equations. In the quantized version of this formulation of the theory one would look for an operator representation for the $\bar{F}$'s which would reproduce the classical commutator algebra between the various $\bar{F}$'s obtained from their Pb's. Since now one has many more observables than degrees of freedom the observables are not all independent of one another and so one has certain consistency conditions to satisfy that are not present when one eliminates degrees of freedom from the theory directly. It is not clear at present whether or not one can satisfy these consistency requirements. If they can be satisfied then the BK procedure would have the advantage over the other schemes of quantization that it does not require a weak-field approximation procedure to obtain its results.

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Four-Dimensional Formulations of Newtonian Mechanics and Their Relation to the Special and the General Theory of Relativity

PETER HAVAS

Department of Physics, Lehigh University, Bethlehem, Pennsylvania

CONTENTS

I. INTRODUCTION AND SUMMARY

As is well known, the fundamental equations of Newtonian mechanics are invariant under the transformations of the Galilei group, those of the special theory of relativity[1] under the transformations of the Lorentz group, and those of the general theory of relativity under all[2] coordinate transformations ("principle of general covariance").[3] The special theory, originally developed in 1905 by Einstein in the usual three-dimensional[4] notation, was given a four-dimensional formulation by Minkowski in 1908[5]; the general theory was formulated in four dimensions from the start.[1,6]

The Minkowskian formulation of special relativistic mechanics leads to a law of motion unifying the laws determining the rate of change of momentum and of energy[7] and to a unification of the conservation laws.

[1] For recent reviews of the special and the general theory see the two articles by P. G. Bergmann in *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1962), Vol. IV.

[2] Subject to certain restrictions not relevant for the discussion of this section; these restrictions are considered in detail in Sec. V.1.

[3] In equating "principle of general covariance" with the requirement of invariance under all coordinate transformations we are following the original terminology of A. Einstein, Ann. Phys. **49**, 769 (1916). Some authors prefer a narrower interpretation of this principle, as discussed in Sec. VII, but the physical considerations involved should be kept apart from questions of terminology.

[4] We use, as customary, the terms "three-dimensional" and "four-dimensional" for brevity to distinguish between notations which treat space and time coordinates on a different or on the same footing, respectively. No *physical* distinction between the formulations is implied by these terms, however; it is one of the main purposes of this paper to clarify the relation between different formulations and the physical content of a theory.

[5] H. Minkowski, Nachr. Akad. Wiss. Göttingen, Math.-physik. Kl.53 (1908); Math. Ann. **68**, 472 (1910).

[6] For a discussion of some aspects of a three-dimensional formulation see A. Peres, Bull. Res. Council Israel **8F**, 179 (1960).

[7] J. L. Synge, in *Encyclopedia of Physics* (Springer-Verlag, Berlin, 1960), Vol. III/1.

General relativity, by combining the principle of general covariance and the principle of equivalence, arrives at a description of gravitation in terms of curved four-dimensional space–time.[1] In ascending from Newtonian mechanics to the special and general theories of relativity and thereby approaching a more perfect agreement with experiment, theoretical physics introduced two successive basic revisions of its fundamental concepts. However, Newtonian mechanics still represents a mathematical limit of the theories of relativity. Shouldn't it then be possible to formulate it directly in terms of concepts much closer to those of relativity than those of Newton or its 19th century refinements, these concepts having meaning within the framework of Newtonian mechanics itself rather than only as remnants of entirely different structures?

Of course there exist some obvious similarities between these theories. All three theories operate in a space–time of four coordinates; the Galilei group is a ten-parameter group of linear transformations just as is the Lorentz group[8]; some aspects of the principle of equivalence are already embodied in Newtonian theory. Furthermore, it was pointed out as early as 1917 by Kretschmann[9] and concurred in by Einstein[10] that the principle of general covariance is devoid of physical content, and any theory whatever can be formulated in a generally covariant form. Nevertheless, the impression left with the reader by most textbooks[11] of relativity and of point mechanics[12] is that there is indeed a fundamental difference between a necessarily three-dimensional Newtonian mechanics and the four-dimensional (Lorentz-invariant or generally covariant) theories of relativity, that the descent from relativity to Newtonian mechanics must be conceptually discontinuous. This is all the more surprising considering that the groundwork for the reformulation of Newtonian mechanics necessary to bring out its similarity to the theories of relativity was laid very soon after the creation of these theories; to some extent the close relation between Newtonian and special relativistic mechanics in a four-dimensional formulation was elaborated by Frank almost immediately after Minkowski's work,[13] and generally covariant formulations of Newton's theory of gravitation were given in the 1920's by Cartan[14] and Friedrichs.[15] This paper is written in the spirit of these early studies. Its purpose is to present formulations of Newtonian mechanics which, while obviously identical in their physical predictions with the usual one, are closely analogous to the four-dimensional formulations of the special and of the general theory of relativity. It is hoped that such a presentation will show better than the conventional approach how closely related the theories are and that Newtonian mechanics can indeed be formulated in such ways that it can be considered as a well-defined conceptual as well as mathematical limit of either the special or the general theory of relativity. It might thereby also help to dispel some of the "mystique" surrounding some of the concepts of relativity in many minds.

Section II contains a very brief summary of those properties of affine and metric spaces which are needed for the subsequent development. Section III consists of a review of the fundamental properties of the Lorentz and Galilei groups and their invariants, and a brief discussion of the significance of these invariants in terms of measurements. It also contains a fundamental lemma which allows us to establish a one-to-one correspondence between co- and contravariant vectors of a certain type in the affine space of the Galilei group. Part 1 of Sec. IV contains a four-dimensional formulation of Newton's Second law analogous to the Minkowskian formulation of Einstein's Second law. The remaining parts of that section are designed to show by selective applications (with particular attention given to the question of conservations laws) that all of Newtonian point mechanics can be re-expressed on the basis of this formulation in close analogy to the mechanics of special relativity, and to separate the formal and the physical aspects of various questions in these theories. Section V contains a generally covariant formulation of Newtonian and of special relativistic dynamics in a flat space–time. In Sec. VI (which is largely independent of Sec. IV) it is shown that Newton's theory of gravitation can be given a generally covariant formulation in a curved space–time; Newton's

[8] F. Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert* (Springer-Verlag, Berlin, 1927), Vol. 2, Chap. 2.

[9] E. Kretschmann, Ann. Physik **53**, 575 (1917).

[10] A. Einstein, Ann. Physik **55**, 241 (1918).

[11] Exceptions are brief discussions of the lack of physical content of the principle of general covariance (with or without mention of the Kretschmann–Einstein argument) in several books, such as L. Silberstein, *The Theory of General Relativity and Gravitation* (D. Van Nostrand Company, New York, 1922); R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, Oxford, England, 1934); V. Fock, *The Theory of Space Time and Gravitation* (English translation: Pergamon Press, Inc. New York, 1959).

[12] Some aspects of a four-dimensional development of point mechanics are contained in the use of homogeneous coordinates in Lagrangian dynamics (compare Ref. 7, part E II); however, this is usually treated as a purely formal device. On the other hand, much work has been done on the generally covariant formulation of the classical mechanics of continua following the early work of Cartan (Ref. 14), and a comprehensive review is included in the monumental article by C. Truesdell and R. Toupin in *Encyclopedia of Physics* (Springer-Verlag, Berlin, 1960), Vol. III/1. Although its general mathematical approach is closely related to the one taken here, the article does not discuss either point mechanics or gravitation, which are our main concern. It contains a detailed generally covariant formulation of Galilei-invariant results, but this is hidden through the use of the term "world-invariant" for "generally covariant" (i.e., admitting the group of all analytic transformations of four coordinates), a term so conducive to confusion because of its association with the Minkowski "world" in a physicist's mind that it is even "translated" as "Lorentz-invariant" in the German index of the volume!

[13] P. Frank, Sitzber. Kgl. Akad. Wiss. Wien, Math-naturw. Kl. **118**, Abt. IIa, 373 (1909).

[14] E. Cartan, Ann. École Norm. **40**, 325 (1923) and **41**, 1 (1924), reprinted in *Oeuvres Complètes* (Gauthier-Villars, Paris, 1955), Vol. III/1, pp. 659 and 799.

[15] K. Friedrichs, Math. Ann. **98**, 566 (1927).

second law now requires that particles move along geodesics of the four-dimensional space, whose affine connections satisfy field equations of the same form as Einstein's. Some implications of these results are discussed in Sec. VII.

While the presentation is such that the Newtonian and relativistic equations are generally displayed in parallel to stress their similarity, and the Newtonian formulas are the mathematical limit of the corresponding relativistic ones, the formulations of Newtonian mechanics given here are in principle independent of the theory of relativity and could have been developed without it.

II. AFFINE AND METRIC SPACES

Throughout this paper we are concerned with affine spaces. Their fundamental concepts and mathematical properties will be assumed known,[16] and we only present a brief summary. We consider an n-dimensional continuum, whose points are labeled by coordinates x^ρ. In the following, Greek indices always run from 0 to $n-1$ and Latin indices from 1 to $n-1$; summation over repeated indices is understood. We now consider coordinate transformations

$$x'^\mu = x'^\mu(x^\rho). \tag{1}$$

A contravariant vector A^ρ is defined as a quantity with n components which transform like the dx^ρ, i.e.,

$$A'^\mu = (\partial x'^\mu/\partial x^\rho)A^\rho. \tag{2}$$

Thus the coordinate differentials themselves form a contravariant vector, but for the finite coordinate differences Δx^ρ this is only the case if the transformation (1) is linear, and for the coordinates x^ρ themselves only if it is also homogeneous.[17] A quantity B_ρ is called a covariant vector if its components transform as

$$B'_\mu = (\partial x^\rho/\partial x'^\mu)B_\rho. \tag{3}$$

In general one can *not* unambiguously associate a covariant and a contravariant vector with each other.

We can define covariant, contravariant, and mixed tensors of any rank by similar expressions, i.e.,

$$T'^{\alpha\beta\cdots}{}_{\mu\nu\cdots} = \frac{\partial x'^\alpha}{\partial x^\kappa}\frac{\partial x'^\beta}{\partial x^\lambda}\cdots\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\cdots T^{\kappa\lambda\cdots}{}_{\rho\sigma\cdots}, \tag{4}$$

and tensor densities of weight w by a transformation law which in addition to the factors included in (4) contains a factor D^{-w}, where

$$D \equiv |\partial x'^\rho/\partial x^\mu| \tag{5}$$

is the Jacobian of the transformation. A vector is a tensor of rank 1, a scalar (invariant) one of rank 0.

We can also define addition and subtraction for tensors of the same rank and multiplication and contraction for tensors of any rank. To define differentiation maintaining the tensor character of any expression, we have to introduce an *affine connection* $\Gamma^\sigma{}_{\kappa\lambda}$ with the transformation law

$$\Gamma'^\rho{}_{\mu\nu} = \frac{\partial x'^\rho}{\partial x^\sigma}\frac{\partial x^\kappa}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\nu}\Gamma^\sigma{}_{\kappa\lambda} + \frac{\partial x'^\rho}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu}; \tag{6}$$

thus $\Gamma^\sigma{}_{\kappa\lambda}$ does not transform as a tensor except under *linear* transformations. Then we can define covariant derivatives by

$$\begin{aligned} T^{\kappa\lambda\cdots}{}_{\rho\sigma\cdots;\nu} &= \partial_\nu T^{\kappa\lambda\cdots}{}_{\rho\sigma\cdots} + \Gamma^\kappa{}_{\alpha\nu}T^{\alpha\lambda\cdots}{}_{\rho\sigma\cdots} + \Gamma^\lambda{}_{\alpha\nu}T^{\kappa\alpha\cdots}{}_{\rho\sigma\cdots} \\ &\quad + \cdots - \Gamma^\alpha{}_{\rho\nu}T^{\kappa\lambda\cdots}{}_{\alpha\sigma\cdots} - \Gamma^\alpha{}_{\sigma\nu}T^{\kappa\lambda\cdots}{}_{\rho\alpha\cdots} - \cdots, \\ \partial_\nu &\equiv \partial/\partial x^\nu. \end{aligned} \tag{7}$$

The affine connection can be arbitraily assigned in one coordinate system; it determines the meaning of parallel displacement in the space considered. In the following we are only concerned with symmetric connections $\Gamma^\sigma{}_{\kappa\lambda} = \Gamma^\sigma{}_{\lambda\kappa}$. It is clear from Eq. (6) that if we have two different affine connections, their difference transforms like a tensor.

If we form the second covariant derivatives by successive applications of Eq. (7), then in general the differentiations with respect to different coordinates do not commute. This property of the affine space is called curvature and is described by the Riemann–Christoffel curvature tensor $R^\kappa{}_{\mu\lambda\nu}$ and the contracted curvature tensor $R_{\mu\nu}$, defined as

$$\begin{aligned} R^\kappa{}_{\mu\lambda\nu} &\equiv \partial_\lambda \Gamma^\kappa{}_{\mu\nu} - \partial_\nu \Gamma^\kappa{}_{\mu\lambda} + \Gamma^\kappa{}_{\rho\lambda}\Gamma^\rho{}_{\mu\nu} - \Gamma^\kappa{}_{\rho\nu}\Gamma^\rho{}_{\mu\lambda}, \\ R_{\mu\nu} &\equiv R^\lambda{}_{\mu\lambda\nu}. \end{aligned} \tag{8}$$

$R_{\mu\nu}$ is symmetric provided

$$R^\mu{}_{\mu\lambda\nu} = 0. \tag{9}$$

A geodesic in affine space is defined as the straightest line. As a function of a parameter τ which is defined up to a linear transformation it satisfies the equation

$$\frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{10}$$

In general length can only be defined along a geodesic, and the lengths along different geodesics can not be compared. Such a comparison becomes possible if the space admits a symmetric tensor of rank two $g_{\mu\nu}$ with vanishing covariant derivatives, the metric tensor.[16] Then the distance ds is determined by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{11}$$

which by suitable choice of the arbitrary factor in τ agrees with the distance along geodesics.

[16] For a brief introduction well suited for our purposes see E. Schrödinger, *Space-Time Structure* (Cambridge University Press, Cambridge, England, 1950).

[17] This introduces a slight notational inconsistency which could be avoided only at great inconvenience. A similar standard inconsistency is the notation for the $\Gamma^\sigma{}_{\kappa\lambda}$ introduced below, which transform according to Eq. (6).

If the determinant of the $g_{\mu\nu}$ vanishes, the metric is called singular; if it does not vanish anywhere, the space is called metric or Riemannian.[18] In such a space the geodesics are the shortest (or at least extremal) as well as the straightest lines; the equations $g_{\mu\nu;\rho}=0$ can be solved for the affine connections which are given by

$$\Gamma^{\rho}{}_{\mu\nu}=\left\{{\rho \atop \mu\nu}\right\},$$
$$\left\{{\rho \atop \mu\nu}\right\}\equiv\tfrac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma}+\partial_\nu g_{\sigma\mu}-\partial_\sigma g_{\mu\nu}), \qquad (12)$$

where $\left\{{\rho \atop \mu\nu}\right\}$ is called the Christoffel three-index symbol and $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, defined by

$$g_{\mu\rho}g^{\rho\nu}=\delta^\nu{}_\mu, \qquad (13)$$

where $\delta^\nu{}_\mu$ is the Kronecker δ (which transforms as a mixed tensor). With the help of these two tensors we now can associate with each contravariant vector A^ρ a covariant vector $A_\mu=g_{\mu\rho}A^\rho$, and conversely $A^\rho=g^{\rho\mu}A_\mu$.

Thus *mathematically* we can always associate a metric space with a given affine space, provided certain conditions are satisfied[16]; in particular, this is always possible if the space is flat. Whether this is a *physically* meaningful association is discussed later.

Any symmetric tensor of rank two such as $g_{\mu\nu}$ which can be associated with a quadratic form can be characterized by its signature, i.e. by the difference between the number of positive and negative terms after the quadratic form has been diagonalized. This number is an invariant by Sylvester's "Law of Inertia of Quadratic Forms."[19]

III. FUNDAMENTAL PROPERTIES OF THE GALILEI AND LORENTZ GROUPS

1. Definition and Structure of the Groups

The Special Theory of Relativity is based on Einstein's principle of relativity (I) and his principle of the constancy of the velocity of light (II). They are; (I) If properly formulated, the laws of physics are of the same form in all inertial systems; (II) There exists a maximum signal velocity in nature, the velocity of light in empty space, which has the same value c in all inertial systems.

In the above, inertial systems are defined as those systems in which Newton's first law is valid. The existence of a maximum signal velocity leads to the results that the concept of simultaneity within a single inertial system involves an element of definition [the particular definition implied in (II) being that clocks are synchronized by means of light signals, whose velocity is taken to be independent of direction], and this combined with principle (I) implies the nonexistence of an absolute time. The two principles allow the determination of the group of transformations relating the space and time coordinates in different inertial systems. We consider instead first the wider group of linear transformations of the Cartesian space coordinates x^1, x^2, x^3 and the time $t=x^0$

$$x'^\mu=\alpha^\mu{}_\rho x^\rho+\xi^\mu. \qquad (14)$$

We have

$$\partial x'^\mu/\partial x^\rho=\alpha^\mu{}_\rho,$$
$$\partial x^\sigma/\partial x'^\nu=(\alpha^{-1})^\sigma{}_\nu,$$
$$\alpha^\mu{}_\rho(\alpha^{-1})^\rho{}_\nu=\delta_\nu{}^\mu. \qquad (15)$$

The full inhomogeneous Lorentz group (also called Poincaré group) is the group of transformations (14), restricted by the condition[20]

$$\eta_{\mu\nu}\alpha^\mu{}_\rho\alpha^\nu{}_\sigma=\eta_{\rho\sigma}, \qquad (16a)$$

where $\eta_{\mu\nu}$ is a nonsingular symmetric tensor with signature -2. From Eqs. (15) and (16a) we obtain

$$\eta_{\mu\nu}=(\alpha^{-1})^\rho{}_\mu(\alpha^{-1})^\sigma{}_\nu\eta_{\rho\sigma},$$

and by comparison with Eq. (4) it follows that $\eta_{\mu\nu}$ has numerically the same components in all allowed coordinate systems. We can define its inverse by

$$\eta_{\mu\rho}\eta^{\rho\nu}=\delta^\nu{}_\mu \qquad (17)$$

and shall take the nonvanishing components of these tensors to be

$$\eta_{00}=1, \qquad \eta_{11}=\eta_{22}=\eta_{33}=-c^{-2}, \qquad (18a)$$
$$\eta^{00}=1, \qquad \eta^{11}=\eta^{22}=\eta^{33}=-c^2. \qquad (18b)$$

It follows immediately from Eqs. (18) that

$$\eta^{\mu\nu}\alpha^\rho{}_\mu\alpha^\sigma{}_\nu=\eta^{\rho\sigma}, \qquad (16b)$$

and from Eq. (16) that the determinant D defined by Eq. (5) equals ±1. As only six of the $\alpha^\mu{}_\rho$ are independent, the group contains ten independent parameters.

Newtonian mechanics is in agreement with principle (I). However, it assumes the possibility of infinite signal velocities and of an absolute time. The full inhomogeneous Galilei group is defined as the group of transformations (14), restricted by the conditions[8]

$$\alpha^0{}_0=\pm1, \qquad \alpha^0{}_r=0, \qquad \alpha^m{}_r\alpha^n{}_r=\delta_m{}^n; \qquad (19)$$

the last one of these implies $\alpha^m{}_r\alpha^m{}_s=\delta_s{}^r$ by the well-known properties of orthogonal transformations. These conditions are nontensorial; however, they are equiva-

[18] Some properties of spaces with singular metric tensor are discussed by G. Vranceanu, *Leçons de Géometrie Differentielle* (Bucarest, 1947; reprinted by Gauthier-Villars, Paris), Vol. I; this book includes such spaces among Riemannian ones, contrary to the customary terminology.

[19] Compare M. Bôcher, *Introduction to Higher Algebra* (The Macmillan Company, New York, 1907), Sec. 50. (The term "rank" is used here with a different meaning.)

[20] See, e.g., E. M. Corson, *Introduction to Tensors, Spinors, and Relativistic Wave Equations* (Blackie & Son Limited, London, 1953), Chap. I.

lent to the tensor equations

$$g_{\mu\nu}\alpha^{\mu}{}_{\rho}\alpha^{\nu}{}_{\sigma}=g_{\rho\sigma}, \tag{20a}$$

$$h^{\mu\nu}\alpha^{\rho}{}_{\mu}\alpha^{\sigma}{}_{\nu}=h^{\rho\sigma}, \tag{20b}$$

with a suitable choice of $g_{\mu\nu}$ and $h^{\mu\nu}$. From Eqs. (15) and (20a) we have

$$g_{\mu\nu}=(\alpha^{-1})^{\rho}{}_{\mu}(\alpha^{-1})^{\sigma}{}_{\nu}g_{\rho\sigma}$$

and thus $g_{\mu\nu}$ has numerically the same components in all coordinate systems; for $h^{\mu\nu}$ this property follows directly from Eq. (20b). We choose for the nonvanishing components

$$g_{00}=1, \tag{21a}$$

$$h^{11}=h^{22}=h^{33}=-1. \tag{21b}$$

Thus these tensors are singular and satisfy the equation

$$g_{\mu\rho}h^{\rho\nu}=0. \tag{22}$$

Because of this the relations (20a) and (20b) had to be postulated independently, whereas the corresponding relations (16a) and (16b) imply each other.

Equation (20a) with (21a) implies $(\alpha^0{}_0)^2=1$ and $\alpha^0{}_r=0$. Equation (20b) with (21b) implies $\alpha^r{}_m\alpha^s{}_m=\delta^r{}_s$. Thus Eqs. (20) and (21) are equivalent to Eq. (19). As they again restrict $\alpha^{\mu}{}_{\rho}$ to six independent components, the Galilei group too is a ten-parameter group. The conditions also imply again that D equals ± 1.

Thus in complete analogy to the structure of the Lorentz group[20] the Galilei group consists of four parts, corresponding to the four combinations of the signs of D and $\alpha^0{}_0$. The part with $D=\alpha^0{}_0=+1$ forms a subgroup, the proper orthochronous Galilei group, as does the proper orthochronous Lorentz group defined by $D=$ sign $\alpha^0{}_0=+1$. The physical requirement of invariance under these subgroups is more basic than that under the full groups, because it is these subgroups which correspond to uniform relative motion of frames of reference without reflections and time reversal, i.e., which express the equivalence of all inertial systems. The transformations with sign $\alpha^0{}_0=-1$ are called antichronous.

If we restrict ourselves to orthochronous Galilei transformations, the quantity

$$w_{\mu}=(1,0,0,0) \tag{23}$$

transforms like a covariant vector, with numerically the same components in all coordinate systems; under antichronous transformations it changes sign. Clearly we have

$$h^{\mu\nu}w_{\nu}=0, \tag{24a}$$

$$g_{\mu\nu}=w_{\mu}w_{\nu}. \tag{24b}$$

If we introduce a tensor

$$H^{\mu\nu}=c^{-2}\eta^{\mu\nu}, \tag{25}$$

Eq. (17) implies

$$\eta_{\mu\rho}H^{\rho\nu}=c^{-2}\delta^{\nu}{}_{\mu}, \tag{26}$$

a relation which degenerates as $c\rightarrow\infty$, $H^{\mu\nu}$ and $\eta_{\mu\nu}$ becoming independent and equal to the tensors $h^{\mu\nu}$ and $g_{\mu\nu}$ introduced by Eq. (21). Equation (26) then reduces to Eq. (22), and the Lorentz transformations to the Galilei transformations.

The structure of the two groups (14) with (16) or (20) determines all of the space–time structure and much of the physical content of the theories required to be invariant under these groups. However, it does not determine the physical content entirely. There may be several specific laws which satisfy the invariance requirements, and the decision between them must be based on experiment. The familiar forms of Newtonian and special relativistic mechanics are *specific* theories satisfying the *more general* invariance requirements of the two groups. In the next section we discuss these two mechanics and indicate some possibilities of generalization. In this section we restrict ourselves to general considerations following from the group structures alone.

2. Intervals

One possibility of associating an affine space with the coordinates x^{ρ} used in the Lorentz or Galilei transformations (14) is to choose all Γ's equal to zero in one coordinate system; from Eq. (6) and the linearity of (14) they then vanish in all allowed coordinate systems, and thus this choice does not disturb the physical equivalence of all inertial systems. This space is *flat*, i.e., its curvature tensor $R^{\kappa}{}_{\mu\lambda\nu}$ vanishes everywhere.

In the space–time of the Lorentz transformations, we can then take the tensor $\eta_{\mu\nu}$ as the metric tensor, since the connections (12) formed with this tensor all vanish. The four-dimensional distance is then determined by

$$ds^2=\eta_{\mu\nu}\,dx^{\mu}\,dx^{\nu}, \tag{27}$$

which is of the same form in all coordinate systems, $\eta_{\mu\nu}$ having the same components. The space associated with (27) is Riemannian, since $\eta_{\mu\nu}$ is nonsingular.

We could also try to introduce a metric tensor in the four-space of the Galilei group. The tensor $g_{\mu\nu}$ introduced by Eq. (21a) is singular, and thus the space associated with the four-dimensional distance

$$ds^2=g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} \tag{28}$$

is not Riemannian. Other tensors can be introduced which are nonsingular,[21] but their components do not have the same numerical values in all coordinate

[21] The requirement of vanishing of the Γ's in all coordinate systems is satisfied by choosing any symmetric tensor whatever whose components are constant in one coordinate system because of Eq. (12) and the linearity of the Galilei transformation.

systems.[22,23] This is mathematically permissible. But such a "length" is not the same numerical combination of the coordinate differentials in all inertial systems; we say that it is not form-invariant. It takes a particularly simple form in one system Σ in terms of the dx^μ, and while it has the same numerical value in all other systems, this value can only be determined if the transformation from Σ to the other systems is known. Physically the "length" then is not measured by the same procedure in all frames of reference, and a preferred inertial system is introduced artificially.

The four-dimensional expression (28) actually refers to a pure time interval because of the form (21a) of $g_{\mu\nu}$, and assigns a separation zero to any two simultaneous events. A nontrivial characterization of the interval of such events (whose four-dimensional form will be given in Subsection 4) is provided by the invariant three-dimensional length

$$dl^2 = dx^m\, dx^m, \qquad dx^0 = 0. \tag{29}$$

The length and time intervals defined by (29) and (28) are those associated with the usual length and time measurements in Newtonian physics. For later convenience we have written these two intervals only in terms of infinitesimal separations, although similar expressions can be defined for finite separations.

Instead of postulating Eq. (16) and deducing the form invariance of (27), we could have taken the form invariance of (27) as our basic postulate, and deduced Eq. (16) and thus the Lorentz transformations from it.

Similarly it can be shown that we can regain Eqs. (20) from a requirement of form invariance of Eqs. (28) and (29) under linear transformations. More generally, if we stipulate this form invariance for Eqs. (28) *only* (equivalent to the requirement of the validity of Euclidean geometry for instantaneous measurements within any inertial system), it can be shown that this leads to a group of transformations (14) characterized by a constant of the form $\pm c^{-2}$, where c is a velocity.[24] The choice $c=\infty$ [equivalent to Eq. (29)] leads to the Galilei group; a finite c with one choice of sign gives the Lorentz group, while the other sign gives a group which has not found any application in physics.[24]

Dropping the requirement of form invariance, we could generalize the allowed coordinate transformations by treating the quantities $h^{\mu\nu}$, $g_{\mu\nu}$ and $\eta_{\mu\nu}$ as tensors under these transformations, and require only that they reduce to the particular values (21) and (18) for inertial systems and Cartesian coordinates. However, the analogies and distinctions between Newtonian and special relativistic mechanics are already fully apparent with the linear transformations (14) and nothing is added to these features by considering nonlinear transformations. Thus we discuss this generalization only briefly in Sec. V, principally as an introduction to another generally covariant reformulation of Newtonian mechanics given in Sec. VI. 3, which is compared with Einstein's general theory of relativity.

3. World Lines

We now consider the world line of a particle whose four coordinates we denote by z^ρ and which can be expressed as functions of an arbitrary parameter. If they transform under the Lorentz group, it is convenient to take as this parameter the proper time τ defined by

$$d\tau^2 = \eta_{\mu\nu} dz^\mu\, dz^\nu. \tag{30}$$

We can define the four-velocity v^ρ and four-acceleration a^ρ by

$$v^\rho = dz^\rho/d\tau, \qquad a^\rho = d^2 z^\rho/d\tau^2, \tag{31}$$

and the equivalent covariant vectors $v_\mu = \eta_{\mu\rho} v^\rho$ and $a_\mu = \eta_{\mu\rho} a^\rho$. From the definitions (30) and (31) it follows that

$$v^\rho v_\rho = 1, \qquad v^\rho a_\rho = v_\rho a^\rho = 0. \tag{32}$$

In the case of the Galilei group we similarly take the z^ρ as functions of a parameter τ defined by

$$d\tau^2 = g_{\mu\nu}\, dz^\mu\, dz^\nu. \tag{33}$$

Clearly τ differs from $z^0 = \pm t$ at most by an additive constant. However, we prefer to distinguish the two quantities notationally. First, this serves to emphasize that τ is a parameter on the world line rather than a coordinate (and thus could have been chosen differently from t), second, this renders the formulas of Newtonian and special relativistic mechanics much more symmetric and third, this distinction is needed in considering general coordinate transformations in Secs. V and VI. The factors in the definitions are chosen so that Eq. (30) reduces to Eq. (33) in the limit $c\rightarrow\infty$. We can thus define a Newtonian four-velocity and four-acceleration by the same Eqs. (31) as before. The spatial parts of these contravariant vectors are the usual three-velocity $\mathbf{v}$ (or $-\mathbf{v}$ for antichronous transformations) and acceleration $\mathbf{a}$. In the absence of a nonsingular metric we can not automatically define any corresponding covariant vectors. However, the vector w_μ defined in Eq. (23) can be written

$$w_\mu = g_{\mu\rho} v^\rho \tag{34}$$

and satisfies the relations

$$v^\rho w_\rho = 1, \qquad w_\rho a^\rho = 0 \tag{35}$$

[22] One such possibility was suggested by E. Cartan, Bull. Math. Soc. Roum. Sci. **35**, 69 (1933), reprinted in *Oeuvres Complètes* (Gauthier-Villars, Paris, 1955), Vol. III/2, p. 1239.

[23] The numerically constant tensors of the Galilei group and the invariants which can be constructed with their help are discussed in a paper in preparation.

[24] A detailed exposition of this (with some references to the literature) has been given recently by H. M. Schwartz, Am. J. Phys. **30**, 697 (1962) for the case of the proper orthochronous homogeneous groups; the extension to the full groups requires only minor modifications. An essentially equivalent axiomatization was given by E. Hahn, Arch. Math. Phys. **21**, 1 (1913).

analogous to (32); while independent of v^ρ, it is numerically equal to the limit $c\to\infty$ of the special relativistic v_μ and formally plays the same role. A method which can be used to define a covariant counterpart to a^ρ is developed in the following subsection.

Now we consider *two* events z_1^ρ and z_2^ρ which may lie on the same or on different world lines, and the corresponding four-velocities v_1^ρ and v_2^ρ. In the case of the Lorentz group, we can define in addition to the four-dimensional distance ds^2 (27) the following independent invariants out of these four quantities:

$$\kappa_1=(z_2^\rho-z_1^\rho)v_{1\rho},\qquad \kappa_2=(z_1^\rho-z_2^\rho)v_{2\rho},\qquad \omega=v_1^\rho v_{2\rho}. \quad (36)$$

In the case of the Galilei group we also have three independent invariants[23] in addition to the "distance" (28), which are more conveniently written in three-dimensional notation as

$$\Delta\mathbf{v}^2=|\mathbf{v}_2-\mathbf{v}_1|^2,$$
$$\mathbf{s}^2=|\mathbf{z}_2-\mathbf{z}_1-\tfrac{1}{2}(\mathbf{v}_1+\mathbf{v}_2)(t_2-t_1)|^2,$$
$$K=[\mathbf{z}_2-\mathbf{z}_1-\tfrac{1}{2}(\mathbf{v}_1+\mathbf{v}_2)(t_2-t_1)]\cdot(\mathbf{v}_2-\mathbf{v}_1). \quad (37)$$

If the events are simultaneous, the second of these reduces to the form (29) for a finite separation.

4. A Lemma

We now consider a three-vector $\mathbf{B}$ (not necessarily associated with a world line), defined as a quantity whose components transform as[25]

$$B'^m=\alpha^m{}_rB^r. \quad (38)$$

It is obvious from (19), and also trivial, that we can associate a contravariant four-vector B^ρ with $\mathbf{B}$ by adjoining a 0-component which is zero; this vector can be characterized invariantly by

$$B^\rho w_\rho=0. \quad (39)$$

Equation (39) implies (38).

It is fundamental for our later considerations that we can also associate a covariant four-vector B_ρ with $\mathbf{B}$ or B^ρ. We prove the following

Lemma: Given a contravariant four-vector B^ρ subject to condition (39) and a three-velocity $\mathbf{v}$ (with $v^m=dz^m/dz^0$), the quantity

$$B_\rho=(\mathbf{B}\cdot\mathbf{v},\ -\mathbf{B}) \quad (40)$$

satisfying the condition

$$B_\rho v^\rho=0 \quad (41)$$

[with v^ρ defined in Eq. (31)] transforms as a covariant vector under the Galilei group. B^ρ is related to B_ρ by

$$B^\rho=h^{\rho\sigma}B_\sigma,\qquad B_\rho=k_{\rho\sigma}B^\sigma, \quad (42)$$

where the tensor $h^{\rho\sigma}$ is defined in Eq. (21b) and the nonvanishing components of $k_{\rho\sigma}$ are

$$k_{00}=-|\mathbf{v}|^2,\qquad k_{0r}=k_{r0}=v^r,\qquad k_{11}=k_{22}=k_{33}=-1. \quad (43)$$

Any vectors related by Eqs. (42) automatically satisfy (39) and (41).

From the transformation law (38) for $\mathbf{B}$ and the one for $\mathbf{v}$ following from Eq. (14) the transformed components of B_ρ are

$$\alpha^m{}_rB^r(\alpha^m{}_0+\alpha^m{}_nv^n)\alpha^0{}_0,\ -\alpha^m{}_rB^r. \quad (44)$$

On the other hand, a covariant four-vector must transform according to Eq. (3). The transformation coefficients $\partial x^\rho/\partial x'^\mu$ follow from Eqs. (14) and (19). They are

$$\partial x^0/\partial x'^0=\alpha^0{}_0,\qquad \partial x^0/\partial x'^m=0,$$
$$\partial x^r/\partial x'^0=-\alpha^0{}_0\alpha^k{}_r\alpha^k{}_0,\qquad \partial x^r/\partial x'^m=\alpha^m{}_r. \quad (45)$$

Therefore

$$B'_\rho=(\alpha^0{}_0B^kv^k+\alpha^0{}_0\alpha^k{}_r\alpha^k{}_0B^r,\ -\alpha^m{}_rB^r),$$

which agrees with (44).

The vanishing of $B_\rho v^\rho$ follows from the definitions of the two vectors. Conversely, condition (41) and the definition of v^ρ imply the form (40) of B_ρ. Equation (42) is also readily verified from the definitions. The fact that $k_{\rho\sigma}$ transforms as a tensor can also be verified directly, using the same method as for B_ρ. Furthermore, $k_{\rho\sigma}$ satisfies $v^\rho k_{\rho\sigma}=0$ from its definition and $h^{\rho\sigma}$ satisfies Eq. (24a), which proves the last part of the Lemma.

The Lemma is important because it allows us to establish a one-to-one correspondence between co- and contravariant vectors of a certain type, a correspondence which in general is only possible in a metric space.

The transformation law (38) is satisfied for the coordinate differentials (or differences) of two events which are simultaneous, a condition which can be expressed four dimensionally in the equivalent forms

$$ds^2=g_{\mu\nu}\,dx^\mu\,dx^\nu=0\quad\text{or}\quad w_\rho\,dx^\rho=0. \quad (46)$$

Thus by virture of the Lemma we can associate a covariant vector $dx_\rho=k_{\rho\sigma}\,dx^\sigma$ with these events. This allows us to write the three-dimensional length (29) in a four-dimensional form as

$$dl^2=-h^{\mu\nu}k_{\mu\rho}k_{\nu\sigma}\,dx^\rho\,dx^\sigma;$$

however, from definitions (21b) and (43) we have $h^{\mu\nu}k_{\mu\rho}k_{\nu\sigma}=k_{\rho\sigma}$, and thus

$$dl^2=-k_{\rho\sigma}\,dx^\rho\,dx^\sigma, \quad (47)$$

subject to condition (46).

The acceleration $\mathbf{a}$ and the usual (velocity-independent) Newtonian force $\mathbf{F}$ also transform according

[25] This is the usual law for orthogonal coordinate transformations in a three-space. However, note that it is *not* the law of transformation of the space coordinates under the Galilei group, and that the transformation law for $\mathbf{v}$ differs from either.

to Eq. (38) and thus we can define covariant four-vectors

$$a_\rho = (\mathbf{a}\cdot\mathbf{v}, -\mathbf{a}), \qquad F_\rho = (\mathbf{F}\cdot\mathbf{v}, -\mathbf{F}) \tag{48}$$

satisfying condition (41). However, no covariant four-vectors can be constructed from either the position vector $\mathbf{z}$ or the velocity $\mathbf{v}$. Thus a_ρ is not the derivative of a four-vector.

IV. SPECIAL RELATIVITY AND NEWTONIAN MECHANICS

1. The Second Law of Motion

The three-dimensional forms of Newton's and Einstein's second law of motion for a single body are[26]

$$m_0 d\mathbf{v}/dt = \mathbf{F}, \tag{49N}$$

$$m_0(d/dt)[\mathbf{v}/(1-v^2/c^2)^{\frac{1}{2}}] = \mathbf{F}, \tag{49S}$$

respectively, or alternatively

$$d\mathbf{p}/dt = \mathbf{F}, \qquad \mathbf{p} = m\mathbf{v}, \tag{50}$$

with

$$m = m_0, \tag{51N}$$

$$m = m_0/(1-v^2/c^2)^{\frac{1}{2}}. \tag{51S}$$

The relations between rate of change of energy and rate of doing work can be deduced from these equations in a well-known manner, and we obtain

$$d\bar{E}/dt = \mathbf{F}\cdot\mathbf{v}, \tag{52}$$

with

$$\bar{E} = \tfrac{1}{2}mv^2 + \text{const}, \tag{53N}$$

$$\bar{E} = mc^2 + \text{const}. \tag{53S}$$

The constant of integration is of course not determined by the differential equations; its choice is mathematically completely arbitrary, and physically irrelevant in the case under consideration of a single body. While many authors take the constant in (53S) to be zero in a desire to obtain a relation between mass and energy, it can not be emphasized too strongly that the equivalence of mass and energy can not be *derived* from the law of motion (49S) at all, and is a physically empty statement as long as only a single body is being considered.[27] We return to this question in Sec. IV.5.

It is formally convenient, however, to choose the constants equal to zero in Eq. (53N) and (53S). Following Minkowski,[5] we can then combine the four equations (49S) and $(52)/c^2$ with (53) in a single equation relating two contravariant vectors in a four-dimensional flat space:

$$m_0 a^\rho = dP^\rho/d\tau = F^\rho,$$
$$a^\rho = d^2z^\rho/d\tau^2, \qquad P^\rho = m_0 v^\rho, \tag{54}$$

where

$$a^\rho v_\rho = 0, \qquad F^\rho = \left[\frac{\mathbf{F}\cdot\mathbf{v}}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}, \mathbf{F}\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right] \tag{55S}$$

[with τ defined by Eq. (30)]. It has also been noted and commented upon frequently that we can (trivially) write a corresponding equation (54) for Newtonian mechanics [with τ defined by Eq. (33)], with

$$a^\rho w_\rho = 0, \qquad F^\rho = (0, \mathbf{F}), \tag{55N}$$

which thus contains only Eq. (49N), but *not* the energy equation. However, (except for the neglected early work of Frank[13]) it seems to have escaped notice that we can write a relation between *covariant* vectors[28]

$$m_0 a_\rho = F_\rho, \qquad a_\rho v^\rho = 0, \tag{56}$$

which contains the energy equation (52) as well as the law of motion (49) (the latter with changed signs), in *both* the Newtonian and the special relativistic case. Equations (54) and (56) are obviously equivalent in the special relativistic case, and equivalent by virtue of the Lemma of Sec. III.4 in the Newtonian case; for the latter case, the explicit form of the quantities appearing in Eq. (56) is given in Eq. (48).

The difference between the two cases in Eq. (56) is that while in special relativity we have $a_\rho = d^2(\eta_{\rho\sigma}z^\sigma)/d\tau^2$, the Newtonian a_ρ can not be written as a second derivative; furthermore, while we can write this equation as

$$dP_\rho/d\tau = F_\rho \tag{57}$$

in both cases, the transformation properties of P_ρ depend on the choice of the constants in Eqs. (53N) and (53S).

If we choose them as appropriate for a *kinetic* energy T which should vanish for $\mathbf{v}=0$, we have[28]

$$T = \tfrac{1}{2}mv^2, \qquad P_\rho = (T, -\mathbf{p}), \tag{58N}$$

$$T = (m-m_0)c^2, \qquad P_\rho = (T, -\mathbf{p})/c^2, \tag{58S}$$

and P_ρ does *not* transform as a covariant vector in either case (and is thus also independent of the previously defined P^ρ); it is only dP_ρ or a finite ΔP_ρ which

[26] In this section we denote corresponding formulas of Newtonian and special relativistic mechanics by N and S, respectively, and (except as noted otherwise) all other formulas are valid for both theories. The subscript 0 in m_0 is not necessary in the Newtonian case, but is introduced to allow writing a single formula for both theories in many cases.

[27] For a discussion of this problem see, e.g., C. Møller, *The Theory of Relativity* (Oxford University Press, Oxford, England, 1952), Sec. 30.

[28] Because of the definition of a covariant vector B_ρ associated with a given contravariant vector B^σ by Eq. (40) in the Newtonian case and by $\eta_{\rho\sigma}B^\sigma$ with the values (18a) for the components of the metric in the relativistic case, the Newtonian and the relativistic a_ρ differ by a factor c^2, as do the F_ρ, and the P_ρ introduced below; although this could have been avoided by a different choice of the metric in Eq. (18), this would only have shifted the appearance of the factor c^2 to some other equations. To compensate this difference and thereby maintain the symmetry of the equations containing the potential U introduced in the next subsection, it is understood in the following that in the relativistic equations U is to correspond to $1/c^2$ times its Newtonian counterpart.

has the proper transformation property. If, on the other hand, we choose the constant in (53S) to be zero, we have

$$dP_\rho/d\tau = F_\rho, \qquad P_\rho = (E, -\mathbf{p})/c^2, \qquad E = mc^2, \quad (59)$$

and P_ρ now is the covariant four-vector corresponding to P^ρ in Eq. (54). No choice of the constant in Eq. (53N) allows us to define a true covariant four-vector P_ρ in Newtonian mechanics.

The different choices of the constants in Eqs. (53N) and (53S) are physically irrelevant and Eqs. (52)–(59) do not contain any more than Eqs. (49) or (50). It is only through new assumptions in the case of several bodies that the choice of constants acquire physical significance, as is discussed in Sec. IV.5.

The considerations of this subsection are valid whether the system under consideration is closed or open, i.e., whether the source of the force is considered to be part of the system or not. No further statements can be made for the mechanics of a single body without further specification of the force. The question of specific force functions for a single body is discussed in the next subsection, and the problem of several bodies in the subsequent subsections.

2. Specific Force Functions

Since the covariant acceleration satisfies the relation $a_\rho v^\rho = 0$, Eq. (56) implies that we must also have

$$F_\rho v^\rho = 0. \quad (60)$$

This is automatically satisfied from the definition of F_ρ in terms of the three-dimensional force $\mathbf{F}$ in both the Newtonian and the relativistic case. However, a four-dimensional formulation does not need such constant reference to a three-dimensional one; instead Eq. (54) can be taken as the fundamental law of motion, and its consequences for the four-force can be studied directly.

Equivalently, if we start from Eq. (56), the relations replacing Eq. (60) are

$$F^\rho w_\rho = 0, \quad (61\text{N})$$

$$F^\rho v_\rho = 0. \quad (61\text{S})$$

In the following, we restrict the discussion to Eq. (60).

The most important forces we have to consider are those derivable from a scalar potential U. The covariant vector $\partial_\rho U$ is not orthogonal to v^ρ, however; we must therefore take as our equation of motion, denoting the interaction constant by g,

$$m_0 a_\rho = g(\partial_\rho U - w_\rho v^\sigma \partial_\sigma U), \quad (62\text{N})$$

which can also be written, using the notation of Eq. (58N),

$$d/d\tau (P_\rho + g w_\rho U) = g \partial_\rho U, \quad (63\text{N})$$

since

$$v^\sigma \partial_\sigma U = dU/d\tau. \quad (64)$$

Here U and its derivatives are evaluated at the position of the particle. It can easily be verified that Eqs. (62N) or (63N) are indeed equivalent to the usual Newtonian equations in a force field $\mathbf{F} = -\nabla U$.

In the relativistic case a possible form of the equations is [28]

$$m_0 a_\rho = g(\partial_\rho U - v_\rho v^\sigma \partial_\sigma U); \quad (62\text{S})$$

another, analogous to (63N), but *not* equivalent to (62S) is [in the notation of *either* Eq. (58S) *or* (59)]

$$d/d\tau \ (P_\rho + g v_\rho U) = g \partial_\rho U. \quad (63\text{S})$$

Both of these reduce to the same Newtonian limit.

For later reference we also need the contravariant form of Eqs. (63), which follows by contractions with $h^{\rho\sigma}$ and $\eta^{\rho\sigma}$, respectively:

$$dP^\rho/d\tau = g h^{\rho\sigma} \partial_\sigma U, \quad (64\text{N})$$

$$d/d\tau \ (P^\rho + g v^\rho U) = g \eta^{\rho\sigma} \partial_\sigma U. \quad (64\text{S})$$

As is well known from electrodynamics, Eq. (60) is identically satisfied for a force of the form

$$F_\rho = e F_{\rho\sigma} v^\sigma, \qquad F_{\rho\sigma} = -F_{\sigma\rho}, \quad (65)$$

which has the structure of the Lorentz force; thus the equations of motion of a charged particle in an external electromagnetic field are

$$m_0 a_\rho = e F_{\rho\sigma} v^\sigma \quad (66)$$

in both cases.[29]

In the Newtonian case it is an immediate consequence of our Lemma that in addition to a^ρ itself any derivative $d^n a^\rho/d\tau^n$ leads to a covariant vector

$${}^n a_\rho = k_{\rho\sigma} \, d^n a^\rho / d\tau^n \quad (67)$$

satisfying condition (41) and thus any force proportional to ${}^n a_\rho$ can be used in Eq. (56); ${}^n a_\rho$ does *not* equal $d^n a_\rho / d\tau^n$. In particular we can use ${}^1 a_\rho$ for the Lorentz radiation damping force. Similarly, forces depending on the acceleration and its derivatives satisfying Eq. (60) can be constructed in the relativistic case by successive differentations of $a^\rho v_\rho = 0$, but no simple general formula analogous to (67) can be given.

3. Interactions

The considerations of the last two subsections referred to a single particle subject to an arbitrary force. If this

[29] We are not concerned here with possible field equations determining $F_{\rho\sigma}$. Galilei-invariant equations can easily be formulated, but they can clearly not be equivalent to the Lorentz-invariant Maxwell equations. In our context we should thus in the Newtonian case consider Eq. (66) as a possible equation of motion for a particle in some Galilei-invariant field, which happens to be applicable to electromagnetism if the fields are given as functions of x^ρ. The linking of the electric and magnetic fields in a four-dimensional tensor (rather than considering them as two independent three-vectors) leads through the transformation law (4) to effects which are the correct limit $c \to \infty$ of the relativistic ones (unlike the three-vector interpretation): An observer moving with velocity $\mathbf{v}$ with respect to a frame of reference in which the fields have the values $\mathbf{E}$ and $\mathbf{B}$, will note an additional electric field $\mathbf{v} \times \mathbf{B}$.

force is due to other particles, which in turn are acted upon by the particle under consideration, the usual assumption of Newtonian mechanics is that the forces are determined by the *simultaneous* positions (and possibly their derivatives) of the particles, and that they are related by Newton's third law. No such assumption is possible in special relativity since simultaneity is not an invariant concept in that theory. However, as discussed in Sec. III.3, even in Newtonian mechanics $\Delta t=0$ is only one of several possible two-body invariants, and thus there exists the possibility of introducing Galilei-invariant forces other than those ordinarily considered, in closer analogy to possible special relativistic interactions; this point will be discussed in detail elsewhere.[30] In the following we restrict ourselves as an example to the simplest types of forces in both theories, those which can be derived from a scalar potential satisfying a Poisson or d'Alembert equation; extension to the Helmholtz or Klein–Gordon equations requires only minor modifications.

We have then[28,31,32]

$$h^{\mu\nu}\partial_{\mu\nu}U=-4\pi\rho, \tag{68N}$$

$$\eta^{\mu\nu}\partial_{\mu\nu}U=-4\pi\rho, \tag{68S}$$

where $\partial_{\mu\nu}=\partial_\mu\partial_\nu$ and ρ is the source density. In the case of n point sources of strength g_i we can take

$$\rho=G\sum_{i=1}^{n}g_i\int_{-\infty}^{\infty}\delta^4[s_i{}^\rho(\tau_i)]\,d\tau_i,$$
$$s_i{}^\rho\equiv x^\rho-z^\rho{}_i(\tau_i), \tag{69}$$

where δ^4 denotes a fourfold product of Dirac δ-functions, and G is a constant. In the case of gravitation, G is the gravitational constant, and because of the universal proportionality of inertial and gravitational masses the g_i can be taken as equal to the m_{0i}.

The solutions of Eqs. (68) are given by

$$U(x^\mu)=-4\pi\int\rho(x'^\mu)G(x'^\mu,x^\mu)\,d^0x', \tag{70}$$

where d^4x' is the four-dimensional volume element and $G(x'^\mu,x^\mu)$ is the Green's function appropriate for each particular equation.[33] The three-dimensional Poisson equation is of elliptic type and for the infinite three-dimensional domain there exists only one Green's function $|\mathbf{x}'(t)-\mathbf{x}(t)|^{-1}$; thus we can write for its four-dimensional equivalent (which involves an additional integration over x^0), using Eq. (47),

$$G(x'^\mu,x^\mu)=\frac{1}{4\pi}\frac{\delta(x'^0-x^0)}{[-k_{rs}(x'^r-x^r)(x'^s-x^s)]^{\frac{1}{2}}}, \tag{71N}$$

and we get from Eq. (70) with (69)

$$U(x^\mu)=-G\sum_i g_i\int_{-\infty}^{\infty}\frac{\delta(s_i{}^0)}{(-k_{rs}s_i{}^rs_i{}^s)^{\frac{1}{2}}}d\tau_i, \tag{72N}$$

which we can write

$$U(x^\mu)=-G\sum_i g_i/R_i,\qquad R_i=(-k_{\rho\sigma}s_i{}^\rho s_i{}^\sigma)^{\frac{1}{2}}|_{x^0=z^0_i}. \tag{73N}$$

The d'Alembert equation is hyperbolic and the choice of the Green's function is not unique. In view of later applications we write down only the time-symmetric (half-retarded, half-advanced) Green's function given by

$$G(x'^\mu,x^\mu)=\frac{1}{8\pi c^3}\frac{\delta[(x'^0-x^0)+R]+\delta[(x'^0-x^0)-R]}{R}$$
$$=\frac{1}{4\pi c^3}\delta[\eta_{\kappa\lambda}(x'^\kappa-x^\kappa)(x'^\lambda-x^\lambda)],$$
$$R=[-\eta_{rs}(x'^r-x^r)(x'^s-x^s)]^{\frac{1}{2}}. \tag{71S}$$

Thus we obtain from Eq. (70) with (69)

$$U(x^\mu)=-G\sum_i\frac{g_i}{c^3}\int_{-\infty}^{\infty}\delta(\eta_{\kappa\lambda}s_i{}^\kappa s_i{}^\lambda)\,d\tau_i, \tag{72S}$$

which equals

$$U(x^\mu)=-\frac{G}{2c^3}\sum_i g_i\left[\frac{1}{\kappa_i(\tau_{ir})}-\frac{1}{\kappa_i(\tau_{ia})}\right],\qquad \kappa_i=\eta_{\rho\sigma}s_i{}^\rho v_i{}^\sigma, \tag{73S}$$

where τ_{ir} and τ_{ia} are the retarded and advanced proper times, determined from $\eta_{\mu\nu}s_i{}^\mu s_i{}^\nu=0$, with $s_i{}^0>0$ and <0, respectively.

As Eqs. (68) contain *all* particles as sources, the potentials (73) introduce a spurious self-interaction into the equations of motion (63) (regrettably as usual). Omitting these terms, we get for the kth particle

$$\frac{d}{d\tau_k}\left\{P_{k\rho}-Gg_kw_\rho\sum_{i\neq k}\frac{g_i}{R_i}\right\}=-Gg_k\partial_\rho\sum_{i\neq k}\frac{g_i}{R_i}, \tag{74N}$$

$$\frac{d}{d\tau_k}\left\{P_{k\rho}-\frac{G}{2c^3}g_kv_{k\rho}\sum_{i\neq k}g_i\left[\frac{1}{\kappa_i(\tau_{ir})}-\frac{1}{\kappa_i(\tau_{ia})}\right]\right\}$$
$$=-\frac{G}{2c^3}g_k\partial_\rho\sum_{i\neq k}g_i\left[\frac{1}{\kappa_i(\tau_{ir})}-\frac{1}{\kappa_i(\tau_{ia})}\right]. \tag{74S}$$

These equations could of course have been postulated directly without any reference to field equations, as was indeed the case for Newton's theory of gravitation.

[30] P. Havas and J. Plebański (to be published shortly); possible relativistic interactions were discussed in Bull. Am. Phys. Soc. **5**, 433 (1960).

[31] Eq. (68 S) is a special case of the scalar meson equation (with rest mass zero), which has been studied extensively in meson theory; compare P. Havas, Phys. Rev. **87**, 309 (1952), and references given there.

[32] For the case of gravitation, Eq. (68S) has been studied by O. Bergmann, Am. J. Phys. **24**, 38 (1956).

[33] See, e.g., D. Iwanenko and A. Sokolow, *Klassische Feldtheorie* (German translation: Akademie-Verlag, Berlin, 1953). The d'Alembert operator used by these authors differs from that of Eq. (68S) by a factor $-c^2$.

4. Variational Principles

For forces derivable from a scalar or a vector potential, the Lagrangians leading to the three-dimensional Eqs. (49) are well known. In the presence of a scalar potential alone we have the variational principle

$$\delta J=0, \qquad J=\int_{-\infty}^{\infty} L\, dt, \tag{75}$$

with

$$L(z^r, dz^r/dt)=\tfrac{1}{2}m_0v^2-gU, \tag{76N}$$

$$L(z^r, dz^r/dt)=-m_0c^2(1-v^2/c^2)^{\frac{1}{2}}-gUc^2. \tag{76S}$$

To obtain the four-dimensional equations (63) directly we must replace the principle (75) by

$$\delta J=0, \qquad J=\int_{-\infty}^{\infty} L\, d\tau. \tag{77}$$

If we take

$$L(z^\rho, dz^\rho/d\tau)=\tfrac{1}{2}m_0v^2/w_\mu v^\mu-gw_\mu v^\mu U, \tag{78N}$$

we obtain Eq. (63N), as can be verified easily. The canonically conjugate momenta are

$$p_0=-\tfrac{1}{2}m_0v^2/w_\mu v^\mu-gU, \qquad p_r=m_0v^r/w_\mu v^\mu,$$

which can be written

$$p_\rho=-[P_\rho/w_\mu v^\mu+gw_\rho U] \tag{79N}$$

in the notation of Eq. (58N).

In the relativistic case we have

$$L(z^\rho, dz^\rho/d\tau)=-c^2(v_\mu v^\mu)^{\frac{1}{2}}(m_0+gU). \tag{78S}$$

We then obtain Eq. (63S), and the canonically conjugate momenta are given by

$$p_\rho=-\frac{c^2}{(v_\mu v^\mu)^{\frac{1}{2}}}(P_\rho+gv_\rho U) \tag{79S}$$

in the notation of Eq. (59).

If we construct the Hamiltonian from the Lagrangians (78) as usual by

$$H=p_\rho dz^\rho/d\tau-L, \tag{80}$$

we obtain expressions which vanish identically. This property of Lagrangians of the type (78) is well known[34]; it is due to the fact that the four equations of motion are not independent, and the Lagrangians are homogeneous of the first degree in the velocities. Nevertheless, a Hamiltonian formalism can still be developed,[34] but it will not be presented here.

If we consider the particular scalar fields described by Eqs. (68), we can write a single variational principle which entails both the equations of motion and the field equations. We take

$$J=\sum_i\int\frac{1}{2}\frac{m_{0i}v_i^2}{w_\mu v_i^\mu}\,d\tau_i-\sum_i g_i\iint w_\mu v_i^\mu U\delta^4(s_i^\rho)\,d^4x\,d\tau_i+\frac{1}{8\pi G}\int h^{\mu\nu}\partial_\mu U\partial_\nu U\,d^4x, \tag{81N}$$

$$J=-\sum_i\int m_{0i}c^2(v_{i\mu}v_i^\mu)^{\frac{1}{2}}\,d\tau_i-\sum_i g_i\iint c^2(v_{i\mu}v_i^\mu)^{\frac{1}{2}}U\delta^4(s_i^\rho)\,d^4x\,d\tau_i+\frac{c^2}{8\pi G}\int \eta^{\mu\nu}\,\partial_\mu U\partial_\nu U\,d^4x, \tag{81S}$$

and put $\delta J=0$. Then variation with respect to z_k^ρ and $dz_k^\rho/d\tau_k$ will give the equations of motion of the kth particle (including the spurious self-action terms) as before, and variation with respect to U and its derivatives will give the field equations (68).

We can avoid the spurious self-action terms by using a Fokker-type variational principle,[35] which is expressed entirely in terms of particle variables. We take[36]

$$J=\sum_i\int\frac{1}{2}\frac{m_{0i}v_i^2}{w_\mu v^\mu}\,d\tau_i+4\pi\sum_{i<k}Gg_ig_k\iint w_\rho v_i^\rho w_\sigma v_k^\sigma G(z_i^\mu, z_k^\mu)\,d\tau_i\,d\tau_k, \tag{82N}$$

$$J=-\sum_i\int m_{0i}c^2(v_{i\mu}v_i^\mu)^{\frac{1}{2}}\,d\tau_i+4\pi\sum_{i<k}Gg_ig_k\iint c^2(v_{i\rho}v_i^\rho)^{\frac{1}{2}}(v_{k\sigma}v_k^\sigma)^{\frac{1}{2}}G(z_i^\mu, z_k^\mu)\,d\tau_i\,d\tau_k, \tag{82S}$$

where the $G(z^\mu{}_i, z^\mu{}_k)$ are the Green's functions (71), and put $\delta J=0$. Variation with respect to z_k^ρ and $dz_k^\rho/d\tau_k$ leads directly to Eqs. (74). In addition we can define the adjoint fields U_k (where $k=1, \cdots n$), by

$$U_k(x^\rho)=-4\pi Gg_k\int w_\sigma v_k^\sigma G(x^\mu, z_k^\mu)\,d\tau_k, \tag{83N}$$

$$U_k(x^\rho)=-4\pi Gg_k\int (v_{k\sigma}v_k^\sigma)^{\frac{1}{2}}G(x^\mu, z_k^\mu)\,d\tau_k, \tag{83S}$$

and apply the operators $h^{\mu\nu}\partial_{\mu\nu}$ and $\eta^{\mu\nu}\partial_{\mu\nu}$, respectively. Since the Green's functions satisfy Eqs. (68) with

[34] See Ref. 7, Part E II, and the first article of Ref. 1, Secs. 22 and 25. The discussion there is based on an explicitly parameter-independent formulation of the equations of motion and the Lagrangian. Even though we are using a particular parametrization in this paper, our Lagrangians (78) are manifestly parameter-independent, and thus the arguments of these references apply to our case.

[35] First introduced for electrodynamics by A. D. Fokker, Z. Physik **58**, 386 (1929).

[36] For Eq. (82 S) compare Ref. 31, Eq. (27b) (with $\chi=0$) and the subsequent discussion.

$\rho = -\delta^4(s_k^\mu)/4\pi$ by definition, we obtain

$$h^{\mu\nu}\partial_{\mu\nu}U_k = -4\pi G g_k \int \delta^4(s_k^\rho)\, d\tau_k, \qquad (84\text{N})$$

$$\eta^{\mu\nu}\partial_{\mu\nu}U_k = -4\pi G g_k \int \delta^4(s_k^\rho)\, d\tau_k. \qquad (84\text{S})$$

The introduction of an adjoint field (83N) then shown to satisfy (84N) is essentially the historical procedure which led from Newton's law of gravitation to the introduction of a potential satisfying Poisson's equation.

5. Conservation Laws

We first consider the case of a single free particle. Then from Eqs. (54) and (56) we have for the four-momentum

$$P^\rho = \text{constant}, \qquad P_\rho = \text{constant}, \qquad (85)$$

which from the definitions of these quantities implies that the mass and velocity of the particle are constant.

Since P^ρ is a four-vector, while this is not necessarily the case for P_ρ (see Sec. IV. 1), we have to distinguish between them in the following discussion.

The existence of conservation laws involving the total four-momentum for interacting particles depends on the nature of the forces. In particular, the fact that there is no invariant meaning to the concept of forces acting instantaneously at a distance in special relativity leads to difficulties to be discussed below. We therefore next consider the special case of point particles with momenta P_i^ρ interacting only at the instant of collision, i.e., at zero distance, since then these difficulties do not arise. We want to discuss the problem of the kinds of covariant conservation laws which are mathematically possible; their actual existence is a question of experiment. For simplicity we do not discuss angular momentum.

We first consider the possible covariant conservation laws involving P_i^ρ. Taking the difference $\Delta\sum_i P_i^\rho$ of the momenta at two different times, we *could* have a conservation law

$$\Delta\sum_i P_i^\rho = 0. \qquad (86)$$

In the case of Newtonian mechanics this implies by the definitions (54) and (34) in three-dimensional notation

$$\Delta\sum_i m_i = 0, \qquad (87)$$

$$\Delta\sum_i \mathbf{p}_i = 0. \qquad (88)$$

However, we could instead of Eq. (86) formulate the more restricted covariant conservation law for the total four-momentum

$$\Delta w_\rho \sum_i P_i^\rho = 0, \qquad (89\text{N})$$

which only implies (87).

Since P_i^ρ is itself a four-vector, it is not necessary that the total number of particles be the same at the two instances of time considered, and thus Eqs. (86)–(89) might also be valid for collisions which do not conserve the total number of particles. On the other hand, if we start from P_i^ρ, it is only the difference of two momenta which is a four-vector; furthermore, this is only true provided the mass does not change, since it is the transformation law for the velocities rather than the momenta which is responsible for this property.[37] Thus we can only formulate a covariant conservation law if neither the total number of particles nor their mass changes. We then might have a law

$$\Delta\sum_i P_{i\rho} = 0, \qquad (90)$$

which implies Eq. (88) and also

$$\Delta\sum_i T_i = 0, \qquad (91)$$

and where Eq. (87) is implied in the formulation of Eq. (90) by the previous discussion.

We can *not* pass from Eq. (86) to Eq. (90) in the Newtonian case by using our Lemma, since the separate vectors P_i^ρ do not satisfy its conditions, and its application to the four-vector $\Delta\sum_i P_i^\rho$ (all of whose components vanish) only leads to a trivial zero vector. On the other hand, we *can* pass from Eq. (90) to Eq. (86) (with conservation of particle number) by constructing $\Delta P_i^\rho = h^{\rho\sigma}\,\Delta P_{i\sigma}$. Thus we can summarize the Newtonian case as follows: If the total number of particles is not conserved, we could have (a) no conservation law, (b) conservation of mass alone, (c) conservation of mass and momentum. If the total number of particles is conserved, we could have the same three alternatives, and also (d) conservation of mass, momentum, and kinetic energy. Conversely, conservation of kinetic energy implies that of momentum, mass, and the number of particles, and conservation of momentum implies that of mass.[38]

We can proceed similarly in special relativity. If we require Eq. (86), this again implies the conservation laws (87) and (88) by the definitions (54) and (31); however, it should be noted that the masses m_i now depend on the velocities. The two laws could hold again even if the number of particles is not conserved. There is no possibility of a conservation law analogous to (89N) involving only the total four-momentum;

[37] This can be readily verified from the Galilei transformation (14). In the previous discussions of P_ρ in Sec. IV.1 the problem of a change in mass did not arise.

[38] For similar considerations in three-dimensional formulation see the first article of Ref. 1, Sec. 19.

nstead, we could have

$$\Delta \sum_i v_{i\rho} P_i^{\rho} = 0, \tag{89S}$$

which implies

$$\Delta \sum_i m_{0i} = 0. \tag{92}$$

Thus we could have conservation of rest mass, but not of mass, without conservation of momentum. In the Newtonian case there is of course no such distinction.

If Eq. (90) is assumed and use is made of the definition (58S) for $P_{i\rho}$, which then is not a four-vector, we again must have conservation of the number of particles; however, now it is not necessary that the rest masses of the particles be unchanged individually, but only Eq. (92) is required. Then Eq. (90) implies Eqs. (88) and (91). However, we can also choose the definition (59); then $P_{i\rho}$ is a four-vector and thus we do not need conservation of particle number. Eq. (90) then implies Eq. (88) and the conservation of mass, Eq. (87). This latter equation appears multiplied by c^2, i.e., in the form of a law of conservation of energy,

$$\Delta \sum_i E_i = 0. \tag{93}$$

The physical content of the "equivalence of mass and energy" does not consist in the appearance of the factor c^2, but is due to the fact that neither Eq. (87) nor the "equivalent" Eq. (93) require the conservation of *rest* mass (92) or of kinetic energy (91). Therefore they allow a system of particles with point interactions to gain rest mass at the expense of kinetic energy, and conversely.[39]

With definition (59), Eqs. (86) and (90) are equivalent even if the number of particles is not conserved (unlike the Newtonian case). In the case of definition (58S), however, we can only pass from Eq. (86) to Eq. (90) and conversely if the number of particles and the sum of all the rest masses is conserved.

Thus in the relativistic case we have the following alternatives: If the total number of particles is not conserved, we have (a) no conservation laws, (b) conservation of the sum of all rest masses, (c) conservation of mass (and equivalently of total energy) and of momentum. If the total number of particles is conserved, we could have the same three alternatives, and also (d) conservation of kinetic energy and of the sum of all rest masses separately (and thus also of total mass and energy) and of momentum. Conversely, conservation of kinetic energy implies that of momentum, the sum of all rest masses, and the number of particles, and conservation of momentum and of mass (energy) imply each other.

These are the mathematical possibilities in the case of point interactions, which are all consistent with the second law of motion. The physical assumption made in both Newtonian and special relativistic mechanics is that for a system of "elementary" particles whose number and individuality is conserved it is alternative (d) which exists in nature, and for a system of *any* kind of particles whose number is not necessarily conserved it is alternative (c), i.e., Eq. (86). The fundamental difference between the two theories is that this latter alternative, which in both cases follows from the assumption of conservation of momentum alone, does not imply conservation of energy in Newtonian mechanics, while it does so in special relativity because of the equivalence of mass and energy for each particle.

In *both* theories the stipulation of the conservation law (86) for a system of particles whose number is *not* conserved implies that the energy of any one of these particles is no longer only determined up to an arbitrary constant. Unlike the case of a *single* particle subject to an arbitrary force, we now *must* choose the constants in Eqs. (53) to be zero. The total energy of the *entire* system is of course still only determined up to a constant.

To obtain a general equivalence of mass and energy for a general nonmechanical system requires the further assumption of a conservation law (86) including all forms of energy and momentum, and a study of the interaction of the general system with a system of elementary free particles.[27]

To proceed beyond point interactions in full generality is a problem which has not been investigated to any extent. If we restrict ourselves to systems of a fixed number of particles preserving their rest masses, it is simplest to start from variational principles of the types (82N) and (82S). Since they are both invariant under a ten-parameter group [up to a divergence only in the case (82N)], the Galilei and Lorentz group, respectively, they both imply ten conservation laws by Noether's theorem,[40] i.e., ten quantities which do not vary in the course of time. However, there is a fundamental difference between the expressions which follow from the Newtonian principle (82N)[41] and those which follow from a relativistic principle such as (82S).[42] In the Newtonian case these expressions depend only on the time under consideration, while in the relativistic case they involve integrals over the past and future motion of the particles, and thus are not conservation laws in the usual sense.

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[39] This derivation of the equivalence of mass and energy (in three-dimensional language) from the requirement of a covariant conservation law (90) with (59) for a system of particles is given in P. Frank's review article "Relativitätsmechanik," Sec. 7, in *Handbuch der physikalischen und technischen Mechanik*, edited by F. Auerbach and W. Hort (J. A. Barth, Leipzig, 1930), Vol. 2, and is based on considerations by H. A. Lorentz in *Das Relativitatsprinzip* (Beihefte zur Zeits. für mathematischen und naturwissenschaftlichen Unterricht 1, B. G. Teubner, Leipzig, 1914).

[40] E. Noether, Nachr. Akad. Wiss. Göttingen, Math.-phys. Kl. 235 (1918); for a recent review see E. L. Hill, Rev. Mod. Phys. **23**, 253 (1951).

[41] These were first obtained by Noether's method by E. Bessel-Hagen, Math. Ann. **84**, 258 (1921).

[42] The general form of these laws was given by J. W. Dettman and A. Schild, Phys. Rev. **95**, 1057 (1954). See also the first article of Ref. 1, Sec. 24.

6. Conservation Laws and Laws of Motion

We now return to the case of a free point particle with the law of motion

$$m_{0i}\, d^2 z_i^\rho / d\tau^2 = 0. \tag{94}$$

Since the space of the x^ρ is flat, these are the Eqs. (10) of a geodesic in this space. As was shown elsewhere,[43] this implies that there exists a symmetric tensor $P^{\rho\sigma}$ satisfying the conservation law

$$\partial_\sigma P^{\rho\sigma} = 0, \tag{95}$$

which has the form

$$P^{\rho\sigma} = \sum_i \int_{-\infty}^{\infty} m_{0i} v_i^\rho v_i^\sigma \delta^4(s_i^\mu)\; d\tau_i. \tag{96}$$

Conversely, if we make the requirement of the existence of a conservation law (95) our starting point, with $P^{\rho\sigma}$ of the form

$$P^{\rho\sigma} = \sum_i \int_{-\infty}^{\infty} p_i^{\rho\sigma}(\tau_i)\, \delta^4(s_i^\mu)\; d\tau_i, \tag{97}$$

with $p_i^{\rho\sigma}$ as yet undetermined, it then *follows* that $P^{\rho\sigma}$ must have the form (96) and that the law of motion takes the form (94), both in Newtonian and in special relativistic theory.

These considerations can be generalized in the presence of a force field for which one can define a symmetric tensor $T_f^{\rho\sigma}$ whose divergence vanishes everywhere except at the position of the particle which is itself one of the sources of the field. Restricting ourselves to the fields described by Eqs. (68) with (69), we can take

$$T_f^{\rho\sigma} = (4\pi G)^{-1}(h^{\rho\mu}h^{\sigma\nu}\partial_\mu U \partial_\nu U - \tfrac{1}{2}h^{\rho\sigma}h^{\mu\nu}\partial_\mu U \partial_\nu U), \tag{98N}$$

$$T_f^{\rho\sigma} = (4\pi G)^{-1}(\eta^{\rho\mu}\eta^{\sigma\nu}\, \partial_\mu U \partial_\nu U - \tfrac{1}{2}\eta^{\rho\sigma}\eta^{\mu\nu}\, \partial_\mu U \partial_\nu U), \tag{98S}$$

with

$$\partial_\sigma T_f^{\rho\sigma} = G^{-1}\rho h^{\rho\sigma}\, \partial_\sigma U, \tag{99N}$$

$$\partial_\sigma T_f^{\rho\sigma} = G^{-1}\rho \eta^{\rho\sigma}\, \partial_\sigma U. \tag{99S}$$

We can then stipulate as the basic requirement of our theories the conservation of all forms of energy and momentum

$$\partial_\sigma T^{\rho\sigma} = 0, \qquad T^{\rho\sigma} = P^{\rho\sigma} + T_f^{\rho\sigma}, \tag{100}$$

instead of Eqs. (74), because it implies these equations. To show this we can follow the procedure of Ref. 43[44,45] except for using an energy–momentum tensor $P^{\mu\nu} + T_f^{\mu\nu}$ instead of just $P^{\mu\nu}$; the distinction made there between tensors and tensor densities is not relevant here. The additional tensor leads to an additional term

$$\sum_i \int \partial_\nu T_f^{\mu\nu} \xi_\mu\, d^4x$$

in Eqs. (3) and (4) of that reference (with $n=4$ in our case), where ξ_μ is an arbitrary function. We then insert the form (99) with (69) for the divergence and integrate over x^ρ to obtain

$$\sum_i g_i \int h^{\mu\nu}\partial_\nu U \xi_\mu (d\tau_i/d\lambda_i)\, d\lambda_i$$

and

$$\sum_i g_i \int \eta^{\mu\nu}\partial_\nu U \xi_\mu (d\tau_i/d\lambda_i)\, d\lambda_i$$

in the Newtonian and the relativistic case, respectively, where λ_i is a parameter along the ith world line which is arbitrary except for increasing monotonically with z_i^0. These terms will be carried along in Eq. (9) of Ref. 43; we then obtain instead of Eq. (12) of that reference

$$(D/d\lambda_i)(M_i\, dz_i^\mu/d\lambda_i) + g_i h^{\mu\nu}\partial_\nu U\; d\tau_i/d\lambda_i = 0, \tag{101N}$$

$$(D/d\lambda_i)(M_i\, dz_i^\mu/d\lambda_i) + g_i \eta^{\mu\nu}\partial_\nu U\; d\tau_i/d\lambda_i = 0, \tag{101S}$$

with the covariant derivative $D/d\lambda_i$ reducing to an ordinary derivative in our case. This can be written

$$\frac{D}{d\lambda_i}\left(M_i \frac{d\tau_i}{d\lambda_i}\right)\frac{dz_i^\mu}{d\tau_i} + M_i\left(\frac{d\tau_i}{d\lambda_i}\right)^2 \frac{d^2 z_i^\mu}{d\tau_i^2} + g_i h^{\mu\nu}\partial_\nu U \frac{d\tau_i}{d\lambda_i} = 0, \tag{102N}$$

$$\frac{D}{d\lambda_i}\left(M_i \frac{d\tau_i}{d\lambda_i}\right)\frac{dz_i^\mu}{d\tau_i} + M_i\left(\frac{d\tau_i}{d\lambda_i}\right)^2 \frac{d^2 z_i^\mu}{d\tau_i^2} + g_i \eta^{\mu\nu}\partial_\nu U \frac{d\tau_i}{d\lambda_i} = 0. \tag{102S}$$

We now contract these equations with w_μ and $dz_{i\mu}/d\tau_i$, respectively. Using Eqs. (30), (35) and (32) we get

$$d/d\lambda_i(M_i\, d\tau_i/d\lambda_i) = 0, \tag{103N}$$

$$d/d\lambda_i(M_i\, d\tau_i/d\lambda_i) + g_i\, dU/d\lambda_i = 0, \tag{103S}$$

from which we obtain

$$M_i\, d\tau_i/d\lambda_i = m_{0i}, \tag{104N}$$

$$M_i\, d\tau_i/d\lambda_i = m_{0i} - g_i U, \tag{104S}$$

where m_{0i} is a constant of integration. Inserting these expressions into Eqs. (102) we obtain Eqs. (74) [originally including a spurious infinite self-action as in the case of the Lagrangians (81), which however can be removed by standard methods[46]]. From the significance of M_i as introduced in Ref. 43 we also

[43] P. Havas, J. Math. Phys. **5**, 373 (1964). As noted there, no particular physical interpretation of the tensor $P^{\rho\sigma}$ is implied; for want of a better word we refer to it as the energy–momentum tensor, although P^{00} has the dimensions of mass density rather than energy density, and is not equivalent to an energy density in the Newtonian case.

[44] In the relativistic case this can be done more simply, since the space is metric. For a general method, see M. Mathisson, Proc. Cambridge Phil. Soc. **36**, 331 (1940); a modification of this method and application to specific fields including (68S) is given in Ref. 45.

[45] P. Havas, in *Recent Developments in General Relativity* (Pergamon Press–PWN, New York–Warsaw, 1962), p. 259.

[46] These methods have been developed for special relativity (see Ref. 44 and other references given in Ref. 45), but are applicable in the Newtonian case with minor modifications.

obtain that the matter tensor must be of the form

$$P^{\rho\sigma}=\sum_i \int_{-\infty}^{\infty} m_{0i} v_i^\rho v_i^\sigma \delta^4(s_i^\mu)\, d\tau_i, \qquad (105\text{N})$$

$$P^{\rho\sigma}=\sum_i \int_{-\infty}^{\infty} (m_{0i}-g_i U) v_i^\rho v_i^\sigma \delta^4(s_i^\mu)\, d\tau_i. \qquad (105\text{S})$$

V. GENERALLY COVARIANT FORMULATION OF NEWTONIAN AND SPECIAL RELATIVISTIC MECHANICS IN FLAT SPACE-TIME

1. General Coordinate Transformations and Fundamental Tensors

As discussed in Sec. III, we associated a flat affine space with the coordinates x^ρ used in the Lorentz and Galilei transformations by choosing affine connections $\Gamma^\rho{}_{\mu\nu}$ such that

$$R^\kappa{}_{\mu\lambda\nu}(\Gamma)=0. \qquad (106)$$

The coordinate systems we used (Cartesian coordinates and a special choice of the time coordinate x^0) were such that the Γ's actually vanished in all these systems. However, clearly the physical content of the theories is not changed by allowing the use of curvilinear coordinates or of a time coordinate which is not simply proportional to the time as measured by natural clocks; it is only the expression of the predictions of the theories in terms of these coordinates which is changed, as discussed below. As mentioned in the Introduction, it was realized at least as early as 1917 that it should be possible to express any theory whatever in a generally covariant form. Following Cartan's early work,[14] such a formulation was developed in great detail both for Newtonian mechanics and for the special theory of relativity.[12,47,48] We give only a brief outline, mainly to allow comparison with the treatment given in Sec. VI and to give a more detailed discussion of the significance of this formulation in terms of measurements than in the references cited; this discussion is equally applicable for Sec. VI.

We have already prepared the ground for a covariant formulation by writing all our formulas in tensor notation. To transcribe them fully into generally covariant form, we must only replace ordinary derivatives by covariant derivatives [covariant differentiation being commutative just like ordinary differentiation because of the vanishing of the curvature tensor (106)]. Then the formulas must simply be understood to be valid tensor relations not just under the groups of transformations (14) with (16) or (20), but under the group of all analytic coordinate transformations (subject to certain restrictions to be discussed below).

[47] R. A. Toupin, Arch. Rat. Mech. Anal. **1**, 181 (1958), and references given there.

[48] V. Fock, Ref. 11, Chap. IV, and Rev. Mod. Phys. **29**, 325 (1957).

In particular, we must consider $g_{\mu\nu}$, $h^{\mu\nu}$, and $\eta_{\mu\nu}$ as tensors under these transformations. But then they no longer possess the property of having numerically the same components [given by Eqs. (18) or (21)] in all coordinate systems, and we must therefore give an invariant characterization. To achieve this we require that these three tensors be symmetric and of rank four; $h^{\mu\nu}$ should have signature -3 (and thus is necessarily singular); $g_{\mu\nu}$ should have signature 1 (and thus also is singular); $\eta_{\mu\nu}$ should have signature -2 (and be nonsingular). Furthermore we maintain conditions (17) and (22) and require that

$$g_{\mu\nu;\rho}=0, \qquad h^{\mu\nu}{}_{;\rho}=0, \qquad (107\text{N})$$

$$\eta_{\mu\nu;\rho}=0, \qquad \eta^{\mu\nu}{}_{;\rho}=0, \qquad (107\text{S})$$

where each part of Eq. (107S) implies the other by Eq. (17). Equations (106) and (107) imply that there exist coordinate systems in which $g_{\mu\nu}$, $h^{\mu\nu}$, $\eta_{\mu\nu}$, and $\eta^{\mu\nu}$ are constant everywhere; from the requirements on the signatures and Eqs. (17) and (22) it follows that by suitable linear transformations of these systems and possible renumbering of the coordinates we can obtain coordinate systems in which the components of the four tensors take the numerical values (18) and (21).

Introducing the invariant parameters τ by Eqs. (30) or (33) as before, we can still define a contravariant four-velocity

$$v^\rho \equiv dz^\rho/d\tau; \qquad (108)$$

from this we can define a covariant vector w_μ by Eqs. (34) as before, which satisfies

$$w_{\mu;\rho}=0 \qquad (109)$$

from its definition. Alternatively, we could have introduced w_μ rather than $g_{\mu\nu}$ as a basic quantity, defined by the requirements (109) and (24a)[12,47] and then could have defined $g_{\mu\nu}$ by (24b). The existence of coordinate systems in which w_μ has the form (23) follows by the same arguments as used above for $h^{\mu\nu}$ and $g_{\mu\nu}$. By either way of introducing w_μ we have then

$$v^\rho w_\rho=1 \qquad (110)$$

as before.

All the above considerations are valid for arbitrary analytic coordinate transformations, and there is no *mathematical* necessity to restrict these transformations in the following. However, if we want a reasonable description of physical phenomena, we must exclude any coordinate system for which signals emitted at a time t_0 could arrive at some points of the system at $t>t_0$ and at others at $t<t_0$.[49] This leads to different restrictions in the two theories. In special relativity it

[49] It is too strong a requirement to demand $t>t_0$, since this would assign physical meaning to the obviously conventional orientation of the time axis and would exclude even the ordinary antichronous transformations considered in Sec. III. Allowing both signs does not contradict the "causality condition" that a signal should not arrive earlier than it was emitted, which can be looked upon as a definition either of "signal" or of "earlier."

is assumed that there exists a maximum signal velocity equal to c in an inertial system. Thus for any signal we must have in *any* coordinate system

$$ds^2=\eta_{\mu\nu}\,dx^\mu\,dx^\nu\geq 0 \tag{111S}$$

between any two adjacent events along the world line of the signal; conversely, if two events are simultaneous, i.e., have $dx^0=0$, they can not be connected by a signal, and thus for such events ds^2 must be negative definite. This leads to the following conditions on $\eta_{\mu\nu}$[50]:

$$\eta_{00}>0,\ \eta_{mm}<0,\ \begin{vmatrix}\eta_{mm} & \eta_{mn}\\ \eta_{nm} & \eta_{nn}\end{vmatrix}>0,\ \begin{vmatrix}\eta_{11} & \eta_{12} & \eta_{13} & \eta_{14}\\ \eta_{21} & \eta_{22} & \eta_{23} & \eta_{24}\\ \eta_{31} & \eta_{32} & \eta_{33} & \eta_{34}\\ \eta_{14} & \eta_{24} & \eta_{34} & \eta_{44}\end{vmatrix}<0, \tag{112S}$$

(no summation over repeated indices).

These conditions imply a similar set of restrictions on the components of $\eta^{\mu\nu}$.

In Newtonian mechanics it is assumed that there exist signals propagating with infinite velocity. Thus we must now have

$$ds^2=g_{\mu\nu}\,dx^\mu\,dx^\nu\geq 0 \tag{111N}$$

between any two adjacent events along the world line of a signal; in particular, for infinite signal velocity, ds^2 must be zero, and in the special coordinate systems used in the previous sections these events are simultaneous, i.e., have $dx^0=0$. If in some other coordinate system the corresponding time interval dx'^0 would not vanish, this would mean that a signal sent with infinite velocity from the event 1 at time $x'_1{}^0$ would arrive at event 2 at time $x'_2{}^0=x'_1{}^0+dx'^0>x'_1{}^0$, while a similar signal sent from event 2 at time $x'_2{}^0$ would arrive at event 1 at time $x'_1{}^0<x'_2{}^0$. But such a description must be excluded according to our criterion, and thus we must require that only those coordinate transformations be allowed which lead from $dx^0=0$ for *any* pair of events to $dx'^0=0$, independent of position; thus simultaneity is still absolute. To satisfy this requirement the transformations must be of the form

$$x'^0=x'^0(x^0),\quad dx'^0/dx^0>0\ \text{ for all }\ x^0\ \text{ or }\ <0\ \text{ for all }\ x^0,\quad x'^m=x'^m(x^\rho), \tag{113a}$$

and therefore also

$$x^0=x^0(x'^0),\quad dx^0/dx'^0>0\ \text{ for all }\ x^0\ \text{ or }\ <0\ \text{ for all }\ x^0,\quad x^m=x^m(x'^\rho); \tag{113b}$$

they thus only involve a reparametrization of the time coordinate.[51] But then we get from the form (21) of $g_{\mu\nu}$ and $h^{\mu\nu}$ in an inertial system and from the transformation law (4) that we must have in *any* coordinate system

$$g_{\mu\nu}=0\quad\text{unless}\quad \mu=\nu=0, \tag{112aN}$$

$$h^{\mu 0}=h^{0\mu}=0, \tag{112bN}$$

and thus also

$$w_\mu=0\quad\text{unless}\quad\mu=0. \tag{114}$$

Similarly Eqs. (6) and (113) imply that the spatial part k_{mn} of the tensor $k_{\mu\nu}$ introduced by Eq. (43) transforms independently of $k_{\mu 0}$; this combined with (112bN) implies that

$$k_{mr}h^{rn}=\delta_m{}^n \tag{115}$$

in all coordinate systems, since this relation is true in an inertial system and maintains its form under the transformations (113). Thus the spatial parts of $k_{\mu\nu}$ and $h^{\mu\nu}$ are inverses of each other.

As the $\Gamma^\rho{}_{\mu\nu}$ vanish in an inertial system, their values in an arbitrary coordinates system follow from their transformation law (6) with (113). We obtain immediately

$$\Gamma^0{}_{\mu n}=\Gamma^0{}_{n\mu}=0. \tag{116a}$$

Several of the other components can be expressed in terms of the tensors $g_{\mu\nu}$, $h^{\mu\nu}$, and $k_{\mu\nu}$. It can be readily verified by evaluating each side in an arbitrary system by application of the respective transformation laws to the components given in an inertial system that

$$\Gamma^0{}_{00}=\partial_0\log g_{00}{}^{\frac{1}{2}},$$

$$\Gamma^r{}_{mn}=\tfrac{1}{2}h^{rs}(\partial_m k_{ns}+\partial_n k_{sm}-\partial_s k_{mn}). \tag{116b}$$

The last expression together with Eq. (115) shows that $\Gamma^r{}_{mn}$ is the three-dimensional Christoffel symbol formed from k_{mn}.

2. The Relation of the Fundamental Tensors to Measured Quantities

As discussed in Sec. IV.2, the coordinates used in the previous sections had a direct physical significance in terms of length and time measurements; moreover, they had the *same* significance in all allowed coordinate systems. No further information was needed to establish the time or space intervals from a given set of coordinate differentials dx^μ; these intervals were determined by the same *numerical* combinations of the dx^μ in all systems, the fundamental tensors determining them being the same at all points and for all systems. The general coordinates introduced in this section do not have such a direct physical significance. We either must have independent knowledge of the fundamental tensors to determine the intervals or, conversely, we

[50] See, e.g., M. Laue, *Die Relativitätstheorie* (F. Vieweg & Sohn, Braunschweig, 1923), Vol. 2, 2nd Ed., §5, or Ref. 27, §88. Because of our choice (18a) of signs, the signs of all expressions involving an odd number of η's in (112 S) differ from those given in the references quoted.

[51] Some aspects of such a time reparametrization are discussed in A. Grünbaum, *Philosophical Problems of Space and Time* (Alfred A. Knopf, New York, 1963), Chap. 2.

can determine the relevant components of these tensors from suitable measurements.

In the case of special relativity, there exists an invariant four-dimensional interval (27) of the same form as in the general theory of relativity. The problem of relating this expression to space and time measurements in the general theory is discussed in several textbooks[52]; it does not involve the question of the field equations determining the gravitational field at all, but only the use of arbitrary coordinate systems. Thus these discussions apply equally to our case. Therefore we shall not repeat them but only quote the essential results. The length dl of a body at rest in an arbitrary coordinate system as measured by a standard measuring rod (i.e., a rod calibrated to measure length as $dl^2 = -c^2\eta_{mn}dx^m dx^n$ for a body at rest in an inertial system, and assumed not to be affected by acceleration) is related to the metric tensor by

$$dl^2 = \gamma_{mn}\, dx^m\, dx^n,$$
$$\gamma_{mn} = -c^2[\eta_{mn} - \eta_{0m}\eta_{0n}/\eta_{00}], \tag{117S}$$

and thus the six independent components of the spatial metric tensor γ_{mn} can be determined by measurement of six properly chosen line elements at all times. Only in the case of coordinate systems for which all η_{0m} vanish is this tensor proportional to the spatial part η_{mn} of the metric tensor (time-orthogonal systems).

The time interval $d\tau_0$ measured by a standard clock (i.e., a clock calibrated to measure time as x^0 at rest in an inertial system) at rest in an arbitrary system of coordinates and the interval dx^0 measured by a coordinate clock are related by

$$d\tau_0{}^2 = \eta_{00}\, dx^{02} \tag{118S}$$

and thus η_{00} can be determined by comparing the rates of two such clocks. To determine all ten independent components of $\eta_{\mu\nu}$ we still need to measure the one-way velocity of light in three different directions (the round-trip velocity as measured by a standard clock necessarily being equal to c).

In the case of Newtonian mechanics it follows from the existence of the invariant time interval (28) that the time intervals $d\tau_0$ and dx^0 defined as above are related by

$$d\tau_0{}^2 = g_{00}\, dx^{02}. \tag{118N}$$

Because of (112N) the tensor $g_{\mu\nu}$ is thus completely determined by a comparison of the rates of the two clocks.

For the length dl measured with a standard measuring rod in an inertial system we previously found the relation (47), subject to the requirement of simultaneity (46); as these expressions are invariant, they must hold in an arbitrary coordinate system. But as simultaneity implies $dx^0 = 0$ in all coordinate systems, we have

$$dl^2 = -k_{mn}\, dx^m\, dx^n \tag{117N}$$

for any instantaneous measurement (or equivalently a measurement of a body at rest) in any coordinate system, and the six independent components of k_{mn} can be determined by measurement of six properly chosen line elements at all times. The other components $k_{\mu 0}$ have no significance for measurement within a given frame of reference; furthermore, as discussed in Sec. V.1, they do not enter the transformation law for k_{mn} and therefore also have no significance for relating measurements in different frames (and thus all allowed coordinate systems are time-orthogonal). In addition it follows from Eqs. (115) and (112bN) that the tensor $h^{\mu\nu}$ is determined completely from a knowledge of the components of k_{mn}. Unlike the case of special relativity we thus have two separate tensors $g_{\mu\nu}$ and k_{mn} (or equivalently $h^{\mu\nu}$), which transform independently, determining the temporal and spatial metric, respectively.

3. Laws of Motion, Field Equations, and Conservation Laws

The laws of motion of Sec. IV can be transcribed into generally covariant form as discussed in Sec. V.1. We can use the parameters τ discussed there and the four-velocities v^ρ defined by Eq. (108); the four-accelerations now must be defined by

$$a^\rho \equiv Dv^\rho/d\tau \equiv dv^\rho/d\tau + \Gamma^\rho{}_{\mu\nu}v^\mu v^\nu, \tag{119}$$

where $D/d\tau$ denotes covariant differentiation. Instead of Eq. (43) we have then

$$m_0 a^\rho = DP^\rho/d\tau = F^\rho, \qquad P^\rho = m_0 v^\rho, \tag{120}$$

with a^ρ and the force vectors F^ρ satisfying condition (41). Thus, comparing Eq. (120) with Eq. (10), we see that in the absence of forces the particle describes a goedesic in space–time.

If the forces are derivable from a scalar potential, Eqs. (120) become

$$DP^\rho/d\tau = gh^{\rho\sigma}U_{;\sigma} \tag{121N}$$

$$(D/d\tau)(P^\rho + gv^\rho U) = g\eta^{\rho\sigma}U_{;\sigma}. \tag{121S}$$

The generally covariant field equations for these scalar potentials, corresponding to Eqs. (68) of Sec. IV. 3, are

$$h^{\mu\nu}U_{;\mu;\nu} = -4\pi\rho, \tag{122N}$$

$$\eta^{\mu\nu}U_{;\mu;\nu} = -4\pi\rho. \tag{122S}$$

Since the left-hand sides of these equations are scalars, this must also be the case for ρ and thus for the integral appearing in its definition previously given by Eq. (69) (assuming the constants g_i and G to be scalars). But δ^4 has the transformation property of a scalar density of weight one. In the considerations of

[52] See, e.g., Ref. 50, §14, or Ref. 27, Chap. VIII. A very detailed discussion of this and related questions is given by H. Arzeliès, *Relativité Généralisée. Gravitation* (Gauthier-Villars, Paris, 1961), Fasc. I.

Sec. IV we did not have to distinguish between tensors and tensor densities, because the transformation determinant (5) entering the definition of a tensor density (and also the expression for the transformation of a volume element) equals 1 for all proper Galilei and Lorentz transformations. This is no longer the case for the general transformations considered in this section. Therefore we must replace Eq. (69) by

$$\rho = G\sum_i g_i \int_{-\infty}^{\infty} |g|^{-\frac{1}{2}} \delta^4[s_i^\sigma(\tau_i)]\, d\tau_i, \qquad (123)$$

where $|g|^{-\frac{1}{2}}$ is a scalar density of weight -1 (with the invariant volume integral over ρ given by $\int \rho\, |g|^{\frac{1}{2}}\, d^4x$). If the affine space considered admits such a density with vanishing covariant derivative, then the invariant volume integral can be defined unambiguously (up to a trivial constant factor).[53] The necessary and sufficient condition for this is the vanishing of expression (9), which in our case is trivially satisfied because of Eq. (106). For the space of special relativity (or any other Riemannian space) we must identify g with the determinant of the metric tensor. For the space of Newtonian mechanics we can also construct a function with the desired properties, which we can normalize to be equal to one in a Cartesian inertial system; we do not need its explict form here. With these choices Eq. (123) reduces to (69) in an inertial system.

In a general coordinate system the integrals of Eqs. (122) do not in general have such a simple form as those of Eqs. (68), and we do not consider them any further here. In such coordinate systems we also can not introduce any simple conservation laws of the types considered in Sec. IV. 5, as there is no meaning to adding vectors at different points, and the parallel transfer to a single point of vectors separated by finite distances leads to unwieldy expressions. We can, however, introduce local conservation laws of the type considered in Sec. IV. 6, and can relate these to the laws of motion in close analogy to the considerations given there. We now start from a generally covariant conservation law for the total energy-momentum tensor

$$T^{\rho\sigma}{}_{;\sigma} = 0 \qquad (124)$$

rather than Eq. (100). The tensors $T_f{}^{\rho\sigma}$ are given by Eqs. (98) as before; by virtue of Eqs. (122) it is their covariant rather than their ordinary divergence which is now given by Eqs. (99). As Eq. (9) is satisfied in the spaces under consideration, Eqs. (124) also imply similar conservation laws for the corresponding tensor densities. Using these rather than Eqs. (124) as a starting point, we can again carry through the procedure of Ref. 43 as discussed in Sec. IV.6, arriving at the same Eqs. (101), and can proceed from there to Eqs. (121) in complete analogy; however, now it is the tensor densities rather than the tensors which have the form (105). Instead, the tensors are given by

$$P^{\rho\sigma} = \sum_i \int_{-\infty}^{\infty} |g|^{-\frac{1}{2}} m_{0i} v_i^\rho v_i^\sigma \delta^4(s_i^\mu)\, d\tau_i, \qquad (125\text{N})$$

$$P^{\rho\sigma} = \sum_i \int_{-\infty}^{\infty} |g|^{-\frac{1}{2}} (m_{0i} - g_i U) v_i^\rho v_i^\sigma \delta^4(s_i^\mu)\, d\tau_i. \qquad (125\text{S})$$

4. "Inertial Forces"

The factual content of the law of motion (120) in four-dimensional, generally covariant form is of course still the same as that of the three-dimensional law (49) valid only in an inertial system. The latter in the Newtonian case equates the mass times the three-acceleration to the (external) force; similarly the former equates the mass times the four-acceleration to the four-force. However, the four-acceleration (119) was defined as the *covariant* rather than the ordinary derivative of the four-velocity v^ρ. It is frequently felt to be covenient to designate the *ordinary* derivative by the word acceleration, and still to have a terminology available which permits a description of the law (120) as "mass times acceleration equals force." To this effect the word "inertial force" is introduced for the term $-m_0\Gamma^\rho{}_{\mu\nu}v^\mu v^\nu$ in this law. This terminology is in agreement with much of the literature of general relativity; however, the term in the law (120) covered by this word is not necessarily equivalent to the term designated by the same word in most of the literature of Newtonian mechanics. The difference arises naturally from the difference of a vectorial and a coordinate formulation of the laws. In a vectorial formulation of Newtonian mechanics only the difference between the vectors $m\mathbf{a}'$ and $m\mathbf{a}$, which is entirely due to the *acceleration* of a noninertial frame of reference with respect to an inertial one, is labeled inertial force. In a formulation of mechanics in terms of coordinates (whether in three or four dimensions) it is the difference between an arbitrary $m_0\, dv'^\rho/d\tau$ and an $m_0\, dv^\rho/d\tau$ expressed in *Cartesian coordinates* as well as in an inertial system which is called inertial force. Thus no distinction is made between terms arising purely because of the use of non-Cartesian spatial coordinate systems and those arising from more general transformations. Such a distinction could be made only at the expense of much of the formal simplification introduced by allowing the use of arbitrary coordinate systems.

From their very definition, inertial forces and external forces have different transformation properties, only the latter being four-vectors.

VI. NEWTON'S THEORY OF GRAVITATION AND GENERAL RELATIVITY

1. Einstein's Theory of General Relativity

The General Theory of Relativity is based on Einstein's principle of general covariance (I) and his

[53] See, e.g., L. P. Eisenhart, *Non-Riemannian Geometry* (American Mathematical Society, New York, 1927), Sec. 5.

principle of equivalence (II). They are usually stated as: (I) If properly formulated, the laws of physics are of the same form in all coordinate systems; (II) The local effects of a gravitational field are equivalent to those appearing in the description of physical phenomena relative to an accelerated frame of reference.

It is frequently asserted or implied in expositions of the theory of relativity that these two principles by themselves lead to Einstein's theory of general relativity. However, as mentioned in the Introduction, principle (I) is a statement without physical content, and any theory, including Newtonian mechanics, can be fitted to it. Principle (II) is usually introduced by illustrations such as "Einstein's elevator" which show that in Newtonian mechanics *gravitational* forces can be replaced by properly chosen *inertial* forces (because of the universal proportionality of gravitational and inertial mass) and it is then asserted that (II) only extends this equivalence to nonmechanical phenomena.

If this is all that the two principles are understood to contain, then, unlike the relation between the fundamental principles of Newtonian mechanics and of special relativity, the basic principles of Newtonian mechanics and general relativity are compatible; indeed, we have in Sec. V already given one formulation of Newtonian mechanics consistent with these principles, as is discussed in Subsection 2.

On the other hand, it is also frequently asserted (or tacitly assumed) that *any inertial* force can be replaced by a suitably chosen *gravitational* force. This statement does not follow from Einstein's elevator or similar illustrations, and indeed is not true either for the usual three-dimensional formulations of Newtonian mechanics or for the four-dimensional formulation of Sec. V. Nevertheless, another formulation of Newtonian mechanics (identical in physical content with the usual one) can be given which does conform to this interpretation of (II), as is shown in Subsection 3. Thus even with this more stringent interpretation of the principle of equivalence we find that Newtonian mechanics is compatible with the basic principles of general relativity.

Actually, what is usually called "General Thoery of Relativity" is a *specific* theory in conformity with (I) and (II) (similar to the situation for the special theory discussed in Sec. III). An essential requirement needed to obtain this specific theory is that in the absence of gravitational fields it should reduce to the special theory of relativity, in particular to its space–time structure. From (II) it follows that there always exist an infinity of frames of reference (the inertial systems) in which the effects of a gravitational field have been transformed away in the neighborhood of a point; these are the systems in which special relativity is required to hold. Clearly this specific general theory of relativity can not agree with Newtonian physics except in the previously considered limit $c\rightarrow\infty$. But in addition it shows features not generally associated with either Newtonian or special relativistic theory. It describes the effects of gravitation in geometric terms, relating them to the affine connections of space–time, rather than in terms of forces; nongravitational effects exert their influence on the geometry and on the motion of particles through their energy–momentum tensor $T_f^{\rho\sigma}$; the equations of motion are a consequence of the covariant conservation law for the total energy–momentum tensor, which itself is a consequence of the field equations; in the absence of nongravitational fields the particles move along geodesics. In this section we compare this specific general theory (to be referred to as Einstein's general theory) and Newtonian mechanics, but it should be clearly understood that the formulations of Newtonian mechanics to be exposed are themselves "general relativistic" in the sense discussed above. We find that many of the features of Einstein's general theory, often thought to be unique, actually have their counterpart in a suitably reformulated Newtonian general theory.

In Einstein's theory both the effects of gravitation and the geometry of space–time are described in terms of the metric tensor $g_{\mu\nu}$. In the absence of nongravitational fields this tensor is determined by the equations[54]

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=\kappa g_{\mu\rho}g_{\nu\sigma}P^{\rho\sigma} \tag{126}$$

where $R_{\mu\nu}$ is the contracted curvature tensor (8), and

$$R=g^{\rho\sigma}R_{\rho\sigma}, \qquad \kappa=8\pi G; \tag{126E}$$

$P^{\rho\sigma}$ is the energy–momentum tensor of matter. An equivalent form of the fundamental equation is

$$R_{\mu\nu}=\kappa(g_{\mu\rho}g_{\nu\sigma}P^{\rho\sigma}-\tfrac{1}{2}g_{\mu\nu}P),$$
$$P=g_{\rho\sigma}P^{\rho\sigma}, \qquad \kappa=8\pi G. \tag{127}$$

If nongravitational fields are present, $P^{\rho\sigma}$ has to be replaced by $T^{\rho\sigma}=P^{\rho\sigma}+T_f^{\rho\sigma}$ in the above equations.

The metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ defined by Eq. (11) both have vanishing covariant derivatives:

$$g_{\mu\nu;\rho}=0, \tag{128aE}$$

$$g^{\mu\nu}{}_{;\rho}=0. \tag{128bE}$$

The expressions for the Γ's as functions of the metric tensor can be obtained from either of the two sets of equations (128E) by resolution, making use of Eq. (11), and yield the Christoffel symbols (12) for the affine connections. With these particular connections, Eq. (9) is automatically satisfied, and $R_{\mu\nu}$ is symmetric. The left-hand side of Eq. (126) in its contravariant form satisfies the contracted Bianchi identity

$$(R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R)_{;\nu}=0. \tag{129}$$

This relation can of course also be regained from Eq. (127). Eqs. (126) or (127) were postulated by Einstein

[54] Analogous to Sec. IV, in this Section we denote corresponding formulas of Newtonian mechanics and of Einstein's theory by N and E, respectively, and (except as noted otherwise) all other formulas are valid for both theories.

as the basic law determining the gravitational and geometric properties of space because he wanted the covariant conservation law for energy and momentum

$$P^{\rho\sigma}{}_{;\sigma}=0 \qquad (130)$$

to be an automatic consequence of the field equations [which it is by Eq. (129)] and because he wanted these differential equations to be of second order, in analogy to the Newtonian Poisson equation (68N).

The value of κ in Eqs. (126) and (127) is determined both by physical and by formal considerations. From the definitions (8), (12), and (13) it follows that the dimensions of the left-hand side of the field equations are those of $[x^\mu x^\nu]^{-1}$; then the form of their right-hand side together with definition (11) imply that the dimensions of κ are those of $[ds^2 g_{\rho\sigma}P^{\rho\sigma}]^{-1}$. Throughout this paper we have used conventions such that $x^0=t$ and ds have the dimensions of time, g_{00} is dimensionless, and P^{00} is a mass density; then κ has the same dimensions as G. Its numerical value follows from the requirement that in an appropriately chosen limit the solutions of the field equations should reduce to the Newtonian gravitational potential; with our particular conventions we obtain the value given above. Another convention, used in most expositions of Einstein's theory, is to take $x^0=ct$ and ds to have dimensions of length, and P^{00} to be an energy density; then κ equals $8\pi Gc^{-4}$. An advantage of our conventions is that they allow us to write the field equations in a form not containing c. This constant then enters the theory only through the physical requirement that in the absence of gravitation the metric should reduce to that of the special theory of relativity, as discussed above. We return to the implications of this result in Sec. VII.

In his original work Einstein postulated the law of motion of a test particle in the absence of nongravitational fields to be that of a geodesic

$$dv^\rho/d\tau+\Gamma^\rho{}_{\mu\nu}v^\mu v^\nu=0, \qquad (131)$$

with

$$d\tau^2=g_{\mu\nu}\,dz^\mu\,dz^\nu. \qquad (132)$$

It was later realized by him and others that this law need not be postulated, but is a consequence of the conservation law (130), and that it is valid even if the mass of the particle under consideration is not negligible compared to that of other bodies present.[55]

Eq. (131) has the form of the law of motion of a particle moving under the influence of "inertial forces" only, as discussed in Sec. V.4. Thus in Einstein's theory the gravitational effects are not considered to be due to an external force, but as inertial effects; conversely, any "inertial force" is equivalent to a gravitational one, and principle (II) in its most stringent interpretation is satisfied.

[55] For a simple derivation and references to earlier work see P. Havas and J. N. Goldberg, Phys. Rev. **128**, 398 (1962).

2. Covariant Formulation of Newton's Theory of Gravitation in Flat Space-Time

In Sec. V we presented a generally covariant formulation of Newtonian and special relativistic mechanics. In this formulation the field equations of Newton's theory of gravitation are Eqs. (122N) and (123) with $g_i=m_{0i}$; the equations of motion are given by (121N).

We now investigate whether this formulation, designed to satisfy principle (I), also conforms to the principle of equivalence (II). Using Eqs. (120), (108), and (132), we can write the equations of motion as

$$dv^\rho/d\tau+(\Gamma^\rho{}_{\mu\nu}-g_{\mu\nu}h^{\rho\sigma}U_{;\sigma})v^\mu v^\nu=0. \qquad (133)$$

Thus the effect of a given gravitational field $U_{;\sigma}$ on any particle regardless of velocity can be transformed away at a given event $x_0{}^\lambda$ (i.e., at a given point in space at a given time only) if a physically acceptable coordinate system S' can be found in which

$$\Gamma^\rho{}_{\mu\nu}-g_{\mu\nu}h^{\rho\sigma}U_{;\sigma}$$

vanishes at $x_0{}^\lambda$. Since this expression transforms like an affine connection, its vanishing at $x_0{}^\lambda$ can always be achieved[16]; it is sufficient to take

$$x^\rho=x'^\rho-\tfrac{1}{2}(\Gamma^\rho{}_{\mu\nu}-g_{\mu\nu}h^{\rho\sigma}U_{;\sigma})_{x_0{}^\lambda}x'^\mu x'^\nu \qquad (134)$$

for the inverse transformation. Because of the restrictions (112N) and (116) this transformation is of the form (113b) as required. In particular, if the original coordinate system was an inertial one, the new one uses the same time coordinate, and has spatial coordinates in constant acceleration with respect to the inertial ones.

On the other hand, the effect of a given affine connection on a particle with arbitrary velocity *cannot* in general be replaced by that of a suitably chosen gravitational field. This would require that $\Gamma^\rho{}_{\mu\nu}$ be replaced by $-g_{\mu\nu}h^{\rho\sigma}U_{;\sigma}$, but our Eqs. (116b) allow a nonvanishing $\Gamma^r{}_{mn}$ while Eq. (112aN) excludes a nonvanishing g_{mn}. Thus only for very special coordinate systems (those whose spatial coordinates are obtainable from those of an inertial system by a fixed linear transformation plus a constant acceleration) is such a replacement possible.

Even if we do not require that such a replacement be possible regardless of the velocity of a particle, but that only the effects on particles at rest should be the same, we can not always find an equivalent gravitational field. This would imply replacing $\Gamma^\rho{}_{00}$ by $-g_{00}h^{\rho\sigma}U_{;\sigma}$; but Eq. (112bN) excludes a nonvanishing h^{00}, while Eqs. (116b) allow a nonvanishing $\Gamma^0{}_{00}$. It vanishes only in a coordinate system whose time coordinate differs from that of an inertial system at most by a linear transformation.

The fact that we can not have a general equivalence between gravitational and inertial forces is due to their different transformation properties; the gravitational forces were assumed to transform as four-vectors in Eqs. (125), unlike the inertial ones. Therefore a general equivalence requires a different assumption as to the behavior of gravitational forces under general coordinate transformations. Such an assumption must of course be compatible with the physical content of the theory in its original form.

Aside from satisfying principle (II) only in a restricted sense, the covariant formulation of the theory presented in Sec. V has a structure completely different from that of Einstein's general theory. It is a theory in flat rather than in curved space; whereas in Einstein's theory the space–time structure becomes simpler if the gravitational field has been transformed away (reducing to the well-understood structure of the special theory), here it becomes more complicated; unlike Einstein's theory, there is no simple geometrical significance to either the equations of motion or the gravitational field.

3. Covariant Formulation of Newton's Theory of Gravitation in Curved Space–Time

To obtain a formulation of Newton's theory more akin to Einstein's, we start from the law of motion. In Einstein's theory, in the absence of nongravitational fields, a particle's motion is determined by the geodesic law (131) with (132). We can consider our previous generally covariant law of motion (133) as the equation of such a geodesic by simply considering $\Gamma^{\rho}{}_{\mu\nu}-g_{\mu\nu}h^{\rho\sigma}U_{;\sigma}$ rather than $\Gamma^{\rho}{}_{\mu\nu}$ to be the affine connection of our space. We shall introduce this change together with a new notation. We now denote the *new* affine connection by $\Gamma^{\rho}{}_{\mu\nu}$ and put

$$\Gamma^{\rho}{}_{\mu\nu}=\Omega^{\rho}{}_{\mu\nu}+\Lambda^{\rho}{}_{\mu\nu}, \tag{135}$$

where $\Lambda^{\rho}{}_{\mu\nu}$ denotes our *previous* affine connection $\Gamma^{\rho}{}_{\mu\nu}$, and where

$$\Omega^{\rho}{}_{\mu\nu}=-g_{\mu\nu}h^{\rho\sigma}U_{;\sigma} \tag{135N}$$

transforms as a tensor. Thus the previous Eq. (106) now becomes

$$R^{\kappa}{}_{\mu\lambda\nu}(\Lambda)=0 \tag{136}$$

and all components of $\Lambda^{\rho}{}_{\mu\nu}$ vanish in a Cartesian inertial system (to be referred to as a special coordinate system in the following).

Our law of motion is now in the form (131) which allows it to be interpreted as the law of motion of a particle under the influence of "inertial forces" only, and conversely any "inertial forces" can be considered as gravitational ones, just as was the case in Einstein's general theory. At first sight this might appear to imply a difference in the physical content of the formalism developed here and of that discussed in the previous subsection. Actually only a change in language is involved; we are describing the same law of motion (133) or (131) by different words, now considering the entire expression (135) as responsible for gravitational or equivalently for inertial effects, while previously we reserved the word gravitational for the term $\Omega^{\rho}{}_{\mu\nu}$ and the word inertial for the term $\Lambda^{\rho}{}_{\mu\nu}$.

Although the Riemann–Christoffel curvature tensor formed from the Λ's vanishes, this is in general not the case for the curvature tensor formed from the Γ's. Therefore the space-time of the x^{ρ} with the affine connection (135) is *curved*, just like that of Einstein's theory. However, unlike that of Einstein's theory, the curved space of Newtonian theory is not metric, since $g_{\mu\nu}$ is singular; the tensor $h^{\mu\nu}$ is not its inverse, but is related to it by Eq. (22). Nevertheless, the covariant derivatives of both these tensors vanish (as can be easily verified in a special coordinate system; being tensor relations, these equations must therefore hold in all systems). Explicitly they are given by

$$g_{\mu\nu;\rho}\equiv\partial_{\rho}g_{\mu\nu}-\Gamma^{\sigma}{}_{\mu\rho}g_{\sigma\nu}-\Gamma^{\sigma}{}_{\nu\rho}g_{\mu\sigma}=0, \tag{128aN}$$

$$h^{\mu\nu}{}_{;\rho}\equiv\partial_{\rho}h^{\mu\nu}+\Gamma^{\mu}{}_{\rho\sigma}h^{\sigma\nu}+\Gamma^{\nu}{}_{\rho\sigma}h^{\mu\sigma}=0. \tag{128bN}$$

However, now these equations *cannot* be solved for the Γ's since $g_{\mu\nu}$ and $h^{\mu\nu}$ are singular.

It can be easily verified that the expressions analogous to (128N), but formed with the Λ's (which we shall denote by a comma instead of a semicolon), which correspond to our previous relations (107N), also vanish:

$$g_{\mu\nu,\rho}\equiv\partial_{\rho}g_{\mu\nu}-\Lambda^{\sigma}{}_{\mu\rho}g_{\sigma\nu}-\Lambda^{\sigma}{}_{\nu\rho}g_{\mu\sigma}=0, \tag{137a}$$

$$h^{\mu\nu}{}_{,\rho}\equiv\partial_{\rho}h^{\mu\nu}+\Lambda^{\mu}{}_{\rho\sigma}h^{\sigma\nu}+\Lambda^{\nu}{}_{\rho\sigma}h^{\mu\sigma}=0. \tag{137b}$$

In general we can define comma derivatives of arbitrary tensors in complete analogy to covariant derivatives by Eq. (7), with all Γ's replaced by Λ's. Since the Λ's as well as the Γ's transform according to Eq. (6), the comma derivatives also are tensors.

Now we can impose the necessary requirement on our basic quantities without *first* having to introduce a special coordinate system, just as in Einstein's theory. In the latter theory, we can not accept just any metric tensor $g_{\mu\nu}$, but we must require that at any given point there exist a frame of reference in which the space–time structure is that of the Lorentz group, i.e., in which $g_{\mu\nu}$ and $g^{\mu\nu}$ take on the values (18). Similarly we must now require that only those $g_{\mu\nu}$ and $h^{\mu\nu}$ are admissible for which at any given point there exists a frame of reference in which the space–time structure is that of the Galilei group, i.e., in which these tensors take on the values (21) and $\Lambda^{\rho}{}_{\mu\nu}$ vanishes. Up to a linear transformation this condition on $g_{\mu\nu}$ and $h^{\mu\nu}$ can be expressed invariantly in terms of the signatures exactly as in Sec. V. But the vanishing of $\Lambda^{\rho}{}_{\mu\nu}$ at one point implies because of Eq. (136) that there exists a frame of reference in which $\Lambda^{\rho}{}_{\mu\nu}$ vanishes *everywhere*, and Eq. (137) then guarantees that $g_{\mu\nu}$ and $h^{\mu\nu}$ have the values (21) *everywhere* unlike the case of Einstein's theory, where (except in the trivial case of flat space) there does not exist a frame of reference in which $g_{\mu\nu}$ can be reduced to $\eta_{\mu\nu}$ everywhere.[56]

Just as in the covariant formulation of the theory given in Sec. V, there is no mathematical need to introduce restrictions on the allowed coordinate transformations; however, the same physical considerations apply, and we can therefore again only consider transformations of the form (113) to be physically admissible. Then we are led to the same results (112N), (114), and (115) concerning the fundamental tensors, regardless of the fact that space now is curved rather than flat. However, the particular form (116) found for the components of $\Gamma^{\rho}{}_{\mu\nu}$ now applies to the components of $\Lambda^{\rho}{}_{\mu\nu}$ rather than $\Gamma^{\rho}{}_{\mu\nu}$ because of our change in notation.

The considerations of Sec. V.2 apply whether space is curved or not, and thus the relation of the fundamental tensors to measured quantities is the same now as in the case previously considered.

It remains to find the equations satisfied by the affine connection $\Gamma^{\rho}{}_{\mu\nu}$. We first calculate $R_{\mu\nu}$ as given by (8). In the special coordinate systems this equals $-g_{\mu\nu}h^{\rho\sigma}\partial_{\rho\sigma}U$, which in turn can be written as either

$$R_{\mu\nu}=-g_{\mu\nu}h^{\rho\sigma}(\partial_{\rho\sigma}U-\Lambda^{\kappa}{}_{\rho\sigma}\partial_{\kappa}U)=-g_{\mu\nu}h^{\rho\sigma}U_{,\rho,\sigma} \qquad (138a)$$

or

$$R_{\mu\nu}=-g_{\mu\nu}h^{\rho\sigma}(\partial_{\rho\sigma}U-\Gamma^{\kappa}{}_{\rho\sigma}\partial_{\kappa}U)=-g_{\mu\nu}h^{\rho\sigma}U_{;\rho;\sigma}. \qquad (138b)$$

The first of these relations follows because $\Lambda^{\rho}{}_{\mu\nu}$ itself vanishes in the special coordinate systems, and the second because $h^{\rho\sigma}\Gamma^{\kappa}{}_{\rho\sigma}$ vanishes.

In these special systems Poisson's equations must hold if Newton's theory of gravitation is to be valid. But then the right-hand side of Eqs. (138) equals

$$g_{\mu\nu}4\pi G\sum_i\int_{-\infty}^{\infty}|g|^{-\frac{1}{2}}\,\delta^4(s_i{}^{\rho})\,d\tau_i \qquad (139a)$$

from Eqs. (122N) and (123). Using Eq. (125N), we can write this as

$$4\pi Gg_{\mu\nu}g_{\rho\sigma}P^{\rho\sigma}; \qquad (139b)$$

from the condition (21a) on the value of $g_{\mu\nu}$ in the special coordinate system this is equal to

$$4\pi Gg_{\mu\rho}g_{\nu\sigma}P^{\rho\sigma}. \qquad (139c)$$

Eqs. (138) and (139c) allow us to postulate as our generally covariant form of Poisson's equation the same Eq. (126) as in Einstein's theory, but with

$$R=h^{\rho\sigma}R_{\rho\sigma}, \qquad \kappa=4\pi G; \qquad (126N)$$

here R vanishes because of Eq. (22), and the constant κ is half that of Einstein's theory. If we use Eqs. (138) and (139b, c), however, we could equivalently postulate Eq. (127) as our basic equation, with the *same* value of κ as in Einstein's theory. Because of the vanishing of R the left-hand side of Eqs. (126) and (127) has the same value in the Newtonian case.

Since $g_{\mu\nu}$ is singular, we can not obtain either a Bianchi identity (129) or a conservation law (130) from the Newtonian equations (126) or (127). We can, on the other hand, either postulate that $P^{\mu\nu}$ should have the form (125N) which satisfies Eq. (130) and implies Eq. (131), or we could postulate Eq. (130) as a separate requirement. The latter procedure is preferable, as it permits consideration of more general distributions than (125N).

Given the conservation law (130), we can in both Einstein's and Newton's theory derive the law of motion (131) with (132) from it, as well as the form (125N) of the energy–momentum tensor, provided we restrict ourselves to simple poles of the gravitational field, i.e., to a form (97) for each particle. For Einstein's theory this has been known for some time[55]; for the Newtonian case it was shown recently in Ref. 43. Similarly we can derive the law of motion and the form of the energy–momentum tensor for particles with an intrinsic dipole moment in both cases.[43]

If we do not have singular distributions, but are dealing with continua, we can still maintain the field equations (126) or (127) and the form (130) of the conservation law. It still contains the equations of motion, but their form now depends on the equation of state of the matter under consideration; the Newtonian problem does not differ in principle from the Einsteinian one, which has been investigated for many special cases.[57]

As noted in the Introduction, the Newtonian mechanics of continua has been given a generally covariant formulation some time ago and a comprehensive review was given in Ref. 12. This formulation was that of the flat space of Sec. V only, implying consideration of gravitation as a body force. However, it can be generalized along the lines of this subsection without difficulty; all that is required to allow treating

[56] Up to this point the mathematical considerations of this subsection are essentially those of K. Friedrichs, Ref. 15, except that the analysis of Sec. V allowed us to start from a generally covariant four-dimensional formalism rather than from the original three-dimensional Newtonian formalism. The following analysis of the allowed coordinate transformations and of the significance of the fundamental tensors, as well as the introduction of the field equations (126) and (127) go beyond that paper, in which no sources and thus also no tensor $P^{\rho\sigma}$ were introduced. Cartan's work (Ref. 14) also contains a generally covariant formulation of Newton's theory of gravitation (including continuous sources). Although differing in many mathematical details, its results are basically equivalent to those presented here. The results obtained recently by A. Trautman, Compt. Rend. **257**, 617 (1963), are equivalent to those of Ref. 15.

[57] See, e.g., R. C. Tolman, Ref. 11; V. Fock, Ref. 11; G. C. McVittie, *General Relativity and Cosmology* (Chapman & Hall Ltd., London, 1956).

gravitation as a geometric concept is to interpret the affine connections $\Gamma^\rho{}_{\mu\nu}$ of Ref. 12 (originally introduced as those of flat space as in Sec. V) as describing the curved space of this subsection, and thus as being determined by the field equations (126) or (127).[58]

The field equations are of course nonlinear in the Newtonian case as in Einstein's theory because of the structure (8) of $R_{\mu\nu}$; however, for the Newtonian theory this nonlinearity must be spurious. Mathematically this is evident from Eqs. (138) because in the special coordinate system the factor $g_{\mu\nu}h^{\rho\sigma}$ reduces to a constant.

It might be instructive to consider some applications of the equations considered here before embarking on a detailed comparison of the two theories. In the field of a single body of mass M the nonvanishing independent Γ's are given in Einstein's theory from the Schwarzschild metric as[59]

$$\Gamma^0{}_{10}=(2b)^{-1}\,db/dr,\qquad \Gamma^1{}_{00}=(c^2/2a)\,db/dr,$$
$$\Gamma^1{}_{11}=(2a)^{-1}\,da/dr,\qquad \Gamma^1{}_{22}=-r/a,$$
$$\Gamma^1{}_{33}=\Gamma^1{}_{22}\sin^2\vartheta,\qquad \Gamma^2{}_{12}=\Gamma^3{}_{13}=r^{-1},$$
$$\Gamma^2{}_{33}=-\sin\vartheta\cos\vartheta,\qquad \Gamma^3{}_{23}=\cot\vartheta,$$
$$b=1-2GM/rc^2,\qquad a=b^{-1},\tag{140E}$$

in Schwarzschild coordinates $(r, \vartheta, \varphi, t)$. In the Newtonian case we have to evaluate Eq. (135). The potential U equals $-GM/r$; the calculation of the Λ's (arising from the use of polar coordinates) can be simplified by applying Eqs. (116). The nonvanishing independent Γ's are given by

$$\Gamma^1{}_{00}=GM/r^2,\qquad \Gamma^1{}_{22}=-r,$$
$$\Gamma^1{}_{33}=\Gamma^1{}_{22}\sin^2\vartheta,\qquad \Gamma^2{}_{12}=\Gamma^3{}_{13}=r^{-1},$$
$$\Gamma^2{}_{33}=-\sin\vartheta\cos\vartheta,\qquad \Gamma^3{}_{23}=\cot\vartheta.\tag{140N}$$

Clearly in the limit $c\rightarrow\infty$ the Schwarzschild Γ's reduce to the Newtonian ones, regardless of the magnitude of M. Thus the curvature tensor of the Schwarzschild field also reduces to that of the Newtonian field in the same limit. We do not give the explicit expressions for the former; for the latter, the only independent nonvanishing components are, from (140N) and (8),

$$R^1{}_{010}=-2R^2{}_{020}=-2R^3{}_{030}=-2GM/r^3.\tag{141}$$

The geodesics corresponding to bound orbits show the well-known advance of the perihelion in the first case, and no such effect in the second.

To find the paths of light rays is a problem beyond the scope of mechanics. However, in the case of Einstein's general theory these paths are geodesics of zero length [which must be described by a parameter different from (132)]; this leads to the well-known deflection of light rays in the field of a large mass. If we use the same definition in the Newtonian case, taking $ds^2=0$ in (111N), this corresponds to infinite speed and to straight lines in the special coordinate systems. If we take instead the geodesics (131) corresponding to particles moving with speed c (which are null geodesics in Einstein's theory, but not here), we get half the deflection following from Einstein's theory.[60]

The similarity between the general relativistic Newtonian and Einsteinian field equations (126) or (127) allows a new approach to the subject of Newtonian cosmology,[61] which will be discussed elsewhere.[62]

4. Inclusion of Nongravitational Fields and Comparison with Einstein's Theory

In the formulation of Newtonian mechanics given in the previous subsection we did not consider any nongravitational forces. In Einstein's theory such forces are included by adding their energy–momentum tensor $T_f{}^{\rho\sigma}$ to that of matter $P^{\rho\sigma}$ in the field equations (126) or (127), as noted before, and it is natural to follow the same procedure in the Newtonian case; this implies no new physical effects, as is discussed below. In Einstein's theory the field equations then lead to a conservation law of the form (124) for matter plus nongravitational field energy; such a conservation law has to be postulated separately in the Newtonian case. The equations of motion can again be obtained from this conservation law [43,63]; in the Newtonian case the procedure is the same as that of Sec. V.3, only the interpretation of the $\Gamma^\rho{}_{\mu\nu}$ being different, which now include the gravitational effects.

The tensors $T_f{}^{\rho\sigma}$ to be included in Eqs. (126) and (127) must be obtained by a suitable generalization of the special relativistic and the Newtonian tensors, respectively, which in turn follow from the particular field equations satisfied by the nongravitational fields under consideration. In the case of Einstein's theory, the requirement imposed (as discussed in Sec. VI.1) is that the equations should reduce to those of special relativity in an inertial system; such a system is defined as one in which all the components of $\Gamma^\rho{}_{\mu\nu}$ vanish at a point x^λ and the components of $g_{\mu\nu}$ reduce to the values

[58] Compare also Sec. 238 of Ref. 12; some of the generalizations mentioned there are formal extensions of mechanics to generalized spaces and thus differ from Newtonian mechanics in their physical content, whereas the generalization considered here is simply a *transcription* of Newtonian mechanics including gravitation.

[59] See, e.g., Ref. 27, Secs. 122 and 123.

[60] J. Soldner, Berl. Astron. Jahrbuch 161 (1804), reprinted in Ann. Phys. **65**, 593 (1921).

[61] For a recent review see O. Heckmann and E. Schücking in *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1959), Vol. LII, p. 489.

[62] P. Havas (to be published).

[63] For the methods available in Einstein's general theory see L. Infeld and J. Plebański, *Motion and Relativity* (Pergamon Press–PWN, New York–Warsaw, 1960) and references given there; P. Havas, Phys. Rev. **108**, 1351 (1957) (a detailed account of the application to nongravitational fields along the lines described in Ref. 55 is in preparation).

(18a). Thus an inertial system can in general only be defined locally. On the other hand, it is generally considered to be a particularly gratifying feature of Einstein's general theory that it allows a definition of an inertial system which avoids the difficulty posed in Newtonian mechanics; there an inertial system is defined as one in which Newton's first law is valid, but a test of this law requires the absence of a resultant force, whereas Newton's law of gravitation proclaims the omnipresence of gravitational forces.

In the case of the generally covariant form of Newtonian theory under discussion, we have of course global systems of reference available (the ones used in Sec. IV, called inertial systems there in agreement with customary usage, and called special systems in this section), in which the usual results of Newtonain mechanics hold in their usual form. Thus we must choose our nongravitational field equations and the corresponding $T_f^{\rho\sigma}$ to agree with the equations of Sec. IV in these systems. This seems both to disagree with the procedure of Einstein's general theory, and to deprive us of an important advantage of this theory. Actually, this is not the case, as will now be shown.

In the present context, we must use a *local* description of Newtonian nongravitational forces by means of field equations such as Eq. (68N). Newtonian mechanics assumes that forces originating in physically different sources (such as gravitating masses and electric charges) are additive, and thus a field equation for a nongravitational field in the special coordinate systems can not contain any gravitational fields, and conversely.

This implies that the inclusion of a nongravitational $T_f^{\rho\sigma}$ in Eqs. (126) and (127) is only allowed if it is purely formal, but does not have any effect on the equations determining the $\Gamma^\rho{}_{\mu\nu}$. This is indeed the case for the $T_f^{\rho\sigma}$ of the scalar field considered in Sec. IV; it is given by Eq. (98N), and thus $g_{\mu\rho}T_f^{\rho\sigma}$ vanishes. We can expect a similar behavior for any other $T_f^{\rho\sigma}$ which might be defined from Newtonian scalar field equations other than (68N) which are associated with instantaneous action-at-a-distance forces, both for physical reasons and because no contravariant tensor other than $h^{\mu\nu}$ is available for the construction of $T_f^{\rho\sigma}$; then Eq. (22) guarantees the vanishing of $g_{\mu\rho}T_f^{\rho\sigma}$.

In the differential equations for the nongravitational fields, any gravitational terms are described by the $\Gamma^\rho{}_{\mu\nu}$. Thus the requirement that such terms should be absent in the special coordinate system implies that the equations for the fields must be the *same* whether they are written in terms of ordinary derivatives [such as Eq. (68N)] or (as required for a generally covariant formulation) in terms of covariant derivatives. [This is indeed the case for Eq. (68N), since the additional term $h^{\rho\sigma}\Gamma^\kappa{}_{\rho\sigma}\partial_\kappa U$ introduced by using covariant differentiation vanishes in the special system, as noted before (with a different meaning of U!) in deriving Eq. (138c).] But this is also the case in coordinate systems in which the $\Gamma^\rho{}_{\mu\nu}$ themselves vanish. As any scalar potential remains unchanged under arbitrary coordinate transformations, it only remains to be shown that there are coordinate systems with vanishing $\Gamma^\rho{}_{\mu\nu}$ in which the fundamental tensors have the same numerical values as in the special coordinate systems to establish that we could adopt a procedure completely analogous to that of Einstein's theory. This is accomplished by the same procedure by which it was shown in Sec. VI.2 that a given gravitational field can be transformed away. The same transformation (134) can be used, except that due to our change of notation the term in square brackets now is just $\Gamma^\rho{}_{\mu\nu}$. We still simply have a transformation of the form (113b), with the same time coordinate used in both coordinate systems, and with Cartesian spatial coordinates in constant acceleration with respect to each other. But from the considerations of Sec. V.1 it follows immediately that such a coordinate transformation does not change the numerical values of $g_{\mu\nu}$ and $h^{\mu\nu}$.

Thus we are free to define an inertial system as one in which all the components of $\Gamma^\rho{}_{\mu\nu}$ vanish at a point x^λ and the components of $g_{\mu\nu}$ and $h^{\mu\nu}$ reduce to the values (21); we can then require that in such a system the equations of Newtonian mechanics (in their four-dimensional form as presented in Sec. IV) should be valid locally. This definition and requirement are thus completely analogous to those of Einstein's theory.[64] The inertial systems defined in this manner again have only a local significance; on the other hand, they provide the same means of escape from the Newtonian problem of defining such a system as Einstein's theory, and by the same expedient of relegating gravitation to the status of an inertial force.

Thus the essential difference between the two theories in their treatment of frames of reference in a curved space is not in the definition of inertial systems, but in the fact that Newtonian theory allows the establishment of global frames of reference in which the components of the fundamental tensors $g_{\mu\nu}$ and $h^{\mu\nu}$ take the same values everywhere. The existence of these frames of reference implies the existence of an affine connection $\Lambda^\rho{}_{\mu\nu}$ satisfying the condition (136) for a flat space, unlike the affine connection $\Gamma^\rho{}_{\mu\nu}$ which is associated with the gravitational field and which has no direct connection with the fundamental tensors.

In Einstein's general theory as originally formulated, on the other hand, there is only one affine connection $\Gamma^\rho{}_{\mu\nu}$, which is the Christoffel symbol (12) formed from the fundamental metric tensor $g_{\mu\nu}$. However, it was proposed some time ago by Rosen[65] to formulate

[64] A similar redefinition of inertial systems was suggested by O. Heckmann and E. Schücking, Z. Astrophys. **38**, 95 (1955), in connection with (three-dimensional) considerations on Newtonian cosmology.

[65] N. Rosen, Phys. Rev. **57**, 147 and 150 (1940); Ann. Phys. (N. Y.) **22**, 1 (1963). As the notation of these papers and that of Ref. 15 are irreconcilable, it seemed least confusing to use a third notation in this section, maintaining as much of standard notations of the literature as possible.

Einstein's theory in terms of two affine connections $\Gamma^\rho{}_{\mu\nu}$ and $\Lambda^\rho{}_{\mu\nu}$ related by Eq. (135) as above, with $\Lambda^\rho{}_{\mu\nu}$ again satisfying Eq. (136). The tensor $\Omega^\rho{}_{\mu\nu}$ is defined by him as

$$\Omega^\rho{}_{\mu\nu}=\tfrac{1}{2}g^{\rho\sigma}(g_{\mu\sigma,\nu}+g_{\nu\sigma,\mu}-g_{\mu\nu,\sigma}), \qquad (135\text{E})$$

where the comma derivative is defined as in Eq. (137). From the definitions (8) and (137) and from condition (136) it can be seen easily that the curvature tensor is given in both theories by

$$R^\kappa{}_{\mu\lambda\nu}=\Omega^\kappa{}_{\mu\nu,\lambda}-\Omega^\kappa{}_{\mu\lambda,\nu}+\Omega^\kappa{}_{\rho\lambda}\Omega^\rho{}_{\mu\nu}-\Omega^\kappa{}_{\rho\nu}\Omega^\rho{}_{\mu\lambda}, \qquad (142)$$

and thus the contracted curvature tensor by

$$R_{\mu\nu}=\Omega^\lambda{}_{\mu\nu,\lambda}-\Omega^\lambda{}_{\mu\lambda,\nu}+\Omega^\lambda{}_{\rho\lambda}\Omega^\rho{}_{\mu\nu}-\Omega^\lambda{}_{\rho\nu}\Omega^\rho{}_{\mu\lambda}. \qquad (143)$$

Thus in terms of $\Omega^\rho{}_{\mu\nu}$ (as well as of $\Gamma^\rho{}_{\mu\nu}$) the left-hand side of the field equations (127) is of the same form in both cases.

In Rosen's formulation the connection $\Gamma^\rho{}_{\mu\nu}$ is as usual the Christoffel symbol (12) formed from the metric tensor $g_{\mu\nu}$. But in addition a second nonsingular tensor $\lambda_{\mu\nu}$ is introduced such that the Christoffel symbol formed with $\lambda_{\mu\nu}$ gives the connection $\Lambda^\rho{}_{\mu\nu}$:

$$\Lambda^\rho{}_{\mu\nu}=\tfrac{1}{2}\lambda^{\rho\sigma}(\partial_\nu\lambda_{\mu\sigma}+\partial_\mu\lambda_{\nu\sigma}-\partial_\sigma\lambda_{\mu\nu}),$$
$$\lambda_{\mu\rho}\lambda^{\rho\nu}=\delta_\mu{}^\nu, \qquad \lambda_{\mu\nu,\rho}=0, \qquad \lambda^{\mu\nu}{}_{,\rho}=0. \qquad (144)$$

This tensor furnishes a second fundamental quadratic form $d\sigma^2$ in addition to ds^2 given by Eq. (11):

$$d\sigma^2=\lambda_{\mu\nu}\,dx^\mu\,dx^\nu. \qquad (145)$$

The introduction of these quantities can be regarded as simply a formal device which leads to a formulation of the equations of Einstein's theory more convenient for certain purposes discussed by Rosen than the customary one. For some of these purposes both quantities are needed. However, the only aspect of Rosen's discussion relevant in the context of this section is the possibility of distinguishing between "true" gravitational forces and inertial forces through the introduction of $\Lambda^\rho{}_{\mu\nu}$. In effect this is the analog in Einstein's theory of retracing our steps from the formulation of Newtonian theory given in Sec. VI.3 to that considered in Secs. V and VI.2. For this purpose alone it would not be necessary to introduce the tensor $\lambda_{\mu\nu}$; all that is required is that $\Gamma^\rho{}_{\mu\nu}$ and $\Lambda^\rho{}_{\mu\nu}$ should approach the same limit far away from the sources $P^{\rho\sigma}+T_f{}^{\rho\sigma}$ of the gravitational field. This in itself is sufficient to tie Einstein's theory to a flat space, and whether we then wish to interpret it as a flat space or a curved space theory becomes a question of semantics; exactly as in the case of the Newtonian theory, as discussed in detail above, it depends only on which affinity we want to consider as the affinity of the "true" space.

This question of a formal interpretation of Einstein's theory as a flat-space theory has to be distinguished from another question raised by Rosen. As an alternative to the purely formal introduction of the tensor $\lambda_{\mu\nu}$ he suggested the possibility of considering this tensor rather than $g_{\mu\nu}$ as the actual metric of the physical space–time, and to identify it (in a suitable class of coordinate systems) with the Minkowski metric $\eta_{\mu\nu}$. This amounts to a different assumption on the behavior of clocks and measuring rods than that adopted both in Einstein's general theory and in Sec. V. The problems connected with this interpretation are discussed in detail by Rosen.

In the formulation of Newtonian theory given here the connection $\Lambda^\rho{}_{\mu\nu}$ is indispensable, but there is no need to relate it to a tensor $\lambda_{\mu\nu}$. However, we are free to do so by Eqs. (144) if we desire to carry the analogy with Rosen's formalism as far as possible.

As far as the formal structure of both theories is concerned, $\lambda_{\mu\nu}$ is completely arbitrary except for the requirement of flatness (136). It thus might appear tempting to try to imitate Rosen's suggestion and to use $\lambda_{\mu\nu}$ to introduce a Minkowski metric into Newtonian theory. Formally this is indeed possible; however, any attempt to interpret it as the actual metric of space–time must fail, since the conditions imposed on signals by the Minkowski metric and by the Newtonian instantaneous action-at-a-distance forces, discussed in Sec. V.1, are incompatible.

It remains to discuss the precise nature of the transition from the Einsteinian to the Newtonian form of the general relativistic equations and conversely. This problem has been fully discussed by Friedrichs[15]; it can be answered somewhat more briefly if we use Rosen's formulation of Einstein's theory. To proceed from the Newtonian theory contained in Eqs. (127), (135), (135N), (136), and (128N) to that of Einstein, we drop Eqs. (135N) and (128bN), and require that the metric $g_{\mu\nu}$ be nonsingular, with signature -2; then a tensor $g^{\mu\nu}$ can be defined by Eq. (13) which automatically satisfies Eq. (128E), and Eq. (135E) is similarly satisfied.

To achieve the transition from the Einstein theory in Rosen's formulation contained in Eqs. (127), (135), (135E), (136), and (128E) to that of Newton, we first introduce a tensor $H^{\mu\nu}=c^{-2}g^{\mu\nu}$ satisfying

$$g_{\mu\rho}H^{\rho\nu}=c^{-2}\,\delta_\mu{}^\nu. \qquad (146)$$

We then require first that $H^{\mu\nu}$ be replaced by a tensor $h^{\mu\nu}$ of signature -3 satisfying Eq. (146) and that the determinant $|H^{\mu\nu}|$ go to zero as c^{-2} for $c\rightarrow\infty$, and second that there exist a scalar U such that the connection defined by

$$\left\{ {\rho \atop \mu\nu} \right\}+g_{\mu\nu}H^{\rho\sigma}\,\partial_\sigma U \qquad (147)$$

converge toward a connection $\Lambda^\rho{}_{\mu\nu}$ satisfying Eq. (136) in the same limit; then the Christoffel symbols $\left\{ {\rho \atop \mu\nu} \right\}$ of Einstein's theory converge to the Newtonian $\Gamma^\rho{}_{\mu\nu}$. The proof is given by Friedrichs[15,66]; the significance of this result is discussed in the next Section.

VII. DISCUSSION

In the course of this study we have first presented in Sec. IV a formulation of Newtonian mechanics which parallels Minkowski's formulation of the special theory of relativity. Both of these theories were developed in a four-dimensional space–time with Cartesian spatial coordinates; in such a formulation they are invariant under the Galilei and the Lorentz group, respectively. Then these theories were rewritten in Sec. V in a form which is invariant under arbitrary coordinate transformations (restricted only by the requirement of a reasonable description in time of the propagation of signals, whose round-trip velocity has no upper limit in Newtonian theory and a limit c in special relativity). Finally in Sec. VI Newtonian mechanics was rewritten to make gravitation a geometric feature, described by affine connections $\Gamma^{\rho}{}_{\mu\nu}$ which are determined by field equations identical with those of Einstein's general theory of relativity; the distinction is in the requirements imposed on the solutions. In the case of Einstein's theory these are that locally in an inertial system defined as a frame of reference in which all Γ's vanish at a point x^{ρ}, the special theory of relativity and in particular its space–time structure as described in Secs. III and IV should hold; in Newton's theory, they are that locally in an inertial system, defined in the same way as in Einstein's theory, Newtonian mechanics and its space–time structure as described in Secs. III and IV should hold. These requirements imply that in Einstein's theory the $\Gamma^{\rho}{}_{\mu\nu}$ are the Christoffel symbols formed from the metric tensor $g_{\mu\nu}$; in Newtonian theory a condition relating the $\Gamma^{\rho}{}_{\mu\nu}$ to a scalar potential U must be imposed separately.

Before discussing these formulations further, we should contemplate the amazing flexibility of mathematical formalisms. Let us consider specifically Newton's second law combined with his law of gravitation. Their original formulation was given as a three-dimensional law of motion and a similar action-at-a-distance force law. Everything which can be said about the motion of a system of gravitating particles can be deduced from these equations; none of the formulations developed later adds anything to the physical predictions of the theory. But how many different ways of putting the fundamental laws there are! Already in the three-dimensional formulations we can replace the original laws by variational principles, and the action-at-a-distance force law by a field description governed by a Poisson equation. In the four-dimensional formulation within the framework of the space–time of the Galilei group we can restate the second law so as to contain in addition to the law of motion also the relation between work and kinetic energy [Eq. (63N)]; the law of gravitation can be restated as a four-dimensional action-at-a-distance law [Eq. (74N)] or Poisson equation [(68N)]; this equation and the law of motion can be derived from a variation principle (81N) involving particles and fields; the law of motion combined with the law of gravitation can be derived from a Fokker-type variational principle (82N) for the particles alone, or from a conservation law (110) involving both particles and fields. Generalizing the four-dimensional formulation by first allowing arbitrary coordinate transformations in flat space–time, we have the second law in the form (120) and Poisson's equation in the form (122N); the law of motion combined with the law of gravitation can be derived from a covariant conservation law (124) involving both particles and fields. In curved space–time the law of motion becomes the geodesic law (131) and the equations determining the gravitational field via the affine connection are given by Eqs. (126) or (127); the law of motion can be derived from the conservation law (130) for the matter tensor alone.

The variety of ways, in part based on entirely different sets of concepts, in which we can express the fundamental laws for gravitating matter, all leading to identical physical predictions, should caution us not to put undue stress on the supposed implications of a particular formulation of a theory, even if other formulations might not be available at a given time.

As to the formal relation of Newton's theory in its various four-dimensional formulations to the similar formulations of Einstein's special and general theory of relativity, the main difference is that the space–time of the latter theories is metric, whereas that of the former is only affine, or more precisely is affine with a singular metric. Physically this distinction arises from the existence of a limiting signal velocity c in the theories of relativity. The formal analogy would have been brought out even more strongly if in the theories of relativity we had not used the contravariant tensors $\eta^{\mu\nu}$ and $g^{\mu\nu}$ [defined as the inverse of $\eta_{\mu\nu}$ and $g_{\mu\nu}$ by Eqs. (17) and (13), respectively], but instead only the tensors $H^{\mu\nu}$ defined by Eqs. (26) and (146), respectively; then it would have been clearer how the formulas of these theories degenerate to those of Newtonian theory in the limit $c\rightarrow\infty$, $H^{\mu\nu}$ now corresponding to the singular Newtonian tensor $h^{\mu\nu}$. We refrained from this because this would have put the formulas of Einstein's theories into an unfamiliar form.

In Secs. IV and V nothing more than such a straightforward limiting process is required to obtain the Newtonian formulas from those of special relativity except for some of the formulas involving variational principles. The slight complications involved in this case are due to the fact that we must be careful in taking

[66] The choice of constants and signs in Eqs. (13) and (21) differs from that of Ref. 15; thus the transition conditions stated here show a similar difference, and the proof requires corresponding trivial modifications.

the limit because the covariant four-velocity v_μ reduces to the *constant* vector w_μ.

The relation of Newton's and Einstein's general theories presented in Sec. IV involves more than the limit $c\rightarrow\infty$, as already noted. In Einstein's theory there exists a close relation (12) between the affine connections $\Gamma^\rho{}_{\mu\nu}$ and the metric tensor $g_{\mu\nu}$, which is broken in the limit. But then the ten independent field equations (126) or (127) are not sufficient to determine the 40 independent components of $\Gamma^\rho{}_{\mu\nu}$. A new restriction has to be imposed, which is provided by the introduction of the scalar potential U in the expression (147) and imposition of the convergence requirement discussed at the end of Sec. V.4.

A similar separation of the affine connection $\Gamma^\rho{}_{\mu\nu}$ and the metric of the physical space is introduced into Einstein's general theory by Rosen's suggestion of using a tensor $\lambda_{\mu\nu}$ rather than $g_{\mu\nu}$ for this metric. This tensor is introduced in the theory through the artifice of having two affinities $\Gamma^\rho{}_{\mu\nu}$ and $\Lambda^\rho{}_{\mu\nu}$ which are the Christoffel symbols formed with $g_{\mu\nu}$ and $\lambda_{\mu\nu}$, respectively. While this suggestion offers an alternate interpretation of Einstein's theory, there is no such choice available in Newton's theory, which does not permit the association of a physically meaningful metric tensor with $\Gamma^\rho{}_{\mu\nu}$.

The separation of affine connections and metric in Newtonian theory necessitates a reexamination of the relations of the concepts of curvature, metric, and non-Euclidean geometry. The curvature of an n-space is defined in terms of the affine connections $\Gamma^\rho{}_{\mu\nu}$ by Eq. (8). If the n-space is metric, this curvature can also be expressed in terms of the metric tensor $g_{\mu\nu}$, and a knowledge of this tensor allows us to determine whether the n-space is curved or not. If it is flat, we can introduce a global coordinate system in which the square of the n-dimensional line element is a diagonal quadratic form; if all terms are positive, the geometry of the n-space is Euclidean, otherwise it is pseudo-Euclidean. However, geometry in the ordinary sense refers to properties of the three-dimensional subspace $x^0=$ constant of the four-dimensional space–time. This geometry may be non-Euclidean even if the 4-space is flat. A case in point is the space–time of special relativity. If we allow arbitrary coordinate transformations, the square dl^2 of the three-dimensional line element is given by Eq. (117S). This involves more than just the spatial components of the metric tensor, and is not necessarily reducible to a sum of squares everywhere by a transformation of the spatial coordinates alone; thus the spatial geometry may be non-Euclidean, the geometry of a rotating disk being the most familiar example.

In Newtonian theory the curvature tensor (8) formed from the $\Gamma^\rho{}_{\mu\nu}$ can not be expressed in terms of a metric tensor, and the knowledge of the fundamental tensors $g_{\mu\nu}$ and $h^{\mu\nu}$ does not allow us to determine whether the 4-space is curved or not. Furthermore, the geometry of the three-dimensional subspace is determined by Eq. (117N), which in turn is determined by h^{mn} from Eq. (115). But by Eq. (136) there always exist global coordinate systems in which h^{mn} takes the values (21b) whether or not the 4-space is curved, and in which dl^2 is the sum of squares; since the values of the components of h^{mn} in any coordinate system can be obtained from those in these special systems alone, and conversely, it follows that the geometry of the subspace $x^0=$ constant is always Euclidean.

Thus Einstein's and Newton's theory of general relativity both ascribe curvature to space–time, and both consider this curvature to correspond to gravitational effects. While in both theories the curvature tensor can be determined from a study of gravitational effects,[67] its determination from a study of metric effects is possible only in Einstein's theory.

The above considerations concerned local properties of space–time. The global properties are closely connected with cosmological problems, and will be discussed elsewhere.[63]

The close resemblance between Einstein's and Newton's theory in the form developed in Sec. VI might be of advantage in the further exploration of Einstein's theory in two respects. First, the transcription of known Newtonian results into the formalism of Sec. VI.3 might be helpful as the first step in an approximation procedure for the solution of the corresponding problems in Einstein's theory (or to take an educated guess at the form of exact solutions). Second, this resemblance might help in the understanding of some features of Einstein's theory which are intimately connected with its four-dimensional structure and until now appeared to have no counterpart in Newtonian theory.

In our discussion of the principle of general covariance we equated this principle to the requirement of invariance under arbitrary coordinate transformations; as mentioned in the Introduction, in this form the principle imposes only a formal requirement, but has no physical content. This appears to be the main reason why some authors prefer a narrower interpretation by adding requirements such as "that within the theory no privileged set of frames can be constructed"[1] to the formulation of the principle. There seems to be little to be gained in arguing about terminology. It is more important to investigate whether

[67] For a direct determination by means of the "geodesic deviation" in the motion of test particles see F. A. E. Pirani, Acta Phys. Polon. **15**, 389 (1956). In that paper the method is developed for Einstein's theory, but the extension to Newton's theory is immediate; indeed, from a study of the three-dimensional Newtonian equations a tentative suggestion is made there for a four-dimensional generalization of Poisson's equation as an approach to Einstein's theory, which bears some resemblance to some of the mathematical considerations at the beginning of Sec. VI.3.

such an additional requirement would indeed provide a better guide in constructing "generally covariant" theories than the original one. It would certainly seem to deprive the Newtonian theory presented in Sec. VI. 3 of the label "generally covariant," since Eq. (136) allows the construction of privileged frames of reference, although it has no bearing on the fact that this theory is closely analogous to Einstein's theory. But Rosen in reformulating Einstein's theory also introduced Eq. (136). Did this deprive Einstein's theory of general covariance? If Rosen's suggestion is treated as a formal one, this seems to lead to a purely semantic argument. If it is treated as the physical statement that the metric tensor of physical space is $\lambda_{\mu\nu}$ rather than $g_{\mu\nu}$, it does appear to introduce physically privileged reference systems. But the actual physical consequences of the new interpretation are identical with the old ones, only the language of the description has been changed.

These arguments might be dismissed because there is after all no need to introduce Eq. (136) into Einstein's theory. Then the question arises whether privileged frames of reference can be constructed within the theory as originally formulated. This seems to come down to a semantic argument about the word "privileged." Certain coordinate systems can be singled out locally by some special properties; these are the "intrinsic coordinates" used in the construction of observables.[1] Globally a class of coordinate systems is singled out by the usual boundary conditions which require the metric to be Minkowskian at infinity, as has been remarked by many authors.[68] Thus, if taken literally, the narrower definition of the principle of general covariance might deprive Einstein's theory of the label "generally covariant" too.

As for the principle of equivalence, in Sec. VI we carried through a reformulation of Newtonian mechanics to conform it to this principle in its most stringent form. Up to this section we had treated Newtonian and special relativistic mechanics in parallel. We could also have transcribed special relativity by a similar change in language as introduced in Sec. VI.3 for Newtonian mechanics, and could thereby also have arrived at a geodesic law of motion (131). However, the field equations of the special relativistic scalar potential U, when transcribed in terms of the curvature tensor, show no resemblance to Einstein's field equations (126) or (127), even though the metric structure of the two theories is identical, and therefore we did not include this formulation of the special theory here. It appears from this result, perhaps not surprisingly, that Newton's theory of gravitation is a more legitimate forerunner of Einstein's than a special relativistic theory would be. Supporting this conclusion is another consideration based on the fact that Einstein's general theory of relativity has only two constants, c and G, available for studying limiting cases, and that c enters the theory through the requirements imposed on the solutions for the field equations rather than through the equations themselves. If we let c approach infinity, we can arrive at Newton's theory of gravitation by changing the requirements imposed in the manner discussed earlier. On the other hand, if we let G approach zero, but maintain the requirements on the solutions, we obtain as one possible solution the global space–time structure of special relativity, and this solution together with the conservation law (124) implicit in the field equations leads to the equations of motion of the special theory without gravitation, as discussed in detail in Ref. 45.

Thus Einstein's general theory of relativity contains the special theory of relativity without gravitation and Newton's theory of gravitation in a four-dimensional generally covariant form as two distinct straightforward limits. Historically, the emphasis in the development of Einstein's theory was on the generalization of the space–time structure of the special theory of relativity to incorporate the principles of general covariance and of equivalence. The field equations were obtained by trying to build a four-dimensional analog of Newton's theory in its three-dimensional form, and establishing the exact correspondence requires a somewhat awkward double limiting process.[1] If on the other hand we choose an approach from the four-dimensional form of Newton's theory, the incorporation of Einstein's two principles leads directly to field equations of the same form as in Einstein's theory; to obtain this theory we only have to change the requirements to be imposed on the local space–time structure in an inertial system. For some purposes this alternate approach appears to be preferable.

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[68] Especially in connection with Mach's principle; for a recent brief discussion and references to the literature see Ref. 51, Chap. 14. Compare also V. Fock, Refs. 11 and 48, and the article on invariance groups by P. G. Bergmann in *Fundamental Topics in Relativistic Fluid Mechanics and Magnetohydrodynamics*, edited by R. Wasserman and C. P. Wells (Academic Press Inc., New York and London, 1963).