Empirical Formula for High-Energy Nuclear Cross Sections*

ROBERT W. WILLIAMS

Physics Department, University of Washington, Seattle, Washington

I. INTRODUCTION

The cross sections of high-energy nucleons, pions, and other strongly interacting particles for collision with various nuclei are often needed in the course of high-energy experimental work. For example: Targets may perforce be complex nuclei, as in nuclear emulsion work; collision cross sections must be fed into nucleonic cascade calculations for cosmic-ray and shielding problems; particle removal caused by counters and degraders must be calculated in many experiments. Although much experimental data exists, and several accurate cross section calculations for specific situations have been published, physicists generally use a crude "geometrical" formula, $\sigma=\pi r_1^2 A^{\frac{2}{3}}$, to estimate nuclear cross sections.[1] One reason probably is that this formula gives results at least as good as those of the square-well nucleus transparency calculation,[2] which is the only other simple analytical formula available. The more accurate transparency calculations are not difficult in principle,[3] and we show below that the results agree with the measurements as well as the measurements agree among themselves. However, the calculations are numerical, and somewhat lengthy, particularly if one includes the finite range of elementary interactions.

In this paper we present an expression for inelastic high-energy cross sections which takes advantage of the empirical success of the geometrical formula, and which proves to agree with the more accurate calculations over a wide range of nuclei and of elementary cross sections. The calculations are compared with a collection of experimental results.

II. ANALYSIS

We restrict the discussion to bombarding particles of $\gtrsim 1$ BeV, and to the inelastic part of the total cross section (also called absorption, reaction, interaction, or collision cross section.) At these energies the elastic scattering is essentially all in a forward diffraction peak. The inelastic cross section measurements are usually done by transmission techniques, extrapolating to zero degrees under the diffraction peak.[4] The calculations referred to here were all done by the method given in Refs. 3 and 4: Consider a nucleus of nuclear density distribution $\rho(r)$ (this distribution has been shown[5,6] to be essentially the same as the charge density measured by electron scattering). As the bombarding particle passes through the nucleus, the probability of interaction in distance ds is $\rho_e(r)\bar{\sigma}\,ds$, where $\bar{\sigma}$ is the effective average elementary cross section. The inelastic cross section is then given by integrating over the impact parameter b, as

$$\sigma_{\text{inel}}=\int_0^\infty\left\{1-\exp\left[-\bar{\sigma}\int_{-\infty}^{\infty}\rho_e\left((b^2+s^2)^{\frac{1}{2}}\right)\,ds\right]\right\}2\pi b\,db.$$

The effective density $\rho_e(r)$ differs from $\rho(r)$ in regions where $\rho(r)$ changes appreciably over the range of the interaction between the bombarding particle and a nucleon. Describing that range by a function $F(r')$, we have

$$\rho_e(r)=\int F(\mathbf{r}-\mathbf{r}_{\text{nuc}})\rho(r_{\text{nuc}})\,d^3r_{\text{nuc}}.$$

The space-distribution function F is normalized so that $\int\rho_e\,d^3r=A$.

Some small effects mentioned in Sec. III, below, are disregarded in all these calculations. In addition, small-energy-transfer inelastic collisions constitute an ambiguity both in the calculations and in the experiments, but experiments by different techniques agree fairly well, indicating that the elastic–inelastic separation is in practice fairly clean.

It has been emphasized repeatedly[3–5] that the calculations referred to are in general agreement with the facts. An empirical formula is now presented which relates σ_{inel} to $\bar{\sigma}$ and A, and which agrees with available calculations. The formula is then compared with a compilation of recent high-energy measurements.

The extensive work of Cronin, Cool, and Abashian[4] on the nuclear scattering of high-energy pions shows clearly how the inelastic cross section varies as $\sim A^{\frac{2}{3}}$ for an elementary cross section $\bar{\sigma}\approx 33$ mb.[7] Figure 1

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[1] For example, Table II of "Data for Elementary-Particle Physics," W. H. Barkas and A. H. Rosenfeld, UCRL 8030 Rev (1961) (Unpublished).

[2] S. Fernbach, R. Serber, and T. Taylor, Phys. Rev. **75**, 1352 (1949).

[3] R. W. Williams, Phys. Rev. **98**, 1387 (1955).

[4] J. Cronin, R. Cool, and A. Abashian, Phys. Rev. **107**, 1121 (1957).

[5] L. R. B. Elton, Nucl. Phys. **23**, 681 (1961).

[6] A. Abashian, R. Cool, and J. Cronin, Phys. Rev. **104**, 855 (1956).

[7] The surprisingly good fit of an $A^{2/3}$ law for $\bar{\sigma}=33$ mb was noted by Ashmore *et al.* [A. Ashmore, G. Cocconi, A. N. Diddens, and A. M. Wetherell, Phys. Rev. Letters **5**, 576 (1960)] who show an empirical fit to their 23-BeV proton data. It is an accident of the variation of transparency of the nuclear density distribution: heavy nuclei are more transparent, and light nuclei less transparent, than a uniform-density model would lead us to suspect. See also N. R. Steenberg, Nucl. Phys. **32**, 281 (1962), for a reference to this point.

TABLE I. The empirical formula [Eq. (1)] is compared with available accurate calculations. Inelastic cross sections, in millibarns, are given for various $\bar{\sigma}$ and A values. Results of the accurate calculations are shown in parentheses.

	$\bar{\sigma}$ (mb)	20	27	33	40	50
Element						
	N	197 (192)	242 (230)	272 (256)	297 (275)	
	O	215 (216)	261 (260)	292 (291)	318 (316)	
	Al			428 (434)		
	Fe	584 (588)	654 (670)	705 (720)	745 (767)	785 (828)
	Sn			1187 (1190)		
	Pb	1549 (1503)	1661 (1630)	1740 (1710)	1805 (1780)	1860 (1850)

shows a reproduction of their data (Fig. 8 of Ref. 4) with the cross sections divided by $\pi r_1^2 A^{\frac{2}{3}}$. Also plotted are a square-well transparency curve and Cronin's accurate transparency curve, both evaluated at $\bar{\sigma}=33$ mb, and a power law, $\sigma_{\text{inel}}=44A^{0.69}$ mb, which agrees both with the data and with Cronin's calculation.

Starting with this power law for $\bar{\sigma}=33$ mb, we have fitted a power series in $(\bar{\sigma}-33)$ to some published accurate calculations[8,9] of σ_{inel} as a function of $\bar{\sigma}$. Two terms proved sufficient, over the limited range for which calculations are available, which is also the range of practical interest. The empirical formula is

$$\sigma_{\text{inel}}=44A^{0.69}[1+0.039A^{-\frac{1}{3}}(\bar{\sigma}-33) -0.0009A^{-\frac{1}{3}}(\bar{\sigma}-33)^2], \quad (1)$$

where both σ_{inel} and $\bar{\sigma}$ are in millibarns. The range of validity in $\bar{\sigma}$ is 20 mb$\leq\bar{\sigma}\leq$50 mb; in A, all elements heavier than Li seem to be well-represented, although N is the lightest for which good calculations exist. Nuclear structure effects such as shell closure and large-distortion regions are of course ignored by such a gross formula. These effects, though visible in the electron scattering experiments, have not been conspicuous in nucleon or pion scattering results.

Table I compares Eq. (1) with various published calculations. The calculations for N and O used the actual density parameters as determined by electron scattering for each element; heavier elements were approximated by the well-known standard density distributions,[10] with the "Fermi shape" which has a uniform central density and a smooth taper. The empirical formula misses the nitrogen calculation by as much as 8%, but is very good for oxygen, and also agrees with the standard-density calculations[4] for $A=12$ and $A=8$, $\bar{\sigma}=33$ mb. The nitrogen density distribution is sufficiently nonstandard to cause it to be out of line. Otherwise, the values all agree to 5% or better. We take the results shown in Table I as evidence that Eq. (1) represents the accurate calculations over the range claimed.

III. COMPARISON WITH EXPERIMENT

Counter measurements of inelastic cross sections of various nuclei are now available at energies up to 23 BeV. To compare these with Eq. (1)—and therefore with the calculations whose results it represents—we must fix the value of $\bar{\sigma}$, the effective elementary (nucleon–nucleon or pion–nucleon) cross section. If we identify $\bar{\sigma}$ with the total elementary cross section we include in σ_{inel} the (perhaps unobserved) small-momentum-transfer events associated with the diffraction-scattering part of σ(elem); we also ignore the inhibiting effect of the Pauli principle on the small-

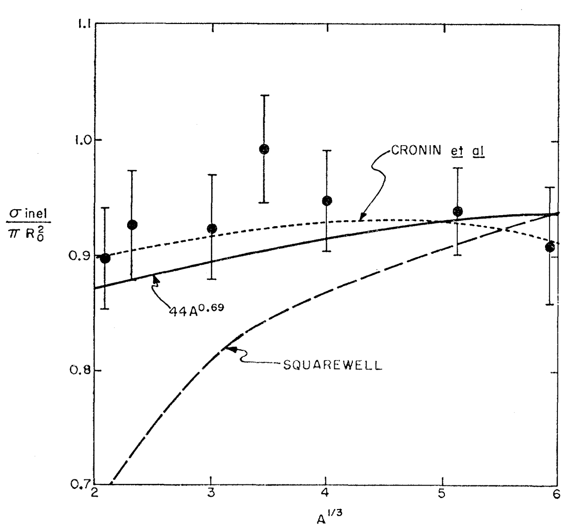

FIG. 1. The experimental data of Cronin *et al.* (Ref. 4) compared with Cronin's accurate calculation, with a transparency curve obtained from a square well of $R=1.3A^{\frac{1}{3}}F$, and with the approximation $\bar{\sigma}_{\text{inel}}=44A^{0.69}$ mb. All cross sections are divided by πR_0^2 with $R_0=1.3A^{\frac{1}{3}}F$.

[8] For N and O we used R. W. Williams, Nuovo Cimento 16, 762 (1960); for Fe, Ref. 9; for heavy elements, Ref. 3. At $\bar{\sigma}=33$ mb these differ from Cronin's numbers by less than 2%.

[9] A. E. Brenner and R. W. Williams, Phys. Rev. 106, 1020 (1957).

[10] B. Hahn, D. Ravenhall, and R. Hofstadter, Phys. Rev. 101, 1131 (1956).

TABLE II. Some values of elementary cross sections, total and inelastic ["inelastic" includes charge exchange in $(\pi^- p)$ collisions]. These are smoothed values chosen from the compilations of Ref. 12. Cross sections in millibarns. In computing average cross sections, inelastic (np) cross sections are assumed to scale from (pp) in the same ratio as the total cross sections, and charge symmetry is invoked to get $(\pi^- n)$ or (nn) cross sections.

	0.97 BeV		0.9–1.4 BeV		5 BeV		8.3 BeV		23 BeV	
	Total	Inelastic	Total	Inelastic	Total	Inelastic	Total	Inelastic	Total	Inelastic
$\pi^- P$	46	26								
$\pi^+ P$	24	11								
pp			47	26	43	35	41	32	39	31
np			37		37		40		40	
					average value					
$\bar{\sigma}$ Light nuclei	35	19	42	23			40	32		
$\bar{\sigma}$ Heavy nuclei π^- or p beam	33	17	41	22			40	32		

momentum-transfer events. If we take $\bar{\sigma}$ to be only the inelastic (i.e., meson-producing) part of the elementary cross section we neglect some quasi-elastic collisions which disrupt the nucleus and lead to considerable energy and momentum transfer. In either case we neglect other effects of perhaps comparable magnitude: the correlation of nucleons in nuclear matter,[11] and, for pion beams, the three-body absorption process.[12] We appeal to experiment, listing some measured elementary cross sections σ_{total} and $\sigma_{\text{meson production}}$ in Table II.[13] Table III is a collection of measurements of nuclear inelastic cross sections in the BeV region; these are the principal sources known to the author.

For convenience in comparing the calculated with the observed values of the cross section we have grouped the values in Table III according to similar values of $\bar{\sigma}$ (from Table II), and have averaged the experimental values in each of the three groups—nucleons ~1 BeV, pions ~1 BeV, and nucleons ≥5 BeV—weighting by the quoted errors. The error we assign is simply the smallest quoted error for each set of measurements. This represents in our view a fair, or even optimistic,

TABLE III. Inelastic cross sections found in various high-energy experiments, with the errors reported by the experimenters. All are transmission measurements with the diffraction scattering eliminated by an extrapolation from "bad-geometry" values (except that there appears to be no extrapolation correction to the 8.3-BeV neutron data). All cross sections in millibarns.

Beam	0.86 BeV p^a	0.9 BeV p^b	1.4 BeV n^c	0.97 BeV π^{-d}	5.0 BeV n^e	8.3 BeV n^f	23 BeV p^g
Element							
Be	169±15		187±12	197±9			180±5
C	209±22	230±20	201±13	252±13	235±16	218±8	225±5
Al	394±10	370±29	414±23	442±20	381±27	380±13	400±10
Cu	728±17	740±50	674±34	806±30	586±25	626±29	740±20
Sn	1110±30	1133±22[h]	1158±63	1199±52		1218±50	1221±26[i]
Pb	1680±40	1660±50	1727±45	1690±100	1670±79	1713±66	1750±30

[a] F. F. Chen, C. P. Leavitt, and A. M. Shapiro, Phys. Rev. **99**, 857 (1955).
[b] N. E. Booth, B. Ledley, D. Walker, and D. H. White, Proc. Phys. Soc. (London) **70A**, 209 (1957).
[c] T. Coor, D. Hill, W. Hornyak, L. Smith, and G. Snow, Phys. Rev. **98**, 1369 (1955).
[d] Reference 4.
[e] J. H. Atkinson, W. N. Hess, V. Perez-Mendez, and R. Wallace, Phys. Rev. **123**, 2054 (1961).
[f] V. S. Pantuev and M. N. Khachaturyan, Zh. Eksperim. i Teor. Fiz. **42**, 909 (1962) [English transl.: Soviet Phys.—JETP **15**, 626 (1962)].
[g] Reference 7.
[h] Converted from Sb by the factor 0.981.
[i] Converted from Cd by the factor 1.036.

[11] R. J. Glauber, Physica **22**, 1185 (1956).
[12] See C. J. Batty, Proc. Phys. Soc. (London) **76**, 577 (1960), and Ref. 4.
[13] A very complete compilation of elementary cross sections will be found in the review by V. S. Barashenkov and V. M. Maltsev, Fortschr. Physik **9**, 549 (1961). [An earlier compilation is given by W. N. Hess, Rev. Mod. Phys. **30**, 368 (1958).] More recent (pp) and (np) data are given by A. N. Diddens, E. Lillethun, G. Manning, A. E. Taylor, T. G. Walker, and A. M. Wetherell, Phys. Rev. Letters **9**, 32 (1962).

Table IV. Weighted average of the inelastic cross sections given in Table III, in groups of similar $\bar{\sigma}$ values. For each group the difference between the calculated values (from Eq. (1)) and the measured values is given using $\bar{\sigma}=\bar{\sigma}_{tot}$ and $\bar{\sigma}=\bar{\sigma}_{inel}$ from Table II.

	0.86 BeV p, 0.9 BeV p, 1.4 BeV n			0.97 BeV π^-			5 BeV n, 8.3 BeV n, 23 BeV p		
		$\sigma_{calc}-\sigma_{obs}$			$\sigma_{calc}-\sigma_{obs}$			$\sigma_{calc}-\sigma_{obs}$	
Element	σ_{obs}(Av)	$\bar{\sigma}=\bar{\sigma}_{tot}$	$\bar{\sigma}=\bar{\sigma}_{inel}$	σ_{obs}	$\bar{\sigma}=\bar{\sigma}_{tot}$	$\bar{\sigma}=\bar{\sigma}_{inel}$	σ_{obs}(Av)	$\bar{\sigma}=\bar{\sigma}_{tot}$	$\bar{\sigma}=\bar{\sigma}_{inel}$
Be	180±12	27%	−14%	197±9	6%	−33%	180±5	26%	9%
C	210±13	30%	−8%	252±13	0	−34%	224±5	20%	7%
Al	394±10	19%	−9%	442±20	−1%	−26%	391±10	18%	8%
Cu	720±17	15%	−6%	806±30	−5%	−23%	702±20[a]	16%	9%
Sn	1120±22	12%	−6%	1199±52	−1%	−19%	1220±26	2%	−3%
Pb	1690±40	7%	−7%	1690±100	3%	−12%	1740±30	4%	−1%

[a] The 5-BeV n value was omitted from this average as it is over four standard deviations below the average of the other two values.

assessment of the error in the final results, since there are systematic uncertainties in all these measurements. Table IV presents these average cross sections, along with the difference between the calculated values, from Eq. (1), and the observed values. The calculated values were obtained using both sets of $\bar{\sigma}$ values from Table II, one set corresponding to the total elementary cross section, and the other set to the inelastic elementary cross section.

Inspection of Table IV leads to a definite prescription for choosing $\bar{\sigma}$ so that Eq. (1) reproduces the data to about 5%: (a) For pions near 1 BeV, $\bar{\sigma}$ is obtained from total cross sestions. (b) For nucleons near 1 BeV, $\bar{\sigma}$ is obtained from the inelastic cross sections plus about one third of the diffraction (i.e., elastic) cross section. (c) For nucleons above 5 BeV, $\bar{\sigma}$ is obtained from the inelastic cross sections only.

IV. DISCUSSION

The above rules for determining $\bar{\sigma}$ from the elementary cross sections seem reasonable in two respects: (1) pions, having extra modes of interaction in nuclei, are expected to have a larger effective elementary cross section than nucleons; (2) at very high energies the

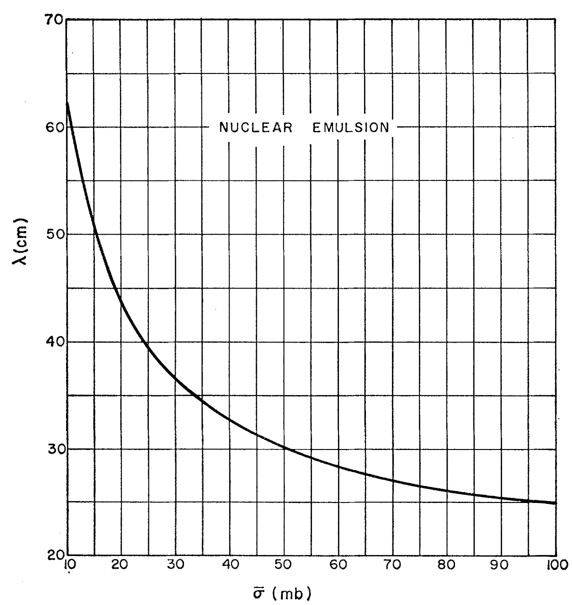

Fig. 2. Mean free path for interactions in nuclear emulsion, as a function of $\bar{\sigma}$. Ordinate is mean free path in cm of G-5 emulsion of density 3.85 g cm^{-3}. Adapted from Barachenkov and Maltsev, Ref. 13.

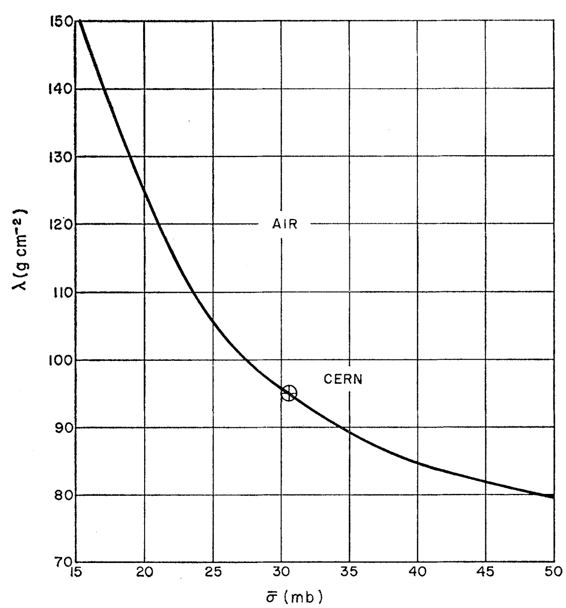

Fig. 3. Mean free path for interactions in air, as a function of $\bar{\sigma}$. The cross is a point interpolated from experimental data at 23 BeV (Ref. 7).

diffraction part of the elementary cross section would represent fractionally very small momentum transfer events which would be unlikely to be detectable.[14]

The tendency for all high-energy inelastic cross sections to be about the same (as one can see from Table III) is a consequence of the lack of variation in $\bar{\sigma}$ for pions or nucleons above 1 BeV. It would be interesting to have some good nuclear cross sections for K^+ beams (small $\bar{\sigma}$) and antiproton beams (large $\bar{\sigma}$).[15]

For rare processes (e.g., hyperon scattering; collisions at ultrahigh energies) one may have available only a mean free path in nuclear emulsion. This leads to a determination of $\bar{\sigma}$ essentially equivalent to the one discussed in this paper, although the experimental conditions are different. Barashenkov and Maltsev[13] have calculated the relevant inelastic cross section as a function of $\bar{\sigma}$ for the elements of a *G*–5 type of emulsion, using the standard nuclear density distributions. We have checked their curve at several points, using Eq. (1), and find good agreement; our values for the mean free path are lower by about 2%. Figure 2 is a reproduction of their curve, lowered by 2%.

A related problem is that of the mean free path in air for inelastic collisions of high-energy particles. This was in fact the motivation for the N and O cross section calculations in Ref. 8. Figure 3 reproduces the mean free path curve of Ref. 8, extended to 50 mb; also shown is the point corresponding to interpolation in the 23-BeV data of Ashmore *et al.*[7] It is noteworthy that the value often assumed in cosmic-ray work[16] for the inelastic mean free path, $\lambda=75$ g cm^{-2}, lies far outside the range of $\bar{\sigma}$ observed in the energy region accessible to clear-cut experiments.

V. SUMMARY

A very simple expression, Eq. (1), has been presented which adequately reproduces the results of detailed calculations relating the inelastic nuclear cross sections of strongly-interacting high-energy particles to the elementary interaction with free nucleons. When this expression is compared with a collection of experimental cross sections, it agrees fairly well provided the elementary interaction $\bar{\sigma}$ is chosen to be: (a) For nucleons over 5 BeV, the inelastic cross section; (b) For pions around 1 BeV, the total cross section; and (c) For nucleons around 1 BeV, the inelastic plus $\frac{1}{3}$ the elastic cross section.

For pions and other strongly interacting particles over 5 BeV, we conjecture that the inelastic cross section alone is the appropriate elementary interaction.[17]

The range of $\bar{\sigma}$ covered by the available data is small, so that the $\bar{\sigma}$ dependence of the calculations which the formula represents is not tested well by the comparison with experiment.

Finally, curves are presented which relate the interaction mean free path in nuclear emulsion, and in air, with the elementary cross section, $\bar{\sigma}$. From these curves one sees that at the typical value $\bar{\sigma}=35$ mb the error in a mean-free-path measurement is multiplied by 2.7 (nuclear emulsion) or 2.2 (air) when used as a determination of $\bar{\sigma}$.

[14] The shrinking diffraction pattern observed in some high-energy elastic scattering experiments [e.g., K. J. Foley *et al.*, Phys. Rev. Letters **10**, 376 (1963)] means that even the absolute (average) momentum transfer may go down. The Regge pole interpretation of this phenomenon suggests that also the momentum transfer in inelastic events is weakened as the energy increases, i.e., that the collisions take place over a growing region of space. In terms of the calculation outlined in Sec. II this corresponds to a growing range-of-interaction correction, until in the limit it dominates the calculation, and one finds $\sigma_{\text{inel}}=A\bar{\sigma}$. A quantitative treatment is given by B. M. Udgaonkar and M. Gell-Mann [Phys. Rev. Letters **8**, 346 (1962)], who show that the presumed effect would occur only at energies presently out of reach.

[15] Note that the empirical formula, Eq. (1), is not meaningful above $\sigma=55$ mb, the point at which it has zero slope.

[16] See, e.g., B. Peters, *1962 International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962), p. 623.

[17] Some support for this conjecture is given by 4.6-BeV pion data of M. F. Likhachev, V. S. Stavinskii, H. Yün-Ch'ang, and C. Nai-Sen, Zh. Eksperim. i Teor. Fiz. **41**, 38 (1961) [English transl.: Soviet Phys.—JETP **14**, 29 (1962)]. They find cross sections, on light elements, which are less than any given in Table III, and some K^+ cross sections which are very small indeed. Unfortunately, few experimental details are given.