Critical Current and Field in Nonideal Superconductors*

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I. INTRODUCTION

Abrikosov[1] and Goodman[2] have developed phenomenological theories which describe bulk superconducting alloys in high fields as a periodic array of many microscopic superconducting and normal regions. Their predicted critical fields, at which the mixed state appears and disappears, depends on only two phenomenological constants: the thermodynamic field, H_c, and a nondimensional surface energy parameter. The latter specifies the free energy of the interface or transition zone between a normal and superconducting region. Abrikosov's surface energy parameter is inversely proportional to κ, the phenomenological constant of the Ginzburg–Landau[3] theory. Gorkov[4,5] and Shapoval[6] have shown that κ can be calculated directly from two easily accessible experimental properties of the material: the electronic heat capacity coefficient γ and the normal electrical conductivity (at low temperatures) σ. Bardeen, Cooper, and Schrieffer[7] had already shown that H_c can be calculated from γ and T_c, the transition temperature. Hence, γ, T_c, and σ completely determine the magnetic properties of superconducting alloys. In this paper, we show how certain current-carrying properties of these materials are likewise determined by these three quantities based on a laminar model similar to London's[8] and Goodman's.[2]

II. EXPERIMENTAL RESULTS

The test specimens were cold-rolled strips (0.008 in. × 0.025 in.). Measurements were made in a dc magnetic field applied parallel to the major axis of the strip cross section at 4.2°K. We also measured the normal electrical conductivity of each sample slightly above the critical temperature to obtain the theoretical upper critical field.

The alloy composition, critical temperature, electronic heat capacity coefficient, normal electronic conductivity, thermodynamic critical field (0°K), and upper critical field (0°K) are listed in Table I. The critical fields are calculated from Eqs. (5) and (1), respectively.

The upper critical field could be measured only for the alloys vanadium with 15% or less titanium, whose upper critical fields were less than 50 kG, the maximum field available in the experiments.

TABLE I. Composition and properties of alloys.

Alloy	T_c[a,b] °K	$\gamma \times 10^{-3}$ ergs/deg²/cm³	$\sigma \times 10^{-4}$ [c,d] mhos/cm	H_c^0 (kG) calculated	H_{c2}^0 (kG) calculated
V	5.2	10.9[a]	29.70	1.29	11
V.93–Ti.07	6.4	12.5	7.92	1.74	56
V.90–Ti.10	6.8	13.1	6.50	1.88	76
V.88–Ti.12	7.0	13.3	5.00	1.96	103
V.85–Ti.15	7.2	13.6	3.95	2.02	138
V.75–Ti.25	7.5	14.2	2.50	2.18	236
V.67–Ti.33	7.7	13.2	1.77	2.14	319
V.60–Ti.40	7.5	12.4	1.56	2.03	331
Nb.75–Zr.25	10.4	9.5[b]	4.0	2.50	137

[a] C. H. Cheng, K. P. Gupta, E. C. Van Reuth, and P. A. Beck, Phys. Rev. **126**, 2030 (1962) (for V–Ti data).
[b] See Ref. 8.
[c] This work.
[d] Multiply by 9 × 10¹¹ to obtain esu conductivity for use in Eq. (2).

The upper critical field, H_{c2} at 4.2°K for V–0%, –7%, –10%, –12%, and –15% Ti alloys is shown in Fig. 1. The dashed curve in Fig. 1 is the upper field calculated from the Gorkov–Shapoval formula. Their formula for the upper critical field at 0°K is the following (in esu units):

$$H_{c2}^0 = (3/2\pi)ec\gamma T_c/k\sigma \,. \qquad (1)$$

The constants e, c, and k are the electronic charge, the velocity of light, and Boltzmann's constant, respectively. γ is the electronic heat capacity coefficient, T_c is the critical temperature, and σ is the normal electrical conductivity.

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1 A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **32**, 1442 (1957) [English transl.: Soviet Phys.—JETP **5**, 1174 (1957)].
2 B. B. Goodman, Phys. Rev. Letters **6**, 597 (1961).
3 V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950).
4 L. P. Gorkov, Zh. Eksperim. i Teor. Fiz. **36**, 1918 (1959) [English transl.: Soviet Phys.—JETP **9**, 1364 (1959)].
5 L. P. Gorkov, Zh. Eksperim. i Teor. Fiz. **37**, 833 (1959) [English transl.: Soviet Phys.—JETP **10**, 593 (1960)].
6 E. A. Shapoval, Zh. Eksperim. i Teor. Fiz. **41**, 877 (1961) [English transl.: Soviet Phys.—JETP **14**, 628 (1962)].
7 J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
8 F. London, *Superfluids* (Dover Publications, Inc., New York, 1961), Vol. 1.

The upper critical field H_{c2}^T at a temperature T is given as follows[6]:

$$H_{c2}^T \begin{cases} = H_{c2}^0 1 - 2(T/T_c)\ln 2\,, & T \ll T_c \\ = H_{c2}^0 (1 - T/T_c) 8/\pi^2\,, & T_c - T \ll T_c\,. \end{cases} \quad (2)$$

From Fig. 2, it is clear that the measured values

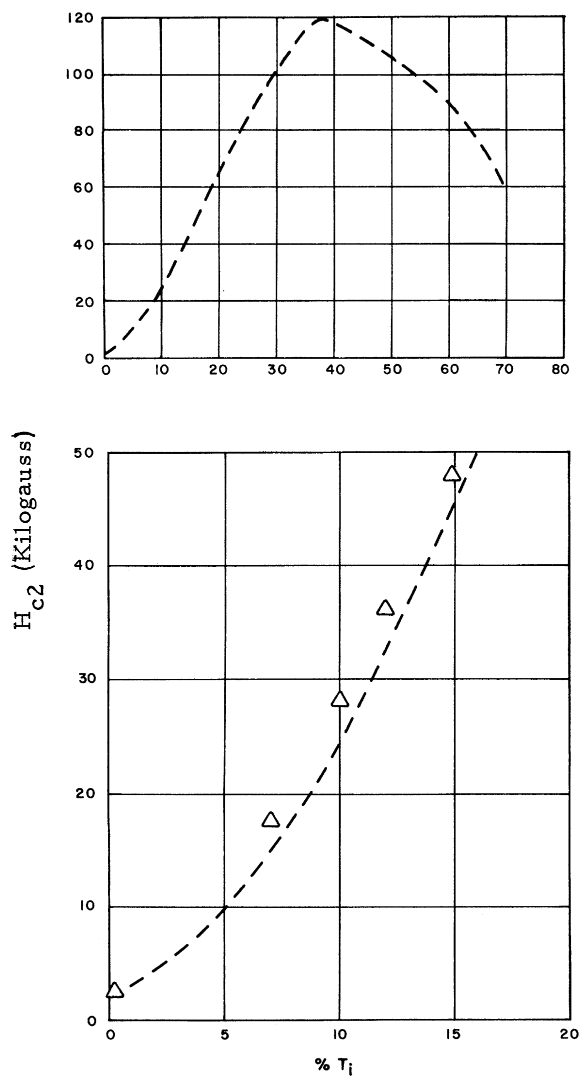

Fig. 1. The upper critical field H_{c2} for the V–Ti system at 4.2°K. Experimental points are compared to the calculated values (dashed curve) obtained from the Shapoval formula. Beyond 40% Ti, extrapolated resistivity data was used.

of H_{c2} are in good agreement with the Shapoval formula up to the highest fields available (50 kG). Previous experiments[9] with Nb–25% Zr also agreed with Shapoval's upper critical field at 4.2°K.

[9] A. El Bindari and M. M. Litvak, J. Appl. Phys. **34**, 2913 (1963).

The measured values of the critical current for zero applied fields substantially agree with the value

$$I_c = 2dH_c c/4\pi\,. \quad (3)$$

I_c is the Silsbee current, based on the thermodynamic critical field H_c, for a strip with a very small aspect ratio (thickness t divided by width d) provided that negligible currents flow near the ends.

The thermodynamic field H_c, from which the Silsbee current is calculated, is obtained from the following formula[6,7]:

$$H_c^T \begin{cases} = H_c^0 [1 - 1.07(T/T_c)^2]\,, & T \ll T_c \\ = H_c^0 1.74(1 - T/T_c)\,, & T_c - T \ll T_c \end{cases} \quad (4)$$

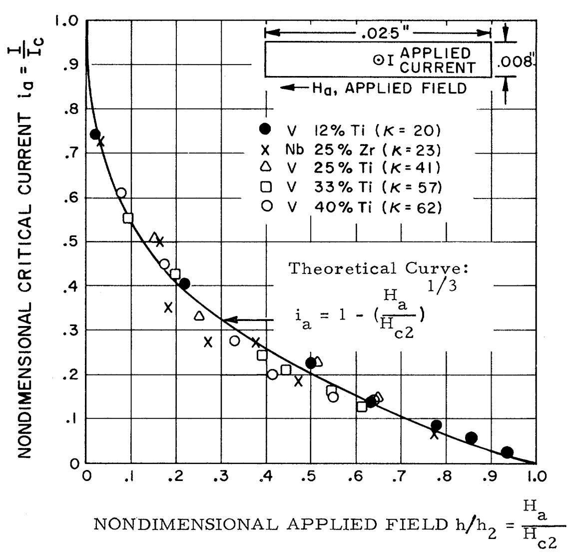

Fig. 2. The nondimensional critical current $i_a = I/I_c$ vs the nondimensional applied field H_a/H_{c2} for various alloys at 4.2°K. $I_c = 2dH_c c/4\pi$, the Silsbee current based on the sample width and the thermodynamic critical field. H_{c2} is the upper critical field. The solid line is the theoretical curve: $i_a = 1 - (H_a/H_{c2})^{1/3}$, derived from the mixed state free energy.

where

$$H_c^0 = \gamma^{1/2} T_c/(0.17)^{1/2}(\mathrm{G})\,. \quad (5)$$

We obtain a single curve for the various samples tested when we plot the critical current I divided by the Silsbee current I_c as a function of the applied field H_a divided by the upper critical field H_{c2}^T. This is shown in Fig. 2. Each sample has a different value of the ratio of H_{c2}^T to H_c^T, given by

$$\frac{H_{c2}^T}{H_c^T} \begin{cases} = \kappa 3.03[1 - 2(T/T_c)\ln 2]\,, & T \ll T_c \\ = \kappa\sqrt{2} & T_c - T \ll T_c\,. \end{cases} \quad (6)$$

The values of κ calculated from Eqs. (2)–(6) are indicated in Fig. 2.

III. THEORETICAL CURVE OF CRITICAL CURRENT VS APPLIED FIELD

To predict the critical current according to the laminar model, we have assumed the following:

(a) The alternating layers of normal and superconducting material are parallel to the external field, which is applied in this case parallel to the width (or long side) of the strip's rectangular cross section. (See Fig. 3.)

(b) The applied field has its full value in each of the normal layers. Negligible current flows in them.

(c) The aspect ratio is sufficiently small to neglect end effects.

(d) The magnetic induction **B** and the applied current density **j** in the superconducting material obey the London equation, that is,

$$\nabla^2\mathbf{B} = \lambda^{-2}\mathbf{B} \qquad (7)$$

$$\nabla^2\mathbf{j} = \lambda^{-2}\mathbf{j}\,. \qquad (8)$$

This local electromagnetic theory is applicable for cases of sufficient alloying ($\kappa \gg 1$).[10]

(e) The interfacial free energy is given by $qH_c^2\lambda/8\pi$, where q is Goodman's nondimensional surface energy parameter.

$$q = \frac{H_{c2}}{H_c} + \left[1 - \left(\frac{H_{c2}}{H_c}\right)^2\right]\tanh^{-1}\frac{H_c}{H_{c2}}, \qquad (9)$$

$$q \simeq \tfrac{2}{3}\,(H_c/H_{c2})\,, \qquad (10)$$

for cases of $H_{c2}/H_c \gg 1$ (i.e., $\kappa \gg 1$).

Figure 3 shows our coordinate system: The x axis runs parallel to the thickness t, the y axis, the magnetic field direction, runs parallel to the width d, and the z axis, the current direction, runs parallel to the length.

$H(x)$ and $B(x)$ when nondimensionalized by H_c are denoted by $h(x)$ and $b(x)$, respectively. The applied field H_a and H_{c2} divided by H_c are denoted by h_a and h_2, respectively. The applied current I divided by I_c is denoted by i_a.

From assumptions (b) and (d) above we obtain for $h(x)$ the following:

$$h(x) = h_a + i_a\frac{\sinh\,[x]/\lambda}{\sinh\,[t]/2\lambda}\,. \qquad (11)$$

The associated applied current distribution is shown in Fig. 3.

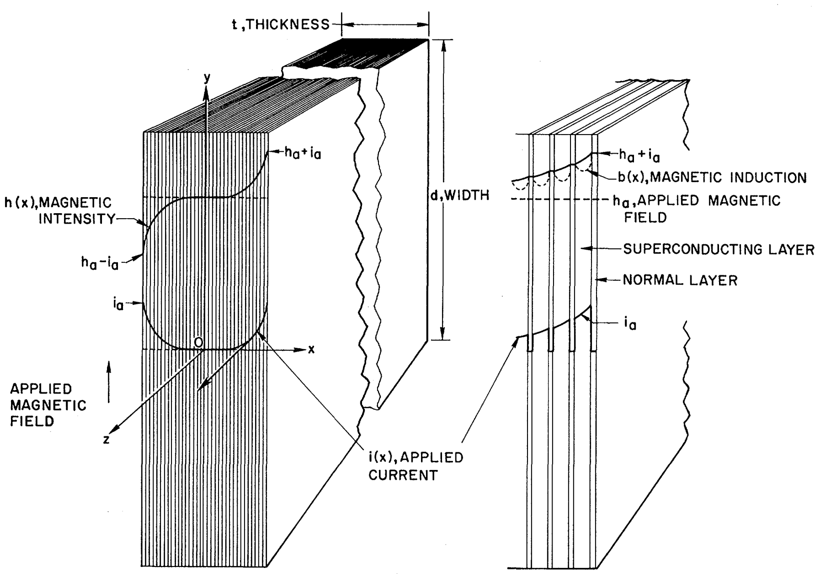

FIG. 3. Magnetic field (intensity and induction) and applied current profiles for a cross section near the center of the sample. The layers of superconducting and normal material are shown.

Solving the London equation for $b(x)$ in a superconducting layer, with the boundary conditions that $b(x)$ is equal to $h(x)$ at either side of the superconducting layer, we obtain for a single layer

$$b(x) = \bar{h}\,\frac{\cosh\xi}{\cosh p} + \Delta h\,\frac{\sinh\xi}{\sinh p}\,, \qquad (12)$$

where $2p\lambda$ is the thickness of the layer. $\xi\lambda$ is the x distance measured from the center of the layer. Also, $\bar{h} = \frac{1}{2}\,[(h(p) + h(-p)]$ and $\Delta h = \frac{1}{2}\,[h(p) - h(-p)]$. $h(p)$ and $h(-p)$ are the field intensities on the right-hand and left-hand sides of the layer, respectively. Equation (11) for $h(x)$ in a superconducting layer can be rewritten as follows:

$$h(x) = h_a + (\bar{h} - h_a)\,\frac{\cosh\xi}{\cosh p} + \Delta h\,\frac{\sinh\xi}{\sinh p}\,. \qquad (13)$$

[10] A. A. Abrikosov and L. P. Gorkov, Zh. Eksperim. i Teor. Fiz. **35**, 1558 (1958) [English transl.: Soviet Phys.—JETP **8**, 1090 (1959)].

The nondimensional magnetization $m(x)$ is equal to

$$m(x) = h_a \left(\frac{\cosh \xi}{\cosh p} - 1 \right) . \quad (14)$$

The average thermodynamic potential contributed by the fields and currents in a single superconducting layer is

$$g_s \frac{H_c^2}{8\pi} = \frac{1}{2p} \int_{-p}^{p} d\xi \left[b(\xi)^2 + \left(\frac{db}{d\xi} \right)^2 - 2h(\xi)b(\xi) \right] \frac{H_c^2}{8\pi} . \quad (15)$$

Evaluating g_s we have

$$g_s = \bar{h}(\bar{h} - 2h_a) \frac{\tanh p - p \operatorname{sech}^2 p}{2p} - \bar{h}^2 \frac{\tanh p + p \operatorname{sech}^2 p}{2p} + (\Delta h)^2 \operatorname{cosech}^2 p . \quad (16)$$

The thermodynamic potential for the normal layer adjacent to this superconducting layer is simply

$$g_{n\pm} \frac{H_c^2}{8\pi} = \left[1 - (\bar{h} \pm \Delta h)^2 \right] \frac{H_c^2}{8\pi} . \quad (17)$$

If d_s is the thickness of the normal layer, then we defined the fraction $\alpha = d_s/(d_s + d_n)$. The free-energy contribution from the two surfaces of the superconducting layer is $\alpha(q/p)H_c^2/8\pi$. The total thermodynamic potential g for superconducting layer, plus either the right-hand or left-hand normal layer, and two interfaces is

$$\begin{aligned} g &= \alpha g_s + (1 - \alpha)g_{n\pm} + \alpha(q/p) \\ &= \alpha \Delta g_\pm + 1 - (h_a \pm \Delta h)^2 \end{aligned} \quad (18)$$

where

$$\Delta g_\pm = -\bar{h}h_a \frac{\tanh p - p \operatorname{sech}^2 p}{p} + (\bar{h} \tanh p \pm \Delta h \coth p)^2 + \frac{q}{p} - 1 . \quad (19)$$

"+" refers to the right-hand normal layer and "−" refers to the left-hand one. For g to be minimum with respect to α ($0 \leqslant \alpha \leqslant 1$), we must have $\alpha = 1$ when $\Delta g \pm, <0$. Note that $\alpha = 1$ up to the normal transition so that the normal layers have negligible thickness until then. When $\Delta g_\pm > 0$, $\alpha = 0$. The critical condition for the disappearance of the superconducting layer obtains when $\Delta g_+ = 0$, since the Δg_- involving the left-hand normal layer is lower by the amount $-4\bar{h}\Delta h$.

The equilibrium condition on the width p of the superconducting layer is obtained by setting the derivative of the thermodynamic potential, with respect to p, equal to zero. However, care must be taken to include only those pressures acting on the layer which tend to expand or contract the layer thickness and not that part of the $1/cr\, \mathbf{I} \times \mathbf{H}_a$ force which tends to move the whole layer bodily to one side. The equilibrium condition for p is then

$$(\partial/\partial p) \left[\tfrac{1}{2} (\Delta g_+ + \Delta g_-) \right] = 0 . \quad (20)$$

Together with the critical condition on g, we now can determine i_a and p for a given h_a. We do this in the approximation that $p \ll 1$. Equation (19) for $\Delta g_+ = 0$ becomes

$$\tfrac{1}{3} (h_a p)^2 + 2(h_a p)i^* + \frac{2h_a/h_2}{3h_a p} + i^{*2} - 1 = 0 . \quad (21)$$

We have used the approximations $q \simeq 2/3h_2$, $\bar{h} \simeq h_a + i^* \tanh x_0/\lambda$, and $\Delta h \simeq i^* p$. We have defined $i^* = i_a[\cosh (x_0/\lambda)/ \sinh (t/2\lambda)]$, where x_0 is the x position of the center of the superconducting layer, measured from the center of the specimen. Equation (20) becomes

$$\frac{2}{3p} \left[(h_a p)^2 - \frac{h_a/h_2}{h_a p} \right] = 0 \text{ or } p = \frac{1}{(h_2 h_a^2)^{\frac{1}{3}}} . \quad (22)$$

Note that the thickness of the superconducting layer, $2p\lambda$, as determined by Eq. (22), is independent of the applied current. The most unstable condition is obtained for layers near the surface of the specimen where the applied current density is highest, i.e., for $x_0 \simeq \frac{1}{2}t$. Henceforth, $i^* = i_a \coth t/2\lambda$. When this outer superconducting layer goes normal, then the next layer and the next must also go normal for this same applied current.

Equations (21) and (22) result in the critical condition:

$$i^* + (h_a/h_2)^{\frac{1}{3}} = 1 . \quad (23)$$

Equation (23) may be rewritten in terms of current densities as follows:

$$j_a + j_s(H_a) = j_c , \quad (24)$$

$$j_a \equiv (I/2d\lambda) \coth (t/2\lambda) , \quad (25)$$

$$j_s = (H_a c/4\pi\lambda) \tanh p \simeq (H_a/H_{c2})^{\frac{1}{3}} j_c , \quad (26)$$

$$j_c \equiv H_c c/4\pi\lambda , \quad (27)$$

where j_a is the maximum local applied current density, j_s is the maximum supercurrent density in a superconducting layer, and j_c is the critical current density discussed by Bardeen.[11]

For all the samples tested, $t/2\lambda \gg 1$ so that $\coth t/2\lambda \simeq 1$. Therefore, we can make a direct com-

[11] J. Bardeen, Rev. Mod. Phys. **34**, 667 (1962).

parison of the "I/I_c vs H_a/H_{c2}" experimental data with the following theoretical curve:

$$I/I_c + (H_a/H_{c2})^{\frac{1}{3}} = 1\,, \tag{28}$$

which is the solid curve in Fig. 2.

IV. THRESHOLD FOR THE MIXED STATE IN A BULK SPECIMEN

We now consider the limit for which the superconducting layers are thick (compared to λ) even though κ may be large. This situation arises just above the threshold for the mixed state. In the previous section we had $p \ll 1$. Now $p \gg 1$.

After omitting terms which are small by factors of $e^{-t/2\lambda}$, the thermodynamic potential difference will then reduce to

$$\Delta g_{\pm} \simeq h_a^2\left(1 - \frac{1}{p}\right) \pm h_a i_a\left(2 - \frac{1}{2p}\right) + i_a^2 + \frac{q}{p} - 1\,. \tag{29}$$

Minimizing g as before, we obtain

$$(h_a + i_a)^2 - 1 - (i_a h_a/2p) \leqslant 0 \tag{30}$$

and

$$-h_a^2 + q = 0\,. \tag{31}$$

From Eq. (31) we determine that the threshold field, regardless of the applied current, is still

$$h_a = h_1 = q^{\frac{1}{2}} \simeq (3/2h_2)^{\frac{1}{2}}\,, \tag{32}$$

as Goodman predicted for no applied current. We note that Eq. (30), with $p \rightarrow \infty$, is the Silsbee rule: $i_a = 1 - h_a$. This rule holds for fields below h_1. For fields above h_1, i_a lies above the Silsbee value because of the added term $i_a h_a/2p$. For $\kappa \gg 1$, the small-p limit is quickly reached as h_a increases. When the threshold behavior is thus limited to very low fields, h_1 being much less than h_2, the I vs H curve closely resembles Eq. (28) even in the low field region.

V. CONCLUSIONS

(1) The Gorkov–Shapoval expression for the upper critical field has been verified over a wide range of compositions and materials. (2) A thermodynamic analysis can predict the current-carrying capacity of a nonideal superconducting strip (with the applied field parallel to the long side of the strip cross section) from a single equation, involving only H_c and H_{c2}. (3) The applied current flows within a penetration depth of the surface. (4) There exists a critical total current density that is independent of the applied field and current. (5) According to the laminar model, the applied current does not affect the threshold field for the mixed state.

ACKNOWLEDGMENTS

We wish to express our gratitude to Dr. Z. J. J. Stekly for many helpful discussions and for a critical reading of the manuscript, to David Colling at Avco RAD, Wilmington, Massachusetts, for the preparation of many of the materials, to Ethan Hoag for many helpful comments and criticisms, and to the others at the Avco-Everett Superconductivity Laboratory.

Discussion 13

T. G. Berlincourt, *Atomics International:* I would like to point out that your values for the upper critical field and for resistivity are in relatively good accord with our own values for the Ti–V system. However, I think you are unjustified in trying to interpret these data in terms of Shapoval on the basis of measurements at only one temperature, 4.2°K, which turns out to be a temperature where the Gor'kov and Shapoval relations show relatively good accord. Our own data published on this system show rather good accord with Gor'kov. We did measure the upper critical field as a function of temperature. Another point. You assume rather than prove that the Lorentz force is stabilized in your calculation. We know also that you can change the current capacity drastically by cold-working the sample, so it is surprising that all your data seem to fit the same curve.

A. E. Bindari, *Avco-Everett Research Laboratory:* When you analyze the model you see that the Lorentz force has to cancel when the field is parallel to the current. If you look at our present case, the Lorentz force moves the body to one side if it is not pinned, but in any case does not affect the size of the lamina. This is not true when we reverse the direction of the field and the field becomes perpendicular to the surface; then the Lorentz force may affect the results.

J. P. McEvoy, *R.C.A. Laboratories:* Was the sample of NbZr annealed?

Bindari: No, all the samples were heavily cold-worked.

McEvoy: Can you estimate the degree of cold work?

Bindari: We rolled it from a $\frac{1}{4}$ in. down to 8 mils.