degenerate spin-zero mesons, belonging to the $(\mathbf{3}^+\otimes\overline{\mathbf{3}}^-+\overline{\mathbf{3}}^+\otimes\mathbf{3}^-)$ representation of $\tilde{S}^{00}$. This symmetry group can be reduced by an SU_3-invariant interaction to $\tilde{S}^0=SU_3(+,-)$, the group of simultaneous SU_3 transformations on the right- and left-handed quarks which is generated by their total SU_3-spin, $F_i=F_i^+ + F_i^-$. F_i is so constructed that it does commute with space-inversion, and the 18 level consequently splits into four, a scalar singlet and octet and a pseudoscalar singlet and octet. Finally, adding an SU_3-breaking interaction splits the octets further into the conventional I-spin multiplets.

An era of model-building clearly lies ahead.

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Rotations and Vibrations in Deformed Nuclei

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A general survey of phenomenological, collective-model theories, especially those based upon the hydrodynamic model of rotations and vibrations of a deformed liquid drop, is presented. Details of models of even–even, odd–odd and odd-A nuclei are given as well as a discussion of electromagnetic moments and transitions in these models. The influence the vibrating and rotating nuclear surface has on alpha and beta decay of deformed nuclei is outlined. The discussion is extended to collective model calculations of the photonuclear process from which information can be obtained on the shape of nuclei in their ground states. Comparison of theory with the results of some recent experiments is given where this has been considered useful.

CONTENTS

I. INTRODUCTION

The need to consider a deformation degree of freedom, and thus the low-energy collective rotations and vibrations of atomic nuclei, occurred some time ago with the first attempts to explain the phenomenon of nuclear fission. Immediately after its discovery, and essentially at the same time, Feenberg (Fe 39) and Bohr and Wheeler (Bo 39) studied the shape and stability of a deformed and charged liquid drop and showed how the total energy changed as a function of deformation from the spherical equilibrium shape. Feenberg's original note was concerned with demonstrating the stability of nuclei against spontaneous fission, while Bohr and Wheeler investigated in detail the theory of the process. The nuclear instability arises from the fact that the total energy considered in this process is the sum of surface and Coulomb energies [ideas originally introduced by Weizzäcker (We 35) in the empirical binding energy formula] and distortions from spherical increase the surface area, and hence the surface energy, while necessarily decreasing the Coulomb energy. Long ago Lord Rayleigh considered a similar classical problem in his studies on the stability of electrically charged liquid drops (St 82), and some of the early work on the hydrodynamical model of the nucleus parallels quite closely his theoretical investigation.

While it has been usual to consider the axially symmetric deformation from the spherical equilibrium shape, a more general treatment is only slightly more involved. Following Feenberg's notation (Fe 39) we may define an ellipsoid of volume equal to that of a sphere of radius R_0 by

$$\lambda x^2+\kappa y^2+z^2/\lambda\kappa=R_0^2, \qquad \text{(I-1)}$$

where the eccentricities may be taken as

$$e_1=(a-c)/R_0\cong 2(1-\sqrt{\lambda})+(1-\sqrt{\kappa}),$$
$$e_2=(b-c)/R_0\cong(1-\sqrt{\lambda})+2(1-\sqrt{\kappa}), \qquad \text{(I-2)}$$

and the semi-axes are made definite by requiring $a>b>c$. For a deformed liquid drop, the change in Coulomb plus surface energies is then, to lowest order

$$\Delta E=(4/45)(2E_s^0-E_c^0)(e_1^2-e_1e_2+e_2^2), \qquad \text{(I-3)}$$

where E^0 is the spherical energy in question. For axial symmetry this reduces to the usual value (Fe 39). Treating the eccentricities as variational parameters we see that the spherical shape is the most stable. From

these arguments one could conclude that the equilibrium and low-energy shape of even–even atomic nuclei was spherical and because of this they would, in general, not display a low-energy rotational level structure.

The next step occurred at the time the shell model was being rapidly developed and a large number of ground-state nuclear electric quadrupole moments had been measured. That a relation between these moments and nuclear shell structure existed was suggested by Gordy (Go 49) who showed that near closed shells the quadrupole moment was small, while in the middle of the shells it was fairly large. However, Townes, Foley, and Low (To 49) demonstrated conclusively that the large quadrupole moments observed near the middle of the shells, especially in the rare-earth region, were not compatible with the shell model as they required contributions from a large number of the protons and not just one. That a simple explanation for these large nuclear polarizations existed was shown by Rainwater (Ra 50) who demonstrated that a single particle moving in a potential well had a lower energy if the well were deformed than if it were spherical. In fact, the change in particle energy due to the distortion is proportional to the eccentricity; hence a nonspherical deformation is possible.

To demonstrate this, we generalize a perturbation calculation by Feenberg and Hammack (Fe 51) and consider a particle moving in an ellipsoidal potential well of the form

$$V(t)=-D \qquad t<R_0$$
$$=0 \qquad t\geq R_0,$$

where R_0 is defined in Eq. (I-1). Then the change in energy due to the deformation of the well from the spherical is just

$$\Delta E_p=E(e_1,e_2)-E(0)$$
$$=\int |\psi_0(xyz)|^2[V(t)-V(r)]\,dx\,dy\,dz$$
$$=DR_0^3R_{nl}\frac{l(l+1)-3m^2}{3(2l-1)(2l+3)}, \qquad \text{(I-4)}$$

where the unperturbed state functions are

$$\psi_0(r,\theta,\phi)=R_{nl}(r)\,Y_{lm}(\theta,\phi),$$

the $Y_{lm}(\theta,\phi)$ being the usual spherical harmonics of order l. The total energy for an odd-A nucleus is then just the sum of Eqs. (I-3) and (I-4) or

$$\Delta E=A(e_1^2-e_1e_2+e_2^2)+B(e_1+e_2).$$

Again treating the eccentricities as variational parameters yields the fact that

$$e_1=e_2=-B/A,$$

that is, the system will assume a deformed but symmetric equilibrium shape.

The development of these ideas and the consequences derived from them have proceeded rapidly in the past decade until now the regions of applicability of this unified, or collective, hydrodynamic model are quite well known and progress is being made on the more fundamental understanding of the parameters of the model. This would seem then to be an opportune time to review the successes and failures of this phenomenological, unified model and to attempt to bring together some of the great mass of experimental information which can be codified by use of the model. In the remainder of this section the theoretical foundations of the model are reviewed, especially the theory of the surface oscillations of a liquid drop. In the following sections, the model is applied to even–even, odd, and odd–odd–A nuclei to discuss the energy-level structure as well as the alpha-, beta-, and gamma-ray transitions in such nuclei.

Finally, a few words are said concerning the role the model plays in the theory of photonuclear reactions in heavy nuclei.

Because of the very rapid advances being made in understanding and applying the effects of pairing forces to nuclear matter, this is perhaps a poor time to include the results and predictions of these calculations in this review. Therefore, only passing reference is made to these calculations in places where they are of fundamental importance. The reader is referred to a recent article (Ki 63) and a book on the subject (La 64).

Since Rainwater's suggestion indicates that a nuclear system of a single particle (or perhaps a group of them) coupled to a core may achieve a relative minimum energy configuration if the core is deformed from spherical, we consider the classical theory of the surface vibrations of a liquid drop. By a liquid drop we mean a system whose deformations are volume preserving ones since nuclear matter is considered essentially incompressible. While keeping the discussion as general as possible, we assume that the oscillations are about a spherical equilibrium shape. In general we follow the treatment by Bohr (Bo 52) and point out where this parallels the classical treatment.

Let the nuclear surface be represented by $S^L(\theta,\phi)$ in a coordinate system fixed in the laboratory. Then if R_0 is the radius of the undistorted nuclear surface of the same volume (assumed spherical), the surface can be expanded in spherical harmonics (St 79)

$$S^L(\theta,\phi)=R_0[\alpha_0+\sum_{\lambda>1,\mu}\alpha^*_{\lambda,\mu}Y_{\lambda\mu}(\theta,\phi)], \qquad \text{(I-5L)}$$

where α_0 differs from unity by small quantities of the second order. On the other hand, the surface shape could be expanded with respect to a coordinate system fixed in the nucleus (the body-fixed system) or

$$S^B(\theta',\phi')=R_0[a_0+\sum_{\lambda>1,\nu}a^*_{\lambda,\nu}Y_{\lambda\nu}(\theta',\phi')]. \qquad \text{(I-5B)}$$

From these expansions we see that the $S^L(\theta,\phi)$ and

$S^B(\theta', \phi')$ represent the magnitude of the radius vector to the nuclear surface. In both expansions, $\lambda=1$ terms vanish as they represent simple translations of the nuclear surface, a type of motion which does not concern us here (see Sec. V).

The laboratory and body-fixed expansion coefficients are related by

$$\alpha_{\lambda,\mu}=\sum_{\nu} D_{\mu,\nu}^{\lambda *}(\theta_i)\, a_{\lambda,\nu}, \tag{I-6}$$

where the $D_{\mu,\nu}^{\lambda}(\theta_i)$ are the $(2\lambda+1)$-dimensional representations of the rotation group defined by Rose (Ro 57) being functions of the Euler angles relating the orientation of the body-fixed system with respect to the laboratory system.[1] The reality of the nuclear surface imposes the condition

$$\alpha_{\lambda,-\mu}=(-1)^{\mu}\alpha^*_{\lambda,\mu} \tag{I-7}$$

and similarly for $a_{\lambda,\mu}$.

Making use of the classical theory of small oscillations we may take the $\alpha_{\lambda,\mu}$ as generalized coordinates so that in the laboratory the kinetic energy is represented by

$$T^L=\tfrac{1}{2}\sum_{\lambda,\mu} B_{\lambda,\mu}\,|\,\dot{\alpha}_{\lambda,\mu}\,|^2. \tag{I-8}$$

On the other hand, assuming the potential energy of the nucleus to consist of surface (attractive) and Coulomb (repulsive) terms we can write it as

$$V=\tfrac{1}{2}\sum_{\lambda,\mu} C_{\lambda,\mu}\,|\,\alpha_{\lambda,\mu}\,|^2. \tag{I-9}$$

The kinetic and potential energy expansion coefficients are independent of μ if the surface is expanded about a spherical equilibrium shape and the nuclear flow is taken to be irrotational.

Since the potential energy is conservative a Lagrangian may be defined in the usual way and from it a momentum $\pi_{\lambda,\mu}$, conjugate to the coordinate $\alpha_{\lambda,\mu}$. This in turn permits one to determine the Hamiltonian for the classical motion:

$$H=\tfrac{1}{2}\sum_{\lambda,\mu}(1/B_\lambda)\,|\,\pi_{\lambda,\mu}\,|^2+\tfrac{1}{2}\sum_{\lambda,\mu} C_\lambda\,|\,\alpha_{\lambda,\mu}\,|^2=\sum_{\lambda} H_\lambda. \tag{I-10}$$

The coefficients for the spherical case are (St 79, St 82)

$$B_\lambda=\rho R_0^5/\lambda, \tag{I-11a}$$

$$C_\lambda=(\lambda-1)(\lambda+2)R_0^2 S-[3(\lambda-1)Z^2e^2/2\pi(\lambda+1)R_0], \tag{I-11b}$$

where ρ is the nuclear density, assumed constant, S the surface tension, and Ze the nuclear charge.

The Hamiltonian of Eq. (I-10) is that for a system of uncoupled harmonic oscillators of frequency $\omega_\lambda=(C_\lambda/B_\lambda)^{\frac{1}{2}}$.

[1] We use the same Euler angles as Rose with $\theta_{1,2,3}=\alpha, \beta, \gamma$ to avoid confusion later, and also the spherical harmonic phases of Condon and Shortley (Co 57).

The angular momentum of the nucleus $\mathbf{L}$ is defined classically from the nuclear velocity field $\mathbf{v}(\mathbf{r})$

$$\mathbf{L}=\int \rho \mathbf{r}\times\mathbf{v}(\mathbf{r})\, d\tau, \tag{I-12}$$

the integral being over the nuclear volume. The assumption of irrotational flow implies the existence of a velocity potential which, because of the constancy of the density, is a solution of Laplace's equation. For small distortions about a spherical equilibrium shape this potential function is (St 79)

$$\chi(\mathbf{r})=-R_0^2\sum_{\lambda,\mu}(\alpha^*_{\lambda,\mu}/\lambda)\,(r/R_0)^{\lambda}Y_{\lambda\mu}(\theta,\phi) \tag{I-13}$$

with

$$\mathbf{v}(\mathbf{r})=-\ \mathbf{grad}\ \chi(\mathbf{r}).$$

Substitution of Eq. (I-13) into Eq. (I-12) yields

$$\mathbf{L}=i\sum_{\lambda\mu\mu'} B_\lambda \dot{\alpha}^*_{\lambda\mu}\alpha_{\lambda\mu'}\langle\lambda\mu'\,|\,\mathbf{L}\,|\,\lambda\mu\rangle, \tag{I-14}$$

where

$$i\langle\lambda'\mu'\,|\,\mathbf{L}\,|\,\lambda\mu\rangle=\int Y^*_{\lambda'\mu'}(\theta,\phi)\,(\mathbf{R}_0\times\nabla\chi)\,Y_{\lambda\mu}(\theta\phi)\,d\Omega$$

has been used which is diagonal in λ.

The theory may be quantized in two different ways. If the number of nucleons outside the deformable core is small, then the zero-point energy of the oscillations is greater than the energy of deformation and the core shape, on the average, does not possess a stable deformation and we must deal with a theory of quantized surface oscillations (Bo 52, Sc 55). On the other hand, for a larger number of particles coupled to the deformable core, the zero-point oscillations are of sufficiently low energy compared with the energy of deformation that the nuclear surface stabilizes about a deformed shape. It is instructive to outline the former before passing on to a more detailed study of the latter.

One can quantize the nuclear surface oscillations given by Eq. (I-10) in a number representation (Sc 55a) by introducing the quantities $b_{\lambda,\mu}$ and $b^*_{\lambda,\mu}$

$$\alpha_{\lambda,\mu}=(\hbar/2B_\lambda\omega_\lambda)^{\frac{1}{2}}(b_{\lambda,\mu}+(-1)^{\mu}b^*_{\lambda,-\mu}),$$
$$\pi_{\lambda,\mu}=i(\hbar B_\lambda\omega_\lambda/2)^{\frac{1}{2}}(b^*_{\lambda,\mu}-(-1)^{\mu}b_{\lambda,-\mu}), \tag{I-15}$$

and requiring the quantization conditions

$$[\pi_{\lambda,\mu},\,\alpha_{\lambda',\mu'}]=-i\hbar\delta_{\lambda,\lambda'}\delta_{\mu,\mu'}$$

so that

$$[b_{\lambda,\mu},\,b^*_{\lambda',\mu'}]=\delta_{\lambda,\lambda'}\delta_{\mu,\mu'},$$

all other commutators being zero. As usual, the number operator is defined as

$$b^*_{\lambda\mu}b_{\lambda\mu}=n_{\lambda,\mu}.$$

The $b^*_{\lambda\mu}$ operating on the vacuum state $|\,0\rangle$ creates a λ surfon in the μ state, while $b_{\lambda\mu}$ operating on a state with a single λ surfon in the μ state gives the vacuum state. Thus the Hamiltonian of the surface oscillations

TABLE I–1. The possible angular momenta to which a number N of λ surfons can couple for $\lambda=2$ (positive-parity states) and $\lambda=3$ (negative-parity states).

N	$\lambda=2$ I	$\lambda=3$ I
0	0	0
1	2	3
2	0, 2, 4	0, 2, 4, 6
3	0, 2, 3, 4, 6	1, 3, 3, 4, 5, 6, 7, 9
.	.	.
.	.	.
.	.	.

becomes

$$H=\sum_{\lambda}\hbar\omega_{\lambda}[N_{\lambda}+(2\lambda+1)/2], \qquad \text{(I–16)}$$

where

$$N_{\lambda}=\sum_{\mu}n_{\lambda\mu}=0,\ 1,\ 2,\ \cdots.$$

Making use of the definitions (I–15) in Eq. (I–14) for the z component of the angular momentum,

$$L_z=\hbar\sum_{\lambda,\mu}\mu n_{\lambda\mu}=\sum_{\lambda}L_z{}^{\lambda}$$

since $\langle\lambda\mu'\mid L_z\mid\lambda\mu\rangle=\mu\delta_{\mu\mu'}$. Thus the λ surfons are field particles with spin λ. Now it is well-known (He 52a) that particles of angular momentum λ have parity $(-1)^{\lambda}$ so that to describe positive-parity nuclear levels we need to take λ even, while to describe negative-parity levels we must take λ odd. Since $\omega_{\lambda i}>\omega_{\lambda j}$ for $i>j$, we should expect in even nuclei the lowest levels to be positive parity while negative-parity levels will first appear at higher energies. In Table I–1 the total angular momenta to which several λ surfons can couple for $\lambda=2$ and 3 are listed and in Fig. I–1 a schematic energy-level diagram of low-lying levels to be expected from nuclei of this type (which occur in what is sometimes called the vibrational or transitional region). The first excited states for $\lambda=2$ are all $I=2$ while the second excited state is a triply degenerate level with $I=0$, 2, 4, etc. An example of this sort of structure is to be found in the energy-level structure of cadmium-114, a nucleus in this transitional region. The ground and first excited states have spins 0+ and 2+ as expected from the model. Instead of a triply degenerate second excited state, four states appear between 1.13 and 1.36 MeV with spin sequence 0+, 2+, 4+, 2+. A negative-parity level, the 3− level, occurs at 1945 keV (Mc 64).

The theory of these quantized surface oscillations has been further developed (Fo 50, Sc 55, Ra 59). In particular Scharff-Goldhaber and Weneser (Sc 55) have coupled a group of four $f_{7/2}$ particles to the vibrating core in weak coupling. Using a Tamm–Dancoff calculation including up to three surfons, they have determined the energy-level structure of transitional nuclei. This coupling does indeed remove the degeneracy of the second excited states yielding a structure similar to that found in Cd^{114}. The important experimental difference between these nuclei and the deformed nuclei is that these have a closely lying triplet for the second excited state with spins 0, 2, 4. As we shall see the deformed systems have either a spin of 2 or 4 for the second excited state and only rarely is there a triplet but with spins 2, 3, 4. This difference is not trivial.

If the deformation energy is very great compared with the vibrational energies we may consider the system as a stable, deformed, rigid rotator which will display a typical rotational spectrum. If the surface stabilizes on the average about some definite nonspherical shape, then the kinetic and potential energies can be referred to a set of axes fixed in the body. This allows us to transform Eq. (I–8) to one involving the $a_{\lambda,\nu}$ parameters by using Eq. (I–6), and assuming small deviations from a spherical equilibrium shape

$$T^{B}=\tfrac{1}{2}\sum_{\lambda}B_{\lambda}\{\sum_{\nu}\mid\dot{a}_{\lambda,\nu}\mid^{2}+\sum_{\nu\nu',kk'}a^{*}{}_{\lambda\nu'}a_{\lambda\nu}\langle\lambda\nu'\mid L_{k'}L_{k}\mid\lambda\nu\rangle\omega_{k'}\omega_{k}+[i\sum_{\nu'\nu,k}\dot{a}^{*}{}_{\lambda,\nu'}a_{\lambda,\nu}\langle\lambda\nu'\mid L_{k}\mid\lambda\nu\rangle\omega_{k}+\text{C.C.}]\}. \qquad \text{(I–17)}$$

The time derivatives of the $D^{\lambda}{}_{\mu\nu}$ are written as

$$\dot{D}_{\mu\nu}{}^{\lambda*}=i\sum_{k=1}^{3}\sum_{\sigma=-\lambda}^{\lambda}\langle\lambda^{\sigma}\mid L_{k}\mid\lambda\nu\rangle D_{\mu\sigma}{}^{\lambda*}\omega_{k},$$

where the ω_k are the body-fixed cartesian components

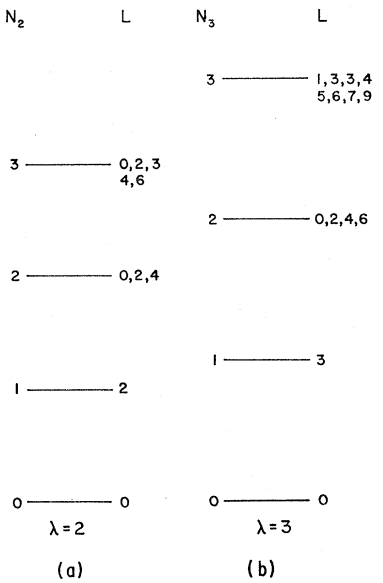

FIG. I–1. A schematic diagram of the energy levels of the pure surface vibrational model from Eq. (I–16). The principle quantum number N_{λ} is to the left and the total angular momenta to which these surfons can couple to the right. The diagram (a) is for surfons of order 2 and represents the positive parity levels of even–even nuclei while diagram (b) represents the negative-parity levels of such vibrating nuclei.

of the angular velocity. It is necessary, when working in the body-fixed system, to recall that the angular momentum operators satisfy (in units of $\hbar$)

$$[L_1, L_2] = -iL_3, \tag{I-18}$$

cyclically. The potential energy becomes just

$$V^B = \tfrac{1}{2}\sum_{\lambda\mu} C_\lambda \mid a_{\lambda\mu} \mid^2. \tag{I-19}$$

The three terms in Eq. (I-17) are a vibration term, a rotation term, and a rotation-vibration cross term, respectively. The rotation term can be placed in a more usual form if the inertial tensor $\mathcal{I}^\lambda{}_{k',k}$ is defined by

$$\mathcal{I}^\lambda{}_{k',k} = B_\lambda \sum_{\nu',\nu} a^*{}_{\lambda\nu'} a_{\lambda\nu} \langle \lambda\nu' \mid L_{k'} L_k \mid \lambda\nu \rangle \tag{I-20}$$

so that the rotational kinetic energy becomes

$$T^B{}_{\text{rot}} = \tfrac{1}{2} \sum_{\lambda,k',k} \mathcal{I}^\lambda{}_{k'k} \omega_{k'} \omega_k. \tag{I-21}$$

At this point in classical rigid-body dynamics it is usual to pick for the body-fixed axes that system which diagonalizes the inertial tensor which is then the principal axis system of the rigid body. It has been shown that a sufficient condition to diagonalize the inertial tensor is to require the $a_{\lambda,\nu}$ to be nonzero only for even or odd values of ν (Wi 62a). This process of course arbitrarily reduces the number of degrees of freedom (except for the case of $\lambda=2$ where it can be shown from the ellipsoidal nature of the surface that $a_{\pm 1}=0$). In essence this means that if we relax the rigidity requirement on the nuclear surface we will then consider only those vibrations which preserve the principal axis system.

One further point, which does not restrict the degrees of freedom, is that we should divide the kinetic and potential energies into two parts, one a sum over even λ, the other other odd λ since it is known that parity is a very good quantum number for nuclear states (Mi 64).

For those cases where we neglect the rotation–vibration cross terms we may define a deformation parameter β_λ by taking $a_{\lambda\mu}$ real and setting

$$a_{\lambda\nu} = \beta_\lambda \epsilon_{\lambda\nu}. \tag{I-22a}$$

The $\epsilon_{\lambda\nu}$ are asymmetry parameters and without loss of generality we may require that

$$\sum_\nu \epsilon^2{}_{\lambda,\nu} = 1. \tag{I-22b}$$

With these substitutions the kinetic and potential energies become

$$T^B = \tfrac{1}{2}\sum_\lambda \{B_\lambda \dot{\beta}_\lambda{}^2 + B_\lambda \beta_\lambda{}^2 \sum_{\nu=-\lambda}^{\lambda} \dot{\epsilon}^2{}_{\lambda\nu} + \sum_{k=1}^{3} \mathcal{I}_k{}^\lambda \omega_k{}^2\} \tag{I-23}$$

$$V^B = \tfrac{1}{2}\sum_\lambda C_\lambda \beta_\lambda{}^2, \tag{I-24}$$

where

$$\mathcal{I}_k{}^\lambda \equiv \mathcal{I}^\lambda{}_{k,k}.$$

If we suppose, as seems physically reasonable, that the form of the nuclear surface for low energies of excitation are well represented by the lowest even or odd λ term in Eq. (I-5L) or Eq. (I-5B) for positive- or negative-parity states then the proper Hamiltonian to use for them will be the sum of a term from each of Eqs. (I-23) and (I-24). That is, we should expect that the Hamiltonian operator to be used to investigate the properties of the low-lying positive-parity levels in even nuclei would be the sum of the $\lambda=2$ terms in these equations while for the odd-parity states one would take the sum of the $\lambda=3$ terms. Indeed the theory for nuclear states whose surface is well-represented by an expansion in terms of the λth spherical harmonics only has been studied under the restrictions discussed before (Wi 62a) and the Hamiltonian used was just

$$H_\lambda = \tfrac{1}{2} B_\lambda (\beta_\lambda{}^2 + \beta_\lambda{}^2 \sum_{\nu=-\lambda}^{\lambda} \dot{\epsilon}^2{}_{\lambda\nu}) + \sum_{k=1}^{3} \mathcal{I}_k{}^\lambda \omega_k{}^2 + \tfrac{1}{2} C_\lambda \beta_\lambda{}^2. \tag{I-25}$$

This equation forms the basis of most of the phenomenological models of deformed nuclei. A detailed examination of the application of Eq. (I-25) to even nuclei is made in the next section.

II. MODELS OF EVEN NUCLEI

A. Energy-Level Systematics

The collective model of deformed nuclei is that of a rotating, deformed, almost rigid body whose inertia tensor $\mathcal{I}$ has in the principal axis system the components $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$. The Hamiltonian is just

$$H = \frac{\hbar^2}{2}\left[\frac{L_1{}^2}{\mathcal{I}_1} + \frac{L_2{}^2}{\mathcal{I}_2} + \frac{L_3{}^2}{\mathcal{I}_3}\right], \tag{II-1}$$

where the body-fixed angular momentum operators[2] satisfy

$$[L_1, L_2] = -iL_3, \tag{II-2}$$

cyclically. The quantum-mechanical system specified by Eqs. (II-1) and (II-2) possesses the symmetry properties belonging to the point group $\mathbf{D}_2$ for which four representations A, B_1, B_2, B_3 exist (La 58). The Hamiltonian (II-1) does not connect states of different representations. Defining an angular momentum representation $\mid LMK \rangle$ diagonal in $\mathbf{L}^2$ and its projections on laboratory and principal Z axes by

$$\mathbf{L}^2 \mid LMK\rangle = L(L+1) \mid LMK\rangle,$$

$$L_z \mid LMK\rangle = M \mid LMK\rangle,$$

$$L_3 \mid LMK\rangle = K \mid LMK\rangle,$$

in units of $\hbar$.[3] The nonvanishing matrix elements of

[2] We use the convention that $\mathbf{L}$ is the total angular momentum of even nuclei and the core angular momentum for other nuclei: that is, the eigenvalues of $\mathbf{L}^2$ are restricted to integral values.

[3] These state functions $\mid LMK\rangle$ are in fact the state functions for a symmetric top $D_{MK}{}^{I*}$ introduced in Eq. (I-6) (Wi 59).

TABLE II-1. Symmetry properties of asymmetric-top state functions, representation of the point group $\mathbf{D}_2$ to which they belong and allowed values of the angular momentum associated with them.

$T_1 \mid LM\rangle$	$T_2 \mid LM\rangle$	K	A_{-K}/A_K	Representation	Allowed L values
+	+	even	$(-1)^L$	A	0, 2, 2, 3, 4, 4, 4, 5, 5,···
−	+	even	$(-1)^{L+1}$	B_1	
+	−	odd	$(-1)^{L+1}$	B_2	1, 2, 3, 3, 4, 4, 5, 5, 5,···
−	−	odd	$(-1)^L$	B_3	

L expressed in the principal axis system are the

$$\langle LMK' \mid L_\mu \mid LMK\rangle = [\mathrm{L}(L+1)]^{\frac{1}{2}} C(L1L;\, K-\mu,\, \mu,\, K)\,\delta_{K',K-\mu}, \quad \text{(II-3)}$$

where

$$L_{\pm 1} = \mp(1/\sqrt{2})\,(L_1 \pm iL_2)$$
$$L_0 = L_3.$$

The Hamiltonian (II-1) thus connects states such that $\Delta K = 0, \pm 2$.

The state functions for an asymmetric top designated by $\mid L, M\rangle$ are

$$\mid LM\rangle = \sum_{K=-L}^{L} A_K \mid LMK\rangle. \quad \text{(II-4)}$$

Thus the rigid rotator Hamiltonian only connects state components for which K is either even or odd. That is, the state functions of the most general top can be initially classified into two broad categories, one being a linear combination of functions for even K, the other for odd K.

The assignment of the labels 1, 2, 3 to the semi-axis of the momental ellipsoid (and thus the choice of the principal axis system) is quite arbitrary and the physical properties of the states must be indifferent to such an assignment. Indeed confining oneself to right-handed systems there are 24 different ways this assignment can be made. The relabeling transformations T_1, T_2, and T_3 which carry out this process are defined in the following way (Bo 52):

(i) T_1 produces the interchange $1 \leftrightarrow -1$, $3 \leftrightarrow -3$ which corresponds to a rotation of π about the 2-axis and $T_1^2 = 1$.

(ii) T_2 produces the interchange $1 \leftrightarrow 2$, $2 \leftrightarrow -1$, which corresponds to a rotation of $\pi/2$ about the 3-axis, so that $T_2^4 = 1$.

(iii) T_3 produces the cyclic interchange $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 1$, thus $T_3^3 = 1$.

The most general relabeling transformation is $T_1^i T_2^j T_3^k$ where $i = 1, 2, j = 1, 2, 3, 4, k = 1, 2, 3$ yielding 24 such transformations. The transformations T_1 and T_2 applied to the state functions $\mid LM\rangle$ give

$$T_1 \mid LM\rangle = \pm \mid LM\rangle,$$
$$T_2^2 \mid LM\rangle = \pm \mid LM\rangle, \quad \text{(II-5)}$$

so that four classes of state functions exist labeled by the sign obtained from these two transformations. These are tabulated in Table II-1 where the notation of Ref. La 58 is used to designate these four different representations of the point group $\mathbf{D}_2$ (see also Ref. Va 54). For the symmetric top functions

$$T_1 \mid LMK\rangle = \exp[i\pi(L+K)] \mid LM-K\rangle,$$
$$T_2^2 \mid LMK\rangle = \exp(i\pi K) \mid LMK\rangle, \quad \text{(II-6)}$$

the latter relation shows that the representations A and B_1 are associated with K even while representations B_2 and B_3 are associated with odd K. Finally relations between A_K and A_{-K} can be obtained from Eqs. (II-4)-(II-6).

(i) A representation (K even):

$$A_{-K} = (-1)^L A_K;$$

(ii) B_1 representation (K even):

$$A_{-K} = (-1)^{L+1} A_K;$$

(iii) B_2 representation (K odd):

$$A_{-K} = (-1)^{L+1} A_K;$$

(iv) B_3 representation (K odd):

$$A_{-K} = (-1)^L A_K.$$

These last four relations show that the energy levels of A-type systems contain one level with $L=0$, no levels of $L=1$, two levels of $L=2$, one level of $L=3$, three levels of $L=4$, two levels of $L=5$, etc. On the other hand, the level systematics for $B_{1,2,3}$-type systems contain no levels of $L=0$, one level each of $L=1$ and 2, two levels, each of $L=3$ and 4, three levels, each of $L=5$ and 6, etc. These spin classifications are quite general, being completely independent of the model involved for the momental ellipsoid and are all tabulated in Table II-1.

Without specifying a model for the momental ellipsoid, the energy eigenvalues of asymmetric tops can be specified for the various representations. For the A representation the ground-state function contains but one component,

$$\mid 00\rangle = \mid 000\rangle,$$

and the energy eigenvalue is zero for all moments of inertia. Therefore, A-type systems *always* have an $L=0$ ground state. Next there are two states for $L=2$

with state functions

$$|2M_2^1\rangle = A_0^{1,2}|2M0\rangle + (A_2^{1,2}/\sqrt{2})(|2M2\rangle + |2M-2\rangle), \quad \text{(II-7)}$$

and with energy eigenvalues

$$E(2^{1,2}) = \hbar^2\left(\frac{1}{\mathcal{I}_1}+\frac{1}{\mathcal{I}_2}+\frac{1}{\mathcal{I}_3}\right) \mp \hbar^2\left[\left(\frac{1}{\mathcal{I}_1}+\frac{1}{\mathcal{I}_2}+\frac{1}{\mathcal{I}_3}\right)^2 - 3\left(\frac{1}{\mathcal{I}_1\mathcal{I}_2}+\frac{1}{\mathcal{I}_1\mathcal{I}_3}+\frac{1}{\mathcal{I}_2\mathcal{I}_3}\right)\right]^{\frac{1}{2}}.$$

The state for $L=3$ also contains essentially one component, which if normalized is

$$|3M\rangle = (1/\sqrt{2})(|3M2\rangle - |3M-2\rangle)$$

with energy eigenvalues

$$E(3) = 2\hbar^2\left(\frac{1}{\mathcal{I}_1}+\frac{1}{\mathcal{I}_2}+\frac{1}{\mathcal{I}_3}\right).$$

One can proceed in a similar manner for higher L values. However, for all A-type rigid rotators the relation

$$E(2^1)+E(2^2) = E(3) \quad \text{(II-8)}$$

is always true. Similar relations hold for higher L values.

For the B-type systems the lowest spin is $L=1$ for which the normalized state functions are

$$|LM\rangle = |LM0\rangle \qquad B_1 \text{ representation,} \quad \text{(II-9a)}$$

$$|LM\rangle = (1/\sqrt{2})(|LM1\rangle + |LM-1\rangle) \qquad B_2 \text{ representation,} \quad \text{(II-9b)}$$

$$|LM\rangle = (1/\sqrt{2})(|LM1\rangle - |LM-1\rangle) \qquad B_3 \text{ representation,} \quad \text{(II-9c)}$$

with energy eigenvalues

$$E(1B_1) = \frac{\hbar^2}{2}\left(\frac{1}{\mathcal{I}_1}+\frac{1}{\mathcal{I}_2}\right),$$

$$E(1B_2) = \frac{\hbar^2}{2}\left(\frac{1}{\mathcal{I}_1}+\frac{1}{\mathcal{I}_3}\right),$$

$$E(1B_3) = \frac{\hbar^2}{2}\left(\frac{1}{\mathcal{I}_2}+\frac{1}{\mathcal{I}_3}\right).$$

$E(1B_2)$ and $E(1B_3)$ can be gotten from $E(1B_1)$ by replacing $\mathcal{I}_2$ by $\mathcal{I}_3$ and $\mathcal{I}_1$ by $\mathcal{I}_3$, respectively. This is generally true or higher values of L.

For the spin-two states, the state functions in the B_2 and B_3 representations will of course contain only $K=1$ terms being simply combinations of $|2M\pm1\rangle$. The state function for the B_1 representation contains only a $K=2$ term,

$$|2M\rangle = (1/\sqrt{2})(|2M2\rangle - |2M-2\rangle),$$

with energy eigenvalues

$$E(2, B_1) = \frac{2\hbar^2}{\mathcal{I}_3}+\frac{\hbar^2}{2}\left(\frac{1}{\mathcal{I}_1}+\frac{1}{\mathcal{I}_2}\right),$$

the others being gotten by the proper interchange of $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$. Higher spin-state functions can be constructed from Eq. (II-1) with the aid of Table II-1; however, the eigenvalues cannot generally be given in closed form if $L>4$ and numerical methods must be used in such cases once values for the moments of inertia have been assigned. The energy eigenvalues for the asymmetric top have been tabulated by several authors (Ki 43, To 55) in terms of moment-of-inertia ratios.

The problem of the rigid symmetric top is analytically simpler as all of its eigenvalues can be given in closed form. For this case let $\mathcal{I}_1=\mathcal{I}_2=\mathcal{I}_0\neq\mathcal{I}_3$; then the Hamiltonian is

$$H_{\text{sym}} = \frac{\hbar^2}{2}\left(\frac{L_1^2+L_2^2}{\mathcal{I}_0}+\frac{L_3^2}{\mathcal{I}_3}\right) = \frac{\hbar^2}{2}\left[\frac{\mathbf{L}^2}{\mathcal{I}_0}+\left(\frac{1}{\mathcal{I}_3}-\frac{1}{\mathcal{I}_0}\right)L_3^2\right]$$

whence

$$E_{\text{sym}}(L) = \frac{\hbar^2}{2}\left[\frac{L(L+1)}{\mathcal{I}_0}+\left(\frac{1}{\mathcal{I}_3}-\frac{1}{\mathcal{I}_0}\right)K^2\right]. \quad \text{(II-10)}$$

Deformed nuclei, as is true of rotating molecules, are not perfectly rigid but tend to distort under rotation. One might expect, in analogy with the classical case, that as these systems rotate their equatorial diameters would increase while their polar diameters would shorten. This would have the effect of lowering each energy level slightly. Indeed this can be seen in the case of symmetric tops in their $K=0$ bands for the centrifugal distortion will increase $\mathcal{I}_0$ and hence decrease $E_{\text{sym}}(L)$. One has in this case that

$$E_{\text{sym}}(L, K=0) = (\hbar^2/\mathcal{I}_0)L(L+1) - bL^2(L+1)^2, \quad \text{(II-11)}$$

where b is positive.[4] The form of b can be given when a model for the momental ellipsoid is picked. This is done later when the hydrodynamical model is discussed in detail.

Without further complications introduced by taking a particular form for the momental ellipsoid we might see to what extent a quantum-mechanical rigid rotator might represent atomic nuclei. We should expect that if such a model works it would work best for nuclei with a sufficiently large number of nucleons to have a fairly rigid structure. The simplest class of these are the even nuclei, most of which have ground-state spin zero, positive parity, and a 2+ first excited state. The zero ground-state spin means that we must pick the A representation for the state functions of the rigid

[4] This relation might be considered as the first two terms of a power series expansion in terms of $L(L+1)$ of the energy. This technique has been used in studying the level structures of Th^{232} and U^{238} by F. S. Stephens *et al.* (St 59a).

rotator which also has as its first excited state one with spin 2. [Assigning the symmetry class or representation from experiment might be objected to on the ground that these nuclei possess "front-to-back" symmetry and hence of necessity belong to the A representation, such assignments being common in molecular studies (He 45). However, such a statement is a model statement depending upon the nature of the nuclear surface. The momental ellipsoid upon which the dynamics of the system depends does of course have "front-to-back" symmetry. However, there would seem to be no *a priori* reason to conclude at this stage that the lower lying levels of even nuclei should belong to the A representation, except that they might belong to the lowest symmetry. It would seem preferable to use the experimental determination of the representation to restrict the models of the nuclear surface.]

Mallman (Ma 61) has based an investigation of the energy-level structure and gamma-ray branching ratios of even nuclei in the range $40 \leq A \leq 250$ on just such a general model as has been developed here. [For early speculations on nuclear rotational spectra see (Te 38, Gu 42, Pr 50).] We review the results of his investigation of the energy levels, deferring until later any discussion on gamma-ray branching ratios.

Mallman makes use of the relation of Eq. (II-8) or a similar one for the first $L=5$ level to test the validity of a rigid rotator model for the low-lying positive-parity levels of even nuclei in the range $40 \leq A \leq 250$. In general he obtains a quite good fit only within the deformed regions, that is for $150 < A < 190$, $A > 228$. A more recent compilation is given in Table II-2 in which the ratios

$$\frac{E(2^1)+E(2^2)-E(3)}{E(2^1)+E(2^2)}$$

and

$$\frac{4E(2^1)+E(2^2)-E(5^1)}{4E(2^1)+E(2^2)}$$

are given for all nuclei with two $L=2$ and at least one $L=3$ or $L=5$ level measured. Here $E(L^n)$ is the nth level of spin L. (Great care must be exercised in using spin and parity assignments reported in various compilations, such as Ref. He 61, which have not been actually measured. Such assignments are often made from model systematics.) This table has been extended to the $2s$–$1d$ shell which is often considered a deformed region (Go 60). Here it is possible to see that the criterion of Eq. (II-8) is quite well satisfied within the deformed regions but in general not outside of them. Another characteristic of these deformed regions, evident from this table, is the fact that the lowest $L=2$ level rises rapidly at the edges of these regions, being quite high outside of them.

The discrepancies in Table II-2 were explained by relaxing the condition of rigidity permitting a small amount of vibration. Mallman used a relation similar to that of Eq. (II-11), but for an asymmetric top (Ma 60a):

$$E_v(L^n) = E(L^n) - bE^2(L^n), \qquad \text{(II-11a)}$$

where the subscript v means the result including vibration, and b was treated as a fitting parameter. It can be shown that b must be positive and small. Thus to fit the low-lying, positive-parity energy levels of even nuclei, Mallman had at his disposal four fitting parameters which he determined for 27 nuclei from Ti^{48} to Cf^{250}. With these values he then calculated the energies of other levels and compared with experiment. Again the model fits very well in the deformed region but less well outside of it. Of the 27 nuclei investigated, six had negative values for b. (Negative values for b are associated with negative values in column six of Table II-2.) There have been suggestions that two other deformed regions might exist one near iron (La 61) and the other near barium (Sh 61) and it is in just these regions that Mallman finds some evidence for rotational levels.

From this rather general approach to nuclear rotations we now turn to a study of other models which are in fact models for the momental ellipsoid and its elastic properties. The most studied and also most successful is the hydrodynamic model first investigated by Aage Bohr and by Bohr and Mottelson in a long series of papers (Bo 51, Bo 52, Bo 53, Bo 53a, Bo 53b, Bo 55a, Me 57). In this work they assumed that the surfaces of deformed nuclei were well represented by the spherical harmonic expansion of Eq. (I-5L). In studying the low-lying positive-parity states of even nuclei it was assumed that the terms with $\lambda=2$ were of major importance so that we have to deal with a pure $\lambda=2$ surface. For the low-lying negative-parity levels Lipas (Li 61) and Davidson (Da 62) have studied the $\lambda=3$ surfaces. Since the bulk of information currently available concerns the positive-parity levels, we discuss the theory of the $\lambda=2$ surface first and touch upon the $\lambda=3$ surfaces later.

The Hamiltonian for the positive-parity levels is given without approximation as the $\lambda=2$ (quadrupole) terms of Eq. (I-25) and it is customary to take the asymmetry parameters, $\epsilon_{\lambda\nu}$, of Eq. (I-22) as

$$\epsilon_{20} = \cos\gamma$$
$$\epsilon_{2\pm2} = (1/\sqrt{2})\sin\gamma \qquad \text{(II-12)}$$

and to suppress the λ subscripts on the deformation parameter β_2 and the mass parameter B_2. In what follows we take the quadrupole deformation and asymmetry parameters as β and γ, but retain the λ subscripts on the mass and potential energy parameters B_2 and C_2. The moments of inertia then are obtained from Eqs. (I-20), (I-22a), (II-12)

$$\mathcal{I}_k^2 = 4B_2\beta^2\sin^2[\gamma-(2\pi/3)k]. \qquad \text{(II-13)}$$

In his paper on the coupling of motion of individual

TABLE II–2. Energies of the first spin-2, 3, and 5 levels and the second spin-2 level in a number of even nuclei and a comparison with the energy systematics of a rigid top. The symbols (?) mean the spin and/or parity is in doubt and probably assigned by level systematics, (U) means upper level of a close doublet of that spin.

Nucleus	$E(21)$ MeV	$E(22)$ MeV	$E(31)$ MeV	$E(51)$ MeV	$\frac{E(21)+E(22)-E(31)}{E(21)+E(22)}$%	$\frac{4E(21)+E(22)-E(51)}{4E(21)+E(22)}$%
$_{12}Mg_{12}^{24}$	1.368	4.24	5.22		6.9	
$_{12}Mg_{14}^{26}$	1.83	2.97	3.97(?)		17	
$_{20}Ca_{22}^{42}$	1.52	2.42(?)	2.751(?)		30	
$_{26}Fe_{30}^{56}$	0.845	2.660	3.445	3.84(?)	17	34
$_{36}Kr_{46}^{82}$	0.777	1.475	2.094 *u*		70	
$_{42}Mo_{54}^{96}$	0.770	1.524	1.850(?)	2.730	19.3	41
$_{44}Ru_{58}^{102}$	0.475	1.105	1.525		3.5	
$_{46}Pd_{60}^{106}$	0.5116	1.1285(?)	1.5575(?)	2.7569(?)	5.0	13
$_{56}Ba_{78}^{134}$	0.604	1.168	1.760(?)		0.68	
$_{62}Sm_{88}^{150}$	0.334	1.167 *u*	1.508		−0.47	
$_{62}Sm_{90}^{152}$	0.122	1.087	1.236		−2.2	
$_{64}Gd_{90}^{154}$	0.123	0.998	1.129		−0.71	
$_{64}Gd_{92}^{156}$	0.089	1.154	1.248(?)	1.622(?)	−0.40	−7.4
$_{66}Dy_{94}^{160}$	0.0867	0.966	1.049		0.38	
$_{68}Er_{98}^{166}$	0.0806	0.788	0.861		0.92	
$_{68}Er_{100}^{168}$	0.0798	0.822	0.897(?)		0.55	
$_{70}Yb_{102}^{172}$	0.0787	1.4675	1.5502 *u*		−0.26	
$_{74}W_{108}^{182}$	0.1001	1.258 *u*	1.331		2.0	
$_{74}W_{110}^{184}$	0.111	0.904	1.006		0.89	
$_{74}W_{112}^{186}$	0.122	0.730	0.85(?)		0	
$_{76}Os_{108}^{184}$	0.125	1.085	1.215		−0.41	
$_{76}Os_{110}^{186}$	0.13715	0.76738	0.91033	1.27530	−0.64	+3.1
$_{76}Os_{112}^{188}$	0.15503	0.63307	0.78999(?)		−0.24	
$_{76}Os_{114}^{190}$	0.1867	0.5572	0.755		−0.5	
$_{76}Os_{116}^{192}$	0.2057	0.4891	0.6909		+0.56	
$_{78}Pt_{114}^{192}$	0.3165	0.6129	0.9208		0.92	
$_{78}Pt_{116}^{194}$	0.3285	0.6220	0.9227(?)		2.9	
$_{78}Pt_{118}^{196}$	0.356	0.689	1.001(?)		4.2	
$_{80}Hg_{120}^{200}$	0.368	1.575	1.776		8.6	
$_{84}Po_{130}^{214}$	0.609	1.281	1.544		18	
$_{90}Th_{138}^{228}$	0.058	0.969	1.023		0.39	
$_{92}U_{140}^{232}$	0.047	0.868 *u*	0.913		0.22	
$_{92}U_{144}^{234}$	0.044	0.922	0.965	1.087(?)	0.10	1.0
$_{94}Pu_{144}^{238}$	0.044	1.030 *u*	1.071		0.28	
$_{98}Cf_{152}^{250}$	0.041	1.032(?)	1.074(?)		−0.093	
$_{100}Fm_{154}^{254}$	0.044	0.692	0.734		0.27	

particles to nuclear surface oscillations (Bo 52), Bohr presented an argument showing that the deformation would stabilize in a configuration for which γ was zero. From the above equation this is seen to yield not only symmetric momental ellipsoids, but ones for which $\mathcal{I}_3^2$ are identically zero. Thus the momental ellipsoid is a long, thin, needle-like object while the surface itself is spheroidal. Rather than discuss symmetric systems as a special case we shall include them, as before, with a more general problem. The vanishing of the third moment of inertia leads to the problem that the energy eigenvalues given in Eq. (II–10) become infinite unless $K=0$. The symmetric model of deformed even nuclei takes the ground state band to be that with $K=0$ in the A representation so that the spin sequence is 0, 2, 4, 6, 8, ··· as observed.

The Hamiltonian is

$$H_{\lambda=2}=\tfrac{1}{2}B_2(\dot\beta^2+\beta^2\dot\gamma^2)+\frac{1}{4B_2\beta^2}\sum_{k=1}^{3}\frac{L_k^2}{\sin^2(\gamma-\frac{2}{3}\pi k)}+\tfrac{1}{2}C_2\beta^2, \quad \text{(II–14a)}$$

the first term being the vibrational kinetic energy, the second the rotational kinetic energy, while the third represents the vibrational potential energy. From the choice of β_λ and $\epsilon_{\lambda\nu}$ in Eq. (I–22a), the vibrational potential energy is not a function of the asymmetry parameters (this requirement dictated the choice of β_λ and $\epsilon_{\lambda\nu}$). For quadrupole surfaces these potentials have been called "γ-unstable" potentials (Wi 56). We return to this point later. The system represented by Eq. (II–14a) is quantized by the usual methods

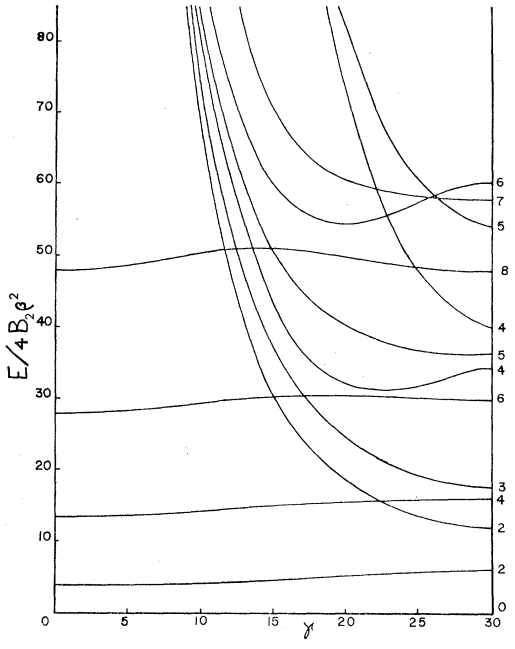

FIG. II-1. The energy eigenvalues in units of $\hbar^2/4B_2\beta^2$ for a rigid, asymmetric top with a quadrupole surface ($\lambda=2$) as a function of the asymmetry parameter γ. The hydrodynamic moments of inertia of Eq. (II-13) have been used.

(Pa 58a) in a five-dimensional, curvilinear, coordinate space $(\theta_1, \theta_2, \theta_3, \beta, \gamma)$. This then leads to the Schrödinger equation:

$$\left\{\frac{\hbar^2}{2B_2}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\left(\beta^4\frac{\partial}{\partial\beta}\right)+\frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\left(\sin 3\gamma\frac{\partial}{\partial\gamma}\right)-\frac{1}{4\beta^2}\sum_{k=1}^{3}\frac{L_k^2}{\sin^2(\gamma-2\pi k/3)}\right]+\tfrac{1}{2}C_2\beta^2\right\}\times\Psi(\theta_i,\beta,\gamma)=E\Psi(\theta_i,\beta,\gamma). \quad \text{(II-14b)}$$

From the way β enters into this equation it is clear that the β dependence can be separated yielding

$$\left[\frac{\hbar^2}{2B_2\beta^4}\frac{d}{d\beta}\left(\beta^4\frac{d}{d\beta}\right)-\tfrac{1}{2}C_2\beta^2-\hbar^2\Lambda/2B_2\beta^2\right]\Phi(\beta)=E\Phi(\beta), \quad \text{(II-15a)}$$

$$\left[\frac{1}{\sin 3\gamma}\frac{d}{d\gamma}\left(\sin 3\gamma\frac{d}{d\gamma}\right)-\tfrac{1}{4}\sum_{k=1}^{3}\frac{L_k^2}{\sin^2(\gamma-2\pi k/3)}\right]\psi_L(\theta_i\gamma)=\Lambda\psi_L(\theta_i\gamma), \quad \text{(II-15b)}$$

with

$$\psi_L(\theta_i\gamma)=\sum_{k=0}^{L}A_K(\gamma)[D_{MK}{}^{L*}(\theta_i)+(-1)^I D_{M-K}{}^{L*}(\theta_i)]. \quad \text{(II-15c)}$$

This system can be solved (Wi 56) and the solution is discussed somewhat later. However, it is instructive to assume that the nucleus is a rigid rotator with both β and γ as fixed parameters. This problem was first considered by Marty (Ma 56, Ma 57) and later in more detail by Davydov and Filippov (Da 58a). In this case, the solution found before can be used with the hydrodynamic moments of inertia (II-13). The range of γ is just $0\leq\gamma\leq\pi/3$ (this follows from the invariance of the system under the relabeling transformations T_3) and the eigenvalues as functions of γ are symmetric about $\gamma=\pi/6$. In Fig. II-1 are shown the energy eigenvalues for different values of $L\leq 8$ as a function of γ. For $\gamma=0$, the eigenvalues are those of a symmetric rotator with $K=0$ [Eq. (II-10)].

Extensive comparisons with this rigid rotator have been made with experiment (Ma 56, Da 58a, Ma 60a) and the slight deviations from theory have been explained by relaxing the rigidity requirement (Ma 60a) as was done in the more general rigid top model [of Eq. (II-11a)]. As an easy-to-use technique to analyze nuclear level structures and aid in assigning level quantum numbers this model is very helpful. However, it would seem best to extend the model consistently and include the various vibrations from the beginning.

This was done by Davydov and Chaban (Da 60a) who assumed that the nucleus was rigid against γ vibrations but not for β vibrations. The Hamiltonian contains but four degrees of freedom which in an angular-momentum representation is just

$$H=\frac{-\hbar^2}{2B_2}\left[\frac{1}{\beta^3}\frac{d}{d\beta}\left(\beta^3\frac{d}{d\beta}\right)-\frac{1}{4\beta^2}\sum_{k=1}^{3}\frac{L_k^2}{\sin^2(\gamma-2\pi k/3)}\right]+\tfrac{1}{2}C_2(\beta-\beta_0)^2, \quad \text{(II-16)}$$

where the possibility that the nucleus may not oscillate about a spherical but perhaps a spheroidal shape, given by β_0, has been included in the potential energy term. This introduces a contradiction into the theory which is a theory of small oscillations about spherical equilibrium.

The Schrödinger equation again separates into two parts, one a rotational part

$$\left[\tfrac{1}{2}\sum_{k=1}^{3}\frac{L_k^2}{\sin^2(\gamma-2\pi k/3)}-\mathcal{E}_{L,N}\right]\psi_{L,N}=0, \quad \text{(II-17a)}$$

and a vibrational part

$$\left[-\frac{\hbar^2}{2B_2}\frac{1}{\beta^3}\frac{d}{d\beta}\left(\beta^3\frac{d}{d\beta}\right)+\frac{\hbar^2}{4B_2\beta^2}\mathcal{E}_{L,N}+\tfrac{1}{2}C_2(\beta-\beta_0)^2\right]\Phi_{L,N}(\beta)=E_{LN}\Phi_{L,N}(\beta). \quad \text{(II-17b)}$$

Here the $\psi_{L,N}$ are the functions of Eq. (II-15c) with the label N an ordering label for the spin L. The eigenvalues $\mathcal{E}_{LN}$ are the ones obtained immediately before and displayed in Fig. II-1. The second and third terms in the bracket of Eq. (II-17b) are a generalized potential energy for the vibrator which gives rise to a new equilibrium deformation. The potential can then be expanded about this new equilibrium position and only the lowest order terms retained.

A stiffness parameter μ (this parameter is often

called the "nonadiabaticity" parameter) is introduced which is a measure of how rigid the nucleus is to β vibrations. When μ is zero the nucleus is rigid and the rotational structure shown in Fig. II-1 is unaltered. As μ increases to one, its maximum effective value, the nuclear surface becomes more elastic. Upon introducing two new quantities Z and Z_1, functions of $\mathcal{E}_{L,N}$ and μ, the new equilibrium deformation $\beta(L, N)$ is given by

$$\beta^4(L,N)-\beta^3(L,N)\beta_0-\beta_0^4(\mu^4/2)(\mathcal{E}_{L,N}+\tfrac{3}{2})=0.$$

A new independent variable, y is defined by

$$y=Z_1[\beta-\beta(L,N)]/\beta(L,N),$$

where

$$Z_1=[Z^4+\tfrac{3}{2}(\mathcal{E}_{L,N}+\tfrac{3}{2})]^{\frac{1}{4}}$$

and Z is the positive real root of

$$Z^4-(1/\mu)Z^3-\tfrac{1}{2}(\mathcal{E}_{L,N}+\tfrac{3}{2})=0.$$

Equation (II-17b) then becomes

$$[d^2D_\nu(\sqrt{2}y)/dy^2]+(2\nu+1-y^2)D_\nu(\sqrt{2}y)=0, \quad \text{(II-18)}$$

where the dependent variable $D_\nu(\sqrt{2}y)$ is defined by

$$\Phi_{L,N}(\beta)=\beta^{-\frac{1}{2}}D_\nu(\sqrt{2}y);$$

the quantity ν is the vibrational quantum number determined from the boundary condition at the origin of the β variable

$$D_\nu(-\sqrt{2}Z_1)=0 \quad \text{(II-19)}$$

and is related to the energy eigenvalue $E_{L,M,n}$, ν_n labeling the roots of this equation with the lowest root being denoted by $n=1$.

Equation (II-18) is Weber's equation for parabolic cylinder functions (Wh 50, Ba 53). As the nucleus becomes quite stiff to β vibrations, that is in the limit of μ going to zero, Z_1 approaches infinity and these Weber's functions become the familiar Hermite functions which are the solutions of the one-dimensional harmonic oscillator. In this limit the ν_n take on integral values:

$$\nu_n=0, 1, 2, \cdots, \qquad \mu=0.$$

For this case the nuclear energy-level structure is formed by bands of rotator levels of Fig. II-1 each separated by an amount $\hbar\omega_0$.

In any event, the total energy including β vibrations and rotations is given by

$$E_{L,N,n}=\hbar\omega_0\{(\nu_n+\tfrac{1}{2})(Z_1/Z)^2 + [(\mathcal{E}_{L,N}+\tfrac{3}{2})/4Z^6](Z^4+\tfrac{1}{2}\mathcal{E}_{L,N}+\tfrac{3}{4})\}. \quad \text{(II-20)}$$

The oscillator energy $\hbar\omega_0$ is usually treated as a scale parameter.

In general there is no simple way to obtain the values of ν_n since Weber's functions are not tabulated and a machine calculation of ν_n from the boundary value condition of Eq. (II-19) is necessary. However, a graphical method of obtaining these ν_n for n equal one and two has been discussed (Wi 62a) which permits a rapid way to compare the model energies of Eq. (II-20) with experiment. This graphical method is good to an accuracy of about 1%.

Certain qualitative features of the level structure are now apparent. The effect of increasing μ in the ground-state vibrational band ($n=1$) is the same as increasing γ. In other words, for a fixed γ, increasing μ causes each spin level to decrease in energy relative to the $L=0$ level. The greater the value of L for the level, the greater the decrease in energy. As μ is increased the higher vibrational bands also come down in energy, but the behavior is somewhat different. The energy ratio R_{LNn}

$$R_{LNn}=(E_{LNn}-E_{011})/(E_{211}-E_{011})$$

is a very strong function of γ.

In Fig. II-2 are shown the behavior of R_{221} and R_{212} as a function of γ. For these it can be seen that for values of μ greater than one the change in the level structure will be very small. Finally, μ is most easily determined when at least one level of a higher vibrational band is known.

For small values of the stiffness parameter μ, the energy eigenvalues $E_{L,N,n}$ of Eq. (II-20) can be put into the form of Eq. (II-11a). One finds then to lowest order in μ that

$$b=\mu^4[1+(57/4)(\nu+\tfrac{1}{2})\mu^2]/2.$$

In Table II-3 the values are shown of the parameters γ and μ resulting in the best fit to the measured energy levels of a number of deformed nuclei. While in general the rms errors between calculated and measured energies are quite small, being all of the order of 1%, this certainly does not mean that we have a comparable understanding of the low-energy structure of these deformed nuclei. For it has been noted in a detailed study of the osmium isotopes that while the deviations between model predictions and experiment are about 1%, the rms experimental uncertainty is two orders of magnitude smaller (Em 63).

Recently an investigation of the dependence of the inertia parameter $\hbar^2/2\mathcal{I}=\hbar^2/6B_2\beta^2$ on L for very large L has been made (St 64) and it is found that this dependence is quite well explained using this model. However, this preliminary study has taken $\gamma=0$ in all cases but it does show that the β vibrations cannot be treated as a perturbation (see below). The study has been restricted to the first levels of a given L, that is to levels labeled (L11). A detailed comparison was reported only for Hf^{170} where the average deviation between experiment and theory was 0.26%.

Finally, for values of $\mu \lesssim 0.25$ the energy eigenvalue predictions of this model differ little from those obtained from Eq. (II-11a) (Kl 61).

We now turn to the problem of including γ vibra-

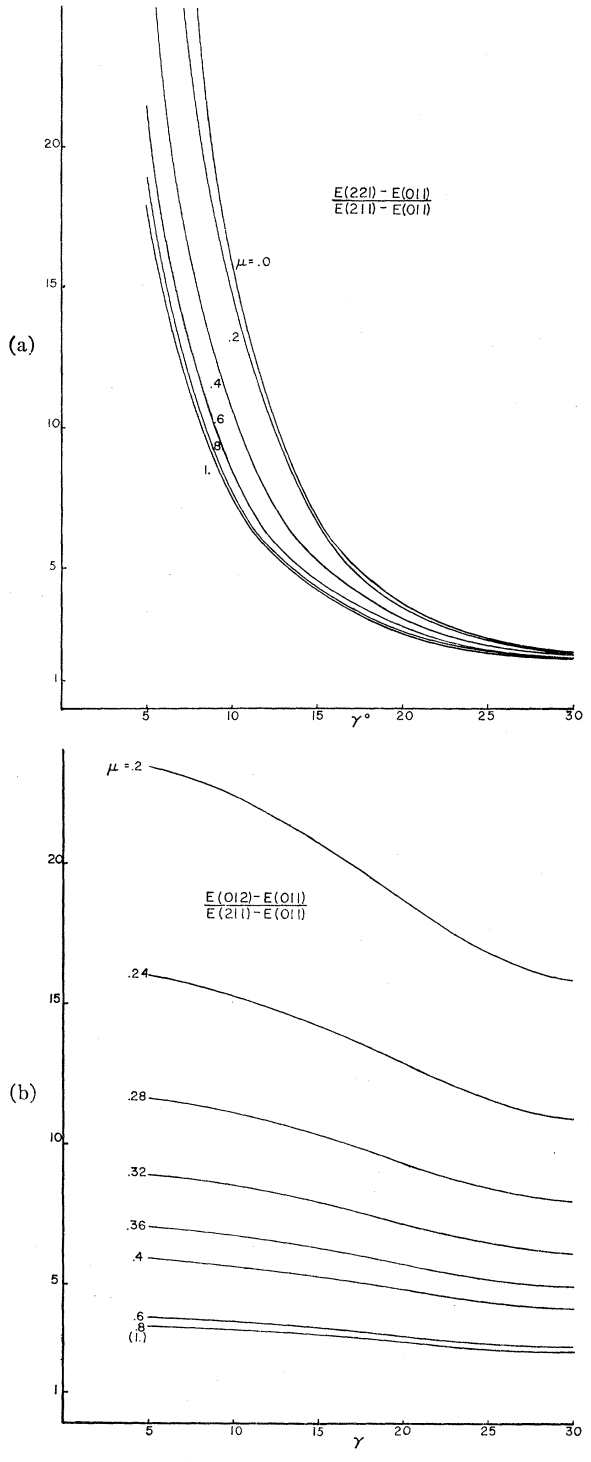

FIG. II-2. The effect the stiffness parameter μ has upon the energy levels in the ground-state vibrational band and in the first excited state vibrational band. In (a) the ratio of the second $L=2$ level to the first $L=2$ level in the ground-state vibrational band (R_{221}) is plotted against γ. In (b) the ratio of the first $L=2$ level in the first excited vibrational band to the first $L=2$ level in the ground state band (R_{212}) as a function of γ is shown. Note in both cases how the value $\mu=1$ represents the effective maximum value of the stiffness parameter.

TABLE II-3. Asymmetric-core model parameters γ and μ for the low-lying positive-parity levels in even-even nuclei.

Nucleus	γ^0	μ	Nucleus	γ^0	μ
Sm^{152}	12	0.4	Os^{192}	25.2	0.1
Gd^{154}	11.8	0.36	Pt^{192}	30	•••
Gd^{156}	10.5	0.26	Pt^{194}	29	0.4
Dy^{160}	11.4	0.26	Ra^{224}	22.3	•••
Er^{166}	12	0.2	Ra^{226}	19.9	•••
Er^{168}	12.3	•••	Th^{228}	9.1	0.30
Yb^{172}	9.0	•••	Th^{230}	10	0.25
Hf^{178}	10	•••	U^{232}	9.1	0.25
W^{182}	10.9	0.28	U^{234}	8.7	0.24
W^{184}	12.5	0.35	U^{238}	10.9	•••
W^{186}	16.6	•••	Pu^{238}	8.3	0.20
Os^{186}	15.9	0.26	Pu^{240}	8.6	0.22
Os^{188}	19.2	0.25	Cf^{250}	7.98	•••
Os^{190}	21.9	0.25	Fm^{254}	10.1	•••

tions so that γ is to be treated as a continuous variable rather than as a fitting parameter. As long as the vibrational potential energy has the form $V(\beta, \gamma)=\frac{1}{2}C_2\beta^2$ used in Eq. (II-14b), the separation of the β and γ variables can be carried out as in Eqs. (II-15a, b). This form of the potential is the proper one for a hydrodynamic model to lowest order in β. The solution of Eq. (II-15a) is straightforward, and is most easily obtained by noting that formally the function $\beta^2\Phi(\beta)$ satisfies the same differential equation as the radial part of the three-dimensional harmonic oscillator. Thus we have that

$$\Lambda=\lambda(\lambda+3), \qquad \lambda=0, 1, 2, 3, \cdots,$$

and the energy eigenvalue solutions of the harmonic oscillator Eq. (I-16) yield

$$E=\hbar\omega(N+\tfrac{5}{2})=\hbar\omega(2n_\beta+\lambda+\tfrac{5}{2}). \qquad \text{(II-21)}$$

Since these solutions are equivalent to those of Eq. (I-16) we can use Table I-1 and the above relations to generate a table relating λ, and L (Wi 56). [Rakavy (Ra 57) gives a group theoretic method of generating this table. Here λ is his "seniority" symbol v.] This is shown in Table II-4, the eigenvalues being given in Fig. I-2a.

The analytic form of $\psi_L(\theta_i\gamma)$ of Eq. (II-15b), or more simply of the $A_K(\gamma)$ of Eq. (II-15c), is quite simple for the cases $L=0$ and $L=3$. The former solutions are $P_{3\lambda}(\cos 3\gamma)$ and $P_{3\lambda}{}^1(\cos 3\gamma)$ for the latter. Some of the solutions for $L=2$ have been calculated in the form of polynomials of sin $n\gamma$ and cos $n\gamma$ and

TABLE II-4. The relation between Λ, λ, and the angular momentum of Eq. (II-15b).

Λ	λ	L
0	0	0
4	1	2
10	2	2, 4
18	3	0, 3, 4, 6
28	4	2, 4, 5, 6, 8

solutions for $L \leq 6$ have also been reported (Ja 59, Be 59a).

To remove the L degeneracy of these solutions the form of the potential function $V(\beta, \gamma)$ must be changed. The requirements of the separation of Eq. (II–14b) imply a γ potential term of the form $V(\beta, \gamma) = \beta^{-2} f(\gamma)$, while the form dictated by a hydrodynamic calculation to second order in β and γ is a term of the form $V(\beta, \gamma) = \beta^2 \sin^2 \gamma$.

An approximate separation has been carried out by Davydov (Da 61, Da 61a) who essentially starts with the set of separated equations (II–15a, b) and adds to the latter a γ-dependent harmonic potential of the form

$$V(\gamma) = (B_2 C_\gamma / \hbar^2) \beta_0^4 (\gamma - \gamma_0)^2. \qquad \text{(II–22)}$$

He then investigated γ vibrations of spherical systems, deformed but symmetric systems, and deformed and asymmetric systems—a different approximation for each being required. For spherical nuclei his results are identical with those given before. It is important to note that $L = 0+$ states are possible for γ vibrations in deformed nuclei. Therefore, where more than one excited 0+ level exists in a nucleus at nearly the same energy one should be associated with the first excited β-vibrational band while the other is associated with the first excited γ-vibrational band.

An example is to be found in ${}_{78}\mathrm{Pt}_{116}{}^{194}$ where two excited 0+ levels have been identified. The level at 1.267 MeV would appear to be the 0+ level associated with the first excited β-vibrational mode while the level at 1.480 MeV may well be the 0+ level of the γ-vibrational first excited state. These two types of 0+ levels can probably be differentiated experimentally by the character of their gamma radiation. A similar situation occurs in Os^{188} which has been discussed in terms of this and other models (Wa 62), and in several other nuclei which have been fitted to this model (Da 61a).

Another much used interpretation of the energy-level structure and other properties of deformed even nuclei is based upon the assumption that the nucleus maintains its axial symmetry so that K is a good quantum number (to a high approximation). The arguments to support this come from one of Bohr's (Bo 52) early papers, and if one uses the hydrodynamic moments of inertia K must be taken zero since $\mathcal{I}_3^2$ is zero as seen in Eq. (II–13) if $\gamma = 0$. For this case then

$$\mathcal{I}_1^2 = \mathcal{I}_2^2 = \mathcal{I}_0^2 = 3 B_2 \beta^2,$$

and from Eq. (II–10) the energy of each level is just

$$E(I) = (\hbar^2 / 6 B_2 \beta^2) L(L+1), \qquad K = 0. \qquad \text{(II–23)}$$

This is in agreement with the lower lying levels in even nuclei where the measured sequence is 0+, 2+, 4+, 6+, $\cdots$. Indeed levels up to about $L = 18$ have been obtained by Coulomb excitation (St 59a), α-particle (Ko 64) and heavy-ion reactions (St 64) and fit this $L(L+1)$ rule fairly well.

As we have seen, the $\lambda = 2$ surface can execute not only rotations but β and γ vibrations about its equilibrium shape. The former vibrations for axially symmetric systems are identical in character with those discussed in connection with Eq. (II–14b) and yield a rotational band given by Eq. (II–23) based on some vibrational frequency $\hbar\omega_\beta$. As "breathing mode" vibrations they preserve in axially symmetric systems that axial symmetry.

In order to explain the systematically occurring second excited 2+ states in these nuclei (if a β-vibrational band occurs then there will be at least three 2+ levels, the upper two probably being quite closely spaced) a vibration of the γ degree of freedom is invoked, forming a rotational band based upon these second 2+ levels and with quantum number $K = |2|$. The band head is given by $\hbar\omega_\gamma$ and the level sequence is $L = 2+, 3+, 4+, 5+, \cdots$. This rotational band is called the γ-vibrational band (Ad 56). [An unfortunate overlap in nomenclature is a source of confusion. By γ vibrations in the symmetric-core theory is meant this rotational band built upon the second 2+ level, while in the asymmetric-core model these levels arise naturally from the asymmetric rotator. However, in this latter model the term γ vibrations means those asymmetry vibrations of the core discussed previously (Da 61).] If more than two levels with even spin appear then it may be necessary to include rotational bands based upon higher values of K, say $K = 4, 6, \cdots$. An example can be found in the spectrum of ${}_{70}\mathrm{Yb}_{102}{}^{172}$ which has five positively identified 4+ levels below 2.1 MeV and one other tentatively identified (Ha 61). The authors of Ref. Ha 61 suggest the assignment of four K bands ($K = 0, 2, 3, 4$). (An alternative assignment of nuclear structure parameters for this nucleus is given in Table II–3.) In Fig. II–3 is shown a schematic drawing of how the nuclear level structure appears in such a model. The three different rotational bands have been separated and no significance should be attached to the relative band-head energies. The different bands are identified by the vibrational quantum numbers n_β, n_γ and K. The quantum numbers for the ground state, β-, and γ-vibrational bands are then $n_\beta = n_\gamma = 0$, $K = 0$); $n_\beta = 1$, $n_\gamma = 0$, $K = 0$; $n_\beta = 0$, $n_\gamma = 1$, $K = 2$, respectively.

This model also includes a rotation–vibration interaction yielding a correction to the energy levels of the form given in Eq. (II–11) so that if each band is assumed to have the same moment of inertia, $\mathcal{I}_0$, then there are four parameters in the theory: $\mathcal{I}_0$, $\hbar\omega_\beta$, $\hbar\omega_\gamma$, and b.

A symmetric-top system with three rotational bands is no more complicated than an asymmetric-top system with β vibrations and the number of parameters in each is the same. For both systems the $\hbar\omega_\beta$ are essentially the same and are associated with the coefficient of the vibrational potential energy C_2. $\mathcal{I}_0$ in the symmetric system can be associated with a similar scale

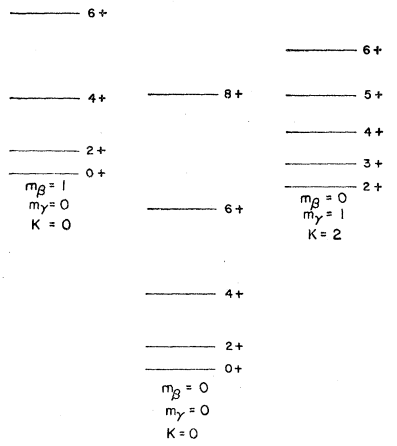

FIG. II–3. A schematic diagram of the energy level structure of a deformed but symmetric nucleus. Three bands are shown: the ground-state rotational band, the β-vibrational band to the left of it, and the γ-vibrational band. The nuclear parameters $\hbar\omega_\beta$ and $\hbar\omega_\gamma$ are the energies of the lowest member of their respective bands.

parameter $\hbar\omega_0$ in the asymmetric system, $\hbar\omega_\gamma$ is replaced by γ, while b as we have seen is replaced by the stiffness parameter μ at least for small μ [for larger values of μ or higher L values it is often necessary to introduce a term $cL^3(L+1)^3$ to get adequate agreement between experiment and theory (St 59a)].

In the hydrodynamic model, the b coefficient in Eq. (II–11) can be related to $\mathcal{I}_0$, $\hbar\omega_\beta$ and $\hbar\omega_\gamma$ and it should be possible to evaluate b from the ground-state rotational band, then predict the positions of the β- and γ-vibrational bands. Such a program has been carried out by Sheline (Sh 60) and the results show that b from the ground-state rotational band is a factor of about two times too small to explain the β- and γ-vibrational band heads. In actual fact Sheline in this study determined the constant b from the spectrum Pu^{238}, then determined a scale factor s so that sb could be used to predict the β- and γ-vibrational band heads when used in conjunction with the moment of inertia, also empirically determined. Finally, assuming that s should remain constant throughout the deformed regions he predicted the β- and γ-vibrational band heads for 26 nuclei. At the time of publication he could compare four of the predictions with experiment and in two cases got very good agreement. While this technique is useful as a tool to aid in determining the energy range to seek out levels of a certain character, more complete experimental evidence in these regions of atomic number, as well as the more recent theoretical developments described before, cast doubt on the validity of the approach. Since the magnitude of the factor s is only given empirically the method would seem to reduce to a two-parameter fitting procedure similar to that used by Mallman (Ma 61) (who used four parameters) which was described earlier.

Following the initial suggestions of Davydov and Filippov (Da 58a) and the very striking agreement with the experimental $E2$ transitions from the second 2+ level (Va 59), several investigations have refined this symmetric-core model to the point where agreement is as good as (Bi 59, Li 61a, Li 62, Fa 62, Pr 63) or for some nuclei better than, the asymmetric-core model. While the details of each of these calculations differ, we follow the discussions of Lipas (Li 61a) because this treatment has been extended to study octupole vibrations and negative parity levels in deformed nuclei (Li 61, Li 62). The method was originally discussed by Birbrair *et al.* (Bi 59) and has been extended by Danos and Greiner (Da 64b) to include giant dipole absorption (see Sec. V). One of the original criticisms of the asymmetric-core model was that there was no mechanism for additional 0+ states above the ground state (Ta 59, Ta60), which comment is of course not valid for the more extended asymmetric models (Da 60a, Da 61).

Restricting motion to small oscillations of β and γ about a spheroidal equilibrium shape defined by $\beta=\beta_0$ and $\gamma=\gamma_0=0$, the hydrodynamic moments of inertia of Eq. (II–13) become

$$\mathcal{I}_1^2\approx\mathcal{I}_2^2\approx 3B_2\beta_0^2=\mathcal{I}_0^2, \qquad \text{(II–24a)}$$

$$\mathcal{I}_3^2\approx 4B_2\beta_0^2\gamma^2=4B_\gamma\gamma^2. \qquad \text{(II–24b)}$$

In lowest order the difference between $\mathcal{I}_1^2$ and $\mathcal{I}_2^2$ is neglected so that K is a good quantum number and the rotational energy is given by Eq. (II–10). One then has two choices in developing the vibrational Hamiltonian, either the rotational term $\hbar^2K^2/2\mathcal{I}_3^2(\gamma)$ can be included in it (Bi 59, Li 61, Pr 63), or it can be placed in the rotation–vibration interaction term (Fa 62). [In this reference the authors do not use β and γ as the vibrational variables, but the a_0, $a_{\pm2}$ of Eq. (I–17). New variables, a'_0 and $a'_{\pm2}$ are then defined with reference to the spheroidal equilibrium conditions as

$$a_0=\beta_0+a'_0\cong\beta_0+(\beta-\beta_0),$$

$$a_{\pm2}=a'_{\pm2}\cong(1/\sqrt{2})\beta_0\gamma^2\ll\beta_0,$$

and the dynamical equations are expressed in terms of a'_0 and $a'_{\pm2}$ and terms through *second* order are used.] In any event for small oscillations about the equilibrium shape the potential is

$$V(\beta,\gamma)=\tfrac{1}{2}C_\beta(\beta-\beta_0)^2+\tfrac{1}{2}C_\gamma\gamma^2.$$

The vibrational Schrödinger equation thus separates

$$\left[\frac{-\hbar^2}{2B_2}\frac{d^2}{d\beta^2}+\tfrac{1}{2}C_\beta(\beta-\beta_0)^2\right]f(\beta)=E_\beta f(\beta) \qquad \text{(II–25)}$$

$$\left\{\frac{-\hbar^2}{2B_\gamma}\left[\frac{1}{\gamma}\frac{d}{d\gamma}\left(\gamma\frac{d}{d\gamma}\right)-\frac{(K/2)^2}{\gamma^2}\right]+\tfrac{1}{2}C_\gamma\gamma^2\right\}\Phi(\gamma)=E_\gamma\Phi(\gamma). \qquad \text{(II–26)}$$

Equation (II–25) is the wave equation for a one-dimensional oscillator while Eq. (II–26) is that for

a two-dimensional oscillator if one defines $m^2=(K/2)^2$. Thus the solutions are

$$E_\beta=\hbar\omega_\beta(n_\beta+\tfrac{1}{2}),$$
$$n_\beta=0, 1, 2, \cdots$$
$$E_\gamma=\hbar\omega_\gamma(n_\gamma+1),$$
$$n_\gamma=\tfrac{1}{2}\mid K\mid+2N, \quad N=0, 1, 2, \cdots, K \text{ even}.$$

[The evenness of K arises from the fact that the regularity of $\Phi(\gamma)$ at the origin restricts m to integral values.] To these must be added the rotational energy

$$E_{\text{rot}}=(\hbar^2/2\mathcal{I}_0)[L(L+1)-K^2]. \qquad \text{(II–27)}$$

Since the β vibrations preserve axial symmetry, such bands have quantum number $K=0$ and spin sequence 0, 2, 4, $\cdots$ and $n_\gamma=0$. The first γ-vibrational band has $n_\beta=0$, $n_\gamma=1$ whence $\mid K\mid=2$ and the spin sequence 2, 3, 4, $\cdots$. The value $n_\gamma=2$ is associated with two bands one having $K=0$ and spin sequence 0, 2, 4, $\cdots$ while the other has $\mid K\mid=4$ and spin sequence 4, 5, 6, $\cdots$. Mixed bands can also occur the lowest having $n_\beta=n_\gamma=1$. While these have been discussed (Ma 60, Sh 60), there seems little experimental evidence for more than the familiar β and γ bands ($n_\beta=1$, $n_\gamma=0$ and $n_\beta=0$, $n_\gamma=1$, respectively) although the large number of 4+ levels in Yb^{172} below 2.1 MeV (Ha 61) would indicate the necessity, from this point-of-view, of bands with $n_\gamma>1$ and perhaps $n_\beta>1$. No such detailed analysis has been made.

For small displacements from axial symmetry, the differences between $\mathcal{I}_1^2$, $\mathcal{I}_2^2$, and $\mathcal{I}_0^2$ cause an interaction between these various bands and, if the asymmetry is small enough, this interaction may be treated by perturbation theory. It is usual to neglect the changes induced in the vibrational part of the Hamiltonian by these slight deviations from axial symmetry so that the perturbation Hamiltonian is

$$H'=T_{\text{rot}}-T_{\text{rot}}^{(0)}=\tfrac{1}{2}\hbar^2\sum_k\frac{L_k^2}{\mathcal{I}_k^2}-\frac{L_1^2+L_2^2}{\mathcal{I}_0^2}-\frac{L_3^2}{(\mathcal{I}_3^2)_0}$$
$$=\tfrac{1}{4}\hbar^2(\mathcal{I}_1^{2-1}+\mathcal{I}_2^{2-1}-2\mathcal{I}_0^{2-1})(L_1^2+L_2^2)$$
$$+\tfrac{1}{4}\hbar^2(\mathcal{I}_1^{2-1}-\mathcal{I}_2^{2-1})(L_1^2-L_2^2). \qquad \text{(II–28)}$$

The first term is diagonal and represents an interaction between bands with the same K while the second term represents an interaction between bands where $\Delta K=2$. Defining δ by

$$\beta_0\delta=\beta-\beta_0,$$

Eq. (II–28) can be written through terms of second order as

$$H'=(\hbar^2/2\mathcal{I}_0^2)[(-2\delta+3\delta^2+2\gamma^2)(L_1^2+L_2^2)$$
$$+(-(2/\sqrt{3})\gamma+(4/\sqrt{3})\delta\gamma)(L_1^2-L_2^2)]. \qquad \text{(II–28a)}$$

The term δ will connect states where $\Delta n_\beta=\pm1$, that is the ground and β-vibrational band, while the second term δ^2 and γ^2 will connect states with $\Delta n_\beta=0, \pm2$, $\Delta n_\gamma=0, \pm2$. The off-diagonal matrix elements in the β and γ space will be quite small since the energy spacing will be $2\hbar\omega$. The diagonal terms will give rise to corrections to the axially symmetric moment of inertia $\mathcal{I}_0^2$. Finally, the term in $(L_1^2-L_2^2)$ gives rise not only to the interaction between the γ vibrations and ground bands, but also to the interaction between the β and γ bands. The renormalized moments of inertia in the lowest three bands are

$$\mathcal{I}_{\text{gr}}=\mathcal{I}_0^2-\tfrac{1}{2}\hbar^2[(9/\hbar\omega_\beta)+(12/\hbar\omega_\gamma)],$$
$$\mathcal{I}_\beta=\mathcal{I}_0^2-\tfrac{1}{2}\hbar^2[(27/\hbar\omega_\beta)+(12/\hbar\omega_\gamma)],$$
$$\mathcal{I}_\gamma=\mathcal{I}_0^2-\tfrac{1}{2}\hbar^2[(9/\hbar\omega_\beta)+(24/\hbar\omega_\gamma)]. \qquad \text{(II–29)}$$

The corrections to the energies of these bands are given by

$$[\Delta E_{\text{gr}}(L)]_\beta=-12\left(\frac{\hbar^2}{2\mathcal{I}_0^2}\right)^3\frac{L^2(L+1)^2}{(\hbar\omega_\beta)^2}, \qquad \text{(II–30a)}$$

$$[\Delta E_{\text{gr}}(L)]_\gamma=-4\left(\frac{\hbar^2}{2\mathcal{I}_0^2}\right)^3\frac{L^2(L+1)^2}{(\hbar\omega_\gamma)^2}, \qquad \text{(II–30b)}$$

$$[\Delta E_\beta(L)]_\gamma=48\left(\frac{\hbar^2}{2\mathcal{I}_0^2}\right)^4\frac{L^2(L+1)^2}{[E_\beta(L)-E_\gamma(L)]\hbar^2\omega_\beta\omega_\gamma}, \qquad \text{(II–30c)}$$

where the terms in $L(L+1)$ have been neglected since they contribute to the renormalization of the appropriate moments of inertia in second and third order. The result (30a) can also be gotten by a classical perturbation calculation (Mo 57). The usual results of a second-order perturbation calculation are that levels are pushed down by those above and up by those below. Hence

$$[\Delta E_\beta(L)]_{\text{gr}}=-[\Delta E_{\text{gr}}(L)]_\beta,$$
$$[\Delta E_\gamma(L)]_{\text{gr}}=-[\Delta E_{\text{gr}}(L)]_\gamma,$$
$$[\Delta E_\beta(L)]_\gamma=-[\Delta E_\gamma(L)]_\beta.$$

These results have made use of second-order terms in what is generally considered to be a first-order theory. Lipas (Li 61a) has discussed this point in some detail and we return to it later.

From the point of view of energy level and spin sequence there would seem to be little reason to choose between either the symmetric model or the asymmetric one. Both contain the same number of parameters which can be related to one another and both are equally successful, in the deformed regions, in explaining the level structures. To base a preference for the symmetric model on the idea that the nucleus, as a liquid drop, somehow cannot be asymmetric has no support classically. It is well-known that the line of stability for a rotating liquid drop passes from spherical shape through the MacLaurin spheroids and the Jacobi ellipsoids into the pear-shaped figures (Ly 53).

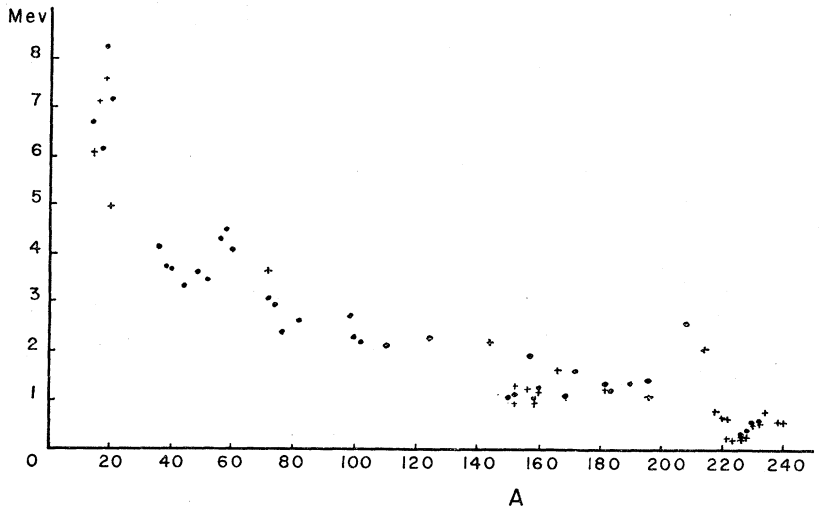

FIG. II–4. The lowest observed negative-parity levels in even nuclei as a function of A. Crosses represent spin-1 levels while dots represent spin-3 levels.

Indeed, as pointed out by Berenger and Knox (Be 61), the hydrodynamics of the problem prohibit such a nuclear system, if in rotation, to be spheroidal, although they expect such shapes to be a good starting point for further approximations. The fact is that at its greatest distortion in the asymmetric model the core is not very far from a spheroid.

We now return to a short discussion of the negative parity states in even nuclei. The occurrence of these states was discovered quite early (St 54) and not long after they were brought within the purview of the collective model (Ad 56) when it was realized that they probably arise from octupole ($\lambda=3$) vibrations of the nuclear surface. These negative-parity levels occur throughout the deformed regions, the lowest such levels being found at about 1.2 MeV in the rare earths and much lower in the actinide series. In Fig. II–4 the lowest observed negative-parity levels are plotted as a function of A. As with the lowest 2+ levels these negative parity levels are relatively low in the deformed regions and are considerably higher outside of them.

Comments made earlier concerning the positive-parity levels are also true here. If the deformation energy is greater than the vibrational energy, rotational spectra are observed while if not, a vibrational spectrum takes its place (Fig. I–2b). In the deformed region the theoretical investigation can proceed along two different lines. On the one hand one can discuss the octupole vibration about a spherical (La 60) or spheroidal (Li 61, 61a) shape assuming the system remains symmetric so that one deals with a $K=0$ and $K=2$ level structure. In the latter study of vibrations about the spheroidal shape the $\lambda=3$ moments of inertia were used for the case of a_{30}, $a_{3\pm2}\neq0$; $a_{3\pm1}=a_{3\pm3}=0$ and the problem studied from the point-of-view of perturbation theory. Thus the vibrational kinetic energy is now

$$T_{\text{vib}}=\tfrac{1}{2}B_2(\dot\beta^2+\beta^2\dot\gamma^2)+\tfrac{1}{2}B_3(\dot a_{30}{}^2+2\dot a_{32}{}^2).$$

Again it is assumed that, in lowest order, these octupole vibrations preserve axial symmetry and are small so that we still have Eq. (II–24a), but Eq. (II–24b) becomes

$$\mathcal{I}_3{}^{2+3}\cong 4B_2\beta_0{}^2(\gamma^2+g^2),$$

where

$$g^2=(2B_3/B_2\beta_0{}^2)\,a^2{}_{32}.$$

By again placing the rotational term $\hbar^2K^2/2\mathcal{I}_3{}^{2+3}(\gamma, g)$ in the vibrational part of the Hamiltonian, one has

$$H_{\text{vib}}=\frac{-\hbar^2}{2}\left[\frac{1}{B_2}\frac{\partial^2}{\partial\beta^2}+\frac{1}{B_3}\frac{\partial^2}{\partial b^2}+\frac{1}{B_\gamma}\left(\frac{\partial^2}{\partial\gamma^2}+\frac{\gamma}{\gamma^2+g^2}\frac{\partial}{\partial\gamma}+\frac{\partial^2}{\partial g^2}+\frac{g}{\gamma^2+g^2}\frac{\partial}{\partial g}\right)\right]+\frac{\hbar^2}{2B_\gamma}\frac{(K/2)^2}{(\gamma^2+g^2)}+V(\beta,\gamma,b,g),$$

where B_γ has been defined before, $b=a_{30}$ and

$$V(\beta,\gamma,b,g)=\tfrac{1}{2}C_\beta(\beta-\beta_0)^2+\tfrac{1}{2}C_\gamma\gamma^2+\tfrac{1}{2}C_b b^2+\tfrac{1}{2}C_g g^2.$$

The β and b variables separate immediately and lead to one-dimensional oscillator equations of the form of Eq. (II–25) with quantum numbers n_β and n_b and $K=0$. For the β band, one takes $n_\beta=1$, $n_b=0$ and this yields the spin-parity sequence $I=0+$, $2+$, $4+$, $\cdots$ while the b band requires $n_\beta=0$, $n_b=1$ and the sequence $I=1-$, $3-$, $5-$, $\cdots$. On the other hand, the γ and g variables can only be approximately separated in lowest order which gives rise to a doubly degenerate system the lowest band of which has $|K|=2$. This degeneracy is removed in the next order leading to γ and g bands both characterized by $|K|=2$ and having the spin sequence $I=2$, 3, 4, $\cdots$. The γ band has even parity and the g band odd.

Interactions between the bands give corrections to the moments of inertia as well as to displacement of the bands relative to their unperturbed positions. The results are an obvious extension of Eqs. (II-29) and (II–30).

This theory has been compared with experiment and

while the experimental data are rather sparse, the qualitative agreement is quite good although there are difficulties, in particular the details of the very rich negative parity spectrum of W^{182} cannot be explained.

Leper (Le 64) points out that this treatment suffers from the defect that the principal axes of the quadrupole and octupole systems are coincident throughout the motion. He generalizes the previous treatment and also shows how to separate the γ and g bands for any frequency of vibration.

On the other hand, one can study the rotations of a pure $\lambda=3$ surface and later add vibrational degrees of freedom to it. In this approach one defines the β_3, $\epsilon_{3\mu}$ parameters from the $a_{3,\mu}$ coefficients of Eq. (I–22a) by

$$a_{30}=\zeta\cos\eta\cos\iota,$$

$$\sqrt{2}\,a_{3\pm1}=\zeta\cos\xi\sin\iota,$$

$$\sqrt{2}\,a_{3\pm2}=\zeta\sin\eta\cos\iota,$$

$$\sqrt{2}\,a_{3\pm3}=\zeta\sin\xi\sin\iota,$$

the generalized coordinates being $\zeta=\beta_3$, η, ξ, ι and the Euler angles. The most detailed study (Da 62) to date of this model has taken $\iota=0$ [this is equivalent to making the body-fixed-axis system the principal system (Wi 62a)]. The octupole moments of inertia

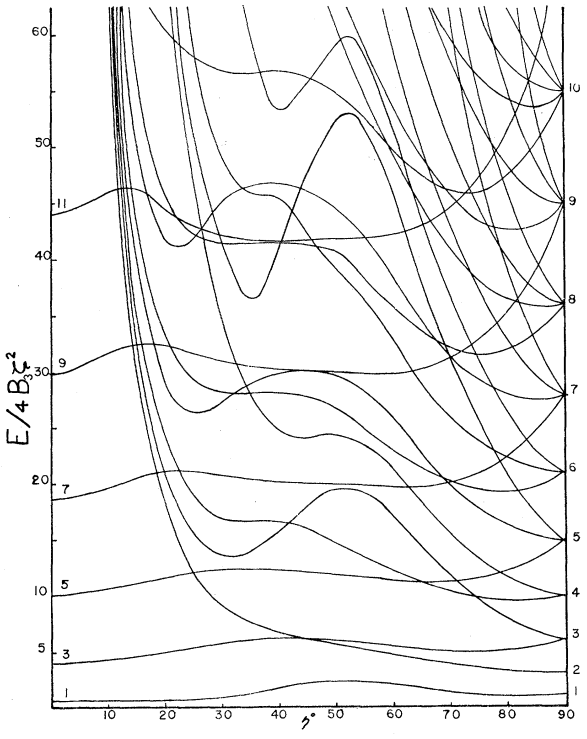

FIG. II–5. The energy levels for a rotating octupole surface with hydrodynamic moments of inertia as a function of the octupole asymmetry parameter η in units of $\hbar^2/4B_3\zeta^2$.

TABLE II–5. Asymmetric-core model parameters η and μ for the negative-parity levels in even-even nuclei.

Nucleus	η^0	μ
W^{182}	83.5	1.00
Th^{228}	12.3	0.258
Th^{230}	14.84	0.375
U^{232}	15.95	0

are then

$$\mathcal{I}_1^3=4B_3\zeta^2(\sin^2\eta\pm\tfrac{1}{2}15^{\frac{1}{2}}\sin\eta\cos\eta+\tfrac{3}{2}\cos^2\eta),$$
$$_2$$

$$\mathcal{I}_3^3=4B_3\zeta^2\sin\eta.$$

The proper representation is B_1 and the effective range of the asymmetry parameter is $0\leq\eta\leq\pi/2$. Figure II–5 shows the energy as a function of η for $I\leq11$ for these rotating octupole shapes. The ζ vibrations can be added in exactly the same way β vibrations were added to the deformed quadrupole rotator, the eigenvalue problem becoming similar to Eq. (II–17). This theory also has been compared with experiment (Da 62, Wi 62a), and agreement is satisfactory for the small amount of experimental data available. In Table II–5 are the assigned octupole surface parameters for deformed even nuclei with at least four known negative-parity levels.

Several relations similar to Eq. (II–8) which are independent of the form of the moments of inertia exist, but are more complicated since the theory does not relate the octupole zero-point energy with the quadrupole zero-point energy. One such relation is

$$[\mathcal{E}(4^1)+\mathcal{E}(4^2)]-[\mathcal{E}(3^1)+\mathcal{E}(3^2)]=4\mathcal{E}(2),$$

where

$$\mathcal{E}(L^n)=E(L^n)-E(1).$$

Finally, the restriction that $\iota=0$ has been relaxed and the model with seven degrees of freedom studied (Da 64a). The level structure is complex and no meaningful comparison with experiment can be made until many more negative-parity levels have been found in deformed even nuclei.

B. Electromagnetic Transitions

Since deformed nuclei have relatively large quadrupole moments it is expected that the off-diagonal matrix elements of the quadrupole moment operator are large in many cases. This is indeed the case and the enhancement of electric quadrupole transitions over the single-particle rate is another test of whether a nucleus is deformed. From the models discussed before it is a simple matter to calculate electric and magnetic transition probabilities to any order since the operators are well known. The comparison of these electromagnetic transition rates and branching ratios with experiment forms a much more stringent test of a given model than does a comparison of the energy-

level structure alone. We discuss the multipole moments and transitions in order of increasing multipole order.

The transition probability per unit time for the emission of a photon of energy $\hbar\omega=\hbar ck$ carrying off angular momentum λ is (Pr 62)

$$T(\lambda)=\frac{8\pi(\lambda+1)}{\lambda[(2\lambda+1)!!]^2}\frac{k^{2\lambda+1}}{\hbar}B(\lambda,\Delta L),$$

where the reduced matrix element $B(\lambda, \Delta L)$ is defined as

$$B(\lambda, L_i\rightarrow L_f)=(2L_i+1)^{-1}\sum_{M_i,M_f}|\langle f|T_{\lambda\mu}{}^L|i\rangle|^2, \tag{II-31}$$

where $T_{\lambda\mu}{}^L$ is the appropriate multipole operator defined in the laboratory. Since these reduced matrix elements have no direct energy or λ dependence, usually they, or their ratios, are the quantities compared with experiment.

The operators inducing the transitions are spherical tensors so that their body and laboratory components are simply related by

$$T_{\lambda\mu}{}^L=\sum_\nu D_{\mu\nu}{}^{\lambda*}(\theta_i)\,T_{\lambda\mu}{}^B. \tag{II-32}$$

The static electromagnetic moments are the diagonal matrix elements of those operators and since parity is a good quantum number in nuclei there are no static electric multipole moments with λ odd or static magnetic multipole moments with λ even.

The lowest multipole order is the monopole or $E(0)$ transitions in which the energy of the transition is carried off by a positron–electron pair formed by internal pair creation (high-energy transitions) or by an atomic electron emitted in internal conversion. In many instances these monopole transitions compete with electric quadrupole transitions in certain deformed even nuclei.

The interaction Hamiltonian is (Ch 60)

$$H_{\rm el}(L=0)=-e\int_0^\infty d\tau_n\rho_n\langle e_f|\frac{1}{r_n}-\frac{1}{r_e}|e_i\rangle,$$

where $|e_i\rangle$ and $|e_f\rangle$ are the initial and final electron wave functions and r_n and r_e the nuclear and electronic radii and ρ_n the nuclear density. The absolute transition probability for electric monopole internal conversion, W is

$$W=\Omega P^2,$$

where

$$P=(1/e)\int\rho_n[(r_n/R)^2+\cdots]\,d\tau_n.$$

The factor Ω is available in graphic form as a function of the transition energy (Ch 56). For a deformed but uniformly charged nucleus whose surface is given by Eq. (I-5L) or (I-5B), the operator inducing the transition becomes

$$T^B(E0)=\frac{3Z}{4\pi}\left(\frac{4\pi}{5}+\beta^2+\frac{5(5/\pi)^{\frac{1}{2}}}{21}\beta^3\cos\gamma\right)$$

so that

$$P=\langle n_f|\,T^B(E0)\,|n_i\rangle.$$

These monopole transitions can take place not only between states of $L=0$ but between any states with the same spin and parity. A theoretical study of the $0'\rightarrow 0$ transitions using a model with β and γ vibrations has been made in Ref. Da 61a. Calculations using the symmetric deformed nuclear model with β vibrations have been done for 22 nuclei (Re 61). These have been compared with experiment where data are available and reasonable agreement with theory is obtained although discrepancies have been noted near the edges of the deformed region (Ra 60).

Figure (II-6) shows a comparison of the monopole conversion transition probability in the k shell compared with the single-particle $M1$ and $E2$ gamma-ray transition as well as the $M1$ and $E2$ conversion probabilities also in the k shell. Since $E2$ gamma-ray tran-

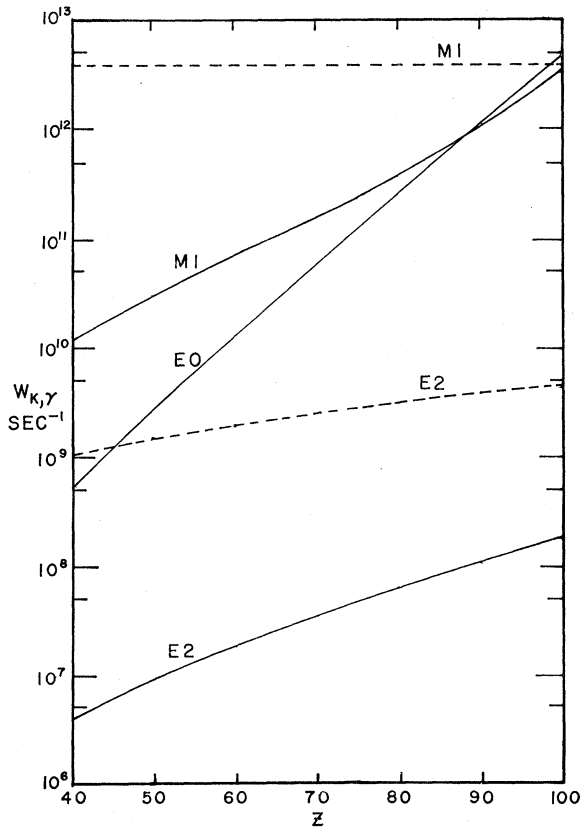

FIG. II-6. Comparison of the $E0$, $M1$, and $E2$ transition probabilities for internal conversion in the k shell for various values of atomic number Z. Also shown are the single particle estimates of the gamma-ray transition probabilities for $M1$ and $E2$ transitions. After Church and Weneser (Ch 56).

TABLE II–6. The electric monopole to electric quadrupole transition probabilities for deformed even nuclei. The calculated values are from Ref. Re 61.

Nucleus	$I=0+$ Calc.	$I=0+$ Obs.	$I=2+$ Calc.	$I=2+$ Obs.	Ref.
Sm^{152}	0.032	0.013	0.077	0.03	Re 61
Gd^{154}	0.034	$\cong 0.02$	···	···	Re 61
Hf^{178}	0.0075	0.0076	···	···	Re 61
W^{182}	···	···	···	3.2×10^{-2}	Sa 62
W^{184}	···	···	···	1.3×10^{-2}	Sa 62
Os^{188}	···	···	···	6×10^{-3}	Sa 62
Os^{190}	···	···	···	2.4×10^{-3}	Sa 62
Pt^{196}	···	···	···	5.8×10^{-4}	Sa 62
Th^{230}	0.60	1.3 ± 0.6	···	···	Bj 63
Th^{232}	···	···	1.3	1.3	Re 61
U^{232}	0.74	1.0	0.60	$\gtrsim 0.5$	Re 61
U^{234}	···	1.1	···	···	Bj 63
U^{236}	0.42	1.33	···	···	Re 61
U^{238}	···	···	$\gtrsim 0.65$	0.09	Re 61
Pu^{238}	0.31	1.0 ± 0.3	···	···	Bj 63
Pu^{240}	0.43	0.1 ± 0.03	···	···	Bj 63

sition probabilities in deformed nuclei are enhanced by several orders of magnitude over the single particle values, it can be seen that for high Z the monopole transitions will compete favorably with the $E2$ gamma-ray transitions.

In Table II–6 are listed the $E0/E2$ ratios as measured and predicted by theory. To date no monopole transitions between negative-parity states have been reported.

The next multipole orders are the electric and magnetic dipole transitions. However, since the electric dipole operator can be directly related to motion of the center of charge and since we have assumed that the center of mass (and also charge) is at rest [this is done by omitting the $\lambda=1$ terms in Eqs. (I–5L, B)] it should be expected that in deformed nuclei the $E1$ transitions should conform to the single-particle estimates (Bl 52). There is, however, one well-known case in which an $E1$ transition is hindered by a very large factor—this is the isomeric transition in Hf^{180m} from the 9− state at 1142.9 keV to the 8+ state at 1085.3 keV. This is an excellent example of K-forbiddenness in the symmetric-core model which is quite applicable here since there is but one $L=2+$ level in this nucleus. The 8+ level is then the fourth excited level of the $K=0$ ground-state band and its state function can be written

$$|\, I=8,\, M,\, K=0+\rangle=\Phi_f(X)\,(17/8\pi^2)^{\frac{1}{2}}D_{M0}^{8}(\theta_i).$$

The initial state must be the first number of a $K=9$-band since no other negative-parity states have been observed at lower energies. This state function is then

$$|\, I=9,\, M,\, K=9-\rangle$$
$$=\Phi_i(X)\,(19/16\pi^2)^{\frac{1}{2}}[D^{9}_{M9}(\theta_i)-D^{9}_{M-9}(\theta_i)],$$

where the functions $\Phi(X)$ represent all other parts of the state function except the rotational part. The transition probability then contains a factor which is the square of the integral over the rotational parts of the state vectors of the form

$$\int D_{Mf0}^{8*}(\theta_i)\,D^{1}_{\mu\nu}(\theta_i)\,[D^{9}_{Mi9}(\theta_i)-D^{9}_{Mi-9}(\theta_i)]\,d\Omega,$$

which is identically zero as $|\,\nu\,|\leq 1$. Thus for the transition to take place at all there must be some slight K-band admixture in either or both states. The failure of the transition to be allowed is due to the fact that K-quantum numbers do not overlap. From the known properties of the Clebsch–Gordan coefficients, integrals of the above type do not vanish if $\Delta K\leq\lambda$. The degree of K-forbiddenness has been characterized by the number ν (Al 55),

$$\nu=\Delta K-\lambda, \qquad \text{(II–32a)}$$

the larger ν the more the transition is inhibited. For the Hf^{180} isomeric transition $\nu=8$ and the transition is much slowed. Other similar, very slow K-forbidden transitions have been reported (Sc 57).

Magnetic dipole transitions are forbidden in even nuclei for these collective models. In this case the operator $\mathfrak{u}$ is

$$\mathfrak{u}=g_R\mathbf{L}$$

in units of the nuclear magneton, where g_R is the gyromagnetic ratio which for the collective model is taken as Z/A as it is for the liquid-drop model (Wa 39). Since the components of the operator $\mathbf{L}$ are stepping operators in an angular-momentum representation we have that for the laboratory components

$$L_\mu\,|\,LM\rangle$$
$$=(-1)^{\mu}[L(L+1)]^{\frac{1}{2}}C(L1L;\,M+\mu,\,-\mu,\,M)\;|\;LM+\mu\rangle \qquad \text{(II–33)}$$

so that for the strict symmetric model, $M1$ transitions are forbidden between levels even when permitted by the spin and parity selection rules. The case for

TABLE II–7. The experimental ratio of transition probabilities for magnetic dipole to electric quadrupole transitions between the (22) and (21) levels in deformed even nuclei. This ratio is sometimes given by δ where

$$\delta^2=0.7[E(22)-E(21)]^5 B(E2{:}22\rightarrow 21)/B(M1{:}22\rightarrow 21).$$

Nucleus	$T(M1)/T(E2)\times 10^{-2}$
Sm^{152}	2.0
Dy^{160}	3.6
W^{182}	6.4
W^{184}	0
Os^{186}	1.0
Os^{188}	0.3
Os^{190}	<25
Pt^{192}	5.3
Pt^{194}	1.0
Pt^{196}	5.3
Th^{228}	0
Th^{232}	0

the asymmetric model is similar: the only possible transitions are for $\Delta L=0$ but then $\Delta K=2$. Equation (II–3) for the stepping properties of L in the principal axis system shows these transitions to be forbidden also. Thus $M1$ transitions are absolutely forbidden in collective models for deformed even nuclei. That this is confirmed to a high degree is to be seen in Table II–7 where the ratio $B(M1;\ 22\rightarrow 21)/B(E2;\ 22\rightarrow 21)$ is very small for such nuclei.

This feature of collective models that the $M1$ transitions are zero for even nuclei depends only upon the stepping properties of the angular momentum operators, Eq. (II–3) and (II–33), and not on their particular form. In their original paper Davydov and Filippov (Da 58a) derived a relation for the ratio $B(M1;\ 22\rightarrow 21)/B(E2;\ 22\rightarrow 21)$ which was not identically zero. Several authors have commented that this result is wrong (Ta 59, Ta 60, Ta 62); however, Lipas (Li 64) in a detailed discussion shows that the error in Ref. Da 58a was to take the magnetic dipole operator to second order in the $\alpha_{2\mu}$ while taking the angular momentum operator only to first order. As a result the ratio of reduced matrix elements, $B(M1)/B(E2)$, is of order $|\alpha_{2\mu}|^2$ as might be expected. Had they taken the angular momentum operator to second order their matrix elements would have vanished. Unfortunately the uncritical use of these results of Ref. Da 58a is found elsewhere (e.g. Pr 63, Gr 63).

The static magnetic dipole moment, $\langle\mu\rangle$, for even nuclei is $\langle\mu\rangle=g_R L$ and is zero for the ground states of even nuclei. Recently, it has become possible to measure the magnetic dipole moments of the excited states of nuclei, which for even nuclei yields a direct experimental determination of the collective gyromagnetic ratio g_R. Table II–8 gives the excited-state gyromagnetic ratios for a number of these nuclei. In general these are for the first excited state (2^1). With few exceptions, the experimental value of g_R is well below the Z/A value usually assumed for collective motion. This difference has also been explained by the effects of the pairing correlation (Ni 61a).

We now turn to electric quadrupole transitions which occur either between states of positive parity or states of negative parity. The part of the operator with the largest matrix elements is (Bl 52)

$$T^B{}_{2\mu}=e\sum_{k=1}^{Z}\int r_k{}^2 Y_{2\mu}(\theta_k,\varphi_k)\psi_f{}^*\psi_i\,d\tau,$$

with the sum running over the protons of the nucleus and the integral over the entire space in which the initial and final nuclear state functions ψ_i and ψ_f are defined. The matrix elements for the transition are

TABLE II–8. The measured gyromagnetic ratios of the excited states of even nuclei. These should be compared with the Z/A value which is also given.

Nucleus	Spin of state	g_R	Z/A	Reference
Fe^{56}	2	$+0.53\pm0.16$	0.464	a
Nd^{150}	2	0.22 ± 0.04	0.400	b
Sm^{152}	2	0.350 ± 0.03	0.408	c
Sm^{154}	2	0.21 ± 0.04	0.403	b
	2	0.31 ± 0.05		c(Fig. 12)
Gd^{154}	2	0.367 ± 0.03	0.416	c
Gd^{156}	2	0.320 ± 0.03	0.410	c
Dy^{160}	2	0.28 ± 0.08	0.412	d
Er^{166}	2	0.31 ± 0.03	0.410	q
Yb^{172}	2	0.304 ± 0.034	0.407	r
Hf^{178}	2	0.29 ± 0.02	0.404	c(footnote 33)
Hf^{180}	2	0.371 ± 0.032	0.400	c
	4	0.5 ± 0.1		c
W^{182}	2	0.285 ± 0.042	0.407	f
	2	0.193 ± 0.018		g
W^{184}	2	0.207 ± 0.016	0.402	g
	2	0.38 ± 0.05		h
W^{186}	2	0.292 ± 0.027	0.398	g
Os^{186}	2	0.316 ± 0.028	0.409	j
	2	0.30 ± 0.08		k
Os^{188}	2	0.29 ± 0.03	0.404	l
		0.20 ± 0.02		n
		0.23 ± 0.03		p
Pt^{192}	2	0.27 ± 0.07	0.406	s
Hg^{198}	2	0.38 ± 0.11	0.404	m

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of the form

$$\langle f \mid T^{L}{}_{2\mu} \mid i\rangle = e\sum_{\nu}(-1)^{\mu-\nu}\langle L_f M_f \mid D^{2}{}_{-\mu-\nu}(\theta_j) \mid L_i M_i\rangle \int \phi_f{}^{*}(\mathbf{r}) \sum_{k=1}^{Z} r_k{}^{2} Y_{2\nu}(\theta_k, \varphi_k)\phi_i(\mathbf{r})\, d\tau.$$

Since all of the rotational models introduced to date have only even K-bands (this is true not only for the models of positive-parity states but those of negative-parity states as well) ν is only even, so these matrix elements become

$$\begin{aligned}\langle f \mid T^{L}{}_{2\mu} \mid i\rangle &= e(-1)^{\mu}\{\langle L_f M_f \mid D^{2}{}_{-\mu 0}(\theta_j) \mid L_i M_i\rangle \int \phi^{*}{}_f(\mathbf{r}) \sum_{k=1}^{Z} r_k{}^{2} Y_{20}(\theta_k, \varphi_k)\phi_i(\mathbf{r})\, d\tau \\ &\quad + \langle L_f M_f \mid D^{2}{}_{-\mu -2}(\theta_j) \mid L_i M_i\rangle \int \phi^{*}{}_f(\mathbf{r}) \sum_{k=1}^{Z} r_k{}^{2} Y_{22}(\theta_k, \varphi_k)\phi_i(\mathbf{r})\, d\tau \\ &\quad + \langle L_f M_f \mid D^{2}{}_{-\mu 2}(\theta_j) \mid L_i M_i\rangle \int \phi^{*}{}_f(\mathbf{r}) \sum_{k=1}^{Z} r_k{}^{2} Y_{2-2}(\theta_k, \varphi_k)\phi_i(\mathbf{r})\, d\tau\} \\ &= e(-1)^{\mu}\{\langle L_f M_f \mid D^{2}{}_{-\mu 0}(\theta_j) \mid L_i M_i\rangle \int \phi^{*}{}_f(\mathbf{r}) \sum_{k=1}^{Z} r_k{}^{2} Y_{20}(\theta_k, \varphi_k)\phi_i(\mathbf{r})\, d\tau \\ &\quad + \tfrac{1}{2}\langle L_f M_f \mid D^{2}{}_{-\mu -2}(\theta_j) + D^{2}{}_{-\mu 2}(\theta_j) \mid L_i M_i\rangle \int \phi^{*}{}_f(\mathbf{r}) \sum_{k=1}^{Z} r_k{}^{2}[Y_{22}(\theta_k, \varphi_k) + Y_{2-2}(\theta_k, \varphi_k)]\phi_i(\mathbf{r})\, d\tau \\ &\quad + \tfrac{1}{2}\langle L_f M_f \mid D^{2}{}_{-\mu -2}(\theta_j) - D^{2}{}_{-\mu 2}(\theta_j) \mid L_i M_i\rangle \int \phi^{*}{}_f(\mathbf{r}) \sum_{k=1}^{Z} r_k{}^{2}[Y_{22}(\theta_k, \varphi_k) - Y_{2-2}(\theta_k, \varphi_k)]\phi_i(\mathbf{r})\, d\tau\}. \quad \text{(II–34)}\end{aligned}$$

This last term is zero since the rotational matrix element

$$\langle L_f M_f \mid D^{2}{}_{-\mu -2}(\theta_j) - D^{2}{}_{-\mu 2}(\theta_j) \mid L_i M_i\rangle$$

vanishes both for the A and the B_1 representations.

If the charge distributions are assumed to be the same in initial and final states we may replace the integrals over the nuclear volume by the components of the quadrupole moment tensor

$$Q_\mu = e(16\pi/5)^{\frac{1}{2}} \sum_{k=1}^{Z} \phi^{*}(\mathbf{r}) r_k{}^{2} Y_{2\mu}(\theta_k, \varphi_k)\phi(\mathbf{r})\, d\tau, \quad \text{(II–35a)}$$

Q_0 then being the intrinsic quadrupole moment of the nucleus. A simplification of Eq. (II–34) is obtained by introducing new quantities $Q_{\mu 0}$

$$Q_{\mu 0} = (5/64\pi)^{\frac{1}{2}}(Q_\mu + Q_{-\mu})$$

which are, of course, model-dependent since theory depends explicitly upon the nuclear charge distribution and shape. (The $Q_{\mu 0}$ are identical with the k_μ defined in Ma 61 and are the same quantities defined in Em 63.) Thus Eq. (II–34) becomes

$$\begin{aligned}\langle f \mid T^{L}{}_{2\mu} \mid i\rangle &= (-1)^{\mu}\{\langle L_f M_f \mid D^{2}{}_{-\mu 0}(\theta_j) \mid L_i M_i\rangle Q_{00} + \langle L_f M_f \mid D^{2}{}_{-\mu -2}(\theta_j) + D^{2}{}_{-\mu 2}(\theta_j) \mid L_i M_i\rangle Q_{20}\} \\ &= \pm(-1)^{\mu}\left(\frac{2L_i+1}{2L_f+1}\right)^{\frac{1}{2}} C(L_i 2 L_f; -M_i, -\mu, -M_f) \\ &\times \left\{\left[\pm\frac{A_0{}^{L_f} A_0{}^{L_i}}{4}[1\pm(-1)^{L_f}][1\pm(-1)^{L_i}]C(L_i 2 L_f; 000) \pm (-1)^{L_i+L_f}\sum_{K_i>0} A_{K_i}{}^{L_f} A_{K_i}{}^{L_i} C(L_i 2 L_f; K_i, 0, K_i)\right] Q_{00}\right. \\ &\quad + \left[\frac{(-1)^{L_i}[1\pm(-1)^{L_f}]^2}{2\sqrt{2}} A_0{}^{L_f} A_2{}^{L_i} C(L_i 2 L_f; 2-20) + \frac{(-1)^{L_f}[1\pm(-1)^{L_i}]^2}{2\sqrt{2}} A_2{}^{L_f} A_0{}^{L_i} C(L_i 2 L_f; 022)\right. \\ &\quad \left.\left. \pm(-1)^{L_i+L_f}\sum_{K_i>0, K_f>0} A_{K_i}{}^{L_i} A_{K_f}{}^{L_f}[C(L_i 2 L_f; K_i, 2, K_f) + C(L_i 2 L_f; K_i, -2, K_f)]\right] Q_{20}\right\}, \quad \text{(II–36)}\end{aligned}$$

where the upper sign is for transitions between states belonging to the A representation (positive parity) and the lower sign is for transitions between states belonging to the B_1 representation (negative parity).

Mallman (Ma 61) has calculated the reduced matrix elements for $E2$ transitions from Eq. (II–36) for $I_{\max} \leq 6$ and has compared his theory of a general asymmetric, rigid rotator with experimental values from 23 nuclei. (As many as 10 branching-ratio predictions have been compared.) In this process he assigns values to Q_{00} and to Q_{20} the latter through the ratio $r=\sqrt{2}Q_{20}/Q_{00}$. In general he finds good agreement with theory and in particular the values of r are in many cases quite close to the hydrodynamic value.

In the hydrodynamic model if one assumes that the nuclear charge is uniformly distributed then the quadrupole moment tensor components of Eq. (II–35a) become

$$Q_\mu = \frac{3Ze}{R_0^2(5\pi)^{\frac{1}{2}}} \int r^2 Y_{2\mu}(\theta, \varphi)\, d\mathbf{r} = \begin{cases} [3ZeR_0^2/(5\pi)^{\frac{1}{2}}]a_{2\mu}, & (\pi^+) \\ -\left(\frac{12}{5}\right)^{\frac{1}{2}} \frac{ZeR_0^2}{\pi} \sum_{\sigma\kappa} a_{3\sigma}a_{3\kappa}C(323;\, \kappa\mu\sigma), & (\pi^-). \end{cases} \tag{II–35b}$$

Thus, for each case

$$Q_{\mu 0} = \begin{cases} \frac{3ZeR_0^2}{8\pi}(a_{2\mu}+a_{2-\mu}), & (\pi^+) \\ -\frac{3ZeR_0^2}{4(3\pi^3)^{\frac{1}{2}}} \sum_{\sigma\kappa} a_{3\sigma}a_{3\kappa}[C(323;\, \kappa\mu\sigma)+C(323;\, \kappa-\mu\sigma)], & (\pi^-). \end{cases}$$

The electric quadrupole operator is a first-order operator in the $a_{2\mu}$ for the quadrupole ($\pi+$ states) surfaces but a second-order operator for the octupole surfaces. The $E3$ operator, is a first-order operator in the $a_{3\mu}$.

Gregoriev and Avotina (Gr 60) have compared the predictions of the asymmetric-core model using the hydrodynamic moments of inertia with experiment and fixed values of β and γ. In this study the asymmetry parameter γ has been picked to fit the measured energy levels. Only two branching ratios

$$B(E2\colon 22\to 21)/B(E2\colon 22\to 01)$$

and

$$B(E2\colon 31\to 41)/B(E2\colon 31\to 21)$$

have been used. Of the 24 nuclei studied, 18 have such measured ratios. In general, the predictions of the model are close to the experimentally observed values with few exceptions (two notable ones are Os^{192} and Pt^{194} where theory and experiment differ by more than 50%). Van Patter (Va 59) in an extensive survey of the $E2$ transitions from the 2+ states in even nuclei finds striking agreement between theory and experiment especially for the branching ratio

$$B(E2\colon 22\to 01)/B(E2\colon 21\to 01).$$

For the case of the symmetric model, transitions within the ground state band ($K=0$) are governed by the first term of Eq. (II–36) while transitions within any other band are governed by the second term. On the other hand, transitions between the ground state and the "gamma vibrational band" are governed by either the third or fourth terms. Thus the branching ratios within a band or from one band to another are proportional to the ratios of squares of Clebsch–Gordan coefficients. Ratios for transitions from different bands [as the $B(E2\colon 22\to 01)/B(E2\colon 21\to 01)$ mentioned above] are not given completely by this model because the elements of the quadrupole tensor $Q_{\mu 0}$ are independent fitting parameters and are not related as they are in the asymmetric model with hydrodynamic moments of inertia. Again surveys show this model is fairly successful in predicting the experimentally observed ratios (Sh 60). However, certain cases are not explained by this simple theory but can be explained by assuming the mixing of the $K=0$ and $K=2$ bands (Gr 59). The reduced matrix elements for the decay between these two bands will then have a contribution from the second term of Eq. (II–36) as well as from the third and fourth terms. If the amount of $K=2$ band mixed into the $K=0$ band is ϵ then clearly a new parameter is needed in the theory involving the ratios of the elements of the quadrupole tensor. This parameter is defined as

$$Z=\epsilon Q_{00}/Q_{20}$$

and the effects of the band mixing on the branching ratios between bands will be

$$\frac{B(E2\colon\to 22\to 20)}{B(E2\colon\to 22\to 00)} = \frac{10}{Z}\left(\frac{1+2Z}{1-Z}\right)^2,$$

where the notation is $B(E2\colon L_iK_i\to L_fK_f)$. (This band mixing is not identical with the K-band mixing occurring naturally in the asymmetric top problem in which all K bands with $K\leq L_{\max}$ enter into the problem. Actually, however, the strength of the higher K bands in the state functions of the lower lying levels in the asymmetric-top model with hydrodynamic moments of inertia is very small so that this technique is a very good approximation to the asymmetric-top model.) Comparison between the model and experi-

ment has been made for 12 nuclei in the deformed regions assigning values of Z to each for various $B(E2)$ ratios with $L_{\max} \leq 4$. The agreement between Z values assigned from different ratios for a given nucleus is fairly good although for some nuclei the agreement is very poor being outside experimental error (i.e., Sm^{152}, Er^{168}, and Th^{228}) (Ni 61).

This band mixing causes a shift of the energy eigenvalue by the amount [Eq. (II–30b)]

$$\Delta E = \epsilon^2 L^2 (L+1)^2 [E(22) - E(21)]$$

and is therefore of the form of the energy shift due to the vibration–rotation interaction. The survey shows that this shift is smaller than the coefficient b of Eq. (II–11).

In general the reduced matrix element for $E2$ transitions can be written as

$$B(E2;\ L_i \rightarrow L_f) = B(E2:\ L_i \rightarrow L_f)_a f(Z,\ L_i,\ L_f)$$

with $B(E2:\ L_i \rightarrow L_f)_a$ being for transitions between pure K bands. Reference Li 62 gives tables of $f(Z,\ L_i,\ L_f)$ for transitions between the β band ($n_\beta = 1$, $n_\gamma = 0$) and the ground-state band ($n_\beta = n_\gamma = 0$) with and without β–γ coupling and for transitions between the γ band ($n_\beta = 0$, $n_\gamma = 1$) and the ground-state band with β–γ coupling. In this reference comparison between theory and experiment is made for the nuclei Gd^{154} and W^{182} the conclusion being that the coupling is not sufficient to account for the $L^2(L+1)^2$ corrections to the rotational energies. However, a similar but much more detailed symmetric-model calculation with this K-band (or rotation–vibration) interaction (Fa 64) does yield results between theory and experiment in the deformed regions comparable with the asymmetric model. Near the edges of the deformed region, especially for the osmium isotopes, the former model gives a somewhat better fit to the reduced gamma-ray transition ratios. This is achieved by including terms through second order in the collective Hamiltonian even though the surface is only expanded in lowest order spherical harmonics. This would seem to be inconsistent since there is evidence that first order $Y_{4\mu}(\theta, \varphi)$ terms are of comparable magnitude to second-order quadrupole terms (see Sec. IV A).

Since the reduced matrix elements are the most model-sensitive measurable quantities, they should yield the most insight into the structure of these nuclei. An extensive comparison of experiment with these several models has been made in the osmium region (Em 63) and the data seems to support best the symmetric top model with a modicum of band mixing. However, the β vibrations alone can cause considerable shifts in these ratios of the reduced matrix elements (Da 63) which effect has not been included in these studies.

Experimental information concerning the branching ratios for $E2$ transitions between negative-parity levels is quite sparse. To date only two nuclei have measured E2 branching ratios between these negative-parity states, these being W^{182} (Ha 61) and Th^{228} (Ar 60). The experimental papers apply the symmetric model to these levels fairly successfully in the case of Th^{228} but considerably less so in W^{182}. In this latter case an attempt to extend the coupling of symmetric octupole and quadrupole surface vibrations to explain the spectrum as well as the branching ratios has been rather unsuccessful (Li 62). The only other study made use of the results of Eqs. (II–36) and (II–35b, π^-), that is the asymmetric-core model with hydrodynamic moments of inertia but without the vibration contributions to the branching ratios (Da 62). The comparisons between theory and experiment are generally

TABLE II–9. Intrinsic quadrupole moment Q_0, deformation parameter β and ratio of experimental to rigid moment of inertia for deformed even nuclei. The rigid moment of inertia is given by $\mathcal{I}_{\text{rig}} = \frac{2}{5} AMR_0^2(1+0.31\beta)$ while the experimental moment of inertia is usually obtained from $\mathcal{I}_0 = 3\hbar^2/E(21)$.

Nucleus	Q_0 (barns)	β	$\mathcal{I}/\mathcal{I}_{\text{rig}}$	Reference
Sm^{150}	3.65	0.184	0.147	El 60a
Sm^{152}	5.86	0.290	0.380	El 60a
Sm^{154}	6.83	0.336	0.551	El 60a
Gd^{154}	5.88	0.280	0.373	El 60a
Gd^{156}	6.79	0.320	0.498	El 60a
Gd^{158}	7.41	0.346	0.547	El 60a
Gd^{160}	7.65	0.354	0.561	El 60a
Dy^{160}	6.72	0.301	0.490	El 60a
Dy^{162}	7.19	0.320	0.512	El 60a
Dy^{164}	7.55	0.334	0.558	El 60a
Er^{164}	7.14	0.306	0.456	El 60a
Er^{166}	7.56	0.323	0.496	El 60a
Er^{168}	7.60	0.320	0.496	El 60a
Er^{170}	7.42	0.310	0.484	El 60a
Yb^{170}	7.48	0.304	0.455	El 60a
Yb^{172}	7.72	0.311	0.477	El 60a
Yb^{174}	7.72	0.308	0.475	El 60a
Yb^{176}	7.60	0.301	0.445	El 60a
Hf^{176}	7.8	0.30	0.41	Ad 56
Hf^{178}	8.1	0.31	0.38	Ad 56
Hf^{180}	7.1	0.27	0.38	Ad 56
W^{182}	7.5	0.28	0.34	Ha 56
W^{184}	6.7	0.25	0.31	Mc 61
W^{186}	7.08	0.259	0.272	Mc 61
Os^{186}	5.61	0.201	0.247	Em 63
Os^{188}	5.33	0.191	0.214	Mc 61
Os^{190}	5.0	0.18	0.18	Mc 61
Os^{192}	4.5	0.16	0.16	Mc 61
Pt^{194}	4.41	0.152	0.0972	Mc 61
Pt^{196}	3.56	0.122	0.0890	Mc 61
Pt^{198}	3.7	0.13	0.076	Ad 56
Ra^{222}	6.63	0.184	0.223	Be 60
Ra^{224}	6.21	0.171	0.291	Be 60
Ra^{226}	7.22	0.197	0.351	Be 60
Ra^{228}	7.79	0.212	0.400	Be 60
Th^{226}	8.25	0.220	0.330	Be 60
Th^{228}	8.47	0.225	0.403	Be 60
Th^{230}	8.80	0.233	0.433	Be 60
Th^{232}	9.25	0.243	0.450	Be 60
Th^{234}	8.93	0.233	0.467	Be 60
U^{230}	9.46	0.245	0.443	Be 60
U^{232}	9.98	0.257	0.470	Be 60
U^{234}	9.77	0.251	0.516	Be 60
U^{236}	10.35	0.263	0.485	Be 60
U^{238}	10.52	0.268	0.480	Be 60
Pu^{238}	10.95	0.271	0.493	Be 60
Pu^{240}	11.26	0.278	0.488	Be 60

within experimental error and thus may also be said to support this model. No extension of the band mixing, symmetric-top model has been proposed for these states. Much more information on the odd-parity levels of even nuclei is needed before a definite choice can be made.

Electric octupole transitions occur between even- and odd-parity bands. A few $E3$ transitions have been measured in the deformed regions (Ha 61) (the preference, of course, being for $E1$ emission followed by $E2$ transitions) and there is also some evidence for $E3$ Coulomb excitations (El 60). Since the even-parity rotational states belong to the A representation and the odd-parity states to the B_1 representation, the reduced matrix elements $B(E3: L_i \rightarrow L_f)$ contain terms of the form

$$\langle L_f M_f \pi+ \mid T_{3\mu}{}^L \mid L_i M_i \pi- \rangle$$

$$= e(-1)^{\mu} \sum_{\nu} (-1)^{\nu} \langle L_f M_f \pi+ \mid D_{-\mu-\nu}{}^3 \mid L_i M_i \pi- \rangle \int \phi^*_f \sum_{k=1}^{Z} r_k{}^3 Y_{3\nu}(\theta_k, \varphi_k) \phi_i \, d\tau$$

$$= e(-1)^{\mu} \Big\{ \langle L_f M_f \pi+ \mid D_{-\mu 0}{}^3 \mid L_i M_i \pi- \rangle \int \phi^*_f \sum_{k=1}^{Z} r_k{}^3 Y_{30}(\theta_k, \varphi_k) \phi_i \, d\tau$$

$$+ \langle L_f M_f \pi+ \mid D_{-\mu -2}{}^3 + D_{-\mu 2}{}^3 \mid L_i M_i \pi- \rangle \int \phi^*_f \sum_{k=1}^{Z} r_k{}^3 [Y_{32}(\theta_k, \varphi_k) + Y_{3-2}(\theta_k, \varphi_k)] \phi_i \, d\tau \Big\}$$

$$= e(-1)^{\mu} \left(\frac{2L_i+1}{2L_f+1} \right)^{\frac{1}{2}} C(L_i 3 L_f: -M_i - \mu - M_f) \Big\{ \Big[\frac{A_0{}^{L_f} A_0{}^{L_i}}{2} [1+(-1)^{L_f}][1-(-1)^{L_i}] C(L_i 3 L_f; 000)$$

$$- (-1)^{L_i+L_f} \sum_{K_f>0} A_{K_f}{}^{L_f} A_{K_f}{}^{L_i} C(L_i 3 L_f; K_f O K_f) \Big] \Omega_{00}$$

$$+ \Big[\frac{A_0{}^{L_f} A_2{}^{L_i}}{2\sqrt{2}} [1+(-1)^{L_f}]^2 C(L_i 3 L_f; 2-20) + \frac{A_2{}^{L_f} A_0{}^{L_i}}{2\sqrt{2}} [1-(-1)^{L_i}]^2 C(L_i 3 L_f; 022)$$

$$- (-1)^{L_f+L_i} \sum_{K_f>0, K_i>0} A_{K_f}{}^{L_f} A_{K_i}{}^{L_i} [C(L_i 3 L_f; K_i, -2, K_f) + C(L_i 3 L_f; K_i, 2, K_f)] \Big] \Omega_{20} \Big\},$$

where

$$\Omega_{\mu 0} = (7/64\pi)^{\frac{1}{2}} (\Omega_{\mu} + \Omega_{-\mu})$$

and

$$\Omega_{\mu} = (16\pi/7)^{\frac{1}{2}} \mid \phi^*_f \sum_{k=1}^{Z} r_k{}^3 Y_{3\mu}(\theta_k, \varphi_k) \phi_i \, d\tau.$$

For surfaces given by Eq. (I–5B) assuming the nuclear charge distribution is constant and the same in the final as the initial state then these octupole terms can be given explicitly. While these integrations should be carried out from the origin to the smaller of the quadrupole or octupole surface, within the approximations generally used it is probably sufficient to carry out the radial integrals from zero to the mean surface (Da 62). In this approximation the operators in lowest order are

$$Q_{\mu} = (3ZeR_0{}^3/8\pi) a_{3\mu}.$$

Tentative $E3$ transition assignments have been made in W^{182} (Ha 61). Generally speaking no higher electric transitions have been observed between the lower lying levels of the deformed nuclei. The theory of these transitions can be developed in a manner analogous to that already given.

Finally, Lipas (Li 63) has made use of the collective model of octupole vibrations about a spheroidal equilibrium shape (Li 61, Li 61a) discussed before to calculate $E2$ and $E3$ transitions involving the various vibrational states. From the empirical data he was able to determine the mass parameters B_2 and B_3. For the former the values calculated for transitions from the beta to ground-state band are in poor agreement with the values calculated for transitions from the gamma to the ground-state band, which is exactly what has been found in more recent experiments by Greenberg *et al.* (Gr 63a) in Sm^{152}. Lipas has also calculated the reduced $E1$ transitions (Li 63) by dropping the requirement that the centers of mass and charge be coincident by postulating the relative displacement of neutrons and protons which is analogous to the models discussed in Sec. V. The calculated transition probabilities are an order of magnitude smaller than experiment.

The static electric quadrupole moments are simply

proportional to the diagonal matrix elements of the operator $T_{2\mu}{}^{L}$. The quadrupole moment as measured in the laboratory, Q, is related to the intrinsic quadrupole moment Q_0 by

$$Q=[L(2L-1)/(L+1)(2L+3)]Q_0$$

so that the measured quadrupole moments of all even–even nuclei in their ground states will be identically zero. The intrinsic quadrupole moment is not necessarily zero, and can be most easily measured by Coulomb excitation (Ad 56). For the symmetric-top model the Coulomb excitation of the first excited state yields the intrinsic quadrupole moment directly,

$$B(E2:01\rightarrow 21)=(5/16\pi)e^2Q_0^2,$$

and is the way most of the moments are determined. Table II–9 shows the values of the intrinsic quadrupole moments of deformed even nuclei measured in this way. Also shown is the value of the deformation parameter β which is related to Q_0 by

$$Q_0=[3/(5\pi)^{\frac{1}{2}}]ZR_0^2(1+0.16\beta)\beta \qquad \text{(II–36)}$$

and a comparison of the experimental moment of inertia (for the symmetric theory)

$$\mathcal{I}_X=3\hbar^2/E(21)$$

with the rigid moment of inertia

$$\mathcal{I}_{\text{rig}}=\tfrac{2}{5}AMR_0^2(1+0.31\beta).$$

The ratio $\mathcal{I}/\mathcal{I}_{\text{rig}}$ should be compared with the ratio

$$\mathcal{I}_{\text{hydro}}/\mathcal{I}_{\text{rig}}=45\beta^2/16\pi(1+0.31\beta).$$

That the experimental moment of inertia is greater than the hydrodynamic moment of inertia arises from the contributions of the pairing force (Be 59). However, in the collective model the moment of inertia is a fitting parameter [in the asymmetric model the similar parameter is $\omega_{\lambda 0}=(C_\lambda/B_\lambda)^{\frac{1}{2}}$].

III. COUPLING OF NUCLEONS TO THE CORE

A. Models of Odd-A Nuclei

In his paper on the collective model, Bohr (Bo 52) discussed the coupling of a single nucleon to the core. Later, he and Mottelson (Bo 53) applied the model to discuss such nuclear properties as beta and gamma transitions as well as static moments. Probably the most successful form of these models are the strong coupling ones. Intermediate coupling has been treated in the above papers and elsewhere.

The core, and hence the potential well in which the extra nucleon moves, is assumed to have a permanent average deformation and any oscillations of shape which take place have sufficiently low frequency to permit the particle to follow adiabatically. In the strong coupling limit the motion of the extra nucleon is exactly the same in the body-fixed frame as that of a nucleon moving in a spatially fixed potential well.

The picture of a single particle coupled to a core formed by all of the other nucleons has the advantage of simplicity; however, a model where several nucleons outside a closed shell are coupled to a deformed core might be equally valid. Since this latter picture is more difficult to treat mathematically in a nontrivial way, it has been seldom used (Fo 53) and we confine our comments to the single-particle model.

The total angular momentum of the nucleus **I**, a constant of the motion, is now the sum of two parts, **L**, the core's angular momentum, and **j**, the particle angular momentum, where

$$\mathbf{I}=\mathbf{L}+\mathbf{j}. \qquad \text{(III–1)}$$

Strong coupling then implies that

$$[I_1, I_2]=-iI_3, \quad \text{cyclically}$$

but

$$[j_1, j_2]=+ij_3, \quad \text{cyclically}.$$

It is customary to take the projections of **I** and **j** along the 3-axis to be K and Ω, respectively, while the z-component of I is M. Since the potential in which the particle moves is, in general, neither spherically nor axially symmetric, $\mathbf{j}^2$, Ω, and K are not constants of the motion. When the system possesses an axis of symmetry it is reasonable to assume that the core will behave like a deformed even nucleus having a similar symmetry for which we saw that, at least for the lower lying levels, $L_3=0$ whence $K=\Omega$ and are constants of the motion.

The Hamiltonian of the system now contains two terms, one being the kinetic energy of rotation of the core (assuming for the moment that core vibrations can be neglected), while the other is the Hamiltonian of the extra-core nucleon:

$$H=\tfrac{1}{2}\hbar^2\sum_{k=1}^{3}(L_k^2/\mathcal{I}_k)+(p^2/2m)+V(\mathbf{r},\mathbf{l},\mathbf{s})$$

$$=(\hbar^2/2)\sum_{k=1}^{3}(I_k^2-2I_kj_k+j_k^2)\mathcal{I}_k$$

$$+(p^2/2m)+V(\mathbf{r},\mathbf{l},\mathbf{s}), \qquad \text{(III–2)}$$

where the possibility that the potential might be dependent upon the extra nucleon's orbital angular momentum **l**, and spin **s**, has been included.

If one further assumes that odd-A nuclei possess an axis of symmetry ($\mathcal{I}_1=\mathcal{I}_2=\mathcal{I}_0$) this equation can be put into a very simple form:

$$H=H_R+H_p+H_c, \qquad \text{(III–3)}$$

where

$$H_R=(\hbar^2/2\mathcal{I}_0)[I(I+1)-2K^2] \qquad \text{(III–4a)}$$

$$H_p=(p^2/2m)+V(\mathbf{r},\mathbf{l},\mathbf{s})+(\hbar^2/2\mathcal{I}_0)\mathbf{j}^2 \qquad \text{(III–4b)}$$

$$H_c=(-\hbar^2/\mathcal{I}_0)(I_1j_1+I_2j_2)=(-\hbar^2/2\mathcal{I}_0)(I_+j_-+I_-j_+), \qquad \text{(III–4c)}$$

the $\pm$ referring to the usual stepping operators $0_\pm=0_1\pm i0_2$. The term H_c has been called the "rotation–par-

ticle coupling" (RPC) term (Ke 56) which it in fact is not since H_R is not the rotator-Hamiltonian nor is H_p the particle Hamiltonian. A true rotation–particle coupling term would be of the form $\mathbf{L}\cdot\mathbf{j}$. However, H_c is of the form $\boldsymbol{\omega}\cdot\mathbf{j}$ which classically can be thought of as a potential term giving rise to the Coriolis force, a characterization which seems preferable here.

If the particle is tightly bound to the core, the particle levels in the potential $V(\mathbf{r}, \mathbf{l}, \mathbf{s})$ are rather widely separated when compared with the spacing of the rotational levels of the core. Under this circumstance the off-diagonal matrix elements of H_c are small and may be neglected in first order. Qualitatively then, the level structure of a symmetric, deformed, odd-A nucleus consists of bands of rotational levels each built upon a separate particle or intrinsic level. The level sequence is determined by $\Omega(=K)$ so that the band will contain levels with angular momentum $I=K, K+1, K+2, \cdots$. In the approximation that H_c can be neglected H_R and H_p commute and the state functions are

$$|EIMK\rangle = |IMK\rangle\,|\Omega=K\rangle,$$

where the $|IMK\rangle = D_{MK}^{I*}$ are the solutions to the rotator problem and the $|\Omega\rangle = \chi_\Omega$ are the solutions of the problem

$$H_p\chi_\Omega = E_p\chi_\Omega. \qquad \text{(III–5)}$$

For a spherically symmetric potential, $\mathbf{j}^2$ is a constant of the motion and the eigenvectors $|j\Omega\rangle$ form a complete set so that we can expand χ_Ω in terms of them

$$|\Omega\rangle = \sum_j C_{j\Omega}\,|j\Omega\rangle = \sum_{j\Omega'} C_{j\Omega}D_{\Omega'\Omega}^{j}\,|j\Omega'\rangle_L, \qquad \text{(III–5a)}$$

where $|j\Omega'\rangle_L$ are the laboratory state functions. Again making use of the symmetry properties under the transformations T_1 and T_2^2 the appropriately symmetrized state functions have the form (Bo 52)

$$|EIMK\rangle = (1/\sqrt{2})\sum_j C_{j\Omega}[\,|IMK\rangle\,|j\Omega\rangle + (-1)^{I-j}\,|IM-K\rangle\,|j-\Omega\rangle] \qquad \text{(III–6)}$$

where in every case $K-\Omega$, $\Omega>0$, must be even (for axial symmetry $K=\Omega$).

The resulting energy is then

$$E(I, K) = (\hbar^2/2\mathcal{I}_0)[I(I+1)-2K^2]+E_p(K) \qquad \text{(III–7)}$$

with $I\geq K=\Omega$ and $E_p(K)$ is an appropriate eigenvalue of H_p defined in Eq. (III–5). It is only necessary to specify the potential function $V(\mathbf{r}, \mathbf{l}, \mathbf{s})$ in H_p to solve the eigenvalue problem for then not only the $E_p(K)$ but the $C_{j\Omega}$ are determined.

It is possible to compare theory with experiment without actually determining the solutions to the particle problem if the extra nucleon is so tightly bound to the core that $\Delta E_p(K)=E_p(K_2)-E_p(K_1)\gg \Delta E(I, K)$, where

$$\Delta E(I, K) = E(I, K)-E(K, K) = (\hbar^2/2\mathcal{I}_0)[I(I+1)-K(K+1)]. \qquad \text{(III–8a)}$$

A nice example in which such a simple description of deformed, odd-A nuclei does agree with observation is found in the low-lying states of U^{233}. With the energies of the various levels one can use Eq. (III–8a) to determine the moment of inertia parameter $\hbar^2/2\mathcal{I}_0$ the mean value being 5.75 keV (it is perhaps more useful to use this parameter, $\hbar^2/2\mathcal{I}_0$ in keV or MeV than the moment of inertia $\mathcal{I}_0$ in g-cm^2). It is interesting to compare this value of the moment of inertia parameter with that of neighboring even nuclei. From Table II–9 we have that for U^{232} $\hbar^2/2\mathcal{I}_0$ has the value 7.8 keV while for U^{234} it is 7.3 keV. This supports the original suggestion of Rainwater that the outer nucleon does indeed polarize the core—in this case it increases the deformation.

Despite the strength with which the odd nucleon is coupled to the core, the Coriolis term H_c does connect states diagonal in $K\neq\frac{1}{2}$ augmenting Eq. (III–8a) by an additional term. The matrix elements of H_c are

$$\langle EIMK'\,|\,H_c\,|\,EMIK\rangle = -(\hbar^2/2\mathcal{I}_0)\sum_j C^*_{j\Omega'}C_{j\Omega}\{[\delta_{K',K-1} + (-1)^{I-j}\delta_{K',-(K-1)}][(I+K)(I-K+1)(j+K)(j-K+1)]^{\frac{1}{2}} + [(I-K)(I+K+1)(j-K)(j+K+1)]^{\frac{1}{2}}\delta_{K',K+1}\}. \qquad \text{(III–9a)}$$

Only for $K=\frac{1}{2}$ bands does the second term contribute and it is customary to write for them

$$\langle EIM\tfrac{1}{2}\,|\,H_c\,|\,EIM\tfrac{1}{2}\rangle = (-1)^{I+\frac{1}{2}}(\hbar^2/2\mathcal{I}_0)(I+\tfrac{1}{2})a, \qquad \text{(III–9b)}$$

where the "decoupling parameter" a is defined as

$$a\equiv\sum_j(-1)^{j+\frac{1}{2}}(j+\tfrac{1}{2})\,|\,C_{j\frac{1}{2}}\,|^2. \qquad \text{(III–9c)}$$

Therefore Eq. (III–8a) becomes, with the addition of these diagonal terms,

$$\Delta E(IK) = (\hbar^2/2\mathcal{I}_0)\{I(I+1)-K(K+1) + a[(-1)^{I+\frac{1}{2}}(I+\tfrac{1}{2})+1]\delta_{K,\frac{1}{2}}\}. \qquad \text{(III–8b)}$$

Equation (III–9a) shows that H_c connects two rotational bands for which $\Delta K=1$. If the extra-core particle is sufficiently tightly bound that these bands are well-separated in energy, then the effect of H_c will clearly be small. However, if the particle is rather loosely bound, then these bands overlap strongly and the corrections due to H_c are important.

TABLE III–1. Parameters for K-band mixing in a number of odd-A nuclei. The parameters are defined in Eq. (III–10). Band parameters for the case of no band mixing in these same nuclei are given in Table III–2a, b.

Nucleus	K	$(3\hbar^2/\mathcal{I}_0)_K$ (keV)	$(3\hbar^2/\mathcal{I}_0)_{K+1}$ (keV)	E_p^K (keV)	E_p^{K+1} (keV)	a	A_K
Gd^{153}	$\frac{3}{2}$	60.00	31.20	7.50	140.0	···	11.75
Tb^{159}	$\frac{3}{2}$	71.04	67.80	6.85	388.4	···	10.00
Er^{163}	$\frac{3}{2}$	71.64	72.90	113.3	46.30	···	5.00
Yb^{177}	$\frac{1}{2}$	92.40	58.80	332.0	706.6	0.185	31.40
Hf^{177}	$\frac{5}{2}$	66.12	90.72	550.2	591.2	···	38.00
W^{182}	$\frac{1}{2}$	97.84	81.30	−1.32	208.3	0.169	23.25
Os^{185}	$\frac{1}{2}$	98.64	92.70	3.21	194.1	3.65×10^{-3}	19.31
Cm^{245}	$\frac{5}{2}$	35.82	37.26	257.0	2.38	···	8.5
Bk^{249}	$\frac{5}{2}$	31.20	31.20	412.5	49.05	···	15.00

To obtain an analytic expression of the effect upon the energy eigenvalues of two such K bands one can simply diagonalize the matrix

$$\begin{pmatrix} H_R & H_c \\ H_c & H_R \end{pmatrix}$$

which leads to the secular equation

$$\begin{vmatrix} E(I,K)-E & \langle K \mid H_c \mid K+1\rangle \\ \langle K+1 \mid H_c \mid K\rangle & E(I,K+1)-E \end{vmatrix} = 0,$$

the solutions of which are

$$E(I)=\frac{1}{2}\left\{E(I,K)+E(I,K+1) \pm \Delta E\left[1+\left|\frac{2\langle K+1 \mid H_c \mid K\rangle}{\Delta E}\right|^2\right]^{\frac{1}{2}}\right\}. \quad \text{(III–10)}$$

$\Delta E = E(I,K)-E(I,K+1)$ and $\langle K+1 \mid H_c \mid K\rangle$ are the matrix elements of Eq. (III–9a) and may be conveniently written as

$$\langle K+1 \mid H_c \mid K\rangle = A_K[(I-K)(I+K+1)]^{\frac{1}{2}} \quad \text{(III–10a)}$$

with

$$A_K = -(\hbar^2/2\mathcal{I}_0)\sum_j C^*_{jK+1}C_{jK}[(j-K)(j+K+1)]^{\frac{1}{2}}. \quad \text{(III–10b)}$$

K is no longer a good quantum number (even though the core still possesses axial symmetry); however, if the extra core particle is lightly bound then ΔE will be large and K will be "approximately" a constant of the motion.

This calculation was first applied to W^{183} with excellent results (Ke 56). The level structure of this nucleus consists of two rather complete bands ($K=\frac{1}{2}^-$ and $K=\frac{3}{2}^-$) and individual members of two others ($K=\frac{7}{2}^-$ and $K=\frac{9}{2}^+$) below about 0.6 MeV, the ground-state band being $K=\frac{1}{2}$ with the head of the $K=\frac{3}{2}$ band being at 0.209 MeV. The other two bandheads are at 0.453 and 0.309 MeV. Thus the lower two bands are connected by H_c and, since they overlap strongly, the effect is important. Figure III–1 shows a comparison between the measured energy-level structure and the values calculated from Eq. (III–10). In Table III–1 are listed the various parameters for a number of odd-A nuclei in the deformed regions. For nuclei with more than two K bands (such as Lu^{175}), a similar calculation can include the interaction between all of the bands which are connected by H_c.

The question of vibrations can be handled for these models of odd nuclei in exactly the same way as even nuclei. Thus Eq. (II–11) can be immediately extended to odd-A nuclei; keeping only diagonal terms in H_c as well as the rotation–vibration term, Eq. (III–7)

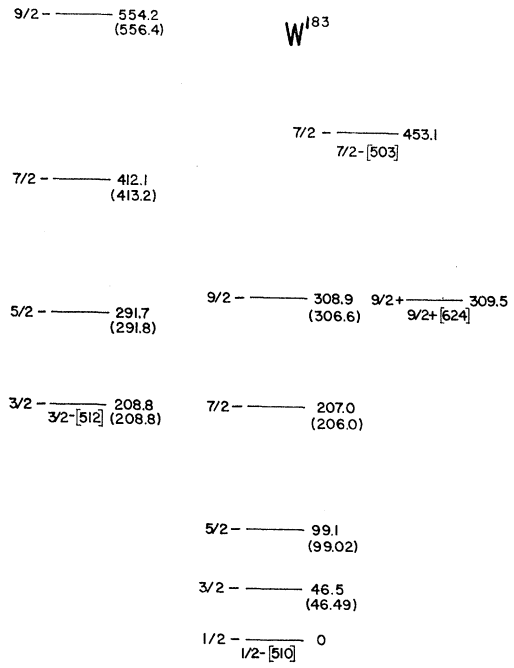

FIG. III–1. Experimental energy levels for W^{183}. The numbers in parenthesis are calculated from Eq. (III–10) with the parameters $(\hbar^2/\mathcal{I}_0)_{\frac{1}{2}}=16.307$ keV, $(\hbar^2/\mathcal{I}_0)_{\frac{3}{2}}=13.550$ keV, $a=0.1691$, $E_{\frac{1}{2}}^p=-1.320$ keV, $E_{\frac{3}{2}}^p=208.27$ keV, $A_{\frac{1}{2}}=23.253$ keV. The experimental values are from Ref. Ha 62.

becomes

$$E(IK)=E_p(K)+(\hbar^2/2\mathcal{I}_0)[I(I+1)-2K^2+(-1)^{I+\frac{1}{2}}(I+\tfrac{1}{2})a\delta_{K,\frac{1}{2}}]-b[I(I+1)-2K^2+(-1)^{I+\frac{1}{2}}(I+\tfrac{1}{2})a\delta_{K,\frac{1}{2}}]^2. \quad \text{(III-7a)}$$

We now turn to the extra-core particle problem. We confine attention to a rather strongly coupled particle moving in an axially symmetric potential. Then the Schrödinger equation defining the problem is given by Eqs. (III-4b) and (III-5), the solution of which depends only upon assigning the form of the potential function $V(\mathbf{r}, \mathbf{l}, \mathbf{s})$. There have been many solutions carried out using various deformed potentials, and the resulting eigenvalues show essentially the same qualitative features. The earliest calculation was that of Hill and Wheeler (Hi 53), the potential being an ellipsoidal one without spin–orbit coupling. Moszkowski (Mo 55) solved the particle problem using an infinitely deep spheroidal potential (the spheroidal box) with and without spin–orbit coupling. Gottfried (Go 56) used a finite spheroidal well with spin–orbit coupling, and recently Lemmer and Green (Le 60) used a nonspherical, velocity-dependent potential with a diffuse edge. This latter calculation was a perturbation calculation in the deviations from spherical symmetry while the rest were exact calculations, the Hamiltonian being diagonalized by machine.

The most used calculation is that of Nilsson (Ni 55) who used an anisotropic harmonic oscillator potential with spin-orbit and $\mathbf{l}^2$ terms of the form

$$V=\tfrac{1}{2}m[\omega_x^2(x^2+y^2)+\omega_z^2z^2]+C\mathbf{l}\cdot\mathbf{s}+D\mathbf{l}^2. \quad \text{(III-11)}$$

The $\mathbf{l}^2$ term has the effect of depressing the high angular momentum states which is in keeping with the notion that for higher A the single-particle potential is more of a square well than an oscillator well. The constants C and D were picked so that the level sequence for zero deformation reproduces that of the shell model with its well-known magic numbers at 2, 8, 20, 50, 82, 126, $\cdots$. The results of this calculation, eigenvalues and eigenvectors, have been tabulated and published for symmetric deformations on the range $|\beta|\leq 0.3$ for all of the oscillator shells with $N\leq 7$ and the level sequence as a function of β has been plotted (Ni 55, Mo 59).

The oscillator frequencies of Eq. (III-11) are related to the deformation of the core from the spherical shape by

$$\omega_x^2=\omega_0^2(1+\tfrac{2}{3}\delta)$$
$$\omega_x^2=\omega_0^2(1-\tfrac{4}{3}\delta).$$

The constant volume condition implies that the product $\omega_x^2\omega_z$ be constant to first order in δ which in fact yields

$$\omega_x^2\omega_x=\omega_0^3.$$

Here δ is a measure of the deformation,[5] and if the nuclear surface is assumed to be an equipotential, then δ can be related to β by

$$\delta=\tfrac{3}{2}(5/4\pi)^{\frac{1}{2}}\beta\cong 0.946\beta$$

(to first order). The oscillator strength is determined by $\hbar\omega_0$ and for an isotropic oscillator it is possible to calculate the rms radius of a particle and relate this to the nuclear radius $R=1.2A^{\frac{1}{3}}\times 10^{-13}$ cm, so that $\hbar\omega_0=\alpha A^{-\frac{1}{3}}$ MeV. Nilsson has taken $\alpha=41$, however, $\hbar\omega_0$ is actually an over-all scale parameter which is to be determined for each nucleus after the levels have been fitted.

Introducing the change of scale by

$$\xi_k=(m\omega_0/\hbar)^{\frac{1}{2}}x_k, \qquad \rho^2=\sum_{k=1}^{3}\xi_k^2, \quad \text{(III-12a)}$$

the Hamiltonian becomes

$$H_0/\hbar\omega_0=\tfrac{1}{2}(-\nabla^2+\rho^2)-\beta\rho^2Y_{20}(\theta,\varphi)=H_{00}+H_\delta. \quad \text{(III-12b)}$$

In picking a basis to diagonalize H, one in which H_{00} is diagonal is taken; however, there is some choice. In one of these, the commuting operators are

$$H_{00_k}=(-1/2)[(d^2/d\xi_k)-\xi_k^2], \qquad k=1, 2, 3$$

and the base vectors are just $|n_1, n_2, n_3\rangle$. Another basis is formed from the commuting operators H_{00}, $\mathbf{l}^2$, l_3 and s_3 which have eigenvalues $(N+\frac{3}{2})$, $l(l+1)$, Λ and Σ, respectively, where

$$N=\sum_{k=1}^{3}n_k=0, 1, 2, \cdots$$
$$l=N, N-2, N-4, \cdots, 1 \text{ or } 0.$$

The base vectors are denoted $|Nl\Lambda\Sigma\rangle$ and this is the system used by Nilsson and subsequent authors.

It is customary to replace the constants C and D of Eq. (III-11) by new ones defined by

$$\kappa=C/\hbar\omega_0,$$
$$\mu=D/\kappa\hbar\omega_0.$$

Then the nondiagonal terms of the single-particle Hamiltonian are just

$$H_\delta+(1/\hbar\omega_0)(C\mathbf{l}\cdot\mathbf{s}+D\mathbf{l}^2)=-\beta\rho^2Y_{20}(\theta,\varphi)-2\kappa\mathbf{l}\cdot\mathbf{s}-\mu\kappa\mathbf{l}^2. \quad \text{(III-13)}$$

These operators will not connect oscillator states with principal quantum number, N, differing by one unit but will connect states in which $\Delta N=2$. In the diagonalization of Eq. (III-13) Nilsson (Ni 55) has neg-

[5] Various authors do not use the same definitions of δ and indeed the definition may be different in different papers by the same author, although to first order they are the same. A table of relations between the various deformation parameters is to be found in Ref. Mo 59. Here we follow the definition of Ref. Ni 55.

lected all matrix elements not diagonal in N. In general this is a very good assumption since levels with N differing by two are separated by $2\hbar\omega_0$. However, for large N and sizable deformation it is found that levels obtained by the diagonalization process and differing by two units in N do in fact cross. Nilsson has shown that there is an alternative representation in which a great deal of the effect of these nondiagonal matrix elements can be taken into account so that such are in fact altered.

Even with this assumption, the nondiagonal matrix elements mix states with different values or l, Λ and Σ although $\Omega=\Lambda+\Sigma=j_3$ and the parity $(-1)^l$ remain good quantum numbers. The diagonalization process was carried out by machine with the values of κ and μ so fixed that for zero deformation the shell-model ordering of the states is obtained (Kl 52); usually κ is considered fixed while μ is a function of N (we use μ_N in what follows). In Ref. Ni 55 are tabulated the energy eigenvalues and eigenvectors (not normalized) resulting from this diagonalization process with $N\leq 6$, $\kappa=0.05$ and $\mu_0=\mu_1=\mu_2=0$, $\mu_3=0.35$, $\mu_4=0.45$, 0.55 (normalized) $\mu_5=\mu_6=0.45$. The calculation is for the range of distortions $|\beta|\leq 0.3$ and the energy eigenvalues are also given in graphical form. In Ref. Mo 59 these tabulations are extended to the $N=5$ and 7 shells with $\mu_5=0.70$ and $\mu_7=0.40$. For light nuclei, a different value of κ has been used with greater success and a graph of the energy eigenvalues for $N\leq 3$, $\kappa=0.08$ and deformation $|\beta|\leq 0.6$ is given in Ref. (Li 58) where the model has been applied to the mirror pair Mg^{25}, Al^{25}.

The Hamiltonian defined by Eqs. (III–12b) and (III–13) which has been diagonalized is not the same as the Hamiltonian H_p of Eq. (III–4b) since the $\mathbf{j}\cdot\mathbf{j}$ term is missing in the former. The effect of deleting this term has been studied (Ch 63b) in the $N=2$ shell where an exact diagonalization of the complete Hamiltonian of Eqs. (III–4) has been carried out as well as the diagonalization with the $\mathbf{j}\cdot\mathbf{j}$ term missing, with the terms of H_c missing, and with both left out. The results for a deformation of $\beta=0.2$ are shown in Fig. III–2. The levels numbered I are the result of the diagonalization of the complete Hamiltonian, those marked II the result of omitting the off-diagonal $\mathbf{l}\cdot\mathbf{j}$ terms, those marked III the result of omitting only the $\mathbf{j}\cdot\mathbf{j}$ terms, while the levels marked IV result from the omission of both. For this shell ($N=2$) and for this particular deformation the effect of omitting the $\mathbf{j}^2$ term is noticeable but does not result in the reordering of the lower lying levels. For higher shells the effect is more pronounced.

For very large deformations the $\mathbf{l}\cdot\mathbf{s}$ and $\mathbf{l}^2$ terms in Eq. (III–13) may be treated as perturbations so that in zero order the eigenvalues are those of a pure anisotropic oscillator. In this limit the energy levels may be labeled by the quantum numbers N, n_z, Λ and Σ. However, since $\Omega=\Lambda+\Sigma$ is always a good quantum

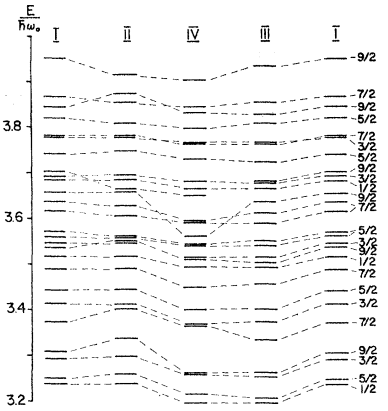

FIG. III–2. The results of an exact diagonalization of the eigenvalue problem defined by the Hamiltonian of Eq. (III–2) for axial symmetry, (denoted by I) as compared with such a diagonalization omitting the off-diagonal $\mathbf{l}\cdot\mathbf{j}$ terms (denoted by II), omitting only the $\mathbf{j}\cdot\mathbf{j}$ terms (III), and omitting both (IV). The total angular momentum of each level is given on the right.

number it is sufficient to give only Ω, N, n_z, Λ. The parity is also usually included and the labeling convention is Ω, $\pm[N, n_z, \Lambda]$ (the parity is redundant being plus or minus as N is even or odd).

As an example, consider again W^{183} which has 109 neutrons. Since each Nilsson level is doubly degenerate, two particles can be put into each. The 109th neutron is then to be placed in a level with spin and parity of $\frac{1}{2}-$ which is only possible for a deformation of $\delta\cong 0.2$ and energy $E/\hbar\omega_0\cong 6.4$. Within a small region around this deformation and energy are to be found the levels $\frac{1}{2}-[510]$, $\frac{7}{2}-[503]$, $\frac{3}{2}-[512]$ and $\frac{1}{2}+[651]$. Thus all of these particle excitations should be observed in the low-lying spectrum of W^{183} and in fact all but the $\frac{1}{2}+$level do occur. The appropriate level assignments are: ground state $\frac{1}{2}-[510]$, 208.8-keV level $\frac{3}{2}-[512]$, 453.1-keV level $\frac{7}{2}-[503]$, and the 309.5-keV level $\frac{9}{2}+[634]$ (here the 108th neutron has been excited to pair with the 109th neutron in the $\frac{1}{2}-[610]$ level leaving the unpaired 107th neutron to give the level characteristics). The other low-lying levels are just the rotational members of the bands built on these various particle excitations. Since the form of the potential has been specified, the $C_{j\Omega}$ coefficients are known from which the decoupling parameter of Eq. (III–9c) can be calculated as well as the parameter A_K of Eq. (III–10b). A table is available (Ni 55) showing the range of a as a function of the spin of the level in the $K=\frac{1}{2}$ band. For W^{183} the empirically determined value for the $\frac{1}{2}-[510]$ level (0.19) falls within the range given in this reference.

Similar assignments can be made for other odd-A nuclei, such assignments depending not only on spin and parity assignments of the levels but alpha, beta and gamma decay systematics and occasionally on reaction data. Table III–2 lists such band-head assignments for a number of odd-A nuclei in the deformed regions.

TABLE III-2. Intrinsic particle orbital assignments, moments of inertia, vibrational parameter and decoupling parameter for (a) odd-proton nuclei and (b) odd-neutron nuclei.

Nucleus	Band-head energy (keV)	Assigned orbital $K\pi[Nn_z\Lambda]$	$3\hbar^2/\mathcal{I}_0$ (keV)	b (keV)	a
		a. Odd-Proton Nuclei			
$_{63}Eu^{153}$	0	$\frac{5}{2}+[413]$	71.5	$+2.6\times10^{-3}$	
	103.18	$\frac{3}{2}+[411]$	83.58	•••	
$_{65}Tb^{155}$	0	$\frac{3}{2}+[411]$	78.0	0.0	
$_{65}Tb^{157}$	0	$\frac{3}{2}+[411]$	73.5	-2.41×10^{-2}	
$_{65}Tb^{159}$	0	$\frac{3}{2}+[411]$	70.0	-2.0×10^{-2}	
	348	$\frac{5}{2}+[413]$	69.4	•••	
$_{65}Tb^{161}$	0	$\frac{3}{2}+[411]$	69.1	-3.3×10^{-2}	
$_{67}Ho^{165}$	0	$\frac{7}{2}-[523]$	62.9	-2.9×10^{-3}	
$_{69}Tm^{167}$	0	$\frac{1}{2}+[411]$	72.2	$+2.8\times10^{-2}$	-0.71
$_{69}Tm^{169}$	0	$\frac{1}{2}+[411]$	71.8	$+3.4\times10^{-2}$	-0.77
	316.19	$\frac{7}{2}+[404]$	84.18	•••	
	379.31	$\frac{7}{2}-[523]$	62.58	•••	
$_{69}Tm^{171}$	0	$\frac{1}{2}+[411]$	72.2	$+2.9\times10^{-3}$	-0.86
$_{71}Lu^{175}$	0	$\frac{7}{2}+[404]$	75.54	-6.31×10^{-3}	
$_{71}Lu^{177}$	0	$\frac{7}{2}+[404]$	80.64	-7.83×10^{-3}	
$_{71}Lu^{179}$	0	$\frac{9}{2}+[624]$	128.5	$+3.5\times10^{-2}$	
$_{73}Ta^{181}$	0	$\frac{7}{2}+[404]$	90.84	-5.20×10^{-3}	
	6.3	$\frac{9}{2}-[514]$	83.2	•••	
$_{75}Re^{183}$	0	$\frac{5}{2}+[402]$	97.68	$+5.36\times10^{-3}$	
	496.1	$\frac{7}{2}-[523]$	111.96	•••	
$_{75}Re^{185}$	0	$\frac{5}{2}+[402]$	107.06	$+2.0\times10^{-3}$	
	646	$\frac{1}{2}+[400]$	144	•••	
$_{75}Re^{187}$	0	$\frac{5}{2}+[402]$	114.78	-3.67×10^{-2}	
$_{77}Ir^{189}$	0	$\frac{3}{2}+[402]$	129.6	$+3.27\times10^{-1}$	
	94.25	$\frac{1}{2}+[400]$	161.8	•••	-0.015
$_{77}Ir^{191}$	0	$\frac{3}{2}+[402]$	147.1	$+3.27\times10^{-1}$	
	82.45	$\frac{1}{2}+[400]$	200.0	•••	-0.035
	539.2	$\frac{5}{2}+[402]$	73.0	•••	
$_{77}Ir^{193}$	0	$\frac{3}{2}+[402]$	161.0	$+2.79\times10^{-1}$	
$_{89}Ac^{227}$	0	$\frac{3}{2}+[651]$	31.8	$+1.65\times10^{-1}$	
	27.5	$\frac{3}{2}-[532]$	54.7	•••	
	327.0	$\frac{1}{2}-[530]$	•••	•••	•••
$_{91}Pa^{231}$	0	$\frac{1}{2}-[530]$	•••	•••	•••
	84.1	$\frac{5}{2}+[642]$	15.6	•••	
	166.2	$\frac{3}{2}+[651]$	20.6	•••	
$_{91}Pa^{233}$	0	$\frac{1}{2}-[530]$	38.0	-2.84×10^{-3}	-1.38
	86.8	$\frac{5}{2}+[642]$	14.8	•••	
	213	$\frac{3}{2}+[651]$	28.5	$+7.14\times10^{-2}$	
$_{93}Np^{237}$	0	$\frac{5}{2}+[642]$	28.5	$+1.49\times10^{-3}$	
	59.57	$\frac{5}{2}-[523]$	37.2	-2.08×10^{-3}	

TABLE III-2 (*Continued*)

Nucleus	Band-head energy (keV)	Assigned orbital $K\pi[Nn_z\Lambda]$	$3\hbar^2/\mathcal{I}_0$ (keV)	b (keV)	a
$_{93}Np^{239}$	0	$\frac{5}{2}+[642]$	...	...	
	74.6	$\frac{5}{2}-[523]$	37.0	-2.4×10^{-3}	
$_{95}Am^{237}$	0	$\frac{5}{2}-[523]$	36.0	...	
$_{97}Bk^{249}$	0	$\frac{7}{2}+[633]$	28.0	-3.33×10^{-3}	
	8.8	$\frac{3}{2}-[521]$	36.8	$+9.5\times10^{-4}$	
	393	$\frac{5}{2}+[642]$	34.3	$+4.0\times10^{-4}$	
		b. Odd-Neutron Nuclei			
$_{62}Sm_{89}^{151}$	0	$\frac{3}{2}+[651]$			
	4.85	$\frac{1}{2}+[660]$			
$_{64}Gd_{89}^{153}$	0	$\frac{3}{2}+[651]$	52.2	7.7×10^{-2}	
	129.2	$\frac{5}{2}+[642]$	46.5		
	212.1	$\frac{3}{2}-[532]$	45.0		
	303.5	$\frac{3}{2}-[521]$	78.1		
$_{64}Gd_{91}^{155}$	0	$\frac{3}{2}-[521]$	72.7	-2.4×10^{-2}	
	86.5	$\frac{3}{2}+[651]$	37.7		
	247.0	$\frac{1}{2}+[660]$	20.8		$+0.88$
$_{64}Gd_{93}^{157}$	0	$\frac{3}{2}-[521]$	65.0	-1.0×10^{-2}	
	64.0	$\frac{5}{2}+[642]$	44.4		
$_{66}Dy_{95}^{161}$	0	$\frac{5}{2}+[642]$	37.5		
	25.6	$\frac{5}{2}-[523]$	66.9		
	74.5	$\frac{3}{2}-[521]$	68.6		
$_{66}Dy_{97}^{163}$	0	$\frac{5}{2}-[523]$	63		
$_{68}Er_{95}^{163}$	0	$\frac{5}{2}-[523]$	72.0		
	104.3	$\frac{3}{2}-[521]$	71.9	-1.0×10^{-2}	
	345.7	$\frac{1}{2}-[521]$	79.7		$+0.47$
$_{68}Er_{97}^{165}$	0	$\frac{5}{2}-[523]$	66.2		
	242.7	$\frac{3}{2}-[521]$	63.8		
	297.2	$\frac{1}{2}-[521]$	76.0		$+0.56$
$_{68}Er_{99}^{167}$	0	$\frac{7}{2}+[633]$	52.9		
	207.8	$\frac{1}{2}-[521]$	67.2	-1.0×10^{-3}	$+0.70$
	585.4	$\frac{5}{2}-[523]$	71.5		
$_{70}Yb_{99}^{169}$	0	$\frac{7}{2}+[633]$	47.2		
	24.3	$\frac{1}{2}-[521]$	70.4	8.0×10^{-3}	$+0.79$
	191.4	$\frac{5}{2}-[512]$	75.4	1.0×10^{-2}	
	570.5	$\frac{5}{2}-[523]$	66.8		
$_{70}Yb_{101}^{171}$	0	$\frac{1}{2}-[521]$	72.4	4.0×10^{-3}	$+0.85$
	95.2	$\frac{7}{2}+[633]$	48.2		
	122.4	$\frac{5}{2}-[512]$	73.5	5.0×10^{-3}	
	835.6	$\frac{7}{2}-[514]$	76.0		
$_{70}Yb_{103}^{173}$	0	$\frac{5}{2}-[512]$	67.6	3.0×10^{-3}	
	351.2	$\frac{7}{2}+[633]$	53.0		
$_{72}Hf_{101}^{173}$	0	$\frac{1}{2}-[521]$	77.1	9.0×10^{-3}	$+0.82$
	107.2	$\frac{5}{2}-[512]$	77.6	7.0×10^{-3}	
$_{72}Hf_{103}^{175}$	0	$\frac{5}{2}-[512]$	70.0	4.0×10^{-3}	
	125.9	$\frac{1}{2}-[521]$	81.0	1.0×10^{-2}	$+0.75$
	348.8	$\frac{7}{2}-[514]$	86.0	3.4×10^{-2}	

TABLE III-2 (*Continued*)

Nucleus	Band-head energy (keV)	Assigned orbital $K\pi[Nn_z\Lambda]$	$3\hbar^2/\mathcal{I}_0$ (keV)	b (keV)	a
$_{72}Hf_{105}^{177}$	0	$\frac{7}{2}-[514]$	76.4	4.0×10^{-3}	
	321.3	$\frac{9}{2}+[624]$	52.9	-1.3×10^{-2}	
	508.9	$\frac{5}{2}-[512]$	82.6		
$_{72}Hf_{107}^{179}$	0	$\frac{9}{2}+[624]$	68.8	-1.9×10^{-2}	
$_{74}W_{107}^{181}$	0	$\frac{9}{2}+[624]$	61.8		
	365.5	$\frac{5}{2}-[512]$	94.6		
	385.2	$\frac{1}{2}-[510]$	87.8		+0.48
	746.1	$\frac{1}{2}-[521]$	90.2		+0.59
$_{74}W_{109}^{183}$	0	$\frac{1}{2}-[510]$	78.1	-3.0×10^{-3}	+0.19
	208	$\frac{3}{2}-[512]$	97.9	-5.0×10^{-2}	
$_{76}Os_{109}^{185}$	0	$\frac{1}{2}-[510]$	71.4	-7.4×10^{-2}	0.02
	127.8	$\frac{3}{2}-[512]$	114.4	3.5×10^{-2}	
$_{76}Os_{111}^{187}$	0	$\frac{1}{2}-[510]$	48.5	-2.0×10^{-2}	−0.60
	74.3	$\frac{3}{2}-[512]$	134.0	-6.0×10^{-2}	
$_{76}Os_{113}^{189}$	0	$\frac{3}{2}-[512]$	83.5		
	36.2	$\frac{1}{2}-[510]$	118		
$_{78}Pt_{111}^{189}$	0	$\frac{3}{2}-[512]$			
$_{78}Pt_{113}^{191}$	0	$\frac{3}{2}-[512]$			
$_{78}Pt_{115}^{193}$	0	$\frac{1}{2}-[510]$	26.0		
$_{78}Pt_{117}^{195}$	0	$\frac{1}{2}-[510]$	117		0.69
		$\frac{3}{2}-[512]$	36.0		
$_{90}Th_{139}^{229}$	0	$\frac{5}{2}+[633]$	36.7	-7.4×10^{-3}	
$_{90}Th_{141}^{231}$	0	$\frac{5}{2}+[633]$	36.0	0	
	185	$\frac{5}{2}-[752]$	17.0	5.4×10^{-2}	
	390	$\frac{7}{2}-[743]$			
$_{92}U_{141}^{233}$	0	$\frac{5}{2}+[633]$	35.6	-4.0×10^{-3}	
	312	$\frac{3}{2}+[631]$	34.8		
$_{92}U_{143}^{235}$	0	$\frac{7}{2}-[743]$	30.8	-3.5×10^{-3}	
	0.08	$\frac{1}{2}+[631]$	35.5	-1.7×10^{-2}	−0.28
	129	$\frac{5}{2}+[633]$	35.9	1.4×10^{-2}	
$_{92}U_{145}^{237}$	0	$\frac{1}{2}+[631]$	46.0	9.5×10^{-3}	−0.56
	145	$\frac{5}{2}+[622]$	47.1	5.0×10^{-3}	
$_{94}Pu_{143}^{237}$	0	$\frac{7}{2}-[743]$			
	145	$\frac{1}{2}+[631]$			
$_{94}Pu_{145}^{239}$	0	$\frac{1}{2}+[631]$	37.5	-6.6×10^{-4}	−0.58
	286	$\frac{5}{2}+[622]$	38.2	-3.2×10^{-3}	
	392	$\frac{7}{2}-[743]$	28.3	-8.1×10^{-4}	
$_{94}Pu_{147}^{241}$	0	$\frac{5}{2}+[622]$			
	172	$\frac{7}{2}+[624]$	36.0	-3.5×10^{-2}	
$_{96}Cm_{145}^{241}$	0	$\frac{1}{2}+[631]$			
$_{96}Cm_{147}^{243}$	0	$\frac{5}{2}+[622]$			
$_{96}Cm_{149}^{245}$	0	$\frac{7}{2}+[624]$	37.3	2.0×10^{-3}	
	255	$\frac{5}{2}+[622]$	35.8	4.0×10^{-3}	
	394	$\frac{9}{2}-[734]$	28.5	1.7×10^{-2}	
$_{98}Cf_{151}^{249}$	0	$\frac{9}{2}-[734]$			
$_{98}Cf_{153}^{251}$	0	$\frac{1}{2}+[620]$	38.8	1.5×10^{-3}	0.29
	106	$\frac{7}{2}+[613]$	40.1	-3.0×10^{-3}	
$_{100}Fm_{155}^{255}$	0	$\frac{7}{2}+[613]$			

An examination of a Nilsson-level diagram in the region between the $g_{9/2}$ and $j_{15/2}$ shells shows a considerable gap at 152 neutrons for deformation of the order of $\beta=0.2$. Such a gap is suggestive of a "magic" number or shell closing. Unlike the more familiar magic numbers, there is no indication of a shell at $N=152$ for spherical nuclei. There is considerable evidence that such a shell (or perhaps better subshell) does exist (Pe 57b, As 64) and it is interesting that the original suggestion (Gh 54) based on alpha decay energies was made before the work of Ref. Ni 55. In general, the energies of emitted alpha particles decrease as the mass number increases; however, at major closed shells ($Z=82$ and $N=126$) this trend is reversed there being a very steep rise in energy with increasing mass number (Pe 50). This reversal can be distinguished at $N=152$ but is far milder than that occurring at the major shells, indicating a subshell closure. The thermal-neutron capture cross section decreases suddenly in the californium isotopes at $N=152$ (being three orders of magnitude less for Cf^{252} than for Cf^{250}) as does the half-life for spontaneous fission (Ma 54) both signs of a change in shells.

The assignment of Nilsson-level quantum numbers also facilitates the investigation of the shape of the nuclear energy surface as a function of mass number. The total nuclear energy $\mathcal{E}$ is the sum of potential and kinetic energies for all A particles and is thus a function of the deformation β. The equilibrium, ground-state deformation β_0 is then determined by

$$(\partial\mathcal{E}(\beta)/\partial\beta)_{\beta_0}=0.$$

For two particle forces $V_i=\sum_{j\neq i}V_{ij}$

$$\mathcal{E}(\beta)=\sum_i T_i+\tfrac{1}{2}\sum_{i\neq j}V_{ij}=\tfrac{3}{4}\sum_i\mathcal{E}_i(\beta)-\tfrac{1}{4}\sum_i(\bar{V}_i-\bar{T}_i). \qquad \text{(III-14)}$$

Since the $\mathcal{E}_i(\beta)$ are known, the evaluation of the expectation value of the last term as a function of β permits the evaluation of the energy surface. For heavy nuclei care must be taken to include the coupling between the different oscillator shells. As mentioned before, Nilsson accomplished this by introducing a pseudospherical basis such that the matrix elements of the "nonspherical" part of the Hamiltonian is exactly diagonal in N. The term $(\bar{V}_i-\bar{T}_i)$ can then be separated into angular momentum-dependent and independent terms for

$$\sum_i(\bar{V}_i-\bar{T}_i)=\sum_i(\bar{V}_i'-\bar{T}_i')+C\mathbf{l}'\cdot\mathbf{s}'+D\mathbf{l}'^2, \qquad \text{(III-15)}$$

the first term being expressed in the new system for which the Hamiltonian is just a harmonic oscillator. For such an oscillator the average value of potential minus kinetic energies is zero, hence

$$\mathcal{E}(\beta)=\tfrac{3}{4}\sum_i\mathcal{E}_i(\beta)-\tfrac{1}{4}\langle C\mathbf{l}'\cdot\mathbf{s}'+D\mathbf{l}'^2\rangle. \qquad \text{(III-16)}$$

This expression neglects not only the Coulomb energy between protons but any residual interactions such as the pairing force. The change in Coulomb energy upon distorting a spherical charged drop is (Fe 39)

$$E_c/E_c^0=1-(4/45)\beta^2.$$

Inclusion of this term thus shifts the equilibrium shape to higher energies. However, this effect is not as great as indicated by the ratio for heavy nuclei since different values of μ have been taken for neutrons and protons in the unfilled shells. Equilibrium-shape studies have been made by Mottelson and Nilsson (Mo 59) in all three deformed regions. These show not only the general trend of β with A but also the preference for prolate shapes in deformed nuclei.

This symmetric-core model has been applied with considerably less success to the light nuclei in the $2s$–$1d$ shell which was the last discovered of the three well-established regions of nuclear deformation. The suggestion was first made in 1956 as a result of an extensive study of Al^{25} in the Mg^{24} (p,γ) Al^{25} reaction (Li 56) and the mirror pair Mg^{25}–Al^{25} are probably the best studied nuclei in the shell. After this initial suggestion most of the other nuclei in the shell below S^{32} have been studied from this point of view (Ra 57a). [O^{18} (Go 59), F^{19} (Pa 57), Ne^{20} (Li 61b), Na^{23} (Pa 58), Mg^{24} (Pe 57a), $A=25$ (Li 58), Al^{27} (Al 60), Al^{28} (Sh 56), $A=29$ (Br 57), P^{31} (Br 58). For a recent review of this region see (Go 60).] While the results vary from nucleus to nucleus, the evidence seems overwhelming that these nuclei are deformed and that the strong-coupling collective model does in fact explain a majority of their ground-state and low-lying excited-state properties.

One of the difficulties with applying the model in this region is that there is a strong overlap of levels belonging to various K bands. This condition vitiates the separation of core and intrinsic parts of the problem with the consequent neglect of the Coriolis term H_c of Eq. (III-4c). In his study of F^{19} Paul (Pa 57) made use of this term in the Hamiltonian to mix the $\frac{1}{2}+[220]$ and $\frac{3}{2}+[211]$ rotational bands (levels 6 and 7 in Fig. III-3). This yields the correct level sequence for the low-lying positive-parity states, but does not reproduce the level spacing well. The first excited state is placed at 0.24 MeV which is about 20% too high and it is a mixture of 81% $K=\frac{1}{2}$ state and 19% $K=\frac{3}{2}$ state. The next state is the $\frac{3}{2}+$ member of the second triplet centered at about 1.5 MeV. It is a 54–46% admixture of these same states. This band-mixing technique has been used with more or less success everywhere possible in the shell; however, for certain nuclei, notably the mirror pair at $A=25$ the bands have $\Delta K=2$ which are not mixed by H_c. The success seems to depend to a large extent on the number of bands mixed. For instance in Na^{23} Paul and Montague (Pa 58) on the one hand mix the $\frac{3}{2}+[211]$, $\frac{1}{2}+[211]$, $\frac{5}{2}+[202]$ (levels 7, 9, and 5 in Fig. III-3), obtaining the second

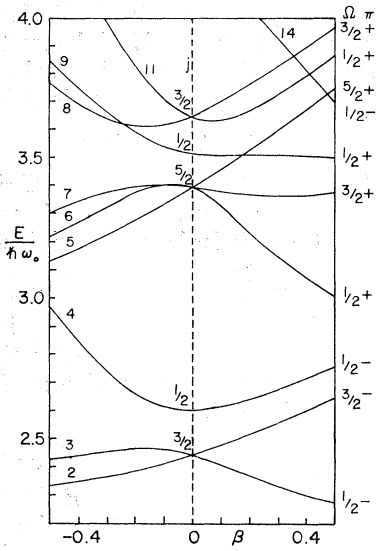

FIG. III-3. The intrinsic particle levels for a deformed but symmetric system whose potential function is given by Eq. (III-11). These levels correspond to the p and $2s$–$1d$ shells (except the level 14 which is the lowest particle state from the $2p$–$1f$ shell).

two by essentially exciting the odd particle from the $\frac{3}{2}+[211]$ level up. This places the second excited state ($\frac{7}{2}+$) about 25% lower than the measured value. On the other hand Clegy and Foley (Cl 62) fit this level (and the first excited $\frac{5}{2}+$ state) very closely by mixing the $\frac{3}{2}+[211]$ and the $\frac{1}{2}+[220]$ (levels 7 and 6) by exciting a particle from the latter state to the former. However, their higher excited states fit poorly. (The important point of their calculations is that low-lying states can be formed with this model by mixing particle and hole configurations.) However, Glöckle (Gl 64) gets a surprisingly good fit to the first six excited states by mixing four bands $\frac{1}{2}+[220]$, $\frac{3}{2}+[211]$, $\frac{5}{2}+[202]$, $\frac{1}{2}+[211]$ (levels 6, 7, 5, and 9). A similar calculation has been carried out by Bunker and Starner (Bu 64), both for Na^{23} and for the mirror pair Ne^{21}–Na^{21} with equal success. In general these calculations do not include the vibrational parameter b of Eq. (III-7a); however, in the mirror pair Mg^{25}–Al^{25}, H_c does not mix the various bands and this contribution has been added (Go 60). In this analysis the spectra are built on the four Nilsson levels $\frac{5}{2}+[202]$, $\frac{1}{2}+[211]$, $\frac{1}{2}+[200]$, $\frac{1}{2}-[330]$ (levels 5, 9, 11, 14 in Fig. III-3) for the ground and first three excited-state bands, but the $\frac{3}{2}+[202]$ band (level 8) is conspicuous by its absence. The decoupling parameters and reduced nucleon widths indicate a deformation parameter β in the range $0.3 \lesssim \beta \lesssim 0.4$. In carrying out the fitting procedure fourteen parameters (including the positions of the band heads) have been used to fit twelve levels. The moment of inertia parameters $\hbar^2/2\mathcal{I}_0$ are different for each band, the difference being as much as a factor of 2. This agrees with the intuitive notion that the moments of inertia of such a rotating system are larger for higher excitations ($\mathcal{I}_0$ increases monotonically with band-head energy). Bhatt (Bh 62) has done a consistent calculation for most of the odd-A nuclei in this shell using band mixing but with the same moment of inertia for each band. While he obtains "remarkable agreement" with a general rotational model the agreement is much poorer than Gove (Go 60) gets with the Nilsson model. However, Bhatt made no attempt to include the effect of deformation vibrations in his calculation.

We now turn to electromagnetic transitions and static moments of the model. The multipole operators are (Bo 53)

$$T(E\lambda)=\sum_j e_j r_j^\lambda Y_{\lambda\mu}(\theta_j,\varphi_j)+(3/4\pi)Z_e R_0^\lambda \alpha^*_{\lambda\mu}, \tag{III-17}$$

$$T(M\lambda)=\frac{e\hbar}{2mc}\sum_j\left[g_s\mathbf{s}+\frac{2}{\lambda+1}g_l\mathbf{l}\right]_j\cdot\nabla_j(r_j^\lambda Y_{\lambda\mu}(\theta_j,\varphi_j))$$
$$+\frac{e\hbar}{mc}\frac{1}{\lambda+1}g_R\int\mathbf{L}(\mathbf{r})\cdot\nabla(r^\lambda Y_{\lambda\mu}(\theta,\varphi))\,dv, \tag{III-18}$$

where in Eq. (III-17) the recoil contribution has been omitted in the first term. Here the sums are the contributions of the extra-core nucleons and run only over those nucleons taking part in the transition. The second terms in each equation refer to the core contributions and their matrix elements are similar to those discussed in Sec. II. From the discussion there the electric dipole transitions are due to single-particle terms in Eq. (III-17) and yield little concerning the model.

Magnetic dipole transitions, on the other hand, are important here and in fact the diagonal matrix elements of Eq. (III-18) were perhaps the first aspects studied of the interaction between surface vibrations and the extra-core particle both in weak coupling (Fo 50) and otherwise (Bo 51, Da 53). Let us consider the magnetic dipole moments first and then the transitions. For this case $\lambda=1$ and the dipole operator of Eq. (III-18) can be simply written as

$$\mu=g_R\mathbf{L}+g_j\mathbf{j}, \tag{III-19a}$$

where g_R can be taken from the measured value of the neighboring even nucleus or for convenience may be taken as Z/A. Making use of Eq. (III-1) we obtain

$$\boldsymbol{\mu}=g_R\mathbf{I}+(g_l-g_R)\mathbf{j}+(g_s-g_l)\mathbf{s}$$
$$=g_R\mathbf{I}+(g_s-g_R)\mathbf{j}+(g_l-g_s)\mathbf{l} \tag{III-19b}$$

since $\mathbf{j}=\mathbf{l}+\mathbf{s}$. The magnetic moment is then

$$\mu=g_R I+[K^2/(I+1)]\{g_K-g_R+(-1)^{I-\frac{1}{2}}(2I+1)$$
$$\times[(g_l-g_R)a+(-1)^l(g_s-g_l)\sum_l a_{l0}^2]\delta_{K,\frac{1}{2}}\}, \tag{III-20}$$

where $Kg_K=\langle NIIK\,|\,g_l l_z+g_s s_z\,|\,NIIK\rangle$, the $a_{l,\Omega\pm\frac{1}{2}}$ are the normalized expansion coefficients of the Nilsson

functions, and the decoupling parameter is defined in (III–9c). This can be written more compactly (Ad 56, Ke 59)

$$\mu = g_R I + [K^2/(I+1)](g_K - g_R) \times [1+(-1)^{I-\frac{1}{2}}(2I+1)b_0\delta_{K,\frac{1}{2}}]. \quad \text{(III–20a)}$$

Thus there are three unknown quantities g_K, g_R and, if $K=\frac{1}{2}$, b_0 which would require three measurements in the same nucleus. If one is willing to assume that the core gyromagnetic ratio is the same in an odd nucleus as in its even–even neighbors, then for $K>\frac{1}{2}$ ground states g_K can be obtained from a measurement of the magnetic dipole moment.

In order to assign all of the magnetic dipole properties of an odd-A nuclcus in a consistent manner it is necessary to make use of two magnetic dipole measurements (or three if $K=\frac{1}{2}$). At present, it is best to take besides the static dipole moment of the ground state the $M1$ transition probability which is

$$T(M1: I_i \rightarrow I_f) = (16\pi/9)(k^3/\hbar)B(M1: I_iK_i \rightarrow I_fK_f). \quad \text{(III–21)}$$

The reduced matrix element is most easily evaluated by introducing the spherical tensor of rank one,

$$G_\mu = (g_l - g_R)j_\mu + (g_s - g_l)s_\mu,$$

which is the only part of the operator $\mathfrak{y}$ that is effective in inducing the transition since we saw in Sec. II that the even–even core does not undergo $M1$ transitions. The reduced matrix element is then

$$B(M1: I_iK_i \rightarrow I_fK_f) = (3/4\pi)(e\hbar/2mc)^2 C^2(I_i 1 I_f; K_i, K_f - K_i, K_f) \times |\langle I_fK_f | G_{K_f-K_i} | I_iK_i\rangle \times [1+(-1)^{I_i+\frac{1}{2}}(2I_i+1)b_0\, d_{I_i}\delta_{K_f,\frac{1}{2}}\delta_{K_i,\frac{1}{2}}]|^2, \quad \text{(III–22)}$$

where

$$d_{I_i} = \begin{cases} (2I_i+1)^{-1}, & I_f = I_i - 1, \\ 1, & I_f = I_i, \\ -(2I_i+1)^{-1}, & I_f = I_i + 1. \end{cases}$$

Therefore the evaluation of this reduced matrix element for any magnetic dipole transition (or two transitions if $K=\frac{1}{2}$) yields enough information (with $\langle\mu\rangle$) to obtain the magnetic parameters. In practice the $B(M1)$ are not directly measurable since the large collective enhancement of the $E2$ transitions causes a great deal of mixing for transitions in which $\Delta I=1$. The experimental quantity which is measurable in such a case is δ^2 the ratio of $E2$ to $M1$ gamma-ray intensities. For the rare-earth-deformed nuclei this ratio has roughly the range (Be 60a) $10^{-2} \lesssim \delta^2 \lesssim 10$. For transitions within a K band, δ^2 is easily evaluated from Eqs. (III–21), (III–22), and (III–23) below:

$$\delta^2(I \rightarrow I-1) = T(E2: I \rightarrow I-1)/T(M1: I \rightarrow I-1) = (3/20)(m\omega Q_0/\hbar)^2\{[(I-1)(I+1)]^{\frac{1}{2}} \times (g_K - g_R)[1+(-1)^{I+\frac{1}{2}}b_0\delta_{k,\frac{1}{2}}]\}^{-2},$$

where Q_0 is the intrinsic quadrupole moment of the nucleus (see below). The sign of δ is defined (Ad 57) as the sign of

$$(g_K - g_R)[1-(-1)^{I+\frac{1}{2}}b_0\delta_{K,\frac{1}{2}}].$$

In practice one must evaluate the additional parameter Q_0 which is most easily done by Coulomb excitation and also measure the intensity ratio of the cross-over to cascade transitions from the second ($I = Ig+2$) rotational level. Knowing $\langle\mu\rangle$ then yields the magnetic parameters. Table III–3 gives these for a number of nuclei in the deformed regions.

There has been some criticism that δ^2 can be determined only with a relatively large error so that the gyromagnetic ratios determined this way (i.e., by Coulomb experiments) are not as reliable as when the mean-life of the first rotational level is determined (Be 62). Gyromagnetic ratios measured in this way are also given in Table III–3, the overlap of observed values being in general within experimental error.

For electric quadrupole transitions in odd nuclei the reduced matrix element for transitions within a K band is

$$B(E2: I_i \rightarrow I_f) = (5/16\pi)e^2Q_0^2 C(I_i 2 I_f; KOK). \quad \text{(III–23)}$$

From this relation come the many intensity rules for $E2$ transitions. For instance for Coulomb excitation from the ground state to the first and second excited states of a rotational band one has

$$B(E2: I_g \rightarrow I_g+2)/B(E2: I_g \rightarrow I_g+1) = 2(I_g+1)/I_g(2I_g+3),$$

while for decays of the type discussed above one has

$$B(E2: I \rightarrow I-1)/B(E2: I \rightarrow I-2) = 2K^2(2I-1)/(I+1)(I-1+K)(I-1-K).$$

Transitions between bands built upon different intrinsic states can also take place. In general the reduced matrix elements for a transition from the state $|E_iI_iK_i\rangle$ to the state $|E_fI_fK_f\rangle$ is

$$B(\lambda: I_iK_i \rightarrow I_fK_f) = |C(I_i\lambda I_f; K_i, K_f - K_i, K_f)\langle K_f | T^B_{\lambda,K_f-K_i} | K_i\rangle + (-1)^{I_i+l_i+\frac{1}{2}}C(I_i\lambda I_f; -K_i, K_i+K_f, K_f) \times \langle K_f | T^B_{\lambda,K_i+K_f} | -K_i\rangle|^2, \quad \text{(III–24)}$$

where the relation involving the coefficients $C_{j\Omega}$ of Eq.

TABLE III-3. Magnetic properties of some odd-A nuclei, the magnetic dipole moments are given in units of nuclear magnetons.

Nucleus	$\langle\mu\rangle$	δ^2	Sign δ	g_R	g_K	Reference
Eu^{153}	1.507	0.47	+	0.452 0.45	0.664	Be 60a Be 62
Gd^{155}	−0.28	0.038	−	0.34	−0.54	Be 60a
Gd^{157}	−0.37	0.040	−	0.22	−0.56	Be 60a
Tb^{159}	1.52	0.02	+	0.24 0.44	1.53 1.37	Be 60a Be 61a
Dy^{161}	−0.375	0.025	−	0.25	−0.311	Be 60a
Dy^{163}	0.52	6.7	−	0.243	0.194	Be 60a
Ho^{165}	3.29	0.031	+	0.30	1.120	Be 60a
Er^{167}	−0.5	0.106	−	0.124	−0.219	Be 60a
Tm^{169}	−0.21	0.020	−	0.38 0.26	−1.79	Be 60a Be 62
Yb^{173}	−0.665	0.047	−	0.20 0.35	−0.453 −0.52	Be 60a Be 61a
Lu^{175}	2.0	0.22	+	0.30 0.31 0.29	0.65 0.65	Be 60a Be 62 Be 63
Hf^{177}	0.61	10±5	−	0.215 0.20	0.162 0.17	Be 60a Be 63
Hf^{179}	0.47	0.15	−	0.203 0.28	−0.172	Be 60a Be 62
Ta^{181}	2.340	0.198	+	0.327 0.31	0.767	Be 60a Be 62
Re^{185}	3.144	0.033	+	0.413	1.597	Be 60a
Re^{187}	3.176	0.030	+	0.413 0.43	1.617	Be 60a Be 62
Ir^{191}	0.17	0.14		0.46	−0.12	Be 63
U^{233}	0.51			0.32	0.16	Al 61

(III-6),

$$C_{j-K}=(-1)^{l+j-\frac{1}{2}}C_{jK},$$

has been used (Ke 59). In (III-24) the first term is the more important for the second term only contributes when $\lambda\geq K_i+K_f$, the less usual case which we shall consider.

For a series of transitions involving the same initial and final intrinsic states the nuclear matrix elements will be identical so that the intensity ratios will be but ratios of the squares of appropriate Clebsch–Gordan coefficients. These interband transitions are then useful in assigning quantum numbers to intrinsic states.

The next higher static multipole moment is the magnetic octupole moment, $\langle\Omega\rangle$, which has been measured for only seven nuclei. There have been two theoretical investigations of $\langle\Omega\rangle$ from the point of view of this model. In one (Su 57), the extra particle angular momentum **j** was taken as a constant of the motion even though the particle potential was not taken as spherical. A self-consistent calculation, within the context of the Nilsson model has been carried out by Williams (Wi 62) who found the octupole moment to be

$$\Omega/\mu_0=-\frac{I(2I-1)(2I-2)}{(2I+4)(2I+3)(I+1)}\times\left[\frac{6}{7}g_R R_0^2+g(3)\frac{\hbar}{m\omega_0}G(I,N,\beta)\right],$$

where R_0 is the nuclear radius, $\hbar/m\omega_0$ the square of the intrinsic oscillator length, and the total particle gyromagnetic ratio $g(l)$ is defined as

$$g(l)=I^{-1}\sum_{L,\sigma}A^2{}_{L,I-\sigma}[\sigma g_s+(2g_L/l+1)(I-\sigma)],$$

the $A_{L,\Omega}$ being those of Nilsson. The function $G(I,N,\beta)$ is a complicated function of its arguments, given in explicit form by Williams. Since only two octupole moments have been measured in nuclei near a deformed region most of the results of this calculation could only be predictive in nature. In Table III-4 are tabulated the results in the deformed regions. Williams has made use of g_s as a fitting parameter. These values are also tabulated.

The electric hexadecapole moments, $\langle Q^4 \rangle$, have also been investigated from the stand-point of the strong-coupling collective model (Su 57a), but again the extra-core particle's angular momentum **j** has been taken as a constant of the motion. To date no electric hexadecapole moments have been measured.

Another set of selection rules governing gamma-ray transitions in odd-A nuclei are those which depend upon the asymptotic quantum numbers and are rigorous in the limit of infinite distortion. However, for actual nuclei they are only approximate, hence if not fulfilled, do not prohibit a transition but only slow or "hinder" it. These selection rules for some of the more common multipole transitions are given in Table III–5 and a more complete table in Ref. Al 57.

A significant detail of the Mg^{25}–Al^{25} spectrum is that $\Delta K=2$. This is a situation for which there is no Coriolis mixing, but such mixing and only such mixing does occur for an asymmetric-rotator Hamiltonian. That such an approach might be fruitful for the deformed region in the $2s$–$1d$ shell can be seen from the asymmetric-model parameters for Mg^{24}. The agreement with this model is not as good as one finds in the rare-earth region; it is sufficiently so to encourage the application of an asymmetric-core model to odd-A nuclei.

Newton (Ne 60, Ne 60a) has extended Nilsson's single-particle calculation to asymmetric-oscillator potentials by relaxing the condition $\omega_x=\omega_y$ in Eq. (III–11). He finds that the preferred shape for model (as distinct from actual) nuclei near $N=Z=12$ and 20 is asymmetric with some evidence for a similar situation near $A=44$–46. This calculation has been extended to the $N=5$ shell (Du 62).

An extension to all of the odd-A nuclei in the $2s$–$1d$ shell has been carried out (Ch 63b) with, in general, much better agreement with experiment than Bhatt (Bh 62) has obtained with the simpler symmetric-core model. Because of the complicated overlapping band structures found in this region, the calculation was carried out initially with only the assumption that the core is rigid against deformation vibrations. The effect of β vibrations has since been examined (Ro 64). The Hamiltonian is that of Eq. (III–2) with the potential identical with Nilsson's [Eq. (III–11)] except that $\omega_x \neq \omega_y$. The nondiagonal part of the Hamiltonian is then

$$(H-H_0)/\hbar\omega_0 = P\sum_{k=1}^{3}\left[\frac{I_k-j_k}{\sin(\gamma-2\pi k/3)}\right]^2 - (\beta r^2/\kappa)\left[Y_{20}\cos\gamma+(Y_{22}+Y_{2-2})\frac{\sin\gamma}{\sqrt{2}}\right] - 2\mathbf{l}\cdot\mathbf{s}-\mu\mathbf{l}^2. \quad \text{(III–25)}$$

The parameter P is a strength parameter for the core defined as

$$P=\hbar^2/8B_2\beta^2\kappa\hbar\omega_0.$$

Initially μ and κ were taken to have the Nilsson values ($\mu_2=0$, $\kappa=0.05$). The model parameters are then P and the usual β and γ which were taken to have ranges

$$P\geq 0, \quad \beta\geq 0, \quad 0°\leq\gamma\leq 60°.$$

The Hamiltonian (III–25) shows that no change in the energy levels can arise, once these parameters have been picked, by changing κ since changing it by

TABLE III–4. The theoretical and experimental values of the magnetic octupole moment, $\langle\Omega\rangle$, in units of nuclear magneton barns, and the deduced value of the single-particle gyromagnetic ratio, g_s, for nuclei in the deformed regions from Ref. Wa 62.

Nucleus	$\langle\Omega\rangle_{\text{exp}}$	$\langle\Omega\rangle_{\text{Thy}}$	g_s
Mg^{25}		−0.16	−3.40
Al^{27}		0.019	5.30
Cl^{35}	−0.020	−0.020	2.75
Cl^{37}	−0.015	−0.015	2.90
Eu^{153}		−0.020	3.80
Gd^{155}		0.044	−2.42
Gd^{157}		0.054	−2.90
Tb^{159}		0.054	3.13
Dy^{161}		0.086	−3.20
Dy^{163}		−0.034	−2.24
Ho^{165}		−0.235	2.01
Er^{167}		0.112	−3.42
Yb^{173}		0.003	−3.11
Lu^{175}		0.007	3.29
Hf^{177}		−0.032	−1.24
Hf^{179}		0.061	−2.30
Ta^{181}		−0.001	3.41
Re^{185}		0.064	3.84
U^{233}		−0.061	−2.01
U^{235}		0.121	−2.10
Np^{237}		−0.409	5.59
Am^{241}		−0.031	3.72

TABLE III–5. Asymptotic selection rules for electromagnetic transitions of multipole order $E1$, $M1$, and $E2$.

Multipole	Operator	μ	$\Delta\pi$	ΔK	ΔN	Δn_z	$\Delta\Lambda$
$E1$	$rY_{1\mu}$	±1	yes	1	±1	0	1
		0	yes	0	1, −1	1, −1	0
$M1$	l_μ	±1	no	1	0, 2	1	1
				1	0, −2	−1	1
	s_μ	±1	no	1	0	0	0
	l_μ, s_μ	0	no	0	0	0	0
$E2$	$r^2Y_{2\mu}$	±2	no	2	0, ±2	0	2
		±1	no	1	0, 2	1	1
				1	0, −2	−1	1
		0	no	0	2, −2	2, −2	0
				0	0	0	±2
				0	0	0	0

TABLE III-6. Comparison of the calculated values with experimental values (where measured) for the ground-state magnetic dipole and octupole and electric quadrupole moments and mean lives of the first excited states. The fitted values of κ are obtained by fitting the electric quadrupole moment, the other values are for $\kappa=0.05$.

Nucleus	P	β	γ	$\hbar\omega_0$ (MeV)	$41A^{-\frac{1}{3}}$ (MeV)	$\langle\mu\rangle$ Exp	$\langle\mu\rangle$ Thy	$\langle Q\rangle$ Exp	$\langle Q\rangle$ Thy	τ_m (n sec) 2nd Exp	τ_m (n sec) 2nd Thy	Fitted κ τ_m	Fitted κ κ'	Fitted κ β'	Fitted κ $\hbar\omega_0'$	$\langle\Omega\rangle$
O^{17}	0.550	0.120	0°	27.78	15.9	−1.9	−1.85	0.026	−0.009	0.255	1.91	0.659	0.15	0.35	9.61	−0.099
F^{19}	0.700	0.320	0°	8.69	15.6	2.627	2.809	0	0	125	342					0
	1st excited state					3.69	4.52	...	−0.06	...						0
O^{19}	0.100	0.090	21.0°	50.57	15.6	...	...	...	...	...	...					...
Ne^{21}	0.074	0.090	20.0°	60.44	14.9	−0.662	−0.567	0.10	0.014	$<10^{-1}$	1.65					...
Na^{21}	0.070	0.090	21.0°	54.60	14.9	...	2.20	...	0.015	...	0.746					...
Na^{23}	0.050	0.085	27.0°	74.27	14.4	2.22	2.14	0.10	0.01	1.8×10^{-3}	2.35	2.00	0.25	0.55	9.40	−0.056
Mg^{25}	0.050	0.080	28.0°	43.35	14.0	−0.855	−0.973	0.15	0.029	3.5	11.9	2.31	0.26	0.41	8.38	−0.013
Al^{25}	0.050	0.080	32.0°	40.77	14.0	...	3.68	...	0.025	...	1.8					...
Si^{29}	0.240	0.150	34.0°	13.74	13.3	−0.56	−1.78	0	0	...	0.034					0
P^{29}	0.270	0.162	33.1°	12.44	13.3	...	2.65	0	0	...	0.11					0
P^{31}	0.180	0.205	31.0°	19.60	13.0	1.13	2.46	0	0	5×10^{-4}	1.1×10^{-2}					0
Cl^{35}	0.020	0.350	30.0°	116.4	12.5	0.8	0.34	−0.08	−0.14	...	9×10^{-4}	1×10^{-3}				...

a factor $1/f$ merely changes β to β/f and $\hbar\omega_0$ to $f\hbar\omega_0$. Thus changing κ only changes the calculated values of the quadrupole moments and $E2$ transition probabilities.

The Hamiltonian was diagonalized in terms of the basis of Eq. (III-6). The Pauli principle was taken into account by a basis truncation procedure, in which basis elements were removed which contained components of filled Nilsson levels. The results of such a calculation are apparently quite sensitive to this procedure since a similar calculation with a different method of truncation (Wa 61) failed to give such close agreement with experiment.

For each odd-A nucleus in the shell, with more than three levels with measured spin and parity, the model parameters were determined from the low energy-level structure. The state functions so obtained were used to calculate the static moments $\langle\mu\rangle$, $\langle Q\rangle$ and $\langle\Omega\rangle$ as well as the mean life of certain of the levels. To improve the agreement between measured and calculated quadrupole moments κ was varied until this value agreed with experiment. In Table III-6 are tabulated experimentally measured values as well as the calculated ones.

Beta vibrations are added by simply augmenting the Hamiltonian of Eq. (III-2) with a vibrational term H_v (Ro 64), where

$$H_v=-\frac{\hbar^2}{2B_2}\frac{1}{\beta^3}\frac{\partial}{\partial\beta}\left(\beta^3\frac{\partial}{\partial\beta}\right)+\tfrac{1}{2}C(\beta-\beta_0)^2.$$

Unfortunately, the equations cannot be separated so that a perturbation approach must be used, making use of the rigid-core calculation as the zeroth-order one. In general, this procedure improves the energy-level fit as well as agreement between theory and experiment. Such agreement is quite striking for the mirror pair Ne^{21}–Na^{21} the parameters for which turn out to be $P=0.061$, $\beta_0=0.088$, $\gamma=20°$, $\mu=0.30$ (Ne^{21}); $P=0.065$, $\beta_0=0.090$, $\gamma=20°$, $\mu=0.30$ (Na^{21}), where μ is the stiffness parameter defined in Sec. II.

This model cannot account for negative-parity levels found in the spectra in this shell. Using the parameters found for the mirror pair Mg^{25}–Al^{25}, the calculation was extended into the $2p-1f$ shell (Ch 63b) and the lowest $N=3$ levels were tens of MeV higher than the negative-parity states found in these nuclei. Such an explanation for these levels has been made with the symmetric-core model (Li 58).

One great difficulty which arises in attempting to extend these calculations into the rare-earth and actinide deformed regions is the fact that a very large number of intrinsic particle levels are available to interact. If the rotational energies are small compared with the single-particle excitations, then a useful simplification would be to neglect the interactions between single particle levels. Hecht and Satchler (He 62) have done just such a calculation for the $N=4$, 5, and 6 oscillator

shells and applied it to nuclei in the upper end of the rare-earth deformed region (specifically to Re^{185}, Ir^{191} and Pt^{195}). They have tabulated the matrix elements of the rotational Hamiltonian [the first terms of Eq. (III–2)] as well as the level ordering for odd-proton and odd-neutron nuclei in this region.

In fitting the model to Pt^{195} they have available three single-particle levels (which is also true of the Nilsson model). The energy-level sequence and spacing resulting from using any of these levels are in good agreement with experiment, at least for the four negative-parity levels below 260 keV. As would be expected the magnetic moment of the ground state is in reasonable agreement with experiment, but only one set of predictions (built on their particle state 115 with $\beta=0.1$ and $\gamma=30°$) yields reasonable agreement with the magnetic dipole transition probabilities. However, the model fails to explain the cross-over $E2$ transitions from the upper levels to the ground and first excited states.

This model has been applied to calculating the magnetic dipole moments of W^{183}, Os^{187}, and Fe^{87} and the energy levels of W^{183} (Ch 63a). The magnetic moment is somewhat improved over the symmetric-core calculation which is perhaps not surprising. The calculation does improve the gamma-ray branching ratios in W^{183}, but values of γ obtained for this nucleus are not consistent. The final result is the negative one that the best state functions do not have $\gamma=0°$.

A similar calculation (Pe 61) has been applied to Cs^{131} but since the region is probably not one of strong equilibrium deformations the result is probably not meaningful.

In an early investigation of odd-A nuclei with asymmetric cores Davydov (Da 59a) coupled a single $j=\frac{1}{2}$ particle to such a rotating core. Here **j** is taken as a constant of the motion but since $\gamma\neq 0$, Ω is not. The results have been compared with W^{183}. Similar calculations for $j=\frac{5}{2}$ and applied to the nuclei Th^{229}, U^{233}, and Np^{237} (Da 61b) and with $j=\frac{7}{2}$ and applied to Th^{231}, Cm^{245}, and Bk^{249} (Da 62a) have been reported. Taking **j** to be a good quantum number is of questionable validity and casts considerable doubt on the results. Davydov has also considered the case of such a model with weak coupling (Da 60), where **j** will be an "almost good" quantum number.

Another similar model of odd-A nuclei is called by its authors the elastic ellipsoid model. The kinematical model was first studied (Os 55, Os 56, Kl 56, Kl 57) and then the dynamical model was developed (Os 58), and is only incompletely studied, as no detailed solution of the dynamical problem has been given and the only eigenfunctions available are empirical ones. As usual in simple collective models the basic assumption is that the nucleus can be thought of as an even–even core the collective behavior of which is described by the three Euler angles and a deformation parameter, δ, which is defined as the ratio of the semi-axis along the 3 direction to the radius of a sphere of the same volume. The extra-core nucleon is assumed not to partake of the collective motion. Also the dynamical properties of the core are determined only by a fraction k of the nucleons. This is similar to the hydrodynamic model where the moments of inertia contributions come only from lobes of the deformed core.

To obtain the model Hamiltonian the kinetic energy is transformed from laboratory to body coordinate systems and then a shrinking transformation is applied to transform the distorted core to a spherical distribution. That transformation matrix is given by

$$S=\begin{pmatrix} \delta^{\frac{1}{2}} & 0 & 0 \\ 0 & \delta^{\frac{1}{2}} & 0 \\ 0 & 0 & \delta^{-1} \end{pmatrix}$$

which is volume conserving. The Hamiltonian can thus be written

$$H_{\text{core}}=H_R+H_v,$$

and H_R is identical with H_{sym} of Eq. (II–10) while H_v depends only upon δ which is an exact measure of the spheroidal deformation and for small deformations can be related to β by

$$\delta\approx 1+\beta(5/4\pi)^{\frac{1}{2}}.$$

For small deformations then and taking $K=0$, this model reduces to the axially symmetric hydrodynamic case. The eigenfunctions are, however, altogether different, being

$$|IM\rangle=(1-a^2)^{\frac{1}{2}}\,|IMLjlN\rangle+a\,|IML'j'l'N'\rangle,$$

where the mixing parameter as well as the quantum numbers $LjlN$ and $L'j'l'N'$ are determined by fitting the observed magnetic dipole and electric quadrupole moments.

The Nilsson calculations have been extended to include the motion of particles in a potential of the form given in Eq. (III–11) but with $Y_{10}(\theta, \varphi)$ and $Y_{30}(\theta, \varphi)$ terms added in order to investigate the octupole deformation and its stability in even nuclei (Du 59, Du 61, Ru 63). These calculations are an extension of earlier suggestions by Lee and Inglis (Le 57) on the octupole deformation. However, since the particle potential with Y_{10} and Y_{30} terms added is not an even function under point reflection, the particle eigenstates must be of mixed parity. This point is not discussed in the particle calculations (Du 59, Du 61, Ru 63) but it is of fundamental importance inasmuch as parity is certainly a good quantum number for nuclear states (Mi 64).

B. Models of Odd–Odd Nuclei

For odd–odd nuclei it can be assumed that the nuclear properties for the most part are determined

by the odd neutron and odd proton, each moving in its own deformed well and associated with the projection quantum number Ω_n and Ω_p. For strong coupling there are two possibilities for the nuclear ground state (Bo 53),

$$K_{1,2}=\Omega_{1,2}=|\Omega_n\pm\Omega_p|, \qquad \text{(III-26)}$$

and the energies of these two states are separated by an energy of the order of rotational energies in the given region. Since such energies are small compared with particle excitation energies, one of these can be expected to be the ground state while the other will be a nearby excited state. Thus assigning the individual quantum numbers Ω_n and Ω_p gives the head of two low-lying rotational bands of the form

$$I_1=|\Omega_n-\Omega_p|,\quad |\Omega_n-\Omega_p|+1,\quad |\Omega_n-\Omega_p|+2,\cdots,$$
$$I_2=\Omega_n+\Omega_p,\quad \Omega_n+\Omega_p+1,\quad \Omega_n+\Omega_p+2,\cdots. \qquad \text{(III-27)}$$

It is quite reasonable to expect that for a given odd–odd nucleus the odd-neutron orbital will be the same as in the neighboring odd-A nuclei with the same number of neutrons, and similarly for protons. This original conjecture (Bo 53) has been well borne out and Peker (Pe 57) has found that the ground states of odd–odd nuclei in all three deformed regions can be determined by this method being either I_1 or I_2 and often the first excited state is then I_2 or I_1.

The next question, of course, is which of the two possibilities in (III–26) yields the ground state, a matter which should give information concerning the interparticle forces in nuclei. This problem is analogous to the one in the j–j shell model which leads to the Nordheim coupling rules (No 50, No 51) which are that

$$I=|j_n-j_p|, \qquad j_n=l_n\pm\tfrac{1}{2}, \qquad j_p=l_p\mp\tfrac{1}{2},$$
$$|j_n-j_p|\leq I\leq j_n+j_p, \qquad j_n=l_n\pm\tfrac{1}{2}, \qquad j_p=l_p\pm\tfrac{1}{2},$$

the former being known as the strong rule. Essentially, these rules state that the intrinsic spins of the last neutron and proton tend to align.

This reasoning has been extended by Gallagher and Moszkowski (Ga 58) to deformed nuclei by assuming that Σ_n and Σ_p, the projections of the intrinsic neutron and proton spins along the symmetry axis, always couple parallel. This assumption depends upon the fact that for very large deformations, Λ and Σ are separately good quantum numbers, while for actual nuclear deformations these are only approximate quantum numbers. However, these coupling rules depend upon the asympototic quantum number assignments. This then leads to the coupling rules.

$$I=|\Omega_n-\Omega_p|; \qquad \Omega_n=\Lambda_n\pm\tfrac{1}{2}, \qquad \Omega_p=\Lambda_p\mp\tfrac{1}{2}$$
$$I=\Omega_n+\Omega_p; \qquad \Omega_n=\Lambda_n\pm\tfrac{1}{2}, \qquad \Omega_p=\Lambda_p\pm\tfrac{1}{2}. \qquad \text{(III-28)}$$

Gallagher and Moszkowski have shown that these coupling rules account in general for the ground-state spins of deformed odd–odd nuclei (they actually considered all odd–odd nuclei and found the coupling rules were violated mainly in regions near closed shells, i.e., not in the deformed regions). They made their particle assignments from the previously discussed Nilsson diagrams, however, there are some cases where the last odd-particle configurations are not the lowest. But even here the lowest lying level is coupled so that the intrinsic spins of odd neutron and odd proton couple parallel (Ga 62a).

One noteworthy breakdown of the coupling rule (III–28) is in Al^{28} in which the prediction is for a 2+ ground state, but 3+ is observed. However, the 2+ state is very close at 31.2 keV (the second excited state is at 974 keV). Sheline (Sh 56) was the first to point out the probable existence of rotational levels in this nucleus, however, some doubt has been cast upon his analysis (Mac 60).

Making use of the coupling scheme (III–28) and the notion that odd neutron and proton move independently each in their own potential well permits us to write the state functions for odd–odd nuclei

$$|EIMK\rangle=[(2I+1)/16\pi^2]^{\frac{1}{2}}\sum_{j_nj_p}C_{j_p\Omega_p}C_{j_nK\mp\Omega_p}$$
$$\times[\chi^{(1)}{}_{j_p\Omega_p}\chi^{(2)}{}_{j_nK\mp\Omega_p}D_{MK}{}^{I*}$$
$$+(-1)^{I-j_p-j_n}\chi^{(1)}{}_{j_p-\Omega_p}\chi^{(2)}{}_{j_n-K\pm\Omega_p}D_{M-K}{}^{I*}], \qquad \text{(III-29)}$$

where the $C_{j\Omega}$ were defined in Eq. (III–5a), and the $\pm$ sign is determined by the particular coupling rule used. From this product wavefunction we obtain for the energy eigenvalues

$$E(IMK)=E_p(\Omega_p)+E_n(\Omega_n)+E_{\rm rot}(I)=E_{\rm pair}+E_{\rm rot}(I) \qquad \text{(III-30)}$$

so that for a given set of particle states the rotational band has the familiar $I(I+1)$ dependence. [We neglect any effects due to vibrations of the core. If necessary we could append a vibrational function $\varphi(\beta)$, say, to (III–29) and add $E_{\rm vib}$ to (III–30).] In Table III–7 are listed the odd–odd nuclei and the assigned odd particle states.

An interesting special case of Eq. (III–29) occurs when both particles couple in such a way that $K=0$. Then Eq. (III–29) is

$$|EIM0\rangle=[(2I+1)/16\pi^2]^{\frac{1}{2}}\sum_{j_nj_p}C_{j_p\Omega_p}C_{j_n-\Omega_p}$$
$$\times[\chi^{(1)}{}_{j_p\Omega_p}\chi^{(2)}{}_{j_n-\Omega_p}$$
$$+(-1)^{I-j_p-j_n}\chi^{(1)}{}_{j_p-\Omega_p}\chi^{(2)}{}_{j_n+\Omega_p}]\,D_{M0}{}^{I*}. \qquad \text{(III-29a)}$$

The two particle terms in the square brackets are not identical so that the enegy spectrum for a given pair of intrinsic states does not display the usual rotational $I(I+1)$ dependence; however, the even-spin and odd-spin bands do and they are displaced slightly one with respect to the other so that the energy eigenvalues can

TABLE III–7. Ground-state spins, odd-neutron and odd-proton states for some deformed odd–odd nuclei. Where known similar information for excited bands is given along with the band-head energy.

Nucleus	Band head (keV)	$I\pi$	Odd-particle configuration: Proton	Neutron
F^{20}	0	2+	$\frac{1}{2}$+[220]	$\frac{3}{2}$+[211]
Na^{22}	0	3+	$\frac{3}{2}$+[211]	$\frac{3}{2}$+[211]
Na^{24}	0	4+	$\frac{3}{2}$+[211]	$\frac{5}{2}$+[202]
	472	1+	$\frac{3}{2}$+[211]	$\frac{5}{2}$+[202]
Al^{26}	0	5+	$\frac{5}{2}$+[202]	$\frac{5}{2}$+[202]
	229	0+	$\frac{5}{2}$+[202]	$\frac{5}{2}$+[202]
Al^{28}	0	3+	$\frac{5}{2}$+[202]	$\frac{1}{2}$+[211]
	31.2	2+	$\frac{5}{2}$+[202]	$\frac{1}{2}$+[211]
P^{30}	0	1+	$\frac{1}{2}$+[211]	$\frac{1}{2}$+[211]
P^{32}	0	1+	$\frac{1}{2}$+[211]	$\frac{1}{2}$+[200]
Cl^{34}	0	0+	$\frac{3}{2}$+[202]	$\frac{3}{2}$+[202]
	143	3+	$\frac{3}{2}$+[202]	$\frac{3}{2}$+[202]
Cl^{36}	0	2+	$\frac{3}{2}$+[202]	$\frac{1}{2}$+[200]
Eu^{152}	0	3−	$\frac{3}{2}$+[411]	$\frac{3}{2}$−[521]
	55	0−	$\frac{5}{2}$−[532]	$\frac{5}{2}$+[642]
Eu^{154}	0	3−	$\frac{3}{2}$+[411]	$\frac{3}{2}$−[521]
Tb^{156}	0	3−	$\frac{3}{2}$+[411]	$\frac{3}{2}$−[521]
Tb^{160}	0	3−	$\frac{3}{2}$+[411]	$\frac{3}{2}$−[521]
Ho^{160}	0	5+	$\frac{7}{2}$−[523]	$\frac{3}{2}$−[521]
Ho^{162}	0	1+	$\frac{7}{2}$−[523]	$\frac{5}{2}$−[523]
	90	6−	$\frac{7}{2}$−[523]	$\frac{5}{2}$+[642]
Ho^{164}	0	1+	$\frac{7}{2}$−[523]	$\frac{5}{2}$−[523]
	139	6−	$\frac{7}{2}$−[523]	$\frac{5}{2}$+[642]
Ho^{166}	0	0−	$\frac{7}{2}$−[523]	$\frac{7}{2}$+[633]
	12	7−	$\frac{7}{2}$−[523]	$\frac{7}{2}$+[633]
	190	3+	$\frac{7}{2}$−[523]	$\frac{1}{2}$−[521]
	419	1+	$\frac{7}{2}$−[523]	$\frac{5}{2}$−[523]
	499	4+	$\frac{7}{2}$−[523]	$\frac{1}{2}$−[521]
Tm^{164}	0	1+	$\frac{7}{2}$−[523]	$\frac{5}{2}$−[523]
Tm^{166}	0	2+	$\frac{1}{2}$+[411]	$\frac{5}{2}$+[642]
Tm^{168}	0	3+	$\frac{1}{2}$+[411]	$\frac{7}{2}$+[633]
Tm^{170}	0	1+	$\frac{1}{2}$+[411]	$\frac{1}{2}$−[521]
Tm^{172}	0	2−	$\frac{1}{2}$+[411]	$\frac{5}{2}$+[512]
Lu^{172}	0	4−	$\frac{7}{2}$+[404]	$\frac{1}{2}$−[521]
Lu^{174}	0	1−	$\frac{7}{2}$+[404]	$\frac{5}{2}$−[512]
	171	6−	$\frac{7}{2}$+[404]	$\frac{5}{2}$−[512]
Lu^{176}	0	7−	$\frac{7}{2}$+[404]	$\frac{7}{2}$−[514]
	∼200	1−	$\frac{7}{2}$+[404]	$\frac{7}{2}$−[514]
Ta^{178}	0	7−	$\frac{7}{2}$+[404]	$\frac{7}{2}$−[514]
	?	1+	$\frac{9}{2}$−[514]	$\frac{7}{2}$−[514]
Ta^{180}	0	9−	$\frac{9}{2}$−[514]	$\frac{9}{2}$+[624]
	212	1+	$\frac{7}{2}$+[404]	$\frac{9}{2}$+[624]
Ta^{182}	0	3−	$\frac{7}{2}$+[404]	$\frac{1}{2}$−[510]
Re^{182}	0	7+	$\frac{5}{2}$+[402]	$\frac{9}{2}$+[624]
	∼200	2+	$\frac{5}{2}$+[402]	$\frac{9}{2}$+[624]
Re^{184}	0	3−	$\frac{5}{2}$+[402]	$\frac{1}{2}$−[510]
Re^{186}	0	1−	$\frac{5}{2}$+[402]	$\frac{3}{2}$−[512]
Re^{188}	0	1−	$\frac{5}{2}$+[402]	$\frac{3}{2}$−[512]
	169	4+	11/2−[505]	$\frac{3}{2}$−[512]
Ir^{188}	0	2−	$\frac{1}{2}$+[411]	$\frac{3}{2}$−[512]
Ir^{192}	0	4−	$\frac{5}{2}$+[402]	$\frac{3}{2}$−[512]
Ir^{194}	0	1−	$\frac{5}{2}$+[402]	$\frac{3}{2}$−[512]
Am^{242}	0	1−	$\frac{5}{2}$−[523]	$\frac{5}{2}$+[622]
	48.6	5−	$\frac{5}{2}$−[523]	$\frac{5}{2}$+[622]
Bk^{250}	0	2−	$\frac{3}{2}$−[521]	$\frac{1}{2}$+[620]
	85.2	7+	$\frac{7}{2}$+[633]	$\frac{7}{2}$+[613]
	99	5−	$\frac{3}{2}$−[521]	$\frac{7}{2}$+[613]
	382	6+	$\frac{5}{2}$+[642]	$\frac{7}{2}$+[613]

be written as

$$E(IMK=0) = E_{\text{pair}}^{(\text{even})} + E_{\text{rot}}(I) \quad I, \text{ even}$$
$$= E_{\text{pair}}^{(\text{odd})} + E(I_{\text{rot}}) \quad I, \text{ odd.} \quad \text{(III–30a)}$$

This displacement of odd and even spin values in $K=0$ rotational bands has been observed in Am^{242} (As 60) where the spin-1 state is displaced downward relative to the spin-0 state. It has also been observed in Ho^{166} where the spin-1 state is displaced upward relative to the spin-0 state making the latter the ground state (He 60). The direction and the amount of this displacement depends upon the details of the interaction between the odd nucleons.

The low-lying level structure of this last nucleus has been studied by the Ho^{165} (d, p) Ho^{166} reaction and some 31 levels below 820 keV have been found (St 63). By using rotational energy systematics alone these authors have been able to fit a large number of these levels into four rotational bands the lowest two being the spin-0 and spin-7 bands built upon the odd proton and odd neutron Nilsson levels $\frac{7}{2}$− [523], $\frac{7}{2}$+ [633], respectively. Two other bands with $K=3$ and $K=4$ are built on the levels $\frac{7}{2}$− [523], $\frac{1}{2}$− [521]. The $K=0$, 7 bands have the same moment of inertia while the $K=3$, 4 bands have a different and smaller moment of inertia. While making spin and parity assignments from energy systematics alone is a doubtful pro-

TABLE III–8. Comparison of experimental and calculated values of the magnetic dipole moment of deformed odd–odd nuclei. Equation (III–31b) has been used for the calculations. The tabulated values of g_R are from either Table II–8 or III–3. When no value is listed Z/A was used.

Nucleus	I	β	Q (barns)	μ_{exp} (nm)	μ_{thy} (nm)	g_R
F^{20}	2	$+\infty$		2.092±0.002	0.886	
Na^{22}	3	0.21		1.746	1.747	
Na^{24}	4	0.27		1.688±0.005	1.69	
P^{32}	1	0.13		−0.2523 ±0.0003	−0.25	
Cl^{36}	2	−0.25	−0.0168	1.28539 ±0.00009	1.28	
Eu^{152}	3	0.27		1.912±0.003	1.91	0.21
Eu^{154}	3	0.22		2.1	2.1	0.35
Tb^{156}	3	0.32	1.4 ±0.5	1.45 ±0.18	1.96	0.367
Tb^{160}	3	0.32	1.9 ±0.5	1.60 ±0.25	1.90	0.28
Ho^{166m}	7	0.32		3.3 ±0.5	3.74	0.31
Tm^{170}	1	$+\infty$	0.61±0.05	0.26 ±0.02	0.26	0.31
Lu^{176}	7	0.32	8.0 ±0.7	3.07 ±0.28	2.81	0.304
Am^{242}	1(K=0)	0.30		0.33	0.39	0.39

cedure, this case is reinforced somewhat by making use of previous spin and parity assignments of a few of the lower levels (He 60, Es 61).

A similar rotational structure has been investigated in the lowest deformed region, in Na^{24} also by the (d, p) reaction on Na^{23} (Da 64). Again Nilsson levels are assigned the odd proton and the odd neutron; however, here most of the levels below about 4 MeV are considered to arise from the assignment of the odd neutron to any one of four different levels (numbers 5, 8, 9, and 11 in Fig. III–3).

In order to explain the energy difference between the ground states of F^{18}, Na^{22}, and Al^{26} which have $T=0$ and the first excited $T=1$ state ($I=0+$), a more specific model has been introduced (Ke 64) in which the two odd particles are coupled to a deformed (but symmetric) even–even, rotating core. The Hamiltonian is then

$$H=(\hbar^2/2\mathcal{I}_0)\mathbf{L}^2+H_{p1}+H_{p2}+V_{12}$$
$$=(\hbar^2/2\mathcal{I}_0)(\mathbf{I}-\mathbf{j}_1-\mathbf{j}_2)^2+H_{p1}+H_{p2}+V_{12},$$

where V_{12} is a residual two-body interaction which in the cited calculation is taken as having a Gaussian radial dependence while the spin–isotopic-spin part has the Rosenfeld (Ro 49) mixture. If ΔE be the energy difference between these lowest $T=0$ and $T=1$ states, then

$$\Delta E=\delta_\Omega-(\hbar^2/\mathcal{I}_0)\Omega,$$

where

$$\delta_\Omega=\langle T=1\mid V_{12}\mid T=1\rangle-\langle T=0\mid V_{12}\mid T=0\rangle.$$

The calculated values of δ_Ω quantitatively reproduce the experimental values of ΔE.

The magnetic dipole moments can be calculated on the basis of the model of noninteracting odd particles by taking the sum of two terms like Eq. (III-20) one for the odd neutrons and one for the odd protons. For the asymptotic quantum number description, the magnetic dipole moment is (Ga 58)

$$\mu=[I/(I+1)][g_R\pm(\Lambda_p+g_s^p\Sigma_p)\mp g_s^n\Sigma_n], \tag{III–31a}$$

where g_s is usually taken as the free proton or neutron value and the upper and lower signs are determined by the coupling rules. Equation (III–31a) with $g_R=Z/A$ has been applied to almost all odd–odd nuclei (Ga 58); however, a calculation only for deformed regions has been done by Hooke (Ho 59) who used the relation

$$\mu=[I/(1+1)][g_R+\langle l_{p3}\rangle+g_s^p\langle s_{p3}\rangle+g_s^n\langle s_{n3}\rangle], \tag{III–31b}$$

taking $g_R=0.40$. In Table III–8 the experimental and theoretical values of the magnetic dipole moments of deformed odd–odd nuclei are tabulated using Eq. (III–31b) for the calculated values. Wherever possible g_R was taken from either a neighboring odd-A nucleus or an even–even nucleus as a simple way to take into account any pairing effects in the nuclear core. The deformation β is obtained from the experimental value of Q_0 or μ and these also are tabulated. The appropriate single-particle Nilsson levels are given in Table III–7.

Gamma-ray transition probabilities can be calculated using the state functions of Eq. (III–29). The result differs in one important respect from the similar case in odd-A nuclei which leads to Eq. (III–24). In

this case there is an integral, called by Gallagher the overlap integral (Ga 60), in addition to the Clebsch–Gordan coefficient and the matrix element of the transition operator. The overlap integral contains only the initial and final nontransforming particle state functions and is zero unless they represent the same state.

Thus, for odd–odd nuclei an additional type of forbiddenness (in addition to K forbiddenness) arises involving two-particle transitions. Such a situation has been called "non-overlap forbiddenness" by Gallagher who has discussed the relatively few measured gamma-ray transitions in deformed odd–odd nuclei (Ga 60).

IV. ALPHA AND BETA TRANSITIONS

A. Alpha Transitions

The collective models described before have been successfully applied to explain some of the puzzling details of alpha-decay fine structure. It was early recognized that the relation between alpha energy and parent mass number showed shell effects very clearly, the energy increasing as the mass decreases until a shell was crossed, at which point there was a precipitous decline in the decay energy (Pe 50). However, away from the shell edges the ground-state–ground-state alpha decay in even–even nuclei was quite well described by the Gamow–Condon–Gurney model of the penetration of charged particles through a spherically symmetric barrier (see Ref. Pe 57b for a general review of alpha-decay theory and a compilation of experimental results). The decay from ground state to excited states in such nuclei did depart from this theory and in some cases to a considerable extent (As 53). The fact that this was not due simply to the added angular momentum carried off by the alpha particle was indicated by the fact that the alpha half-lives to the second excited states were increased by too great an amount relative to the ground state (often hundreds of times longer). It was suggested that a deformed potential barrier might give rise to such half-life departures from simple theory (As 53). Indeed, earlier calculations had been reported in an attempt to explain just such fine-structure details (Pr 49). (For an early discussion of nuclear rotational motion and alpha decay see Te 38.)

Probably the most puzzling feature was to be found in the decays of the odd-A nuclei where the ground-state to ground-state transitions were often strongly inhibited whereas the simple shell model would suggest that their decay rates should be comparable with that of the even–even neighbors. In discussing the alpha spectrum of Am^{241}, Rasmussen (Ra 54) suggested that the ground-state–ground-state transition occurs in a process which leaves the odd nucleon (the ninety-fifth proton) in a state different from the odd nucleon state of the parent nucleus, thus strongly retarding this decay mode. On the other hand, the transitions to the third and fourth excited states leave this odd nucleon in the same state as in the parent. This is now known to be the situation, for the ground state of Am^{241} has the odd proton in the state $\frac{5}{2}-$ [523] while the ground state of Np^{237} is the $\frac{5}{2}+$ [642] state, the $\frac{5}{2}-$ [523] state being at 60 keV in this nucleus (see Table III–2). The notion of favored alpha decays in odd-A nuclei has been defined by Bohr, Fröman, and Mottelson (Bo 55) as being those decays which do leave the odd nucleon in its original state and these decays should be comparable with similar decays in neighboring even–even nuclei. Thus the rate for a given decay is strongly influenced by the alpha formation factor, this factor being greatest when the alpha is formed from a neutron pair and a proton pair, each pair occupying states which differ only in the sign of Ω. Thus these favored transitions will be characterized by the selection rule $\Delta K=0$, no change in parity.

A more detailed discussion of the fine structure is facilitated by introducing the hindrance factor $F(z)$ which is the factor by which the observed alpha half-life is different from that calculated. The ground-state to ground-state decays of even–even nuclei conform to the simple theory. Their half-lives are quite closely given by the Geiger–Nuttall rule (Ge 11)

$$\log t_{\frac{1}{2}}=\mathrm{A}(Z)\,E_{\mathrm{eff}}{}^{-\frac{1}{2}}+B(Z).$$

The hindrance factor is then

$$\log F(Z)=\log t_{\frac{1}{2}}-A(Z)\,E_{\mathrm{eff}}{}^{-\frac{1}{2}}-B(Z), \qquad \text{(IV–1)}$$

where E_{eff} is the effective decay energy when account has been made of electron screening. A table of E_{eff} as well as a table of $A(Z)$ and $\mathrm{B}(Z)$ are in Ref. Pe 57b. A more useful quantity, which might be called the relative hindrance factor $f(Z)$, which is $F(Z)$ but normalized by taking it equal to unity for all ground-state transitions in even nuclei, is

$$f_i(Z)=[F_i(Z)/F_{\mathrm{ground}}(Z). \qquad \text{(IV–2)}$$

The reduced hindrance factor is also useful for it is the hindrance factor with the effects of the centrifugal barrier penetrability and any effects due to the noncentral character of the interaction (important in deformed nuclei) reduced out.

Numerous calculations of the alpha decay hindrance factor have been made for deformed nuclei, which are applicable to the heavy-element alpha emitters as most of them are deformed. Some of these studies have been devoted only to the decay of even–even nuclei (Ra 56, St 57a, Ro 61) while others dealt with both even–even and odd-A nuclei (Fr 57, Ra 62). In general these calculations have taken the deformed nuclear surface as possessing an axis of symmetry, although two have been generalized to asymmetric deformed shapes (Ro 61, Ra 62). We shall outline one of these more general calculations indicating the differences between it and the calculations employing axial symmetry.

The essence of the problem is to calculate the penetration of an alpha particle with given angular momentum through an anisotropic barrier. The system

to be considered is the alpha particle and the daughter nucleus; the Hamiltonian in the center of mass of the two-body system is

$$H=H_\alpha+H_{\text{daughter}}+H_{\text{int}}, \tag{IV-3}$$

where

$$H_\alpha=-(\hbar^2/2M_\alpha)\nabla_r^2 \tag{IV-4a}$$

$$H_{\text{daughter}}=H_{\text{rot}}+H_{\text{part}}+H_{\text{vib}} \tag{IV-4b}$$

and H_{int} is just the electrostatic potential between the alpha particle and the daughter nucleus. The three terms in Eq. (IV-4b) are analogous to Hamiltonians discussed in Secs. II and III.

The state function for this system can be written as

$$|I_iM_i\rangle=\exp(-iEt/\hbar)\sum_{I_fl} r^{-1}f_{lI_f}{}^{I_i}(r)\,\Phi_{M_i}{}^{I_i\tau}(lI_f,\theta,\varphi,\theta_i\beta_\lambda), \tag{IV-5a}$$

where i stands for the parent nuclear quantum numbers and f for the daughter quantum numbers. The variables r, θ, φ are the daughter–alpha radial distance, the angular orientation of the alpha relative to the laboratory, while the θ_i are the Euler angles specifying the orientation of the daughter nucleus and the β_λ are the vibrational deformation parameters. The function $\Phi_{M_i}{}^{I_i}$ in Eq. (IV-5a) is then

$$\Phi_{M_i}{}^{I_i\tau}(lI_f,\theta,\varphi,\theta_i,\beta_\lambda)=[(2I_f+1)/16\pi^2]^{\frac{1}{2}}\Gamma_{\nu_n}(\beta_\lambda)\times\sum_{\substack{M_fm\\K_f,j\Omega_f}} A_{K_f}{}^{I_f\tau}B_{\Omega_f}{}^{j_f}[\chi_{\Omega_f}{}^{j_f}D^{I_f}{}_{M_fK_f}(\theta_i)+(-1)^{I_f-j_f}\chi_{-\Omega_f}{}^{j_f}D^{I_f}{}_{M_f-K_f}(\theta_i)]. \tag{IV-5b}$$

In all these calculations the vibrational term in Eq. (IV-4b) has been neglected and attention focused only on the deformed rotator aspects of the nuclei. However, recent measurements of alpha decay to vibrational states have been reported (Bj 63) indicating that this term might well be considered. The quantum number Ω characterizes the rotational band in the daughter nucleus to which the decay proceeds while ν_n characterizes the vibrational state. In any event by neglecting H_{vib} one sets $\Gamma_{\nu_n}(\beta_\lambda)=1$. The nuclear surface is represented by Eq. (I-5B)

$$S(\theta',\varphi')=R_0[1+\sum_{\mu=-2}^{2}a_\mu Y_{2\mu}(\theta',\varphi')],$$

where the a_μ have been defined in Eq. (II-12). [Fröman (Fr 57) took the surface to be axial and of the form

$$S_a(\theta')=R_0[1+\sum_{\lambda\geq 2}\beta_\lambda Y_{\lambda 0}(\theta')] \tag{IV-6}$$

and kept terms for $\lambda\leq 8$ and even for positive parity transitions.] Corresponding to this surface the electrostatic potential (H_{int}) can be expanded in the appropriate harmonics (Pr 49) as

$$V(\mathbf{r}')=\frac{2(z-2)e^2}{r}+\sum_{\mu=-2}^{2}V_\mu(\beta,\gamma,r)\,Y_{2\mu}(\theta',\varphi')$$

with the expansion coefficients being determined by

$$V_\mu(\beta,\gamma,r)=\frac{8\pi e}{5r^3}\int_{\text{nuclear volume}} r''^2\rho(r'')\,Y_{2\mu}(\theta'',\varphi'')\,d\mathbf{r}''.$$

One method of solving the wave equation is to use a method suggested by Christy (Ch 55) which makes use of a three-dimensional WBK approximation and notions from ray optics. This involves the extremal problem

$$\int_p^{p'} K\,ds=\text{minimum},$$

where p and p' are two points inside the barrier while K can be written as

$$K=K(\mathbf{r}')=k\{[V(r')-\mathcal{E}]/\mathcal{E}\}^{\frac{1}{2}}=K_0(r)+\Delta K(r,\theta',\varphi'),$$

the explicit form of K_0 and ΔK being given in (Ra 62) for the asymmetric core and in (Fr 57) for the symmetric core.

Comparison with experiment is made through the reduced transition probability $c_{l\tau}$ which is the reciprocal of $f_l(Z)$ defined in Eq. (IV-2). Actually, the l dependence of the centrifugal barrier is taken out, yielding a quantity $b_{l\tau}$ (Fr 57, Ra 62) which is compared with the similar experimental quantities.

The actual numerical results depend critically upon the boundary conditions (Ra 62). The condition at large distances is that the wavefunction represents only outgoing waves. However, for the condition at the nuclear surface Fröman (Fr 57), Strutinskii (St 57a), and Rostovskii (Ro 61) take the function constant. Rafiqullah (Ra 62), on the other hand, finds that this assumption does not reproduce the observed Z dependence of the b_{l1} which deviate considerably from the empirical values for large l. He has taken this boundary condition as

$$\bar{\psi}_0(\theta',\varphi')=\psi_0\sum_\mu \epsilon_{2\mu}Y_{2\mu}(\theta',\varphi') \tag{IV-7}$$

and used the empirical data to predict the boundary conditions. [Rasmussen and Segall (Ra 56) solve the coupled set of differential equations for the problem by both outward and inward numerical integration. In the former case, better agreement with the empirical intensities for Cm^{242} are obtained for a nonconstant boundary condition at the nuclear surface. This is also found true for the inward integration.]

For even–even nuclei Rafiqullah (Ra 62) finds that, in general,

$$b_{l1}=\left|\frac{\sum_{K_f}A_{K_f}{}^{l1}(C_{lK_f}+\sum_{\Omega'}\epsilon_{2-\Omega'}h_{lK_f;2-\Omega'})}{C_{00}+\sum_{\Omega'}\epsilon_{2-\Omega'}h_{00;2-\Omega'}}\right|, \tag{IV-8}$$

while for nuclei having $\gamma\leq 15^0$ this can be simplified since to good approximation in the lowest band $A_{K_f}{}^{l1}\simeq\delta_{0,K_f}$. The quantities $h_{lK,2-\Omega}$ and C_{lK_f} are available in numerical form (Ra 62).

TABLE IV–1. The empirical b_{l1} for even–even alpha emitters calculated from the hindrance factors given in Ref. Pe 57b and the fitted values of β, ϵ_{20} and $\epsilon_{2\pm2}$. The calculated values of b_{l1} for $l=8$ are labeled $b_{l1}{}^t$ and the calculated values of Q_{20} and $Q_{2\pm2}$ are from Ref. Ra 62.

Parent nucleus	b_{21}	b_{41}	b_{61}	b_{81}	$\beta=\beta_2$	ϵ_{20}	$\epsilon_{2\pm2}$	$b_{81}{}^t$	Q_{20}	$Q_{2\pm2}$
$_{90}Th^{230}$	1.30	0.77	0.10	···	0.21	4.73	−16.88	0.007	10.30	2.71
$_{92}U^{232}$	1.24	0.60	0.31	···	0.24	−2.40	−72.81	0.11	12.66	1.53
$_{94}Pu^{238}$	1.02	0.22	0.30	0.18	0.24	−1.62	−23.58	0.10	12.65	2.08
$_{96}Cm^{242}$	1.00	0.12	0.36	0.28	0.25	−2.42	−23.08	0.10	14.20	1.44
$_{98}Cf^{246}$	0.77	0.21	0.32	···	0.25	−5.15	−219.1	0.09	14.02	2.55

In Table IV–1 are the results of this calculation where the quantities to be compared with experiment are the b_{81} values. Unfortunately, only for Pu^{238} and Cm^{242} is such a comparison possible. In Table IV–2 are Fröman's results (Fr 57) for these same nuclei where b_{21} and b_{41} have been fitted to the empirical values. The values of β_2 and β_4 are also tabulated. Fröman estimates that β_6 and β_8 will be of the order of 2×10^{-2} that is the same order of magnitude as β_4. These results throw considerable doubt on the validity of expanding the nuclear surface only to lowest order (i.e., $Y_{2\mu}$ terms), but using second- and even third-order terms in the expansion of the moments of inertia about the symmetric shape in the study of the rotation–vibration interaction (Fa 64) (see Sec. II). [Similar values are found by Kjällquist (Kj 58), but larger values have been calculated by Harada (Ha 64).]

Table IV–1 shows that the alpha wavefunction on the nuclear surface is strongly distorted by the nuclear deformation. In general, the wavefunctions are highly elongated at the equator and very much reduced at the poles, a circumstance postulated by Christy (Ch 55) to explain the $l=4$ intensity for Cm^{242}.

Fröman (Fr 57) has also considered odd-parity transitions in even–even nuclei. However, since essentially no empirical data were available, he made only rough theoretical estimates of the hindrance factors.

The favored transitions in odd-A nuclei can be investigated in a manner similar to these transitions in even–even nuclei. However, as mentioned before (Sec. III) nothing is known about the values of γ for nuclei above the $2s$–$1d$ shell or even if an asymmetric-core model is necessary or useful in the other deformed regions. Assuming $\gamma<15^0$ and that the alpha wavefunction on the nuclear surface is given by Eq. (IV–7) the reduced transition probabilities for the favored transition $I_i\rightarrow I_f$ is

$$c_{lr}=\frac{\sum_{lc'l}|\,C(I_ilI_f;\,I_i0I_i)\,|^2}{\sum_{lc'l}|\,C(I_ilI_i;\,I_i0I_i)\,|^2},$$

where the c'_l can be given in terms of the C_{lK_f} and $h_{lK_f;2\Omega'}$ of Eq. (IV–8). Unfortunately even these c'_l cannot be calculated for asymmetric odd-A nuclei so the procedure is to interpolate for the particular Z from the neighboring even–even nuclei. This reduces the results of the asymmetric-core case to those of the

TABLE IV–2. Fröman's (Fr 57) fitted values for β_2 and β_4 and his calculated values for b_{61} and b_{81}. The latter are to be compared with the empirical values of Table IV–1. The estimated values of β_6 and β_8 to fit the theoretical and empirical b_{61} and b_{81} values are $\beta_6\approx\beta_8\approx0.02$ for Pu^{238} and Cm^{242}.

Parent nucleus	β_2	β_4	b_{61}	b_{81}
$_{90}Th^{230}$	0.26	0.041	0.39	0.14
$_{92}U^{232}$	0.26	0.029	0.32	0.11
$_{94}Pu^{238}$	0.26	−0.024	0.14	0.02
$_{96}Cm^{242}$	0.26	−0.041	0.06	0.00
$_{98}Cf^{246}$	0.18	0.000	0.03	0.01

TABLE IV–3. Theoretical reduced transition probabilities c_{If1}, obtained by replacing the c_l' values by the reduced transition probabilities of the neighboring even–even nuclei, interpolated to the Z value of the odd-A nucleus considered.

Parent nucleus	I_i	I_f	c_{If1} (emp.)	c_{If1} (theor.)
$_{91}Pa^{231}$	$3/2^-$	$3/2^-$	1	1
		$5/2^-$	0.32	0.38
		$7/2^-$		0.23
$_{92}U^{233}$	$5/2^+$	$5/2^+$	1	1
		$7/2^+$	0.34	0.31
		$9/2^+$	0.093	0.13
		$11/2^+$		0.018
$_{94}Pu^{239}$	$1/2^+$	$1/2^+$	1	1
		$3/2^+$	0.35	0.26
		$5/2^+$	0.35	0.39
		$7/2^+$		0.0049
$_{95}Am^{241}$	$5/2^-$	$5/2^-$	1	1
		$7/2^-$	0.31	0.24
		$9/2^-$	0.083	0.083
		$11/2^-$	0.0018	0.0016
		$13/2^-$	0.00076	0.0013
Am^{243}	$5/2^-$	$5/2^-$	1	1
		$7/2^-$	0.25	0.24
		$9/2^-$	0.052	0.083
		$11/2^-$		0.0016
		$13/2^-$		0.0013
$_{96}Cm^{243}$	$5/2^+$	$5/2^+$	1	1
		$7/2^+$	0.29	0.23
		$9/2^+$		0.080
$_{99}Es^{253}$	$7/2^+$	$7/2^+$	1	1
		$9/2^+$	0.13	0.11
		$11/2^+$	0.050	0.033
		$13/2^+$		0.0034

symmetric-core problem. Some of these results are tabulated in Table IV–3. The calculated values are in fair agreement with the empirical ones considering the nature of the assumptions made.

Finally, one can consider the unfavored transitions in odd-A nuclei in which the transition connects bands in which the odd nucleon is in different intrinsic states. Here the relative hindrance factor is far more influenced by the formation factor for the alpha particle than by the barrier penetration factor. Such calculations have been done near the "magic numbers" using simple shell-model theory and, in general, the technique is outside the scope of these general phenomenological collective models.

B. Beta Transitions

The operators inducing beta transitions, like those inducing electromagnetic transitions, can be written in terms of spherical tensors (Ro 54) and are also even or odd under point reflection. Calling λ the rank of the spherical tensor operator then, there is an obvious K selection rule (Al 55) which can be formulated as

$$|K_f - K_i| \leq \lambda \qquad \text{(IV–9a)}$$

which is similar to the usual selection rule for the angular momentum

$$|I_i - I_f| \leq \lambda \leq I_i + I_f. \qquad \text{(IV–9b)}$$

These selection rules are similar to the gamma selection rules, and in fact both the beta and gamma decay theories can be cast in the same form (Al 55). However, the selection rule (IV–9b) is rigorous while that of Eq. (IV–9a) is not since it depends upon the K purity of the initial and final states, so violation of this selection rule acts to hinder or retard that particular transition rather than prohibit it. The degree of K forbiddenness is defined as in the gamma-ray case by ν where

$$\nu = \Delta K - \lambda. \qquad \text{(II–32a)}$$

For odd-A nuclei there is a further set of selection rules which depend upon the asymptotic quantum numbers $[N, n_z, \Lambda]$, and in the limit of infinite distortion these selection rules are rigorous. However, since deformed nuclei have relatively small distortions, in fact, these selection rules are also only approximate. Transitions in which all of these selection rules hold are called "unhindered" while those which are allowed by (IV–9a) and (IV–9b) but are forbidden by one or more of the (N, n_z, Λ) selection rules are called "hindered" (Al 55a). These beta-decay selection rules for all of the beta transition operators through second forbidden transitions have been tabulated (Al 57) and we give in Table IV–4 these selection rules for allowed and first forbidden transitions. The notation for the nuclear matrix elements is similar to that of Preston (Pr 62). Several tabulations of ft values exist (Al 55a, Mo 59) which show that, at least for allowed and first forbidden transitions, the hindered and unhindered transitions separate empirically just as the ft values for different degrees of forbidenness are separated by different orders of magnitude. In general these ranges are $a < \log ft < b$ where the limits are $a=4.5$, $b=5.0$ for allowed unhindered; $a=6.0$, $b=7.5$ allowed hindered; $a=5.0$, $b=7.5$ first forbidden, unhindered; $a=7.5$, $b=8.5$ first forbidden hindered.

Another obvious rule is that the ratio of ft values for two transitions to different members of the same rotation band will be in the ratio of the squares of the appropriate Clebsch–Gordan coefficients.

$$\frac{ft(I_i K_i \rightarrow I_{1f} K_f)}{ft(I_i K_i \rightarrow I_{2f} K_f)} = \left[\frac{C(I_i \lambda I_{1f}; K_i, K_f - K_i, K_f)}{C(I_i \lambda I_{2f}; K_i, K_f - K_i, K_f)}\right]^2. \qquad \text{(IV–10)}$$

For a detailed comparison one can calculate the nuclear matrix elements using the Nilsson state functions for a given distortion determined by the level structure or other properties. One can then calculate the ft values which for allowed transitions, using the notation of Ref. Pr 62, is

$$f_0 t = 2\pi^3 \ln 2 [\langle L_0 \rangle (g_V^2 |\, \mathfrak{M}_F \,|^2 + g_A^2 |\, \mathfrak{M}_\sigma \,|^2)]^{-1},$$

where $\langle L_0 \rangle$ is a number near unity and the g_i are the

TABLE IV–4. Asymptotic selection rules for allowed and first forbidden beta transitions in deformed nuclei.

Matrix element	ΔI	$\Delta\pi$	ΔK	ΔN	Δn_z	$\Delta\Lambda$
$\langle 1 \rangle$	0	no	0	0	0	0
$\langle \boldsymbol{\sigma} \rangle$	0, 1 no $0 \rightarrow 0$	no	0, 1	0	0	0
$\langle \mathbf{r} \rangle$	0, 1 no $0 \rightarrow 0$	yes	0	$\left\{\begin{matrix}+1\\-1\end{matrix}\right.$	$\left.\begin{matrix}+1\\-1\end{matrix}\right\}$	0
			1	1	0	1
$\langle \boldsymbol{\alpha} \rangle$	0, 1 no $0 \rightarrow 0$	yes	0	$\left\{\begin{matrix}+1\\-1\end{matrix}\right.$	$\left.\begin{matrix}+1\\-1\end{matrix}\right\}$	0
			1	1	0	1
$\langle \gamma_5 \rangle$	0	yes	0	$\left\{\begin{matrix}+1\\-1\end{matrix}\right.$	$\left.\begin{matrix}+1\\-1\end{matrix}\right\}$	0
				1	0	1
$T_0(1, \boldsymbol{\sigma})$	0	yes	0	$\left\{\begin{matrix}+1\\-1\end{matrix}\right.$	$\left.\begin{matrix}+1\\-1\end{matrix}\right\}$	0
				1	0	1
$T_1(1, \boldsymbol{\sigma})$	0, 1	yes	0	1	0	1
	no $0 \rightarrow 0$ or $\frac{1}{2} \rightarrow \frac{1}{2}$		1	$\left\{\begin{matrix}+1\\-1\end{matrix}\right.$	$\left.\begin{matrix}+1\\-1\end{matrix}\right\}$	0
			1	1	0	1
$T_2(1, \boldsymbol{\sigma})$	0, 1, 2	yes	0	1	0	1
	no $0 \rightarrow 0$ $\frac{1}{2} \rightarrow \frac{1}{2}$ or $1 \leftrightarrow 0$			$\left\{\begin{matrix}+1\\-1\end{matrix}\right.$	$\left.\begin{matrix}+1\\-1\end{matrix}\right\}$	0
			1	1	0	1
				$\left\{\begin{matrix}+1\\-1\end{matrix}\right.$	$\left.\begin{matrix}+1\\-1\end{matrix}\right\}$	0
			2	1	0	1

coupling constants and $\mathfrak{M}_F$ and $\mathfrak{M}_\sigma$ are the Fermi and Gamow–Teller transition operators. Nilsson has discussed this evaluation briefly for mirror transitions and for certain forbidden transitions, the "unique" forbidden transitions, in which the operators are quite analogous to the gamma case.

Bogdan (Bo 62a, Bo 63) has calculated the matrix elements for λth forbidden transitions using Nilsson state functions assuming $(V-A)$ type interactions and has compared them with two or three transitions to even–even nuclei in the rare-earth deformed region.

For beta transitions between even nuclei, Gallagher (Ga 60) has defined two different classes of transition: those in which the initial and final state have the same relative coupling of the two odd nucleons, and those in which the states have different relative couplings. In the former case, the matrix elements in the asymptotic limit of infinite distortion become the product of an overlap integral (c.f. Sec. III B) with the asymptotic selection rules discussed before. For the case of transitions between states of different relative coupling Gallagher found that the odd particles must have $\Lambda_p=\Lambda_n=0$, $\Omega_p=\Omega_n=\frac{1}{2}$ for nonvanishing contributions to the transition matrix elements in allowed and first forbidden decays. He classified twelve such beta transitions in the deformed regions and found that the ranges of the ft values were essentially the same as for the odd-A cases. (This statement is stronger than demanded by the empirical evidence as he was able to classify only one allowed unhindered and one first forbidden hindered decay and five each of allowed hindered and first forbidden unhindered.)

Similar transitions have been discussed from the point of view of the asymmetric model (Da 60b) in which the final states are given quite specifically for any level of angular momentum L(c.f. Sec. II); however, the initial state functions (for the odd–odd parent) are taken as

$$|IM\rangle=\varphi_I(r)\sum_K a_K D_{MK}^I$$

which is a pure rotational model of odd–odd nuclei and neglects any possibility of the interaction between the two odd nucleons. Clearly this calculation corresponds only to transitions between initial and final states having the same relative coupling of the odd nucleons. The a_K are not known, so to proceed it was necessary to take the odd–odd parent as being symmetric (K a good quantum number) and calculate only the ratios of ft products for different final states. In this case one has

$$\frac{ft(I_iK_i\rightarrow L_{1f})}{ft(I_iK_i-L_{2f})}=\left[\frac{\sum_{K_1}A_{K_1}^{L_1}C(I_i\lambda L_{1f};\ K_i,\ K_1-K_i,\ K_1)}{\sum_{K_2}A_{K_2}^{L_2}C(I_i\lambda L_{2f};\ K_i,\ K_2-K_i,\ K_2)}\right]^2$$

which reduces to (IV–10) when K_f is a good quantum number. The data are too sparse to decide upon the merits of this calculation, although it would seem that if one is going to insist on a nuclear model with an asymmetric core the odd–odd case should be handled in a more consistent manner.

V. PHOTONUCLEAR REACTIONS

Immediately after the discovery of the photonuclear giant resonance by Baldwin and Klaiber (Ba 47, Ba 48) several theoretical models to explain the process were suggested by Goldhaber and Teller (Go 48). They invoked collective dipole effects to explain some of the properties of the cross-section curve as a function of energy in particular the energy of the resonance, as well as the integrated cross section. In one of these models the nucleons execute harmonic oscillations each about its equilibrium position the spring constant being the same for all nucleons in all nuclei. Thus, the resonance energy E_m is a universal constant identical for all nuclei which even at that time did not agree with experiment. The second model was a two-fluid model in which compressible proton and neutron fluids oscillate against each other, with the restoring forces proportional to the density gradients and E_m inversely proportional to the nuclear radius or to $A^{-\frac{1}{3}}$. Finally, they suggested that the dipole motion might be caused by the oscillation against each other of the neutrons and protons taken as rigid spheres. This yields E_m proportional to $R^{-\frac{1}{2}}=A^{-\frac{1}{6}}$. For this model they also calculated the integrated cross section.

The second model has been extensively investigated by Steinwedel, Jensen, and Jensen (St 50, St 50a) who solved the two-fluid problem for a spherical nucleus and related the resonance energy E_m to the phase velocity of second-sound u in the fluid by

$$E_m=2.08\hbar u R^{-1}$$

which is proportional to $A^{-\frac{1}{3}}$. This model has been extended by Danos (Da 58) and Okamoto (Ok 58) to spheroidal systems relating, among other things, the width of the resonance to the intrinsic quadrupole moment Q_0. This width is due to the splitting of the resonance because the spheroid has two different eigenfrequencies associated with its semi-axes. These latter calculations were confirmed by Fuller and Weiss (Fu 58) who were able to measure the giant resonance sufficiently accurately in terbium and tantalum to show a double peaked curve.

A major experimental difficulty in these measurements is related to the fact that the giant resonance is studied with a bremsstrahlung x-ray distribution, the unfolding of which from the measured yield curves injects the major uncertainty into the results. Recently it has become possible to obtain essentially monoenergetic x-ray beams of variable energy from high-flux electron linear accelerators which permit very accurate determination of the photonuclear cross-section curves. It would seem that this is now the preferred way to determine the shape of the nucleus in its ground state.

The calculation of Steinwedel, Jensen, and Jensen[6] proceeds from Hamilton's principle

$$\delta\int L\,dt=\delta\int dt\int_\tau \mathfrak{L}\,d\tau=\delta\int dt\int_\tau(\mathfrak{I}-\mathfrak{U})\,d\tau=0,$$

where $\mathfrak{L}$, $\mathfrak{I}$, and $\mathfrak{U}$ are the Lagrangian-, kinetic-, and potential-energy densities, respectively. The fluid velocities are denoted by $\mathbf{v}(t)$, where

$$\mathbf{v}_i(t)=\dot{\mathbf{r}}_i(t),$$

the $\mathbf{r}(t)$ being the generalized coordinate of an element of the fluid, the index i referring to neutron or proton fluid. The variation is carried out with respect to the $\mathbf{r}_i(t)$ and subject to the constraints on the densities ρ_i

$$\rho_0=\rho_p+\rho_n=\text{constant}, \tag{V-1}$$

$$\rho_p+\nabla\cdot\rho_p\mathbf{v}_p=0, \tag{V-2a}$$

$$\rho_n+\nabla\cdot\rho_n\mathbf{v}_n=0, \tag{V-2b}$$

and to the boundary condition

$$\mathbf{r}\cdot\mathbf{v}_i\,|_{r=R_0}=0. \tag{V-3}$$

The condition (V–1) is that of a homogeneous incompressible total fluid while conditions (V–2) are the familiar statement of the conservation of mass (Se 59).

The physical process at hand requires (a) that the total number of nucleons remain fixed during photon absorption, (b) that the electromagnetic field be coupled to the nucleus, and (c) that the dipole oscillations be coupled to other internal degrees of freedom broadening the resonance to some width Γ. Requirement (a) can be entered into the variational calculation through a Lagrange multiplier while (b) requires adding the force on an electric charge $q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$. Finally, the width must be added through a dissipation term which is done in an *ad hoc* fashion. The variational principle then becomes

$$\delta\int L\,dt=\int dt\int_\tau(\delta\mathfrak{I}-\delta\mathfrak{U}+\delta\eta+\delta\mathfrak{A})\,d\tau=0.$$

The potential energy density is just the symmetry energy (Pr 62)

$$\mathfrak{U}=K(\rho_p-\rho_n)^2/\rho_0.$$

Carrying through the variation leads, after linearization, to a wave equation with damping of the form

$$[a\nabla^2-\Gamma(\partial/\partial t)-(\partial^2/\partial t^2)]\rho_p(\mathbf{r}t)=0$$

which can be simplified by assuming a harmonic time dependence for $\rho_p(\mathbf{r},\,t)$

$$\rho_p(\mathbf{r},\,t)=\rho_p{}^{(0)}[1+\eta(\mathbf{r})\,\exp\,(-i\omega t)]$$

whence the wave equation

$$(\nabla^2+k^2)\eta=0. \tag{V-4}$$

[6] The most lucid exposition of this development is to be found in Ref. Da 61c which is unfortunately unpublished.

This equation with the boundary condition of Eq. (V–3) is identical with those associated with the acoustical vibrations of a gas contained in a rigid spherical shell the solutions of which have long been known (St 45). For the particular case at hand the eigenvalue problem may be expressed as

$$\tan kr=2kr/(2-k^2r^2);$$

the lowest nontrivial solution of which was given by Lord Rayleigh (St 45) as $kr=119.26\pi/180=2.08$. This mode leads to the energy

$$E_1=\hbar\omega_1=\alpha A^{-\frac{1}{3}}\,\text{MeV}$$

with $\alpha=70$ and it has been asserted that $\alpha_{\text{exp}}=80$ (Da 61c).

Both Danos (Da 58) and Okamoto (Ok 56, Ok 58) have investigated the relation between the intrinsic quadrupole moment Q_0 of a deformed nucleus and the splitting of the giant resonance cross-section curve into two peaks (Okamoto considered the width of the unresolved resonance peak to be related to Q_0, while Danos takes the resonance widths to be parameters of the theory which is more correct). Okamoto solved Eq. (V–4) with the boundary condition (V–3) in spheroidal coordinates while Danos made use of the $l=1$, $m=0$ solution by MacLauren (Mac 98) of the analogous acoustical problem and the $l=1$, $m=\pm1$ solutions from which he (Danos) showed that to a good approximation the two frequencies for a spheroidal nucleus are related by

$$\omega_b/\omega_a=0.911(a/b)+0.089, \tag{V-5}$$

valid on the range of deformation normally found in nuclei (Da 58). Here a and b are the semi-axes with b normal to the symmetry axis. For a randomly oriented target the average cross section will be

$$\langle\sigma(E)\rangle_{\text{Av}}=\tfrac{1}{3}\sigma_a(E)+\tfrac{2}{3}\sigma_b(E).$$

It was also shown that if the width Γ_i is a function of resonance energy then

$$(\sigma_a)_{\max}/(\sigma_b)_{\max}=\Gamma_b/2\Gamma_a.$$

Equation (V–5) relates the ratios of the resonance energies to that of the semi-axes and hence to the deformation parameter β so that knowing the positions of these peaks the intrinsic quadrupole moment Q_0, can be calculated from Eq. (II–36). These predictions were confirmed by Fuller and Weiss (Fu 58) who observed the splitting in tantalum and terbium and compared the intrinsic quadrupole moments determined in this way with those determined by Coulomb excitation. The agreement was quite good.

The extension of these ideas to ellipsoidal nuclear shapes has been done by Inopin (In 60) and Okamoto (Ok 62). The former has used a variational method to obtain the wave numbers k_i in the form

$$k_i=(2.08/R_i)\,(0.92+0.08R_i/R_0), \tag{V-6}$$

TABLE V–1. Photonuclear cross-section parameters for nuclei in the deformed regions. In the column labeled Type, β refers to a bremsstrahlung gamma-ray beam while γ refers to a monoenergetic gamma-ray beam used in the experimental determination of the listed parameters.

Nucleus	σ_a mb	E_a MeV	Γ_a MeV	σ_b mb	E_b MeV	Γ_b MeV	Q_0 b	Type	Reference
Tb^{159}	258	12.5	2.4	310	16.3	4.0	5.6	β	Fu 58
Tb^{159}	188	12.2	2.67	233	15.6	4.30	7.0	γ	Br 64
Ho^{165}	318	12.2	2.35	328	16.0	4.5	7.6	β	Fu 62
Ho^{165}	200	12.10	2.65	249	15.75	4.4	7.40	γ	Br 63
Er[a]	318	12.2	2.33	328	16.0	4.5	7.6	β	Fu 62
Ta^{181}	308	12.45	2.3	348	15.45	4.4	5.7	β	Fu 58
Ta^{181}	198	12.75	3.0	244	15.50	5.0	6.71	γ	Br 63
U^{238}	510	10.85	2.45	570	14.10	4.00	12.8	γ	Bo 64

[a] Natural erbium.

where R_i is the ith semi-axis of the ellipsoid. Inopin then compared this calculation with the work of Fuller and Weiss (Fu 58) on terbium, reinterpreting their data to yield a resolved three-peak resonance curve with resonance energies at 12.5, 15.0, and 16.8 MeV. Using these and Eq. (V–6) he calculated the deformation and asymmetry parameters $\beta=0.30$ and $\gamma=19°$. This reinterpretation does violence to the cross section determinations of Fuller and Weiss and is not consistent with the reported experimental errors. It also does not take into account fully the $(\gamma, 2n)$ cross section which, from the monoenergetic gamma-ray work, is known to make a large contribution to that part of the cross-section curve above the first resonance peak.

Okamoto (Ok 62) makes use of a geometrical method to calculate the ratios of the resonance energies. He gives a table of predicted resonance energies of 44 nuclei from Ne^{20} to Hg^{200}. No predictions are made in in the actinide deformed region and only the isotopes Os^{190} and Pt^{192}, Pt^{194}, and Pt^{196} in the lower deformed region are given. Since the platinum isotopes may have the greatest asymmetry ($\gamma=30°$) it is interesting to note his predictions of 13.5, 14.2, and 14.9 MeV are considerably closer together than Inopin finds for a much less asymmetric nucleus. Such close resonances are probably not resolvable by present techniques. With regard to the platinum isotopes recent betatron measurements on the separated isotopes Pt^{194}, Pt^{195}, Pt^{196}, and Pt^{198} show a very broad and unresolved resonance for the even-A isotopes and a rather narrow resonance for the odd-A isotope (He 64). The former are not inconsistent with some sort of splitting of the resonance but the latter effect in Pt^{195} indicates that it is not deformed to any great extent.

Much more accurate work can be done using monoenergetic gamma-ray beams for then it is also possible to measure the $(\gamma, 2n)$ cross section. In Table V–1 are tabulated the resonance energies, widths, and derived intrinsic quadrupole moments Q_0 for a number of nuclei in the deformed regions. In general the resonance parameters are obtained by fitting each resonance to a "Lorentz" line of the form

$$\sigma(E)=\sigma_0(E_0)/[1+(E^2-E_0^2)^2/\Gamma^2E^2],$$

where $\sigma_0(E_0)$ is the peak cross section at energy E_0 while Γ is the width of the resonance at half-maximum (Fu 62a).

Since in principle the shape of the giant resonance could resolve the question of whether or not a deformed nucleus is spheroidal or ellipsoidal, one would hope that careful measurements would resolve this question independently of the interpretation of the spectra. Unfortunately, the question is still unresolved. As mentioned before it was originally suggested that the experimental evidence from Tb^{159} indicated it to be ellipsoidal (In 60) and recent measurements with a bremsstrahlung beam (Bo 62b) have been fitted with three Lorentz lines with energies at 12.25, 15, and 17 MeV, widths of 2.5, 3.2, and 2.0 MeV, and heights of 278, 191, and 305 mb. On the other hand, data obtained using a monoenergetic gamma-ray beam rule out the above interpretation of three such widely spaced resonances but do not distinguish between a prolate spheroidal deformation and an ellipsoidal deformation (Br 64). A similar situation exists in Ho^{165} where the betatron data are not accurate enough to distinguish between spheroidal and ellipsoidal shapes (Fu 62). For this nucleus and Ta^{181} the monoenergetic x-ray data could not be fitted with three Lorentz shapes, the conclusion being that these nuclei are "essentially spheroidal" (Br 63).

It has been shown that the coherent photon scattering cross section is related to the total nuclear absorption cross section (Fu 56). The scattering cross section has been measured for a few nuclei in the deformed regions and a discrepancy exists between this scattering and the absorption cross section. This discrepancy is associated with a second component of the scatter-

ing cross section called the "tensor" component (Fu 62) so that the total scattering cross section is

$$d\sigma/d\Omega=(d\sigma/d\Omega)_s+(d\sigma/d\Omega)_t,$$

the first term being the so-called scalar cross section. It has been found that the tensor cross section is sensitive to the shape of the deformed nucleus, being much larger for ellipsoidal nuclei than spheroidal nuclei (Fu 62).

Danos and Greiner have amalgamated the hydrodynamic models of the low energy structure of deformed nuclei and the dipole oscillations of the giant resonance (Da 64c). In this development they have considered the interactions between the dipole oscillations and the rotations and vibrations of an even–even nucleus; however, the effect of the odd particle (or particles) for odd-A (or odd–odd) nuclei should be small for the giant resonance, hence the theory is considered to have validity for all nuclei for which a collective hydrodynamic model is meaningful (i.e., for $A\gtrsim 50$). Further, the calculation is done within the context of the adiabatic approximation, that is, that $E_{\text{rot}}\ll E_{\text{vib}}\ll E_d$ where E_d is the energy of the giant resonance. The particular form of the theory of the low-energy rotation–vibration problem is that one of the rotations and vibrations of an axially symmetric system where the collective parameters are $a_0'\cong(\beta-\beta_0)$ and $a'_{\pm 2}\cong\beta_0\gamma/\sqrt{2}$ mentioned in Sec. II (Fa 62, Fa 62a, Fa 64) although the theory could use other low-energy models as a starting point.

The total Hamiltonian consists then of a term for the rotations, one for the vibrations and one for the dipole oscillations. Calling the total angular momentum **I** the nuclear angular momentum **L** and the angular momentum of the dipole oscillations **j** then

$$\mathbf{I}=\mathbf{L}+\mathbf{j}$$

which is just Eq. (III–1). That is, the dipole oscillations enter the rotational problem in the same way the extra-core nucleon enters the similar odd-A problem. Here **j** is restricted to values appropriate to a dipole oscillation. The total Hamiltonian can then be written as

$$H=H_R+H_{\text{vib}}+H_d+H_{v-R}+H_{v-d}+H_{d-R}$$

where H_{vib} and its associated Schrödinger equation is similar to that of Eqs. (II–25) and (II–26) (except the variables are a_0' and a_2'), H_{v-R} is similar to that of (II–28) and H_{v-d} is identical with H_c of Eq. (III–4c). The dipole oscillation problem is treated in the second quantized representation and is similar to Eq. (I–16).

The wavefunctions are products of a vibrational term, a rotational term, and a dipole term

$$\psi=\varphi_{n_\beta n_\gamma}D_{MK}{}^{I*}\eta_s,$$

respectively, where n_β and n_γ are the vibrational quantum numbers and s indicates the dipole mode, being zero for excitations parallel to the symmetry axis and ± 1 for excitations normal to this axis. These functions must be properly symmetrized under the T_1 and $T_2{}^2$ operators discussed in Sec. II. The energy eigenvalues are then

$$E(I,K,n_\beta,n_\gamma,s)=[I(I+1)-K^2-|s|]E_R+\hbar\omega_s+\hbar\omega_\gamma(n_\beta+\tfrac{1}{2})+\hbar\omega_\gamma(n_\gamma+1).$$

The coupling between the dipole oscillations and the quadrupole vibrations stabilizes the nuclear surface in an asymmetric shape so that the upper resonance is split into two peaks. Further, some of the dipole strength of these two resonances goes into a satellite line of each peak due to the coupling of a single quantum oscillation in the γ mode.

This model has been fitted to the betatron data of the deformed odd-A nucleus Ho^{165} (Fu 62). The parameters E_R, $\hbar\omega_\beta$, $\hbar\omega_\gamma$ were fit to the low-energy structure of the neighboring even–even nucleus Er^{166} while β_0 was determined from the separation of the giant resonance peaks. The widths Γ_k were fit to an expression of the form

$$\Gamma_k=\Gamma_0E_k\delta$$

with $\delta=2.2$. The agreement between experiment and theory is very good indeed (Da 64b) and considerably better than for the so called "static" theories discussed earlier where the cross section was too low on the high-energy side of the giant resonance peaks.

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