,4}{}^1$, the first component of $\mathbf{P}_{i,4}$ is not zero. In the experiment described above this is not the case. We conclude that this quadruple-scattering u–u experiment can not be used to determine any of the elements of $\mathbf{Z}$.

As the second example, consider the same experiment, but let the fourth scattering plane be perpendicular to the third one. Then $P_{i,4}{}^1\neq0$, and will be a linear combination of two terms, one of which is proportional to $Z_3{}^{22}$, the other to $Z_3{}^{32}$, and both of them proportional to $\mathbf{P}_{i3}$, which in turn is proportional to $Z_2{}^{11}$ and $\mathbf{P}_{2,i}$, the latter being the known $\mathbf{P}_{1,f}$. The coefficients of proportionality are linear combinations of certain elements of the $\mathbf{R}_2$ and $\mathbf{R}_3$ matrices.

Questions of the type discussed above can be answered more easily by using a graphical method. This substitutes the drawing of straight lines for writing equations, which leads to considerable time saving. The lines have to be drawn according to a set of simple rules which can be followed mechanically, thus reducing the chance of making an error. A further advantage of the graphs is that their structure is often more easily grasped visually than the structure of equations in which a large number of parameters appear. In the next few paragraphs the rules of how to draw these graphs are given. After that, as an illustration, we will assume that $P_\sigma+P_T$ invariance holds, and discuss elastic multiple-scattering u–u experiments in detail. The graphs corresponding to the two quadruple-scattering experiments which were discussed in the previous paragraph are given in Figs. 6(a) and 6(b). After having constructed the graphs appearing in Fig. 5, one should describe in words the experiments which they represent, just as the experiments corresponding to Figs. 6(a) and 6(b) were described in the previous paragraphs. In the process of doing so, the principle of the graphs (an approximately one-to-one correspondence between a graph and the words describing it) should become quite clear. Therefore, no detailed proof of the rules is given.

To discuss a u–u scattering experiment, with the beam polarizing vector and beam analyzing vector assumed to be known, the rules are as follows:

A graph contains lattice points, denoted by ×'s and ○'s and +'s (Fig. 5). The lattice points are arranged in rows and columns.

There are three rows, numbered from 1 to 3 from top to bottom. They correspond to the three components of polarization of the beam. Each row starts with an × on the right, which is followed by an ○

[17] Unless by some accident $\mathrm{Re}A_{01}=0$.

(to the left of the ×), which in turn is followed by a +, and so on alternately ×, ○, +, ×, ○, +, ⋯.

Each of the ×'s together with the ○'s and +'s immediately to the left of it form a column, together with the ×'s, ○'s and +'s of the other two rows. The ×'s belonging to the same column lie on a vertical line. So do the ○'s and the +'s belonging to the same column. Columns are numbered from 1 to N from *right* to *left*. Here N is the total number of ×, ○, + triplets in a row.

(The ordering from right to left corresponds to the ordering of matrices in a product. For example, in the product **CBA** the matrix **A** acts first, **B** acts second, and **C** acts last.)

We say that a graph and a multiple-scattering u–u experiment correspond to each other if N is equal to the number of successive scatterings in the experiment.

A solid line may connect an × belonging to the nth column with an ○ belonging to the nth column. We say that this solid line belongs to the nth column. We establish a one-to-one correspondence between the solid lines of the nth column and the elements of $\mathbf{Z}_n$, which is the **Z** matrix [see Eq. (34)] of the nth scattering in the experiment under discussion. We say that the solid line belonging to the nth column and connecting an ○ of the sth row with an × of the rth row corresponds to $Z_n{}^{sr}$. Conversely, we say that $Z_n{}^{sr}$ corresponds to the above-described solid line.

If a certain element of $\mathbf{Z}_n$ is not zero, then we call the solid line corresponding to it "allowed." A line is "forbidden" if it is not allowed.

(For example, assume that $P_\sigma+P_T$ invariance holds, and the nth scattering is elastic. Then from rule 10 in Sec. 4.B we know that $Z_n{}^{21}=0$. Therefore in this case the solid line belonging to the nth column, starting in the first row and ending in the second row is forbidden.)

Dashed lines may connect an ○ belonging to the nth column with a + belonging to the nth column. We say that such dashed lines belong to the nth column. Dashed lines which belong to the nth column, and which connect a + of the sth row with an ○ of the rth row correspond to the (sr) element of the matrix $\mathbf{R}_n[(\mathbf{1})_{f,n}, \delta_n]$ [defined in Eq. (38a)].

A dashed line starting in the first row must always end in the first row. It corresponds to a matrix element whose value is unity. A dashed line which starts in the second (third) row may end in the second (third) row, or else in the third (second) row. In the former case it is said to be uncrossed; in the latter case to be crossed. Uncrossed dashed lines correspond to $\cos\delta_n$, crossed ones correspond to $\pm\sin\delta_n$. Therefore, except for special cases (when $\delta_n=0, \pm\pi/2, \pm\pi, \cdots$), both uncrossed and crossed dashed lines are allowed.

(These rules follow from the fact that the matrix $\mathbf{R}[(\mathbf{1})_{f,n}, \delta_n]$ rotates three-space around $(\mathbf{1})_{f,n}$ by an angle δ_n. Consequently, its diagonal element belonging to the first row is always unity. Therefore, the dashed line corresponding to this matrix element—the dashed line starting and ending in the first row—is always allowed and corresponds to unity. The off-diagonal elements of the rotation matrix $\mathbf{R}_n[(\mathbf{1})_{f,n}, \delta_n]$ belonging to its first row or its first column are zero. Therefore, the dashed lines corresponding to these elements —the dashed lines starting (ending) in the first row, but not ending (starting) in the first row—are always forbidden. Those diagonal elements of the rotation matrix which belong to its second or third row have the value $\cos\delta_n$. The corresponding dashed lines are the uncrossed ones belonging to the second and third row. Finally, the crossed dashed lines correspond to the other two elements of the rotation matrix. These elements are $\pm\sin\delta_n$.)

Dotted lines may connect a + in the nth column with an × in the $(n+1)$st column. We say that such a dotted line belongs to the nth column. A dotted line belonging to the nth column and connecting an × in the sth row with a + in the rth row corresponds to the (sr) element of $\mathbf{R}_n[(\mathbf{3})_{f,n+1}, \gamma_n]$ (defined in Eq. (38a)].

A dotted line starting in the third row must always end in the third row. It corresponds to a matrix element whose value is unity. A dotted line belonging to the nth column and starting in the first (second) line may end in the first (second) line, or else in the second (first) line. In the former case it is said to be uncrossed, and corresponds to $\cos\gamma_n$; in the latter case it is said to be crossed, and corresponds to $\pm\sin\gamma_n$. If the $(n+1)$st scattering plane is parallel (perpendicular) to the nth scattering plane, then only uncrossed (crossed) dotted lines are allowed, and they correspond to the value ±1.

(These rules can be proven very simply, just as we have proven the rules for dashed lines.)

When a multiple-scattering experiment is set up in such a way that any two successive scattering planes are either parallel or perpendicular to each other, then we call this a basic arrangement of scattering planes. Any more general arrangement can be treated as a linear superposition of basic ones. It is therefore sufficient to treat a complete set of basic arrangements. In practice basic arrangements are almost exclusively used.

In graphs corresponding to any basic arrangement of a multiple scattering experiment all allowed dotted lines correspond to the values ±1.

A path is a continuous line (as distinguished from a solid line). It contains at least one solid line. If it contains more than one solid line, then it contains one and only one dashed line, together with one and only one dotted line between any two successive solid lines. Two or more paths may partially overlap. A path starts at an ○ of the first column and ends at an × of the Nth column.

Suppose that the interaction to be studied is such that the known polarizing vector does not have a component parallel to the jth axis, and that the known analyzing vector does not have a component parallel

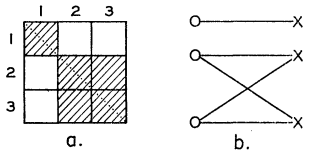

FIG. 4. (a) Scheme of the **Z** matrix for elastic scattering, when $P_\sigma+P_T$ invariance holds. The dotted line along the diagonal indicates that the off-diagonal elements of **Z** are antisymmetric. (b) Allowed solid lines corresponding to the shaded squares in Fig. 4(a).

to the mth axis. If such a j and/or m exist, then we say that any path which starts in the jth, and any path which ends in the mth row is spurious.

(A path which ends in the mth row corresponds to a term which affects the mth component of polarization of the incoming beam in the last scattering. By assumption this component does not influence the differential cross section of the last scattering, and therefore its presence can not be detected. For this reason such a path is called spurious. Similarly, a path which starts in the jth row corresponds to a term which would influence the differential cross section of the last scattering *if* a jth component of polarization could be produced in the polarizing scattering.)

To analyze an nth-order u–u scattering experiment, draw a graph which corresponds to this experiment and draw into it all paths which are not spurious and contain only allowed lines. Note all the $Z_n{}^{rs}$ elements corresponding to all solid lines drawn. These are those elements of $\mathbf{Z}_n$ on which the result of the experiment depend. Furthermore the result depends on terms, each of which is proportional to a product of the $Z_n{}^{rs}$ corresponding to all solid lines in one particular path. The actual value of the numerical coefficient is a product of those elements of $\mathbf{R}_n[(\mathbf{1})_{f,n}, \delta_n]$ and $\mathbf{R}_n[(\mathbf{3})_{1,n+1}, \gamma_n]$ which correspond to all dashed and dotted lines in that particular path. Experiments corresponding to graphs in which only spurious paths are allowed can not yield any information about **Z**.

To analyze a multiple-scattering experiment with polarized beams in general (not just u–u scattering), the rules are similar. The only difference is that the polarizing device is considered to be the first scattering and the analyzing device the last scattering. For example, assume that the polarizing device can produce only a polarization parallel to the first axis, and the analyzing device can detect only the first component of polarization. Draw the graph corresponding to all the experiments with polarized beams in which the beam is polarized by this polarizer, then scattered once, and finally analyzed by the analyzer just described. These graphs are exactly the same as the ones corresponding to a triple-scattering u–u experiment when P_σ invariance holds. The effect of magnetic fields, which rotate the polarization vector, can be taken into account by drawing additional dotted lines.

As an illustration, we discuss the case of elastic fermion–fermion u–u scattering when $P_\sigma+P_T$ invariance holds. As we know from rules 8, 9, 10 in Sec. 4.B, the polarizing vector is parallel to the first axis and the analyzing vector is equal to it. They can be determined, up to a sign, from double scattering experiments. Any path which is not spurious must start and end in the first row. The scheme of **Z**, as given by rule 10, is drawn in Fig. 4(a). The dotted line along the diagonal symbolizes that the off-diagonal part of the **Z** matrix is antisymmetric. The allowed solid lines (they correspond to the shaded squares in the scheme) are shown in Fig. 4(b).

The graphs corresponding to all basic arrangements of triple-scattering experiments are given in Fig. 5. The symbol ($\|\ \perp$) means that the triple scattering in question is such that the first scattering plane is perpendicular to the second, and the second is parallel to the third. (Scatterings are ordered from right to left in graphs.) In Fig. 5(a) there is one path which is allowed and not spurious. It contains a solid line corresponding to $Z_2{}^{11}$. Therefore the triple scattering experiment with the arrangement ($\|\ \|$) will depend on one term which is proportional to $Z_2{}^{11}$. In Fig. 5(b) there are two paths which are allowed and not spurious. They partially overlap; the first dashed line and both dotted lines belong to both paths. One path contains a solid line corresponding to $Z_2{}^{22}$, and the other contains one corresponding to $Z_2{}^{32}$. Therefore the result of a triple-scattering experiment with the arrangement ($\perp\ \perp$) will depend on two terms, one of them proportional to $Z_2{}^{22}$, the other proportional to $Z_2{}^{23}$. In other words, the result will determine a linear combination of $Z_2{}^{22}$ and $Z_2{}^{32}$. In fact, according to Eq. (40a), this linear combination is just $\mathbb{Z}_2{}^{22}$, because it connects the second component of $\mathbf{P}_{i,2}$, measured in the frame $K_{i,2}$ with the second component of polarization of $\mathbf{P}_{f,2}$ measured in $K_{3,i}$. The paths drawn in Fig. 5(c) and Fig. 5(d) are allowed, but spurious. There are no paths which are allowed and not spurious. Therefore, the results of experiments with arrangements ($\perp\ \|$) and ($\|\ \perp$) are independent of **Z**. We conclude that performing triple scattering experiments we can determine $Z_2{}^{11}$ and a linear combination of $Z_2{}^{22}$ and $Z_2{}^{32}$. To achieve this, the arrangements ($\|\ \|$) and ($\perp\ \perp$) have to be used, respectively. The other basic arrangements do not yield any information about **Z**.

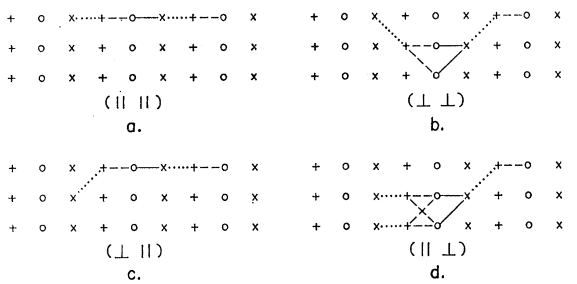

FIG. 5. The graphs shown correspond to all basic arrangements of elastic triple-scattering u–u experiments when $P_\sigma+P_T$ invariance holds. Construction of graphs is discussed in Sec. 4.C.

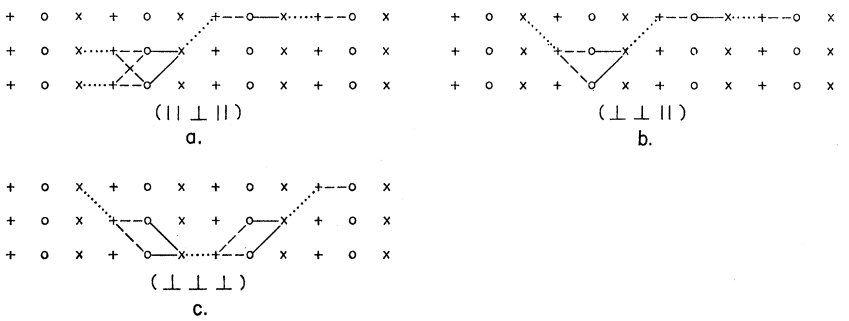

FIG. 6. The graphs shown correspond to three basic arrangements of elastic quadruple-scattering u–u experiments when $P_\sigma + P_T$ invariance holds.

There are eight basic arrangements for quadruple scattering. The graphs corresponding to three of them have been drawn in Fig. 6. The experiments with the basic arrangements (|| ⊥ ||) and (⊥ ⊥ ||) have been described in words before the rules of drawing graphs were given. We see immediately that Fig. 6(a) contains only spurious paths, and there are no allowed paths which are not spurious. Therefore, the arrangement (|| ⊥ ||) can not be used to determine anything about **Z**. In Fig. 6(b) there are two paths. They both contain a solid line corresponding to Z_2^{11}, therefore one can factor out Z_2^{11} from the total expression which corresponds to the two paths in this figure. We remember that Z_2^{11} can be determined by the experiment (|| ||). The other factor of the expression is a linear combination of Z_3^{22} and Z_3^{32}. We observe that this is the same linear combination as the one given by a triple scattering (⊥ ⊥), namely $\mathcal{Z}_3^{22}$. We conclude that the experiment (⊥ ⊥ ||) does indeed yield information about **Z**, but the quantity which it determines is a product of two other quantities both of which can be measured in triple scattering experiments. One can verify that only the basic arrangements (⊥ || ⊥) and (⊥ ⊥ ⊥) can yield information which can not be obtained by triple scattering experiments. The case (⊥ ⊥ ⊥) is illustrated in Fig. 6(c). The graph corresponding to case (⊥ || ⊥) is similar, but contains two additional paths. The quantity corresponding to these additional paths is a product $\mathcal{Z}_3^{22} \cdot \mathcal{Z}_2^{22}$. We know that both of the elements $\mathcal{Z}_2^{22}$ and $\mathcal{Z}_3^{22}$ can be determined from experiments of the (⊥ ⊥) type by appropriately choosing the c.m. energy and angle of the second scattering. Thus the quantity determined by the experiment (⊥ || ⊥), except for terms already known from lower order scattering experiments, is the same as the quantity determined by the experiment (⊥ ⊥ ⊥). This quantity, as the graph shows, is a product of two factors. One is a linear combination of Z_2^{22} and Z_2^{32}, and the other a linear combination of Z_3^{23} and Z_3^{33}. By keeping the c.m. energy and angle of the third scattering fixed, but varying those of the second scattering, we can determine the value of the first factor up to a constant. This, together with the result of the experiment (⊥ ⊥), provides us with two independent linear combinations of Z_2^{22} and Z_2^{32}, so Z_2^{22} and Z_2^{32} can be determined up to a constant. Varying the c.m. angle and energy of the third scattering, the second factor can also be determined up to a constant. Since $Z_n^{32} = -Z_n^{23}$, this gives us the value of Z_3^{33}, up to a constant. This way all elements of **Z** can be determined.

In practice, the graphs here described can be simplified. All the **+**'s can be omitted, and the difference between dashed and dotted lines is not essential. They were introduced only to make the discussion easier. More significant simplification can be achieved in some cases, if solid lines are made to correspond to elements of the matrix $\mathcal{Z}$ (and not **Z**).

D. Fermion–Spin-Zero-Boson Scattering

Fermion–boson scattering does not strictly speaking fall within the scope of this paper. Nevertheless, it is worthwhile to mention the fact that fermion–spin-zero-boson scattering can be treated as a degenerate case of fermion–fermion scattering. We agree to call the boson the target particle. (If it were the beam particle, then the beam could not be polarized. Experiments with polarized beams in this case are trivial and need not be discussed.) All our previous results are still valid if we impose the following two restrictions.

$$P_{ir,n}{}^{kh} = P_{ir,n}{}^{kh} \cdot \delta_{h0},$$
$$P_{fs,n}{}^{kh} = P_{fs,n}{}^{kh} \cdot \delta_{h0}. \tag{41}$$

These restrictions, loosely speaking, mean that a spin-zero boson can be treated as a fermion whose polarization is zero. All elements of the α matrix, except those in the first column, can now be considered to be zero, and we may write

$$\alpha_{\beta\gamma,n} = \alpha_{\beta\gamma,n}\delta_{\gamma 0} \equiv \alpha_{\beta,n}. \tag{42}$$

This means that the α matrix has degenerated into a column vector in index space. The most general fermion–boson (spin-zero) experiments are beam polarization experiments, since Eq. (41) states that the relations (22) and (23) always hold in this case. In other words, "a spin-zero-boson target can never be polarized." Therefore, all experimentally observable information is contained in the **A** matrix. The defining Eq. (26) together with Eq. (42) gives (dropping the subscripts n)

$$A_{\xi\mu} = \alpha_\xi{}^* \cdot \alpha_\mu. \tag{43}$$

Therefore, for example,

$$A_{\xi\mu}=A_{\xi 0}A_{0\mu}/A_{00} \qquad (A_{00}\neq 0) \tag{44}$$

has to hold. One consequence of Eq. (42) is that the **U** matrix in Eq. (27) is now simply a multiplicative phase factor. This means that for fermion–spin-zero-boson scattering, experiments with polarized beams can determine α up to a (physically trivial) phase factor. Of course, this was to be expected, since in this case no other experiments can be performed. The helicity principles defined for f–f scattering degenerate into two cases: either the beam particle switches helicity, or it does not.

Generally speaking, all the other results obtained for f–f scattering remain valid for fermion–spin-zero-boson scattering, and in addition to them Eq. (44) has to hold. For example, in elastic scattering, when P_T invariance holds, Eq. (44) demands that $A_{22}=0$, whereas in fermion–fermion scattering, under the same circumstances, A_{22} need not be zero.

5. EXPERIMENTS WITH POLARIZED TARGETS

Experiments with polarized targets are those in which the polarization vector of the i particle is zero, that of the f particle is not measured, and the polarization three-vectors of the target and beam particles are uncorrelated. Therefore, only those elements of Ω can be determined by these experiments which have the general form $\Omega_{0\eta,0\nu}$ with η and ν arbitrary.

The discussion of target polarization experiments follows the same lines as the discussion of beam-polarization experiments. In fact, all formulas concerning experiments with polarized targets can be obtained immediately from the analogous ones for experiments with polarized beams. To obtain these formulas, one must redefine what one means by the target and beam particle. We may call target (beam) particle that which used to be the beam (target) particle, if at the same time we exchange the order of factors in all direct products of two σ matrices. This exchange is to be performed according to our convention concerning notation as discussed in the second paragraph after Eq. (8). Equation (5) then requires that at the same time the following substitution be performed:

$$P_{ir}{}^{kh}\leftrightarrow P_{ir}{}^{hk}. \tag{45a}$$

Similarly, to satisfy Eqs. (6), (7), (11), and (15) we have to perform together with Eq. (45a) the substitutions

$$P_{fs}{}^{kh}\leftrightarrow P_{fs}{}^{hk}, \qquad \alpha_{\beta\gamma}\leftrightarrow\alpha_{\gamma\beta},$$
$$\Omega_{\xi\eta,\,\mu\nu}\leftrightarrow\Omega_{\eta\xi,\nu\mu}, \qquad [\omega(\xi\eta,\,\mu\nu)]_{\beta\gamma,\epsilon\kappa}\leftrightarrow[\omega(\eta\xi,\,\nu\mu)]_{\gamma\beta,\kappa\epsilon}. \tag{45b}$$

Let us call $\mathbf{P}_r$ the polarization vector of the r particle, $\mathbf{P}_s(uu)$ the polarization vector of the s particle when $\mathbf{P}_r=0$, and $\mathbf{P}_s(up)$ the polarization of the s-particle when $\mathbf{P}_r\neq 0$. Denote by $d\sigma(up)$ the differential scattering cross section when $\mathbf{P}_r\neq 0$. The expressions for $d\sigma(up)$, the kth component of $\mathbf{P}_s(uu)$, and $\mathbf{P}_s(up)$ are given by the right-hand sides of Eqs. (30), (32), and (33), respectively, if one performs in them the substitution (45) and at the same time writes $\mathbf{P}_r$ for $\mathbf{P}_i$. Care must be taken that the substitution (45) be carried out in every term, including Tr **A**, U^k, V^k, and Z^{ik}. This can be done by changing Tr **A**, U^k, V^k, and Z^{ik} into Tr **A**′, U'^k, V'^k, and Z'^{ik}, respectively, where the last three quantities are defined by the right-hand sides of Eqs. (31a), (31b), and (34), respectively, if in these equations we substitute $A'_{\xi\mu}$ for $A_{\xi\mu}$. The $A'_{\xi\mu}$ in turn are defined by the right-hand side of Eq. (26), if we substitute in this equation $\alpha_{\gamma\beta}$ for $\alpha_{\beta\gamma}$,

$$A'_{\xi\mu}=\sum_{\gamma=0}^{3}\alpha_{\gamma\xi}{}^{*}\alpha_{\gamma\mu}=\alpha^{+}\alpha. \tag{46}$$

In analogy with the discussion after Eq. (26) we conclude that experiments with polarized targets can determine all elements $A'_{\xi\mu}$, but nothing more.

Suppose that an α is found which satisfies Eq. (46). Then the most general α' which satisfies Eq. (46) is given by

$$\alpha'=\mathbf{U}\alpha \tag{47}$$

This equation differs from Eq. (27) only in the fact that the **U** appears to the left of α. Once again, if certain invariance principles hold, then the form of **U** may be restricted.

If $\alpha_{\gamma\beta}=\alpha_{\beta\gamma}$, then $A'_{\xi\mu}=A_{\xi\mu}$ and, therefore, experiments with polarized targets can determine only those parameters which can be determined by experiments with polarized beams. This is the case, for example, if we consider nucleon–nucleon scattering and assume that isotopic spin invariance ($\eta_\sigma=1$) and P_σ invariance hold. Then α has to be $P_{12}+P_\sigma$ invariant and this, by Eqs. (17) and (20), implies the symmetry of α.

6. GENERAL POLARIZATION EXPERIMENTS

In the last two sections we discussed two special classes of polarization experiments. Their importance is mainly historical, since in our formalism the discussion of more general experiments is no more difficult than the discussion of these special cases.[18]

In the experiments discussed in this section, both the i particles and the r particles may be polarized and their polarizations may be correlated. After the scattering has taken place, the following quantities may be measured: $d\sigma$, $\mathbf{P}_f$, $\mathbf{P}_s$ and $C_{fs}{}^{kh}$ (k, h=1, 2, 3). Here we use the notation introduced after Eq. (6).

The first line in each of the following equations follows from the definition of the corresponding quantities as given after Eq. (6). The second line in each equation

[18] After this paper had been completed a preprint by L. Durand III and Jack Sandweiss was called to my attention in which some of the questions discussed in Sec. 6 are considered.

follows from Eq. (11). The third line serves to define the coefficients appearing in them.[12]

$$d\sigma = P_{fs}^{00}/P_{ir}^{00}$$
$$= \sum_{h,j=1}^{3} (P_{ir}^{00})^{-1}\{\Omega_{00,00}+\Omega_{00,h0}P_{ir}^{h0}+\Omega_{00,0j}P_{ir}^{0j}+\Omega_{00,hj}P_{ir}^{hj}\}$$
$$= d\sigma(uu)\sum_{h,j=0}^{3}\left\{1+(U^k+V^k)P_i^k+(U'^j+V'^j)P_r^j+\frac{\Omega_{00,hj}}{\Omega_{00,00}}C_{ir}^{hj}\right\}, \quad (47a)$$

$$P_f^k = P_{fs}^{k0}/P_{fs}^{00}$$
$$= \sum_{h,j=1}^{3} (P_{fs}^{00})^{-1}\{\Omega_{k0,00}P_{ir}^{00}+\Omega_{k0,h0}P_{ir}^{h0}+\Omega_{k0,0j}P_{ir}^{0j}+\Omega_{k0,hj}P_{ir}^{hj}\}$$
$$= \frac{d\sigma(uu)}{d\sigma}\sum_{h,j=1}^{3}\left\{P_f^h(uu)+Z^{kh}P_i^h+\frac{\Omega_{k0,0j}}{\Omega_{00,00}}P_r^j+\frac{\Omega_{k0,hj}}{\Omega_{00,00}}C_{ir}^{hj}\right\}, \quad (47b)$$

$$P_s^h = P_{fs}^{0i}/P_{fs}^{00}$$
$$= \sum_{h,j=1}^{3}(P_{fs}^{00})^{-1}\{\Omega_{0i,00}P_{ir}^{00}+\Omega_{0i,h0}P_{ir}^{h0}+\Omega_{h0,0j}P_{ir}^{0j}+\Omega_{0i,hj}P_{ir}^{hj}\}$$
$$= \frac{d\sigma(uu)}{d\sigma}\sum_{h,j=1}^{3}\left\{P_s'^h(uu)+\frac{\Omega_{0i,h0}}{\Omega_{00,00}}P_i^h+Z'^{hj}P_r^j+\frac{\Omega_{0i,hj}}{\Omega_{00,00}}C_{ir}^{hj}\right\}, \quad (47c)$$

$$C_{fs}^{ki} = P_{fs}^{ki}/P_{fs}^{00}$$
$$= \sum_{h,j=1}^{3} (P_{fs}^{00})^{-1}\{\Omega_{ki,00}P_{ir}^{00}+\Omega_{ki,h0}P_{ir}^{h0}+\Omega_{ki,0j}P_{ir}^{0j}+\Omega_{ki,hj}P_{ir}^{hj}\}$$
$$= \frac{d\sigma(uu)}{d\sigma}\sum_{h,j=1}^{3}\left\{\frac{\Omega_{ki,00}}{\Omega_{00,00}}+\frac{\Omega_{ki,h0}}{\Omega_{00,00}}P_i^h+\frac{\Omega_{ki,0j}}{\Omega_{00,00}}P_r^j+\frac{\Omega_{ki,hj}}{\Omega_{00,00}}C_{ir}^{hj}\right\}. \quad (47d)$$

Of the coefficients appearing in the third lines of these equations, the quantities $d\sigma(uu)$, $(\mathbf{U}+\mathbf{V})$, $(\mathbf{U}-\mathbf{V})$, and $\mathbf{Z}$ have been defined in Sec. 4.A. The target polarizing vector $(\mathbf{U}'+\mathbf{V}')$, the target analyzing vector $(\mathbf{U}'-\mathbf{V}')$, and $\mathbf{Z}'$ have been defined in Sec. 5. The coefficient $\Omega_{00,hi}/\Omega_{00,00}$ could be called an element of the analyzing matrix. The quantities $\Omega_{ki,00}/\Omega_{00,00}$ are called the spontaneous correlations, because they determine the C_{fs}^{ki} if both $\mathbf{P}_i$ and $\mathbf{P}_r$ are zero, and if, at the same time, $\mathbf{P}_i$ and $\mathbf{P}_r$ are uncorrelated.

In the rest of this section we restrict the discussion to P_σ invariance, helicity invariance principles cases (a) through (f), and for elastic scattering P_T, $P_\sigma \cdot P_T$ and $P_\sigma + P_T$ invariance. We refer to any one of these invariance principles as IP.

We wish to determine the restrictions imposed by the IP on the quantities which can be measured in general fermion–fermion scattering polarization experiments. This is equivalent to finding the restrictions due to the IP on the elements of the Ω matrix. It is no more difficult to do this in general than in the special cases discussed in Secs. 4 and 5. One simply has to substitute into Eq. (12) the most general form of α permitted by the IP. Once the conditions imposed on Ω are found, they can be used to check experimentally whether or not the IP obtains in nature. The easiest restrictions to check experimentally are those which require that certain elements of Ω be zero. In the rest of this section we limit the discussion mainly to finding these restrictions. That is, we wish to find all those elements of Ω which are zero when a certain IP holds.[19]

To investigate all 256 elements of Ω for each IP would be tedious. Fortunately this can be avoided by the use of four theorems, given below, which are proved in Appendix 3.

To simplify the language, we introduce some expressions and conventions of notation that are used only in this section, and Appendix 3. As everywhere in this paper, all indices can take the values 0, 1, 2, and 3. We denote the pair of indices ξ and μ by (ξ, μ).

We say that (ξ, μ) belongs to the same family of indices as (ξ', μ') if either one of the following two conditions hold:

1. $\xi \neq \mu$ and $\delta(\xi, \mu \mid \xi', \mu') = 1$,[20] or else $\varepsilon_{\xi\mu\xi'\mu'} \neq 0$;
2. $\xi = \mu$ and $\xi' = \mu'$.

The family to which (ξ, μ) belongs is denoted by $\{\xi, \mu\}$. If (ξ', μ') belongs to the same family as (ξ, μ), we write $(\xi', \mu') \subset \{\xi, \mu\}$.

For example (0, 1), (1, 0), (2, 3), and (3, 2) all belong to the same family of indices and $(0, 1) \subset \{1, 0\}$. Similarly (0, 0), (1, 1), (2, 2), and (3, 3) belong to one family of indices. Clearly every pair of indices belongs to one and only one family of indices and every family of indices has four pairs of indices as members.

We say that $\Omega_{\xi\eta,\mu\nu}$ belongs to the same family (of matrix elements) as $\Omega_{\xi'\eta',\mu'\nu'}$ if $(\xi', \mu') \subset \{\xi, \mu\}$ and at

[19] We restrict the discussion to elastic scattering in case of an IP involving P_T invariance, because it is clear that in case of inelastic scattering these IP do not lead to restrictions of the type $\Omega_{\xi\eta,\mu\nu}=0$. For the same reason we do not discuss P_C invariance. However, as pointed out in footnote 14, once the restrictions imposed on α by these IP are understood, the relations between elements of α and α^T, or α and α^C are easily obtained. Consequently, relations between parameters of a scattering process, those of the time reversed and charge conjugate reactions can be derived simply. For example, using the notation of footnote 12, we find that if $P_\sigma \cdot P_C$ invariance holds, then $\Omega_{\xi\eta,\mu\nu}-\Omega_{\xi\eta,\mu\nu}{}^C=0$, whenever $\Omega_{\xi\eta,\mu\nu}=0$ is required by P_σ invariance.

[20] The symbol $\delta(\xi, \mu \mid \xi', \mu')$ has been defined in Eq. (16).

the same time $(\eta', \nu') \subset \{\eta, \nu\}$. The family to which $\Omega_{\xi\eta,\mu\nu}$ belongs is denoted by $\{\Omega_{\xi\eta,\mu\nu}\}$. If $\Omega_{\xi'\eta',\mu'\nu'}$ and $\Omega_{\xi\eta,\mu\nu}$ belong to the same family, we write $\Omega_{\xi'\eta',\mu'\nu'} \subset \{\Omega_{\xi\eta,\mu\nu}\}$.

For example, $\Omega_{02,12}$, $\Omega_{01,11}$, $\Omega_{21,31}$, etc. all belong to the same family. It follows from the definition that every element of Ω belongs to one and only one family. Every family has 16 members. Since the total number of elements of Ω is 256, there are altogether 16 families.

Suppose that we wish to perform a substitution in certain indices in the following manner. Whenever any one of the indices under consideration takes the value a change it to the value a'. Whenever one of the indices takes the value a', we change it to the value a. Similarly, whenever the indices take the value b, c, d (b', c', d'), we change them to b', c', d' (b, c, d), respectively. We denote such a substitution by $a, b, c, d \leftrightarrow a', b', c', d'$. For example, the substitution 0, 1, 2, 3$\leftrightarrow$1, 0, 3, 2, takes α_{01} and $\Omega_{02,03}$ over into α_{10} and $\Omega_{13,12}$, respectively. The substitution 0, 1, 2, 3$\leftrightarrow$2, 1, 0, 3 takes $\alpha_{\xi\mu}$ over into $\alpha_{\xi\mu}$ if $\xi=1, 3$ and $\mu=1, 3$, it takes $\alpha_{\xi\mu}$ over into $\alpha_{\mu\xi}$ if $\xi=0$, $\mu=2$, etc.

If the scheme of the matrix $\alpha_{\beta\gamma}$ is such that a substitution $a, b, c, d \leftrightarrow a', b', c', d'$ performed on the indices β and γ takes shaded (unshaded) squares into shaded (unshaded) squares we say that this substitution takes the scheme of α over into itself.

In this language, four simple theorems can be stated as follows:

Theorem 1. If one of the IP requires that $\Omega_{\xi\eta,\mu\nu}$ be zero, then it requires that $\Omega_{\xi'\eta',\mu'\nu'}$ also be zero whenever $\Omega_{\xi'\eta',\mu'\nu'} \subset \{\Omega_{\xi\eta,\mu\nu}\}$. (Note that if the IP does not require that $\Omega_{\xi\eta,\mu\nu}$ be zero, but by some accident $\Omega_{\xi\eta,\mu\nu}$ *is* zero, then the IP does not require that $\Omega_{\xi'\eta',\mu'\nu'}$ be zero.)

Theorem 2. If one of the IP requires that $\Omega_{\xi\eta,\mu\nu}$ be zero, then it requires that $\Omega_{\xi'\eta',\mu'\nu'}$ also be zero whenever $\Omega_{\xi'\eta',\mu'\nu'} \subset \{\Omega_{\eta\xi,\nu\mu}\}$.

Theorem 3. If the substitution $a, b, c, d \leftrightarrow a', b', c', d'$ takes the scheme of α required by one of the IP over into itself, and this IP requires that $\Omega_{\xi\eta,\mu\nu}$ be zero, then it also requires that $\Omega_{\xi'\eta',\mu',\nu'}$ be zero, where the set ξ', η', μ', ν' is obtained from the set ξ, η, μ, ν by the substitution $a, b, c, d \leftrightarrow a', b', c', d'$.

Theorem 4. Consider two of the IP. Denote them by $(\mathrm{IP})_1$ and $(\mathrm{IP})_2$. Suppose that the scheme of α required by $(\mathrm{IP})_1$ is such that the substitution $a, b, c, d \leftrightarrow a', b', c', d'$ takes it over into the scheme of α required by $(\mathrm{IP})_2$. Suppose further that $(\mathrm{IP})_1$ requires that a certain relation hold between the elements $\Omega_{\xi\eta,\mu\nu}$, $\Omega_{\beta\gamma,\delta\epsilon}$, $\cdots$. Then $(\mathrm{IP})_2$ requires that the same relation hold between $\Omega_{\xi'\eta',\mu'\nu'}$, $\Omega_{\beta'\gamma',\delta'\epsilon'}$, $\cdots$, where ξ', η', μ', ν'; $\beta', \gamma', \epsilon', \delta'$; $\cdots$ is obtained from ξ, η, μ, ν; $\beta, \gamma, \delta, \epsilon$; $\cdots$ by the substitution $a, b, c, d \leftrightarrow a', b', c', d'$. In particular, if $(\mathrm{IP})_1$ requires that $\Omega_{\xi\eta,\mu\nu}$ be zero, then $(\mathrm{IP})_2$ requires that $\Omega_{\xi'\eta',\mu'\nu'}$ be zero.

The first of these theorems shows that in order to find all $\Omega_{\xi\eta,\mu\nu}$ which are required to be zero by a certain IP, it is sufficient to determine whether or not a set of properly chosen sixteen elements $\Omega_{\xi\eta,\mu\nu}$ are zero. This set is properly chosen, if it contains one member of every family. Such a set is, for example, the set $\Omega_{00,\mu\nu}$ $(\mu, \nu=0, 1, 2, 3)$.

As an application of this, consider the case when the IP is such that it requires that the second component of $(\mathbf{U}-\mathbf{V})$ be zero. We know that this term is proportional to $\Omega_{00,20}$. Since $\Omega_{10,30} \subset \{\Omega_{00,20}\}$ and $\Omega_{30,10} \subset \{\Omega_{00,20}\}$ we conclude from Theorem 1 that $\Omega_{10,30}=\Omega_{30,10}=0$. These two elements of Ω are proportional to Z^{13}, Z^{31}, respectively. Therefore, $Z^{13}=Z^{31}=0$ holds in this case, in agreement with conclusions 1 and 3 in Sec. 4.B.

The problem is further simplified by Theorem 2, which relates members of different families. We find that, in general, it is sufficient to determine whether or not a set of ten (not 16) properly chosen elements of Ω is required to be zero by the IP. This set is properly chosen if $\{\Omega_{\xi\eta,\mu\nu}\} \neq \{\Omega_{\xi'\eta',\mu'\nu'}\}$ and $\{\Omega_{\xi\eta,\mu\nu}\} \neq \{\Omega_{\eta'\xi',\nu'\mu'}\}$, where $\Omega_{\xi\eta,\mu\nu}$ and $\Omega_{\xi'\eta',\mu'\nu'}$ are any two of the set of ten elements. For example, the following set is properly chosen:

$$(\Omega_{00,00}), \Omega_{00,10}, \Omega_{00,20}, \Omega_{00,30}, \Omega_{00,11}, \Omega_{00,21}, \Omega_{00,31}, \Omega_{00,22}, \Omega_{00,32}, \Omega_{00,33}. \quad (48)$$

We have put the first element into parentheses, because it is a trivial task to decide whether or not it is required to be zero. The element $\Omega_{00,00}$ is zero if, and only if, all elements of the α matrix are zero, that is, if there is no interaction whatsoever.

An interesting property of this set of elements is that they all appear as coefficients in Eq. (47a). Therefore, each of them can be determined experimentally at any c.m. energy and angle by a measurement of the differential cross section alone at that value of c.m. energy and angle. However, to do this it is necessary to set up the experiment in such a manner that it should be possible to control the correlation between the polarizations of the incoming beam and target particles.

As an application of Theorem 2 consider, once again, the case when the IP is such that it requires that the second component of $(\mathbf{U}-\mathbf{V})$ be zero. Then, using Theorem 2, we conclude that $\Omega_{01,03}$ and $\Omega_{03,01}$ have to be zero, i.e., $Z'^{13}=Z'^{31}=0$.[21]

Theorem 3 has to be applied for the case of each IP, individually. For example, if P_σ invariance holds, then the scheme of α is taken over into itself by any one of the two substitutions: 0, 1, 2, 3$\leftrightarrow$1, 0, 2, 3; 0, 1, 2, 3$\leftrightarrow$0, 1, 3, 2. The second of these substitutions exchanges $\Omega_{00,20}$, $\Omega_{00,22}$, $\Omega_{00,21}$ with $\Omega_{00,30}$, $\Omega_{00,33}$, $\Omega_{00,31}$, respectively. Therefore, in this case it is sufficient to investigate only seven (and not ten) elements of Ω. If we consider elastic scattering and assume that $P_\sigma P_T$ invariance holds, then the scheme of α is taken over

[21] The $\mathbf{Z}'$ has been defined in Sec. 5.

into itself by any one of the following three substitutions: 0, 1, 2, 3↔1, 0, 2, 3; 0, 1, 2, 3↔2, 1, 0, 3; 0, 1, 2, 3↔0, 2, 1, 3. In this case it is sufficient to investigate only four elements of Ω, for example $(\Omega_{00,00})$, $\Omega_{00,10}$, $\Omega_{00,12}$, and $\Omega_{00,11}$.

The fourth theorem explains, for example, the marked similarities in structure between P_σ invariance and helicity invariance case (a). These similarities are a consequence of the fact that the schemes given in Figs. 2(1) and 2(4) are taken over into each other by the following substitution 0, 1, 2, 3↔0, 3, 2, 1. Once conclusions 1, 2, 3 in Sec. 4.B are known, then using Theorem 4, the conclusions 11, 12, 13 follow. In the same way for elastic scattering, Theorem 4 relates P_T invariance and $P_\sigma \cdot P_T$ invariance. We find that conclusions 6 and 7 are consequences of conclusions 4 and 5 if we notice that the schemes shown in Fig. 2(2) and Fig. 3(1) are taken over into each other by (for example) the substitution 0, 1, 2, 3↔0, 1, 3, 2.

Summarizing, suppose we wish to determine which of the 256 elements of Ω are required to be zero by one of the IP. By Theorems 1 and 2 it is always sufficient to check which of a properly chosen set of ten elements are required to be zero by this IP. One such set is given in (48). It is usually possible to reduce further the number of elements to be investigated using Theorems 3 and 4.

The necessary and sufficient condition that an element $\Omega_{\xi\eta,\mu\nu}$ be required to be zero by one of the IP is that all products of the form $\alpha_{\xi'\eta'} \cdot \alpha_{\mu'\nu'}$ be required to be zero by that IP. Here, $(\xi', \mu') \subset \{\xi, \mu\}$ and $(\eta', \nu') \subset \{\eta, \nu\}$. This is a simple matter to check once the scheme of α is known.

In the rest of this section we consider some examples in detail.

P_σ *invariance*

As shown above, in this case we can reduce the number of elements to be investigated to seven. A properly chosen set of seven elements is $(\Omega_{00,00})$; $\Omega_{00,10}$; $\Omega_{00,20}$; $\Omega_{00,11}$; $\Omega_{00,21}$; $\Omega_{00,22}$; $\Omega_{00,32}$. The first three of these were already discussed in Sec. 4, where we found that $\Omega_{00,20}$ is required by P_σ invariance to be zero, but $(\Omega_{00,00})$ and $\Omega_{00,10}$ are not. Checking the other four, the result is that $\Omega_{00,21}$ is required to be zero, but $\Omega_{00,11}$, $\Omega_{00,22}$, $\Omega_{00,32}$ are not. We conclude that an element $\Omega_{\xi\eta,\mu\nu}$ is required to be zero by P_σ invariance if and only if it is a member of one of the following families: $\{\Omega_{00,20}\}$, $\{\Omega_{00,21}\}$, $\{\Omega_{00,30}\}$, $\{\Omega_{00,31}\}$, $\{\Omega_{00,02}\}$, $\{\Omega_{00,12}\}$, $\{\Omega_{00,03}\}$, $\{\Omega_{00,13}\}$. As expected $\Omega_{\xi\eta,\mu\nu}$ is required to be zero if the number of indices which take the value 0 or 1 is odd. The experimental consequences, other than those discussed in Sec. 4 are listed below. As usual we will write $\mathbf{P}_t$ for the polarization of the t particles $(t=i, f, r, s)$. That component of polarization which is perpendicular to the scattering plane is denoted by a subscript $\perp$ while the component of polarization lying in the scattering plane is denoted by a subscript $\|$.

Only the first component of the target polarization vector can be different from zero.

The target analyzing vector is parallel to the target polarizing vector.

(These rules are the analogs of rules 1 and 2 in Sec. 4.B. They apply to target particles.)

Suppose that the incoming (beam) particles are unpolarized and that the polarization of the incoming beam and target particles is uncorrelated. Then neither $\mathbf{P}_{\|i}$ nor $\mathbf{P}_{\|r}$ can influence any of the following four quantities: (1) $\mathbf{P}_{\perp f}$, (2) $\mathbf{P}_{\perp s}$, (3) the correlation between $\mathbf{P}_{\perp f}$ and $\mathbf{P}_{\perp s}$, and (4) the correlation between $\mathbf{P}_{\|f}$ and $\mathbf{P}_{\|s}$. (They may influence $\mathbf{P}_{\|f}$ and $\mathbf{P}_{\|s}$ separately.) Neither $\mathbf{P}_{\|f}$ nor $\mathbf{P}_{\|s}$ can be influenced by either $\mathbf{P}_{\perp i}$ or $\mathbf{P}_{\perp r}$.

Neither the correlation between $\mathbf{P}_{\|i}$ and $\mathbf{P}_{\perp r}$ nor the correlation between $\mathbf{P}_{\perp i}$ and $\mathbf{P}_{\|r}$ can influence any of the following five quantities: (1) the differential cross section, (2) $\mathbf{P}_{\|f}$, (3) $\mathbf{P}_{\|s}$, (4) the correlation between $\mathbf{P}_{\|f}$ and $\mathbf{P}_{\|s}$, and (5) the correlation between $\mathbf{P}_{\perp f}$ and $\mathbf{P}_{\perp s}$.

The spontaneous correlation between $\mathbf{P}_{\|f}$ and $\mathbf{P}_{\perp s}$ is zero. So is the spontaneous correlation between $\mathbf{P}_{\perp f}$ and $\mathbf{P}_{\|s}$.

Neither $\mathbf{P}_{\|f}$ nor $\mathbf{P}_{\|s}$ can be influenced by the correlation between $\mathbf{P}_{\|i}$ and $\mathbf{P}_{\|r}$, nor by the correlation between $\mathbf{P}_{\perp i}$ and $\mathbf{P}_{\perp r}$.

Neither the correlation between $\mathbf{P}_{\|f}$ and $\mathbf{P}_{\perp s}$ nor the correlation between $\mathbf{P}_{\perp i}$ and $\mathbf{P}_{\|r}$ can be influenced by any of the following four quantities: (1) $\mathbf{P}_{\perp i}$, (2) $\mathbf{P}_{\perp r}$, (3) the correlation between $\mathbf{P}_{\perp i}$ and $\mathbf{P}_{\perp r}$, and (4) the correlation between $\mathbf{P}_{\|i}$ and $\mathbf{P}_{\|r}$.

P_T *invariance, elastic scattering*

No elements of Ω are required to be zero.

$P_\sigma \cdot P_T$ *invariance, elastic scattering*

No elements of Ω are required to be zero.

$P_\sigma + P_T$ *invariance, elastic scattering*

This is a special case of P_σ invariance. Therefore it is sufficient to investigate the same seven elements of Ω as in the case of P_σ invariance. For elastic scattering $P_\sigma + P_T$ invariance requires that only those elements of Ω be zero which are required to be zero by P_σ invariance alone.

Helicity invariance case (a)

As shown earlier, the substitution 0, 1, 2, 3↔2, 3, 0, 1 takes the scheme of α required by P_σ invariance over into the scheme required by helicity invariance case (a). Using Theorem 4 we conclude that all the experimental consequences listed under P_σ invariance hold in this case, if we perform in them the following substitution: Write P_t^3 for $\mathbf{P}_{\perp t}$ and, at the same time, write "any linear combination of P_t^1 and P_t^2" for $\mathbf{P}_{\|t}$.

As usual, we have denoted the kth component of $\mathbf{P}_t$ by P_t^k ($k=1, 2, 3$).

Case (a)+P_σ invariance

The only families whose members are not required to be zero are $\{\Omega_{00,00}\}$, $\{\Omega_{00,11}\}$, $\{\Omega_{00,22}\}$, and $\{\Omega_{00,33}\}$. The experimental consequences of this fact could now be listed.

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APPENDIX 1: PROOF OF RESTRICTIONS IMPOSED ON THE α MATRIX BY INVARIANCE PRINCIPLES

In order to derive Eqs. (17), (18), (20), and (21), we consider first any center-of-mass frame of the scattering in question, and call it the basic frame. We will say that a quantity switches (does not switch) sign as a result of a certain operation if it switches the sign of its components in the basic frame under that operation.

Let us first perform P_σ, i.e., reverse all three axes of our basic frame. Under P_σ all polar three-vectors switch sign, and all axial three-vectors do not. The three-momentum being a polar three-vector, we find from Eq. (1) and the definition of K_t that for $t=i$, r, f, s the $(\mathbf{1})_t$ which is the first axis of K_t does not switch sign under P_σ, while $(\mathbf{2})_t$ and $(\mathbf{3})_t$ do switch sign under P_σ. (All K_t are by definition always right-handed, even after a P_σ has been performed in them.) The polarization of a particle measured in its rest frame is an axial three-vector, and therefore does not switch sign. From this it follows that P_t^1, the first component of $\mathbf{P}_t$ measured in K_t, does not change its sign, while the components P_t^2 and P_t^3 do. This result may be summarized by writing

$$P_t^k \xrightarrow{P_\sigma} \epsilon_\sigma^k P_t^k \qquad (k=1, 2, 3), \qquad \text{(A1-1)}$$

where ϵ_σ^k is defined to have the value $+1$ if $k=1$, and -1 if $k=2$, 3. We see from Eq. (3a) that P_t^k is the expectation value of σ_t^k; we conclude that P_σ takes over the expectation value of σ_t^k into $\epsilon_\sigma^k\sigma_t^k$. Equation (3b) shows that the expectation value of σ_t^0 is equal to N_t, a particle density, and therefore does not switch sign under P_σ. The last two conclusions can hold only if any two-component spinor $\varphi=\binom{a}{b}$ is taken over by P_σ into $\varphi_\sigma=\binom{b}{a}$ (times an arbitrary phase factor.) We have thus found the transformation law under P_σ of two-component spinors described in the frames K_t. Using this law together with Eq. (8), which gives the representation of σ_{fi}^μ, σ_{si}^μ ($\mu=0, 1, 2, 3$), we conclude that $\langle\varphi_1 \mid \sigma^\mu \mid \varphi_2\rangle$ goes over under P_σ into $\epsilon_\sigma^\mu\langle\varphi_1 \mid \sigma^\mu \mid \varphi_2\rangle$. Here σ^μ ($\mu=0, 1, 2, 3$) can have either the two subscripts f, i or s, r or, in fact, any subscripts; ϵ_σ^μ is defined to be $+1$ for $\mu=0$, and has been defined above for $\mu=1, 2, 3$ and the φ_1 and φ_2 are two arbitrary spinors. The matrix element of T taken between any initial and any final states is a sum of terms of the form $\langle\varphi_1 \mid \sigma^\beta \mid \varphi_2\rangle\langle\varphi_3 \mid \sigma^\gamma \mid \varphi_4\rangle$. According to the foregoing discussion, any such term will go under P_σ into $\epsilon_\sigma^\beta\cdot\epsilon_\sigma^\gamma$ times itself. If T is to be invariant under P_σ, then the coefficient $\alpha_{\beta\gamma}$ of only those terms can be nonzero for which $\epsilon_\sigma^\beta\cdot\epsilon_\sigma^\gamma=\eta_\sigma$, which is the condition expressed by Eq. (17).

Next we consider the effect of a Wigner time-reversal operation, P_T. This operation will change the incoming (outgoing) state with three-momentum $\mathbf{k}_i$ ($\mathbf{k}_f$) and polarization $\mathbf{P}_i$ ($\mathbf{P}_f$) into an outgoing (incoming) state with three-momentum $\mathbf{k}_i{}^T$ ($\mathbf{k}_f{}^T$) and polarization $\mathbf{P}_i{}^T$ ($\mathbf{P}_f{}^T$) such that

$$\mathbf{k}_i{}^T=-\mathbf{k}_f, \qquad \mathbf{k}_f{}^T=-\mathbf{k}_i,$$
$$\mathbf{P}_i{}^T=-\mathbf{P}_f, \qquad \mathbf{P}_f{}^T=-\mathbf{P}_i. \qquad \text{(A1-2)}$$

The axes of K are defined by Eq. (1). Substituting Eq. (A1-2) into this definition one finds that (1) and (3), the first and third axes of K, both switch sign under P_T, but (2) does not. Remembering the definition of K we conclude that $(\mathbf{1})_t$ and $(\mathbf{3})_t$ do switch sign under P_T but $(\mathbf{2})_t$ does not. From this and the second line of (A1-2) it follows that $P_i{}^{Tk}$, the kth component of polarization of the i particle in the time reversed reaction (measured, as always, in its own rest frame) is $\epsilon_T{}^k$ times $P_f{}^k$, the kth component of polarization of the f particle in the original reaction. Here $\epsilon_T{}^k$ is $+1$ for $k=1$, 3, and is -1 for $k=2$. The expectation value of $\sigma_i{}^0$ which is equal to a density, is changed into $\epsilon_T{}^0$ times the expectation value of $\sigma_f{}^0$, where $\epsilon_T{}^\mu$ is $+1$ for $\mu=0$. An argument very similar to the one employed in the case of P_σ leads to the conclusion that an expression $\langle\varphi_f \mid \sigma^\beta \mid \varphi_i\rangle\langle\varphi_s \mid \sigma^\gamma \mid \varphi_r\rangle$ goes over under P_T into $\epsilon_T{}^\beta\cdot\epsilon_T{}^\gamma\cdot\langle\varphi_i \mid \sigma^\beta \mid \varphi_f\rangle\langle\varphi_r \mid \sigma^\gamma \mid \varphi_s\rangle$. Time-reversal invariance of T requires that only those $\alpha_{\beta\gamma}$ coefficients can be nonzero, for which $\epsilon_T{}^\beta\cdot\epsilon_T{}^\gamma=+1$. From this requirement Eq. (18) follows.

One of the effects of P_{12} is described by Eq. (19a). Since in any center-of-mass frame momentum conservation requires $\mathbf{k}_i=-\mathbf{k}_r$, $\mathbf{k}_f=-\mathbf{k}_s$, Eq. (1) shows that under the operation P_{12} the (2) and (3) switch sign, while (1) does not. Using once again the definition of K_t we conclude that $(\mathbf{2})_t$ and $(\mathbf{3})_t$ switch sign under P_{12} while $(\mathbf{1})_t$ does not. This is just as in the case of space-parity inversion, and as in that case results in multiplying each term on the right-hand side of Eq. (7) by a factor $\epsilon_\sigma^\beta\cdot\epsilon_\sigma^\gamma$.

The other effect of P_{12} is expressed by Eq. (19b). This equation is equivalent to the statement that P_{12} exchanges the spin spaces of the beam and target particles. Thus in every term on the right-hand side of Eq. (7) the operation P_{12} exchanges the two factors in a direct product of two matrices. By renaming the dummy indices in the double sum we see that this is equivalent to exchanging the two indices of $\alpha_{\beta\gamma}$.

The total effect of P_{12} on each term on the right-hand side of Eq. (7) is to multiply it by $\epsilon_\sigma{}^\beta \cdot \epsilon_\sigma{}^\gamma$ and exchange the two indices of α. The condition that a term be invariant under this operation is expressed by Eq. (20).

Turning to the helicity invariance principles, we recall from the discussion after Eq. (4) that the helicity of any particle (as viewed from K) is always the third component of polarization of that particle. If helicity is conserved for the beam (target) particle, then the expectation value of σ^3 referring to the beam (target) particle does not change during scattering. Terms in Eq. (7) in which $\sigma_{if}{}^0$, $\sigma_{if}{}^3$ appear do not change the helicity of the beam particle, while terms in which $\sigma_{if}{}^1$, $\sigma_{if}{}^2$ appear, do. A similar statement can be made about the helicity of the target particle. Eqs. (21) follow from these observations.

APPENDIX 2: THE OPERATION P_{12}

The operations E_{ir} and E_{fs} have been defined in the main text after Eq. (20). We clarify this definition by two examples. In neutron–proton elastic scattering E_{ir} is the operation which changes a neutron into a proton, a proton into a neutron, leaving their momenta and polarizations unchanged, and E_{fs} is the same as E_{ir}. In this case both E_{ir} and E_{fs} may be identified with V_2, the operation which rotates any vector by 180° around the second axis in isotopic-spin space. In the reaction $n+\bar{n}\to\Lambda^0+\bar{\Sigma}^0$ the operation E_{ir} exchanges an n and an $\bar{n}$, while E_{fs} exchanges a Λ^0 and a $\bar{\Sigma}^0$. The E_{ir} can now be identified with charge conjugation, but E_{fs} has no special name.

We denote the incoming two-fermion state by $|i, r\rangle$ and the outgoing two-fermion state by $|f, s\rangle$. The operation $P_{12}\cdot E_{ir}$ completely exchanges the two fermions in $|i, r\rangle$ and thus multiplies the state by a factor (-1). Using the fact that $(P_{12})^2=1$, we find[22]

$$E_{ir}\,|\,i, r\rangle=-P_{12}\,|\,i, r\rangle. \tag{A2-1}$$

Assuming that the S matrix satisfies

$$[\mathcal{E}_{fs}E_{fs}\,|\,fs\rangle]^+S[\mathcal{E}_{ir}E_{ir}\,|\,ir\rangle], \tag{A2-2}$$

where $\mathcal{E}_{ir}$ and $\mathcal{E}_{fs}$ are two suitable operators, we find from Eq. (A2-1) and the similar equation for $|f, s\rangle$

$$[\mathcal{E}_{fs}P_{12}\,|\,f, s\rangle]^+S[\mathcal{E}_{ir}P_{12}\,|\,ir\rangle]=\langle fs\,|\,S\,|\,ir\rangle. \tag{A2-3}$$

If $\mathcal{E}_{ir}$ is the same as $\mathcal{E}_{fs}$ then we may denote both of them by the symbol $\mathcal{E}$. In this special case Eq. (A2-3) says that the matrix elements $\langle fs\,|\,S\,|\,ir\rangle$ is invariant under $\mathcal{E}\cdot P_{12}$.

We illustrate this simple rule by some examples. Consider first the case in which two identical fermions scatter elastically. Both E_{ir} and E_{fs} are now equal to unity and Eq. (A2-1) simply states the Pauli principle. Choosing $\mathcal{E}_{ir}=\mathcal{E}_{fs}=\mathcal{E}$ also to be unity we have $\mathcal{E}E=1$. Since any S matrix is trivially invariant under the unit operation, our rule tells us that in such an experiment any S matrix has to be P_{12} invariant. As another example, consider neutron–proton elastic scattering, which we have discussed in the first paragraph of this appendix. If the S matrix is invariant under V_2, which is usually assumed for strong interactions, then we can choose $\mathcal{E}=1$ and our rule tells us that the S matrix has to be P_{12} invariant. In the case of proton–antineutron scattering, E can be chosen to be P_G, the G-parity inversion operation. Again, if the S matrix is invariant under P_G, then with the choice $\mathcal{E}=1$ we find that the S matrix has to be P_{12} invariant. On the other hand, if the S matrix is not invariant under P_G, then this conclusion may be false. For example, if the S matrix is isotopic-spin invariant, time-reversal invariant, but not invariant under charge conjugation, and if furthermore the CPT theorem holds, then it is not invariant under P_G, but it is invariant under $P_\sigma\cdot P_G$. Choosing $\mathcal{E}=P_\sigma$ our rule tells us that it is also invariant under $P_\sigma\cdot P_{12}$.

APPENDIX 3: PROPERTIES OF THE ω AND Ω MATRICES

This appendix uses the notation introduced in Sec. 6. Any $\omega(\xi\eta, \mu\nu)$ matrix can be written as a direct product

$$[\omega(\xi\eta, \mu\nu)]_{\beta\gamma,\epsilon\kappa}=\omega(\xi\mu)_{\beta\epsilon}\otimes\omega(\eta\nu)_{\epsilon\kappa}, \tag{A3-1}$$

where

$$\begin{aligned}\omega(\xi\mu)_{\beta\epsilon}&=\mathrm{Tr}\,(\sigma^\xi\sigma^\beta\sigma^\mu\sigma^\epsilon)\\&=\{-\delta_{\xi\mu}\varepsilon(\xi\,|\,0)\delta_{\beta\epsilon}\epsilon(\beta\,|\,0)+\delta_{\beta\mu}\delta_{\epsilon\xi}+\delta_{\beta\xi}\delta_{\epsilon\mu}\\&\quad+i\varepsilon_{\xi\beta'\mu\epsilon'}(\delta_{\beta\beta'}\delta_{\epsilon\epsilon'}-\delta_{\beta\epsilon'}\delta_{\epsilon\beta'})\,\varepsilon(0\,|\,\xi\mu)\}.\end{aligned} \tag{A3-2}$$

Equation (A3-2) shows that all sixteen 4×4 $\omega(\xi\mu)$ matrices are Hermitian and unitary. They have one and only one nonzero element in every row and every column. They form a complete linearly independent set of 4×4 matrices. Up to a common unitary transformation they can all be written as a direct product of two σ^μ ($\mu=0$, 1, 2, 3) matrices. We also see from Eq. (A3-2), that

$$\omega(\mu\xi)_{\beta\epsilon}{}^*=\omega(\xi\mu)_{\beta\epsilon}, \tag{A3-3}$$

[22] Equation (A2-1) contains the simple Pauli principle, and what is commonly known as the "generalized Pauli principle" as special cases.

i.e., the ω are "Hermitian" under the exchange of μ and ξ. It is also clear from Eq. (A3-2) that

$$\omega(\xi\mu)_{\beta\epsilon}=\exp(i\varphi_{\beta\epsilon})\omega(\xi'\mu')_{\beta\epsilon} \quad \text{if} \quad (\xi',\mu')\subset\{\xi,\mu\}, \tag{A3-4}$$

where $\varphi_{\beta\epsilon}$ is a phase factor which may depend on the indices ξ, μ, ξ', μ'.

We define $\tilde{\omega}(\xi\eta,\mu\nu)$, the transpose of $\omega(\xi\eta,\mu\nu)$, by the following relation.

$$\tilde{\omega}(\xi\eta,\mu\nu)_{\beta\gamma,\epsilon\kappa}\equiv\omega(\xi\eta,\mu\nu)_{\epsilon\kappa,\beta\gamma}. \tag{A3-5}$$

With this definition, using Eq. (A3-1) and the analogous properties of the $\omega(\xi\eta)$ matrices, we find that the $\omega(\xi\eta,\mu\nu)$ are Hermitian and unitary. They have one and only one nonzero element in each row and each column. They form a complete linearly independent set of 16×16 matrices. Among 16×16 matrices they play a role analogous to the role of the 16 Γ matrices among 4×4 matrices. Equations (A3-3) and (A3-1) give

$$\omega(\mu\nu,\xi\eta)_{\beta\gamma,\epsilon\kappa}{}^{*}=\omega(\xi\eta,\mu\nu)_{\beta\gamma,\epsilon\kappa}. \tag{A3-6}$$

In analogy to Eq. (A3-4) we find

$$\omega(\xi\eta,\mu\nu)_{\beta\gamma,\epsilon\kappa}=\exp(i\varphi_{\beta\gamma\epsilon\kappa})\omega(\xi'\eta',\mu'\nu')_{\beta\gamma,\epsilon\kappa} \tag{A3-7}$$

if $(\xi',\mu')\subset\{\xi,\mu\}$ and $(\eta',\nu')\subset\{\eta,\nu\}$.

From Eq. (15) and the Hermiticity of $\omega(\xi\eta,\mu\nu)$ we find that all $\Omega_{\xi\eta,\mu\nu}$ are real.

To prove the theorems stated in Sec. 6, we notice that the restrictions imposed on α by any of the IP are always such that all elements of α which are not required to be zero are independent of each other. From this fact, it follows that all nonzero terms in the expansion of $\Omega_{\xi\eta,\mu\nu}$ on the right-hand side of Eq. (15) are independent of each other. Therefore, the only way one of the IP can require that $\Omega_{\xi\eta,\mu\nu}$ be zero is to require that all terms in that expansion vanish. Suppose that this is the case.

It follows from Eq. (A3-7) that if $(\xi',\mu')\subset\{\xi,\mu\}$ and, at the same time, $(\eta',\nu')\subset\{\eta,\nu\}$, then all coefficients $\omega(\xi'\eta',\mu'\nu')_{\beta\gamma,\epsilon\kappa}$ in the expansion of $\Omega_{\xi'\eta',\mu'\nu'}$ are zero if and only if all the $\omega(\xi\eta,\mu\nu)_{\beta\gamma,\epsilon\kappa}$ in the expansion of $\Omega_{\xi\eta,\mu\nu}$ are zero. Therefore, in this case, the expansion of $\Omega_{\xi\eta,\mu\nu}$ and $\Omega_{\xi'\eta',\mu'\nu'}$ contain the same terms $\alpha_{\xi''\eta''}\alpha_{\mu''\nu''}{}^{*}$. Since all terms in the expansion of $\Omega_{\xi\eta,\mu\nu}$ are zero by assumption, all terms in the expansion of $\Omega_{\xi'\eta',\mu'\nu'}$ also have to be zero. This proves Theorem 1.

Equation (A3-6) shows that the terms appearing in the expansion of $\Omega_{\mu\nu,\xi\eta}$ are up to a phase factor the same terms which appear in the expansion of $\Omega_{\xi\eta,\mu\nu}$. This proves Theorem 2.

The proofs of Theorems 3 and 4 follow from Eq. (A3-6) and the hermiticity of $\omega(\xi\eta,\mu\nu)$. These two properties imply that

$$\omega(\xi\eta.\,\mu\nu)_{\beta\gamma,\epsilon\kappa}=\omega(\mu\nu,\xi\eta)_{\epsilon\kappa,\beta\gamma}. \tag{A3-8}$$