Some Properties of the Landau Curves

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1. THE LANDAU EQUATIONS

The terms of the perturbation series for a given collision amplitude are of the form

$$F(z)=\lim_{\epsilon\to 0+}\int d^4k_1\cdots d^4k_l\prod_{i=1}^{n}(q_i^2-m_i^2+i\epsilon)^{-1},\quad (1.1)$$

where q_i is the four-momentum of the particle of mass m_i, which corresponds to the ith internal line of a Feynman diagram: q_i depends linearly, via the law of conservation of energy momentum at each vertex, on a set of independent internal momenta k_i and on the external momenta p_i. The symbol z summarizes a total set of independent scalar variables z_i which can be constructed in an invariant manner from the vectors p_i.

The complications of charge and spin dependence have been ignored on the grounds that the factors which these considerations introduce, occurring as they do in the numerator of the integrand, cannot increase the number of singularities of the function $F(z)$. This means that we have discarded important selection rules which we require to impose artificially when necessary. However, as the perturbation approach is intended as a model for a more sophisticated theory, simplicity is of primary importance.

By means of a transformation due to Feynman we obtain

$$F(z)=\lim_{\epsilon\to 0+}\int d^4k_1\cdots d^4k_l\int_0^1\frac{d\alpha_1\cdots d\alpha_n\delta(1-\sum_{i=1}^{n}\alpha_i)}{d(q,\alpha,\epsilon)^n},\quad (1.2)$$

where

$$d(q,\alpha,\epsilon)=\sum_{i=1}^{n}\alpha_i(q_i^2-m_i^2+i\epsilon).$$

The symbols q and α summarize, respectively, the variables q_i and α_i.

In dispersion theory, we wish to locate the singularities of the functions $F(z)$. The sets of values of the variables z_i, both real and complex, which correspond to possible singularities of $F(z)$ are called the *Landau curves*, and may be obtained by processes of elimination from a set of algebraic equations first written down by Landau.[1]

The Landau equations for an uncontracted diagram are

$$\sum_{i=1}^{n}\alpha_i=1,\quad (1.3)$$

$$q_i^2=m_i^2,\quad (1.4)$$

$$\sum{}'\alpha_i q_i=0,\quad (1.5)$$

together with equations which express the law of conservation of four-momentum at each vertex. There exists an equation of the type (1.4) corresponding to each internal line of the diagram. There are l equations of the type (1.5), where the summations are taken round independent closed loops of the diagram.

Equivalent criteria have also been given by Polkinghorne and Screaton.[2] They proceed by first performing the k integrations to obtain

$$F(z)=\lim_{\epsilon\to 0+}\int_0^1\frac{d\alpha_1\cdots d\alpha_n\delta(1-\sum_{i=1}^{n}\alpha_i)f(\alpha,n,l)}{D(z,\alpha,\epsilon)^{n-2l}}.\quad (1.6)$$

Their philosophy is to consider the multiple integral as an integral over a "contour" A in α space (i.e., in the space of the variables α_i). At any point z there exists a set of points $\alpha(z)$ at which $D(z,\alpha,\epsilon)$ vanishes, and as z is varied the analytic continuation of $F(z)$ is obtained by deforming A to avoid the zeros of D. In this way we obtain the singularities of $F(z)$ when such deformations become impossible, i.e., when either two zeros of D pinch the contour between them (Fig. 1) or a zero of D moves up to the fixed boundary of the contour (Fig. 2). The equations of Polkinghorne and Screaton which express these two alternatives are, respectively,

$$\partial D/\partial\alpha_{\nu_i}=0\qquad i=1,\cdots,r$$
$$\alpha_{\nu_i}=0\qquad i=r+1,\cdots,n,\quad (1.7)$$

for any partition of the α_{ν_i} into classes of r and $n-r$ members, $0<r\leq n$. When $r=n$ we talk of the resulting Landau curve as the *leading curve*, and when $r<n$ of a *lower-order curve*. Each partition of the α_{ν_i} in (1.7) corresponds to a complete set of Landau equations with nonzero α_{ν_i}. The leading curve gives the singularities belonging to the uncontracted Feynman graph while the lower-order curves correspond to the contracted diagrams.

[1] L. D. Landau, Nucl. Phys. **13**, 181 (1959).

[2] J. C. Polkinghorne and G. R. Screaton, (a) Nuovo Cimento **15**, 289 (1960); (b) **15**, 925 (1960).

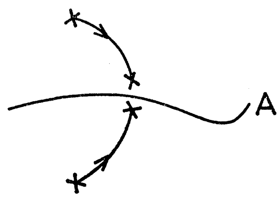

FIG. 1. Pinch configuration in α plane.

2. EFFECTIVE INTERSECTIONS

Suppose that we are at a point of singularity of a function $F(z)$ which corresponds to a pinching of the contour as in Fig. 1. [These figures are to be regarded as 2-dimensional models of the $2n$-dimensional situation. The full justification for drawing such pictures involves performing $n-1$ of the integrations in (1.6)]. If we vary z, while remaining on the Landau curve, Eqs. (1.7) tell us that the zeros of D remain coincident. Thus a mechanism whereby we may move from a region of singularity of $F(z)$ on a Landau curve to one of nonsingularity is that we move up to a boundary of the contour A and the pinch occurs harmlessly, thereafter, as in Fig. 3. This mechanism, for a long time supposed to be the only mechanism for moving from regions of singularity to ones of nonsingularity, is very important. At the point of transition we reach the boundary of the contour and hence the Landau curve on which we were varying our z value has intersected a lower-order curve. We define an intersection between a Landau curve $\sum$ and a lower-order curve $\sum'$, at which the Landau equations for each are satisfied by the same set of α_i values, to be an *effective intersection*. At effective intersections the analytic properties of $F(z)$ may change.

Tarski[3] has proved, *in the special case of single-loop diagrams*, that a Landau curve $\sum$ intersecting a curve $\sum'$ of next lowest order necessarily does so effectively.

We now discuss the validity of the more general assertion that this theorem applies to Landau curves belonging to any arbitrary Feynman diagram.

Consider the set of equations

$$\partial D/\partial\alpha_i=0 \qquad i=1,\cdots,n-1 \tag{2.1}$$

which, as follows immediately from Eqs. (1.2) and (1.6), are homogeneous of degree zero in the variables α_i.

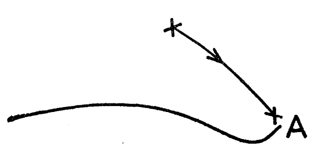

FIG. 2. End-point configuration in α plane.

[3] J. Tarski, J. Math. Phys. 1, 154 (1960).

Suppose, *for the present*, that we may solve Eqs. (2.1), uniquely, in the following fashion:

$$\alpha_1:\alpha_2:\cdots:\alpha_n=A_1:A_2:\cdots:A_n, \tag{2.2}$$

where the A_i are algebraic functions of the variables z_i, from which we have removed any infinities.

The equation of the curve $\sum'$ may be constructed in the following manner. In addition to (2.2) impose the further condition that $\alpha_n=0$. It follows at once that the equation of $\sum'$ is simply given by

$$A_n(z)=0. \tag{2.3}$$

To construct the equation of the curve $\sum$ the condition which must be imposed in addition to (2.2) is $\partial D/\partial\alpha_n=0$. If we now write $\partial D/\partial\alpha_n=E$ then E is homogeneous of degree zero in the α_i variables. It follows that E can be expressed as a function of the ratios of the α_i's only. Hence the equation of $\sum$ is

$$E(A_1,A_2,\cdots,A_n)=0. \tag{2.4}$$

The theorem now follows: our construction ensures that $\sum$ and $\sum'$ have the same set of α_i values at their points of intersection. It should be emphasized that

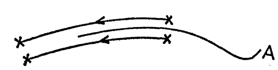

FIG. 3. Transition mechanism in α plane.

the argument does not hold good when more than one of the α_i vanishes and consequently the proof relates only to intersections of curves whose orders differ by unity.

However, it may turn out that the solution (2.2) is *not* unique. In this case the theorem may fail to hold when, corresponding to a given value of z, there exists more than one set of functions $A_i(z)$. If there happens to be an N-fold degeneracy we must rewrite Eqs. (2.2) in the form

$$\alpha_1:\alpha_2:\cdots:\alpha_n=A_1^i:A_2^i:\cdots:A_n^i. \tag{2.2a}$$

where

$$i=1,2,\cdots,N.$$

Now the equations of the curves $\sum'$ and $\sum$ are, respectively,

$$\prod_{i=1}^{N}A_n^i(z)=0 \tag{2.3a}$$

and

$$\prod_{i=1}^{N}E^i(z)=0, \tag{2.4a}$$

where

$$E^i(z)=E(A_1^i,A_2^i,\cdots,A_n^i).$$

There is no longer any guarantee that the intersection of $A_n{}^i(z)=0$ with $E^j=0$ is effective when $i\neq j$, although certainly the intersections of these curves are effective if $i=j$.

The author's conclusion is thus that in the simplest conceivable case, namely when to each point z of a Landau curve there corresponds a unique set of α_i values, curves of consecutive order intersect effectively, regardless of the complication of the Feynman diagram. When the α_i's corresponding to a given z value are degenerate the intersection may or may not be effective: we can give no example of Landau curves of consecutive orders intersecting noneffectively but we can see no reason why such behavior should not occur for some diagram. Further consequences of degenerate behavior are discussed in Sec. 3.

Landshoff, Polkinghorne, and Taylor[4] have shown, in the case of two invariants, that at effective intersections of Landau curves, the curves have parallel tangents.

In the general case the proof of the tangency property is still very simple. It can be shown that the dominator function $D(z, \alpha)$ in Eq. (1.6) can be written as

$$D(z,\alpha)=\sum_{i=1}^{n} f_i(\alpha)z_i+f_{n+1}(\alpha). \qquad (2.5)$$

Now the direction of the normal to the tangent hyperplane of the Landau curve is given by the set of ratios

$$\frac{\partial D}{\partial z_1}:\frac{\partial D}{\partial z_2}:\cdots:\frac{\partial D}{\partial z_n}. \qquad (2.6)$$

Differentiating Eq. (2.5) with respect to z_i we obtain

$$\frac{\partial D}{\partial z_i}=f_i(\alpha)+\sum_j \frac{\partial D}{\partial \alpha_j}\frac{\partial \alpha_j}{\partial z_i}=f_i(\alpha) \qquad (2.7)$$

on both $\sum$ and $\sum'$, using the Landau equations in the form (1.7). Thus, at an effective intersection between $\sum$ and $\sum'$, the ratios (2.6) are identical and the curves touch.

These properties provide a simple geometric way of classifying the Landau curves into families which touch one another in prescribed fashions. Such a classification was given by Tarski in the case of the single-loop scattering diagram. A similar classification for curves corresponding to a five-point single-loop graph was attempted by the present author, and has also been given recently by Cook and Tarski.[5] This latter classification has not, as yet, proved to be of any value whatsoever and so we do not discuss it here. In the former case, however, Tarski proceeded to a satisfying and elegant verification of the truth of the Mandelstam representation, for certain external mass values, for the four-point single-loop graph. Tarski's proof of double dispersion relations for the simplest scattering diagram forms the basis of the more general discussions of Landshoff, Polkinghorne, and Taylor Their methods are inductive, and properties of the scattering diagram to any order in perturbation theory are asserted— the single-loop graph being the starting point of the induction procedure.

An essential feature of the proofs of the Mandelstam representation is that the effective intersections can, in some sense, divide up the Landau curves into regions, each of which corresponds to a definite type of analytic structure for $F(z)$. We ignore, for the present, those points other than effective intersections at which the analytic properties of $F(z)$ may change. One may well ask in what manner this is possible because the set of effective intersections is of too low a dimensionality to divide up the Landau curves! In general, $F(z)$ is a many valued function and our interest is centered upon one specific sheet, namely the physical sheet. Thus, if we perform analytic continuations of $F(z)$ by paths lying on the Landau curves, we must avoid passing through branch cuts on to unphysical sheets, or if we do enter such sheets we must ensure that we return eventually to the correct sheet. It happens that we can sometimes do neither of these things without being forced to pass through an effective intersection —and an effective intersection is a point through which continuations may not be made. A detailed analysis of the mechanisms involved is given in Sec. 4.

3. MODES OF CONTINUATION

Because of the reality of the Landau equations the Landau curves are real in the sense that they are algebraic curves with real coefficients. As a result, a Landau curve can, in some measure, be represented by its *real section*, i.e., that portion of the curve corresponding to choosing all the invariants real. The complex regions of the curve can then be obtained by using a generalization of the "search-line" technique introduced by Tarski for the case of two invariants. For a Landau curve depending on two invariants only, this is to say that all points of a Landau curve are found by taking its real section toegether with the complex intersections of the curve with the set of all possible real lines. In the general case we must take the real section together with the curve's complex intersections with all possible real hyperplanes. By a real hyperplane we mean, in the case of n invariants, any manifold of the form

$$\sum \lambda_i z_i = c,$$

where λ_i and c are real. In Fig. 4, for example, this technique tells us that a whole double region of the curve composed of complex points, joins the two arcs.

[4] P. V. Landshoff, J. C. Polkinghorne, and J. C. Taylor, Nuovo Cimento **19**, 939 (1961).

[5] L. F. Cook and J. Tarski, (a) Phys. Rev. Letters **5**, 585 (1961); (b) J. Math. Phys. **3**, 1 (1962).

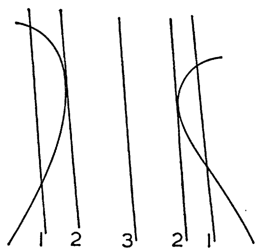

FIG. 4. Search-line method: 1 = real intersections, 2 = coincident intersections, 3 = complex intersections.

A family of search-lines of constant gradient maps out points which sprout off each arc in complex conjugate pairs. The entire complex region of the curve is then obtained by varying the gradient of the search-line.

Evidently enough, the technique is not very informative in the case of Landau curves with no real section (e.g., $x^2+y^2+1=0$ is composed entirely of complex points). If such curves do exist, then unless they are composed entirely of points corresponding to regular behavior, it may be impossible even to write a simple single-dispersion relation in any invariant.

The basic method of determining whether or not a function $F(z)$ is singular is to continue the function from a region of the z plane where it is mathematically well defined to the region of interest, taking into account that a singularity may arise whenever we encounter a Landau curve. In general, the $F(z)$ is many valued and the mode of continuation affects the singularity or nonsingularity at a point.

This last paragraph gives the clue to the true significance of the real section of a Landau curve in the proofs of dispersion relations. The function $F(z)$ always possesses real branch cuts which are the normal cuts—in our continuations we must always take care that we do not leave the physical sheet by passing through one of these (more precisely we must ensure that we end up eventually on the physical sheet). If at some stage we find further branch points we must then decide how to define our physical sheet taking these into account. However, initially the object of our continuations is to look at all the points which may possibly be branch points (i.e., the points of the Landau curves) and decide whether or not they are signular points on the physical sheet as defined by the normal cuts. Now it happens that the Landau curves themselves are often convenient vehicles for analytic continuation so that, in such a continuation, we must

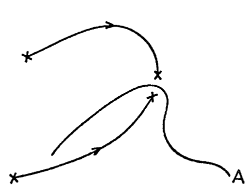

FIG. 5. Coincidence in α plane corresponding to a singularity.

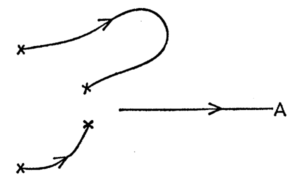

FIG. 6. Coincidence in α plane corresponding to regular behavior.

exercise great care on the real section, because on moving on it we may pass through a normal cut into an unphysical sheet. Thus it is that the real section plays a central role in this theory.

If we continue the function $F(z)$ from a region of the real z plane where $F(z)$ is regular, to a point z_1, then the continuation by the complex conjugate route to z_1^* yields the same type of analytic behavior for $F(z)$ at both z_1 and z_1^*. This as Landshoff, Polkinghorne, and Taylor remark, is because the complex conjugate mode of continuation in the z plane leads us, in the α plane, to the complex conjugate configuration of both the contour A and the zeros $\alpha(z)$ of D.

Let us suppose that we have a real point z_0 at which $F(z)$ is regular, and let us choose a path from z_0 to z_1 which does not intersect any portion of the Landau curve $\sum$ (that it is indeed possible to construct such a path is to be proved later in this section).

Figure 5 depicts a pair of zeros $\alpha(z)$ moving, as z varies from z_0 to z_1, into a coincidence which pinches the contour and produces a singularity of $F(z)$ at $z=z_1$. On the other hand, in Fig. 6, we have a coincidence which corresponds to regular behavior at $z=z_1$. It is fairly evident that the complex conjugate configurations lead to identical behavior at $z=z_1^*$. More complicated situations are clearly conceivable because the zeros of D may move in such a way as to necessitate drastic contour deformation. As a further example, consider a pair of points which move from the initial to final configurations of Fig. 6 but via more tortuous paths. Let us say that the upper point encircles the end point of the contour twice in a clockwise sense while the lower point makes one circuit in an anticlockwise direction. The effective situation is that these points are now encircled by loops of the contour shown in Fig. 7. The result of the complex conjugate continuation is shown in Fig. 8.

Both Fig. 7 and Fig. 8 depict pinches of the type illustrated in Fig. 9, and so both modes of continuation lead to identical types of singularity for $F(z)$.

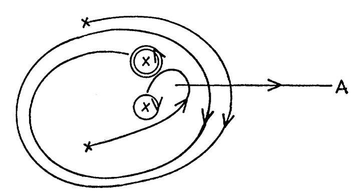

FIG. 7. Paths of zeros in α plane corresponding to a given continuation in z.

Now z_0 was a real point, and so our corresponding zeros $\alpha(z_0)$ occurred, as shown in Figs. 5 and 6, in complex conjugate pairs. There is no need however for this to be true at $z=z_1$ unless z_1 also is real. It is interesting to notice that at a *real* point $z=z_1$, if the singularity of $F(z)$ is due to the simple coincidence of *one* complex conjugate pair of zeros $\alpha(z)$, then the α_i-values at the singularity must be real. By choosing a real path from the point $z=z_0$, one can easily convince oneself that the first genuine singularity which one encounters corresponds to α_i values which are all real and lie between 0 and 1.

If, however, as suggested in Sec. 2, there exists more than one set of α_i values corresponding to a given real point $z=z_1$ of a Landau curve then it is possible for the α_i's to be complex. Figure 10 illustrates such a situation. It is not the coincidence of a complex conjugate pair of points $\alpha(z)$ which gives rise to the pinching of the contour, but a coincidence between two such pairs. It is clear that the complex conjugate configuration also produces a singularity.

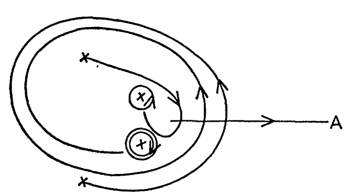

FIG. 8. Paths of zeros in α plane corresponding to the continuation in z complex conjugate to that illustrated by Fig. 7.

All possible situations lead to the same conclusions.

We now proceed to prove that we can, in fact, continue in the complex z plane by paths which never intersect a Landau curve.

Let the equation of the Landau curve be $f(z)=0$ where z summarizes n variables $z_1, z_2, \cdots, z_n$. Essentially there are $2n$ real variables and $f(z)=0$ gives two real equations. Thus the Landau curve is a manifold of dimensionality $2n-2$ in a space whose dimensionality is $2n$. Suppose $z=x$ and $z=y$ are two points of the space such that $f(x)\neq 0$ and $f(y)\neq 0$. Connect these two points by a path having one degree of freedom —call such a path a line. Then the line intersects the Landau curve in some set of points.

This set may be null, in which case the line which we have chosen is a suitable path since it fulfils the requirement of having no intersection with $f(z)=0$.

The set may have dimensionality zero and consist of a finite set of points; we defer the discussion of this case.

The set may have dimensionality unity and, in this case, we simply choose some other line connecting x and y which intersects the Landau curve in a set of points falling into either the first or the second category. It is always possible to do this because if it were

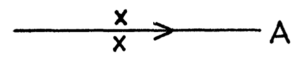

FIG. 9. Pinch configuration in α plane equivalent both to Fig. 7 and Fig. 8.

not possible the manifold $f(z)=0$ would have dimensionality greater than $2n-2$.

Thus our problem is essentially whether or not we can find our way past a single point C on a Landau curve: we assert that it is a trival matter to modify our path throuth C only slightly to avoid C and obtain a line which does not intersect the Landau curve at all. Suppose that the coordinates of the point C are z_c. In the neighborhood of $z=z_c$ the Landau curve has the form

$$f(z_c+\zeta)=f(z_c)+\frac{\partial f}{\partial z}\bigg|_{z=z_c}\zeta=\frac{\partial f}{\partial z}\bigg|_{z=z_c}\zeta=0, \tag{3.1}$$

provided only that $\partial f/\partial z\,|_{z=z_c}\neq 0$ for some of the variables z_c. We can write the equation of the Landau curve locally as

$$\sum_i a_i\zeta_i=0, \tag{3.2}$$

where

$$a_i=\partial f/\partial z_i\,|_{z=z_c}. \tag{3.3}$$

Equation (3.2) is, in general, two real equations which we can write as follows:

$$\sum_i \text{Re } a_i \text{ Re}\zeta_i-\sum_i \text{Im } a_i \text{ Im}\zeta_i=0, \tag{3.4}$$

$$\sum_i \text{Im } a_i \text{ Re}\zeta_i+\sum_i \text{Re } a_i \text{ Im}\zeta_i=0. \tag{3.5}$$

The condition that (3.4) and (3.5) should be the same equation is

$$\text{Re } a_i/\text{Im } a_i=-\text{Im } a_i/\text{Re } a_i \quad \text{for all } i=1, 2, \cdots, n, \tag{3.6}$$

which is simply an expression of the condition

$$|\,a_i\,| = |\,\{\partial f/\partial z_i\,|_{z=z_c}\}\,| =0, \tag{3.7}$$

which we have already assumed untrue in writing Eq. (3.1).

Now (3.4) and (3.5) define a subspace of dimensionality $2n-2$. Let us take a set of basis vectors to span our original $2n$-dimension space, the first $2n-2$ of which actually span the subspace defined by Eqs.

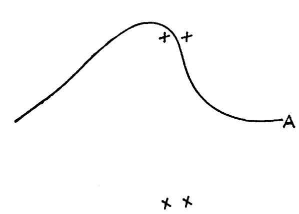

FIG. 10. Pinch configuration in α plane corresponding to complex α values.

(3.4) and (3.5). Then

$$x=(x_1, x_2, \cdots, x_{2n-2}, x_{2n-1}, x_{2n})$$
$$y=(y_1, y_2, \cdots, y_{2n-2}, y_{2n-1}, y_{2n}), \qquad (3.8)$$

where not both of the last two coordinates may vanish since x and y do not lie on the Landau curve. We now construct a path from x to y which does not intersect $f(z)=0$. Several cases must be distinguished. In all cases we first move $(x_1, \cdots, x_{2n-2})\rightarrow(y_1, \cdots, y_{2n-2})$ in a continuous manner.

Case 1

$$x_{2n-1}, x_{2n}, y_{2n-1}, y_{2n}\neq 0.$$

Let $x_{2n-1}\rightarrow y_{2n-1}$, and then $x_{2n}\rightarrow y_{2n}$; or make the variations in the opposite order. In either case it is impossible for both coordinates to vanish simultaneously.

Case 2

$$\left.\begin{array}{ll} x_{2n-1}=0, & x_{2n}, y_{2n-1}, y_{2n}\neq 0 \\ y_{2n}=0, & x_{2n-1}, x_{2n}, y_{2n-1}\neq 0 \\ x_{2n-1}=y_{2n}=0, & x_{2n}, y_{2n-1}\neq 0 \end{array}\right\} .$$

Let $x_{2n-1}\rightarrow y_{2n-1}$ and then $x_{2n}\rightarrow y_{2n}$, the order of the operations being essential in this case to avoid the possibility of both coordinates being zero at once.

Case 3

$$x_{2n-1}=y_{2n-1}=0, \qquad x_{2n}, y_{2n}\neq 0.$$

Let $x_{2n-1}\rightarrow\epsilon\neq 0$, then $x_{2n}\rightarrow y_{2n}$, and finally let $\epsilon\rightarrow 0$. In this way both coordinates can never vanish at the same time.

Thus a path can always be found in the neighborhood of C which connects x and y and which does not intersect $f(z)=0$.

If $\partial f/\partial z_i=0$ for all i at the point C then $f(z)=0$ has a multiple point at $z=z_c$. Unless all neighboring points are also multiple points we can choose an adjacent path and use our previous argument: if, however, *all* neighboring points are multiple points then $f(z)$ must have the form

$$f(z)=g(z)^2h(z),$$
$$\frac{\partial f(z)}{\partial z}=2g(z)h(z)\frac{\partial g(z)}{\partial z}+g(z)^2\frac{\partial h(z)}{\partial z}, \qquad (3.9)$$

and clearly it is the points of the manifold $g(z)=0$ which are apparently blocking our route: intersections with $g(z)=0$ can be avoided by using the above argument again for $g(z)$ instead of $f(z)$. For a long time it has been supposed that Landau curves did not posses multiple points, but, recently Eden, Landshoff, Polkinghorne, and Taylor[6] have given an example of a Landau curve which does possess crunodes, and incidentally also acnodes.

The importance of these results is to facilitate the understanding of the problem of proving the Mandelstam representation to all orders in perturbation theory. A necessary condition for the truth of this representation is the absence, on the physical sheet of our function, of complex singularities. The type of proof which is necessary, as stated in Sec. 2, is an inductive one where at any stage of the induction procedure it is assumed that all lower curves have their complex regions nonsingular on the physical sheet. Thus the problem is to continue the function analytically from a region where it is known to be regular to all points of the leading curve of the diagram being considered. If we establish that a given portion of a Landau curve is nonsingular it is convenient to choose our path of continuation on the curve thereafter. We must always take into account the points at which the analytical behavior of $F(z)$ may change—we may not continue through these points—and also we must notice carefully whether or not we have passed through a cut into an unphysical sheet. An important subset of these points are the effective intersections (see Sec. 2) with lower-order curves. Because of the induction hypothesis these all occur at real points.

4. PROPERTIES OF $F(z)$ ON UNPHYSICAL SHEETS

Suppose that we have a function $F(z)$ of several complex variables $z_1, \cdots, z_n$. By the results of the previous section we know that there exist, at most, 2^{n-1} distinct types of analytic behavior of $F(z)$ as we approach a point of the real plane. For example, if $n=3$, the limits with imaginary parts of z_i having the sign schemes $(+, +, +)$, $(+, +, -)$, $(+, -, +)$, and $(-, +, +)$ are the only possible distinct ones (those obtained by complex conjugation give identical analytic behavior).

We now prove the following theorem: *if the function* $F(z)$ *is singular in only one of the* 2^{n-1} *possible senses, then in any adjacent sheet, the function is singular in only one sense—which sense depending upon which adjacent sheet has been chosen.* For definiteness, we consider the case $n=3$ with $F(z)$ singular only in the $(+, +, +)$ limit and we show that, in the sheets obtained by passing through the z_3 cut, $F(z)$ is singular only in the $(+, +, -)$ limit.

Let P, in Fig. 11, be the point under consideration. As we approach P by path (1), while the imaginary parts of z_1 and z_2 are fixed at positive values, we find a singularity at P. Let us now roll back the z_3 cut, as shown in Fig. 11, and approach P by the path (2). We must also roll back any other cuts which might lie in the way of our path—this is what we mean by "adjacent." It is evident, then, that in the adjacent sheet obtained by going down through the z_3 cut, $F(z)$ is singular in the $(+, +, -)$ sense. If we had fixed the imaginary parts of z_1 and z_2 at negative values and

[6] R. J. Eden, P. V. Landshoff, J. C. Polkinghorne, and J. C. Taylor, (a) Phys. Rev. **122**, 307 (1961); (b) J. Math. Phys. **2**, 636 (1961).

approached the z_3 cut from below the singularity would also appear because the $(+, +, +)$ and $(-, -, -)$ limits are not distinct. Thus, by rolling the z_3 cut up, we would find that $F(z)$ was singular, in the adjacent sheet obtained by going through the z_3 cut, in the sense $(-, -, +)$. Then since $(-, -, +)$ and $(+, +, -)$ are not distinct we can assert that in both adjacent sheets $F(z)$ is singular in the $(+, +, -)$ sense. To complete the proof of the theorem we require that, in these adjacent sheets, $F(z)$ is nonsingular in the senses $(+, +, +)$, $(+, -, +)$ and $(-, +, +)$. This is a trivial matter of fixing the imaginary parts of z_1 and z_2 at suitable values and again rolling back the z_3 cut.

In the particular case of $n=2$ we are dealing with the elastic scattering problem. By considering a real search line of the form $z_1=\lambda z_2+\mu$, it is clear that the complex singularities which sprout off arcs of positive gradient $(\lambda>0)$ have like signs of the imaginary parts of z_1 and z_2: similarly those which sprout off arcs of negative gradient $(\lambda<0)$ have opposite signs of the imaginary parts. One defines that limit on to the real section of a Landau curve in which the imaginary parts of z_1 and z_2 have the same relative sign of the imaginary parts of the Landau curve in that neighborhood to be the *appropriate* limit. That limit which does not satisfy this criterion is called the *inappropriate* limit. *Clearly the above theorem implies that a function singular only in the appropriate (or inappropriate) sense in some sheet is singular only in the inappropriate (or appropriate) sense in an adjacent sheet.* Thus, a curve corresponding to an arc singular only in one sense must lie wholly inside a region which corresponds to cuts of the function in both variables: this is because, in a region below the beginning of a cut, it is immaterial whether we approach the real axis from above or below, and so appropriate and inappropriate behavior must be identical.

In the general case of n invariants there exist several inappropriate limits inside regions which are suitably cut.

At this stage we are now equipped to study the question raised in Sec. 2 of how a set of points, such as the effective intersections, can divide up a Landau curve into regions, each of which corresponds to identical analytic behavior of $F(z)$. We consider specifically the case $n=2$.

Let us define a plot of the Landau curve under consideration on to the z_1 plane as follows: the curve, being 2-dimensional, is capable of being projected on to the z_1 plane. A portion AB (see Fig. 12) of the real z_1 axis corresponds to the real section of the Landau curve, while other points of the plane correspond to complex points of the Landau curve.

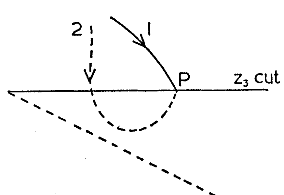

FIG. 11. Modes of approaching the point P in complex z_3 plane.

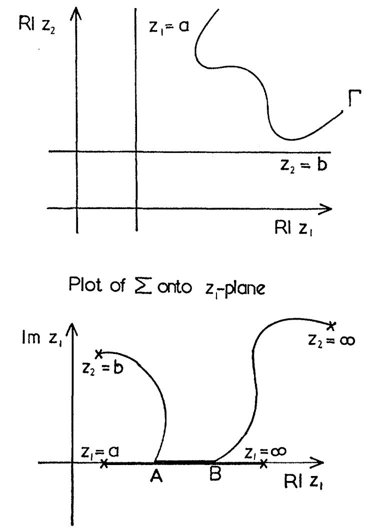

FIG. 12. Graph of Γ, and the plot of Σ onto the complex z_1 plane.

In the example which is illustrated in Fig. 12, the real section Γ of the Landau curve lies wholly inside the region of the cuts in both z_1 and z_2. Thus since Γ is plotted by AB, AB must lie between $z_1=a$ and $z_1=\infty$. Further the line which plots the intersection of the cut $b\leq z_2\leq\infty$ with the Landau curve must also include AB.

Let us now consider the intersection of two Landau curves $\sum$ and $\sum'$, and suppose that we are performing continuations of $F(z)$ on $\sum$, and that $F(z)$ is nonsingular. The intersection with $\sum'$ is a point. Thus, in the plot of $\sum$ on to the z_1 plane the point P of intersection with $\sum'$ appears as a single point. Assuming that all the other branch points are isolated as shown in Fig. 13(a) it is evident that we may thread the branch points in any way we wish—so reaching every sheet of $F(z)$. We may thus continue $F(z)$ on the Landau curve as we wish without any change in the nature of $F(z)$. We conclude, then, that an ordinary intersection *cannot* divide the Landau curve up.

Now let us consider the configuration of Fig. 13(b) where the Landau curves $\sum$ and $\sum'$ have, locally, two intersections. This, while P and P' remain distinct points, is just the situation of Fig. 13(a). However, as $P\rightarrow P'$, and $\sum$ touches $\sum'$, the two branch points in the plot coincide, and paths, such as that sketched in Fig. 13(b), which passed between the branch points P and P' while they were distinct, are now no longer available since we cannot continue through a branch point. Thus *all* sheets of the function $F(z)$ are no longer available. If those we cannot reach include the physical sheet then the touch of two Landau curves has caused the division of the Landau

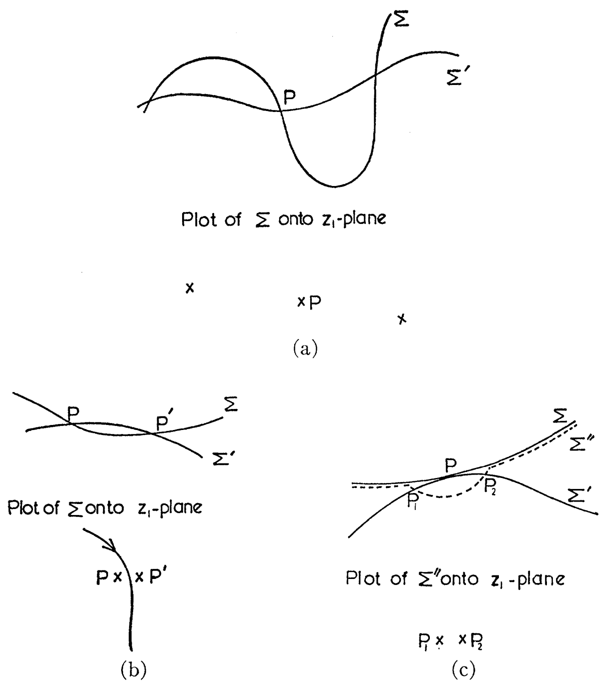

FIG. 13. (a) Intersections of Σ and Σ', and the plot of Σ onto the complex z_1 plane (points of Σ' in this plot are denoted by crosses). (b) Case when Σ and Σ' have locally two intersections. (c) Case when Σ and Σ' touch.

curve $\sum$ into two parts and we cannot continue from one to the other on $\sum$.

Clearly, then, touching of two Landau curves is very significant. In Sec. 2 we proved that if two Landau curves intersected effectively then they touched. *We now prove that the tangency point between two Landau curves can divide up the curve $\sum$ into two parts only if it is an effective touch.* In Fig. 13(c) we have drawn a touching situation between the curves $\sum$ and $\sum'$. Construct a third curve $\sum''$ which coincides with $\sum$ except near the point P of tangency, where it differs only very slightly from $\sum$. In the plot of $\sum''$ on to the z_1 plane the branch point P has split up into two branch points, and the plane is no longer divided, corresponding to the fact that we may find a route past P on the curve $\sum''$ which keeps on the physical sheet. However, in doing this we have left the Landau curve $\sum$ so that, in the α plane, the coincident zeros of D, to which $\sum$ corresponded, are *slightly* separated. If $\sum'$ *is not* a lower-order curve which meets $\sum$ effectively, then the slightly separated pair of zeros in the α plane are never in the neighborhood of an end point (continuity argument) and so no slipping over an end point (see Fig. 3) can occur. Thus when the path of continuation rejoins $\sum$ the zeros of D coincide without causing a pinch. On the other hand, if $\sum'$ *is* an effective intersection, it is quite possible that when the path on $\sum''$ joins $\sum$ again a singularity of the pinch type appear.

In this way effective tangencies can divide up the Landau curves into regions. Strictly the argument presented here applied only to appropriate singularity or nonsingularity. However, the inappropriate case is always the case which is appropriate in some unphysical sheet, and the argument is precisely the same.

The proofs of dispersion relations, and in particular the Mandelstam representation, in the nth order of perturbation theory proceed by using the techniques of analytic continuation to connect up various regions of the Landau curves which correspond to the same type of analytic behavior. In the induction process each leading curve is considered at a stage when all lower-order singularities are real in the physical sheet and various portions of the real section of the leading Landau curve can be shown to correspond to identical analytic properties of $F(z)$ if they can be connected by paths lying on the Landau curves which do not pass through effective intersections with lower-order curves.

Outside the region of the crossed cuts the matter is trivial.

Otherwise two arcs of given slope connected by a single arc of opposite slope can be identified as regards analytic behavior provided that the connecting arc does not effectively intersect an arc of inappropriate singularity. It must be inappropriate because it is of lower order and so is assumed to have no attached complex singularities in the physical sheet, i.e., no arcs which are appropriately singular. The method of constructing a path between two arcs of given slope via adjacent unphysical sheets has been discussed by Landshoff, Polkinghorne, and Taylor. Thus arcs of inappropriate singularity are of fundamental importance in the methods of analytic continuation of $F(z)$, and it is to a discussion of their properties that the following paragraphs are devoted. Such arcs define the boundaries of the Mandelstam spectral functions.

In general, arcs of inappropriate singularity can have no horizontal or vertical tangents. This statement is not intended to include inflectional tangents. Landshoff, Polkinghorne, and Taylor say that a change in the sign of the gradient of the real section of a Landau curve lying wholly inside a region where $F(z)$ has cuts in both z_1 and z_2, implies a change in singularity type from appropriate to inappropriate (or vice versa). The reason for this is that, as we continue on $\sum$ from one side to the other of the point of horizontal or vertical tangency, the relative sign of the imaginary parts of z_1 and z_2 changes because we have gone through either the z_1 or the z_2 cut into an adjacent unphysical sheet. If at the outset our singularity was an appropriate one, we now have $F(z)$ appropriately singular in an adjacent unphysical sheet. By the first result of the present section, this implies that $F(z)$ is inappropriately singular at the corresponding points of the physical sheet. It now follows that an arc which is only inappropriately singular cannot possess horizontal or vertical tangents unless there exist complex singularities in the physical sheet (a circumstance forbidden by the induction hypothesis). This is because, if it did possess such tangents, a contradiction arises: for, by assumption, the arc beyond the tangency point is free from complex sprouts into the physical sheet and is thus appropriately nonsingular. Then, by the above argument, the original arc must have been inappropriately nonsingular which is a contradiction.

The argument presented in the last paragraph is blatantly false in the case when the gradient change is accompanied by an effective intersection with a lower-order singular curve. The single-loop vertex graph exemplifies this situation for certain values of the external masses.

In the case of the elastic scattering problem, there is no serious difficulty arising from this type of behavior. In the remainder of this section we assume, except where explicit statement of the contrary is made, that arcs of inappropriate singularity have no horizontal or vertical tangents.

Arcs, singular only in the inappropriate sense cannot lie across a Landau curve which corresponds to a normal cut and so pass out from the crossed cut region. By drawing a plot of the curve on to the z_1 plane it is clear that, on any curve such as $\sum_1$, whose real section is drawn in Fig. 14(a), a continuation path cannot be blocked by the single intersection with the lower-order curve. Thus we may conclude that $F(z)$ is singular in the appropriate sense to the left of $z_1=a$. This, however, also implies appropriate singularity because we are outside the crossed-cut region. This contradicts the induction hypothesis, so we must exclude curves such as $\sum_1$ from the possible types having arcs of inappropriate singularity. On the curve $\sum_2$ it is not immediately clear that we cannot be blocked, because two branch points coincide in the plot of $\sum_2$ on to the z_1 plane. By the device already employed we can choose a slightly different curve $\sum_2''$ which is not blocked. The intersection is not effective, and so, by the usual argument we may deduce the behavior of $F(z)$ on $\sum_2$ outside the crossed-cut region. It can be argued that if an arc of inappropriate singularity ever meets a normal threshold effectively it does so at infinity. This is because, if it meets a normal threshold effectively, it also meets effectively all curves of intermediate order obtained by setting equal to zero any subset of the α_i which actually vanish at the intersection. The vertex curves which occur in the elastic scattering problem are parallel to the normal thresholds, and since they are the curves of second-lowest order, the result follows because parallel lines meet at infinite points. Just as with $\sum_1$ we find that curves such as $\sum_2$ are, in general, inadmissible. These conclusions would be invalid if, at the point where the arc intersects the normal-threshold curve, there also

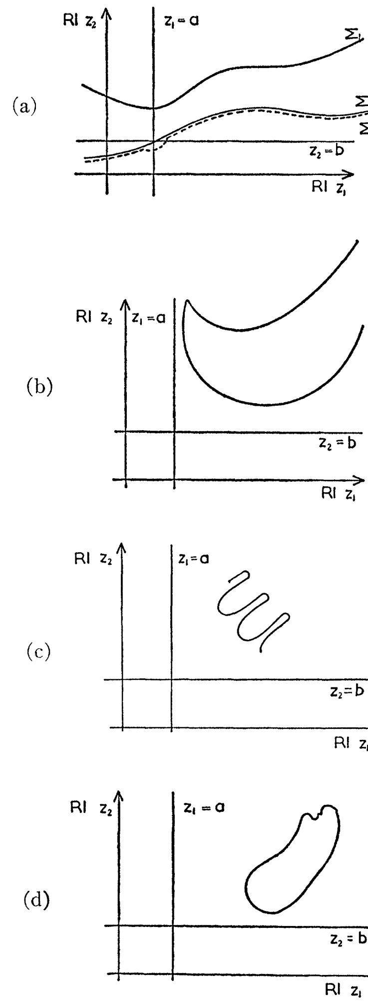

FIG. 14. (a) Γ_1 and Γ_2 lying across normal cuts. (b) Cusplike behavior of Γ. (c) Abrupt termination of Γ. (d) Closure of Γ.

occurs an effective intersection with a lower-order curve. We assume, unless explicit statement to the contrary is made, that this situation does not arise.

A case which is similar to $\sum_2$ passing through the point (a, b), is the case of a curve passing through one or other of the points (a, ∞), (∞, b) which are also coincidences of branch points. This behavior is inadmissible, in general, by the same argument of continuation to the region outside the crossed cuts. This then implies that if an arc of inappropriate singularity does meet a normal threshold curve at infinity it meets it effectively, and this in turn gives us that the directions of the normals at the intersection is the same on both curves.

In order that this latter deduction should be true we must give a reason for excluding the possibility of the type of behavior for $\sum$ which is illustrated in Fig. 14(b): because, in the absence of an effective intersection, there is no way of continuing to that

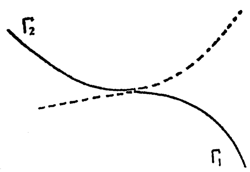

FIG. 15. Composite arc of inappropriate singularity.

part of the real plane outside the crossed-cut region by paths on $\sum$. It was pointed out to the present author by Professor N. Kemmer, that, at cusplike behavior of this sort on a Landau curve, it was possible to find, in the neighborhood of the cusp, points *both* appropriate and inappropriate to the arcs which formed the cusp. Thus the induction hypothesis could be violated by such behavior and so we should exclude such curves from consideration. However, it was very clear that algebraic curves could possess such behavior and we could not see any good reason for supposing that at any stage in the induction such a curve appeared as the leading Landau curve. We set aside this problem in the naive hope that physical situations would not necessitate considerations of cusplike behavior of Landau curves. Subsequently, however, Eden, Landshoff, Polkinghorne, and Taylor have given an explicit example which involves cusplike behavior. In their example, the Mandelstam representation fails to hold.

In general, arcs of inappropriate singularity do extend, in the real plane, as far as the normal-threshold curves.

The fundamental property possessed by arcs of inappropriate singularity is that it is impossible, under the assumptions of the induction hypothesis, to continue $F(z)$ from that unphysical sheet on which it is appropriately singular to the physical sheet via points of the Landau curve. Consider Fig. 12, which depicts a curve, the real section of which lies wholly within the crossed-cut region as we require. In the plot on to the z_1 plane, the real section AB has the properties $a \leq$ A, B and A, B$\leq \infty$. For our argument we require to assume that the branch points actually drawn (marked by crosses in Fig. 12) are the only relevant ones. It is clear that, in the case illustrated, both cuts can be passed through separately, and every sheet of $F(z)$ is accessible on all curves satisfying $a<$A, B and A, B$<\infty$. Thus arcs, such as those sketched in Figs. 14(c) and 14(d), certainly do not do as arcs of inappropriate singularity. The situations in which our route of continuation on the Landau curve is definitely blocked are those in which A and B coincide with $z_1=a$ and $z_1=\infty$ (which also forces them into coincidence with $z_2=b$ and $z_2=\infty$). This proves the stated result.

All this information now tells us that arcs of inappropriate singularity are, in general, arcs of negative slope, with asymptotes parallel to the axes of z_1 and z_2, which lie wholly inside the region of the crossed cuts. We can, in fact, have an arc of inappropriate singularity with these properties composed of several arcs of various orders which do not themselves conform to the general pattern. If, as shown in Fig. 15, arcs Γ_1 and Γ_2 touch effectively and change in nature from arcs of inappropriate singularity to arcs of nonsingularity (in both senses), then the arc composed of the undotted portions of Γ_1 and Γ_2 constitutes an arc of inappropriate singularity, while the dotted portions are irrelevant. In particular, Γ_1 and Γ_2 could have horizontal (or vertical) noninflectional tangents at the intersection while the arc which is essentially the arc of inappropriate singularity does not.

In those cases when our general theorems are invalid, the feeling is that it is not a terribly important matter unless the situation persists for a whole range of values of the external masses of the problem.

These properties, which we have discussed in detail, permit the inductive proof of the Mandelstam representation to proceed. The results are used in the following fashion: assume them true for all but the leading curve and try to prove that there exist no complex singularities of $F(z)$ on the physical sheet, at points associated with the leading Landau curve. It may be impossible to do this if any of the exceptional features, such as cusplike behavior, occur on the leading curve.

No general criteria have yet been found for the exclusion of the exceptional features. Eden, Landshoff, Polkinghorne, and Taylor have given examples when some of the exceptional features arise and it seems likely that the class of Landau curves which fit the pattern of behavior required for the validity of the inductive proof do *not* include *all* interesting physical cases. In the example quoted here, acnodes, crunodes, and cusps, persist for a wide range of values of the external masses.

5. THE SCATTERING AMPLITUDE AND THE PROOF OF MULTIPLE DISPERSION RELATIONS

The work of Eden,[7] of Landshoff, Polkinghorne, and Taylor, and Eden, Landshoff, Polkinghorne, and Taylor has done much to promote the understanding of the analytic properties of collision amplitudes, and, before

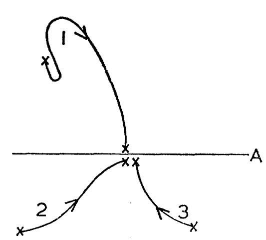

FIG. 16. Triple coincidence mechanism in α plane.

[7] R. J. Eden, (a) Phys. Rev. **119**, 1763 (1960); (b) Phys. Rev. **120**, 1514 (1960); and (c) UCRL Rept. No. 9345 (1960).

the discovery of acnodes (isolated points), crunodes (double points), and cusps on the Landau curves a great deal had been done to imbue field theorists with confidence in the value of dispersion relations, and, in particular, in the truth of the Mandelstam conjecture for the elastic scattering amplitude. Basically their methods of proof consisted in exploiting a limited knowledge of proven dispersion relations and a knowledge of regions of regularity of the amplitudes, by means of the powerful method of analytic continuation. The Landau curves themselves very often provide suitable vehicles for the continuations, because, starting from a point at which the analytic behavior is known, we can move freely on the Landau curves provided we take proper account of the set of points at which $F(z)$ may change its nature—thus far the only members of this set which we have discussed in detail are the effective interactions: the other members of this set were unknown to the author at the time when the present work was in hand and they are associated with the exceptional features such as acnodes, crunodes, and cusps which were discovered by Eden, Landshoff, Polkinghorne, and Taylor. Briefly, the mechanism is that, in the α plane, instead of two zeros of D coinciding and causing a pinch, three zeros coincide. Different modes of continuation in the z plane lead to different pairings of the zeros in the α plane. As (1) and (2) (see Fig. 16) approach the contour they pinch it. As (2) and (3) do so they coincide harmlessly. At a triple coincidence singular and nonsingular behavior may interchange corresponding to (2) changing from its association with (1) to an association with (3) (or vice versa). Such a mechanism can give rise to cusplike behavior of a Landau curve.

The discovery of these new members of the set of points where $F(z)$ may change its nature are associated with the appearance in the physical sheet of complex singularities, and must be regarded as a major set back in the progress towards the goal of a new starting point for the theory of elementary particle phenomena. If, as the work of Polkinghorne[8] suggests, these features occur not only in perturbation theory but in any unitary theory the matter is very serious indeed and requires much further investigation. One hopes that methods can, and will be evolved to cope with these new features of the scattering amplitude.

The reason for the present optimism is that one hopes that the physical data, in which one is interested, are dominated by singularities of the well-understood type, i.e., that integrals over complex branch cuts lead to negligible contributions. In a sense, current experimentation confirms this hypothesis because reasonable results have been obtained by calculations based on the assumption that processes are dominated by a small number of singularities. This sort of outlook tends to downgrade dispersion theory to a mere approximation scheme and hits hard the attitude of mind which has been seeking in the study of analytic properties of collision amplitudes some hint of a fundamental understanding of elementary particle phenomena.

The author feels that altogether too little is known to assert categorically that complex singularities are unimportant in general. They may well be, in the long run, an important and a difficult problem.

ACKNOWLEDGMENTS

The author is indebted to Professor N. Kemmer, Edinburgh, and Dr. G. R. Screaton, Oxford, for helpful discussions in the course of preparing the material contained in this paper.

[8] J. C. Polkinghorne, (a) Nuovo Cimento **23,** 360 (1962); (b) Nuovo Cimento **25,** 901 (1962).