Thus the Born approximation is valid, which suggests an immediate connection between λ^2 and Lindenbaum and Yuan's parameter b in (19). For small t, $d\sigma/dt = A \exp(-bt)$, which is the Born approximation scattering from a Gaussian potential of range

$$\lambda^2 = 1/2b\,. \tag{25}$$

Expressed in $(10^{-13}\text{ cm})^2$, rather than $(\text{BeV}/c)^{-2}$, $b = 0.4080$, and (25) gives $\lambda^2 = 1.225$, in excellent agreement with the value obtained from the numerical integration.

For small r,

$$V(r) = e^{-1.341r}/r\,, \quad r < 0.33\,. \tag{26}$$

For $0.33 < r < 1.1$, $rV(r)$ was calculated by numerical integration of the right side of (21), followed by differentiation to give $rV(r)$. The results are shown in Fig. 3.

In summary, we have shown that it is possible to find an absorptive potential which will represent the data well for both large and small t. For small r, $V(r)$ behaves like a Yukawa potential with a range determined by the width of the scattering curve for large t, while for large r it behaves like a Gaussian with a range determined by the initial rate of decrease of the scattering for small t.

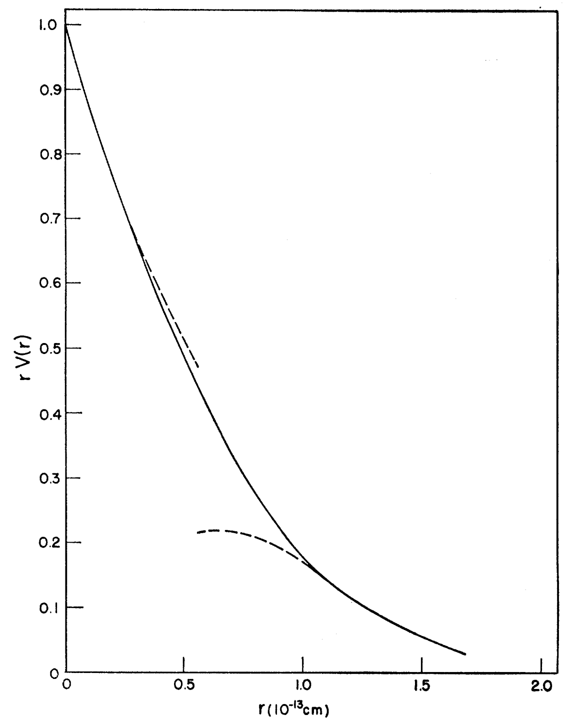

FIG. 3. $rV(r)$ as a function of r. The dashed curves give the continuation of the Yukawa potential (26) (times r) to larger r, and the continuation of the Gaussian potential (24) (times r) to smaller r.

High-Energy Collisions of Strongly Interacting Particles

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1. INTRODUCTION

We are concerned in the present paper with the theoretical analysis of strongly interacting particle collisions in the multi-GeV energy range (laboratory energy above 5 GeV). While one has expected for a long time that the gross features of such collisions would be qualitatively very simple, progress in their detailed experimental and theoretical study has been extremely slow. Reliable and accurate accelerator experiments have started quite recently and have mainly elucidated some of the basic properties of small angle elastic scattering up to 30 GeV. The work on inelastic collisions is only at its most timid beginnings. The systematic study of the most common type of inelastic collisions, often referred to as jets, has hardly started at accelerator energies and is strongly limited by the short-comings of most present detectors (what is required is identification and momentum determination of all pions in the three charge states, for inelastic collisions in pure hydrogen). Regarding our theoretical understanding, fundamental theory has essentially failed to make any progress despite many attempts. It is striking in this context to read over again the early and most interesting paper devoted in 1948 to multiple meson production by Lewis, Oppenheimer, and Wouthuysen,[1] and to compare it with recent attempts at applying field theory or dispersion techniques to this subject. The difficulty to explain even the most predominant features of high-energy collisions remains as great as it was sixteen years ago.

It is in our opinion unavoidable that progress in theoretical understanding of these phenomena will

[1] H. W. Lewis, J. R. Oppenheimer, and S. A. Wouthuysen, Phys. Rev. **73**, 127 (1948).

require a very close interplay between phenomenological analysis of sufficiently accurate experiments, mainly in terms of various tentative models, and application of the few general theoretical principles which seem to be well established. This is the way in which our present knowledge of low-energy nuclear physics was reached both with regard to nuclear structure and nuclear reactions, and it is more likely to be successful than an approach based on some fundamental theory of strong interactions guessed *a priori* in absence of clear experimental clues. The phenomenological approach here advocated was tried in the past in the extreme form of statistical and hydrodynamical models for multiple particle production. That these first guesses met with very limited success means only that high-energy phenomena are not to be understood in full ignorance of the dynamical interactions involved, so that one has good reason to hope that their elucidation will eventually reveal unexpected and interesting dynamical features. As experimental work and theoretical analysis proceed, one can expect to obtain a reasonably clear understanding of high-energy collisions, and this will undoubtedly add an essential part to our knowledge of strong interactions as it is being elaborated at present in the low and intermediate energy region up to a few GeV.

The general theoretical principles on which the analysis of high-energy collisions thus far relies are very few: in addition to Lorentz invariance (which is of limited help because each collision naturally fixes a special system of coordinates, the center-of-mass system) one has only unitarity and crossing symmetry of the S matrix. While the former property simply expresses probability conservation, the latter states certain analyticity properties of the S-matrix elements and certain relations between them, which are believed, and proved in a few special cases, to be consequences of the local nature of basic interactions when expressed in terms of fields.

That a theoretical discussion is possible on the basis of principles so few in number and so general in contents is related to the near constancy at high energy of the total cross section σ_t, as well as of the differential elastic cross section $d\sigma_{el}/dt$ for $0 \leqslant -t \lesssim 1(\mathrm{GeV}/c)^2$ (this is the only domain of t values where $d\sigma_{el}/dt$ is appreciable: as usual, $-t$ denotes the square of the center-of-mass momentum transfer). As has been realized recently, this empirical fact implies, through crossing symmetry and under plausible assumptions, that the elastic scattering amplitude T has a much larger imaginary than real part at high energy (Sec. 2). The latter property, the validity of which is not in disagreement with the rather incomplete experimental evidence on this point, implies in turn that elastic scattering is the shadow of inelastic collisions, and the unitarity condition then allows us to study mathematically how far the leading features of the former can be understood in terms of simple assumptions regarding the latter.

From the theoretical work done up to now on these questions one important indication emerges. Recent experiments have shown that $d\sigma_{el}/dt$ has a t dependence of the form

$$d\sigma_{el}/dt = (d\sigma_{el}/dt)_{t=0} \exp{(at + bt^2)} , \qquad (1.1)$$

with

$$0 < b/a^2 \ll 1 . \qquad (1.2)$$

It is found to hold in an interval $a|t| \lesssim 10$. This simple t dependence can be accounted for theoretically if the secondaries of inelastic collisions (i.e., the particles in inelastic final states) are rather numerous and are emitted with only weak correlations beyond the obvious correlation imposed by energy–momentum conservation. That absence of correlation in the inelastic final state implies approximate linearity of ln $(d\sigma_{el}/dt)$ in t has been verified for various models of jets and is very likely to be quite a general feature of statistical nature. A rather realistic class of models will be discussed in some detail in Sec. 3. Its shadow scattering is calculated and compared with experiment in Sec. 4.

Further progress seems difficult without using more detailed properties of jets than are known at present, but it is clear *a priori* that the approximate independence of $d\sigma_{el}/dt$ upon the energy must limit in some way the energy dependence of multiplicity and momentum distribution in jets. This effect has been studied for jet models in which the transverse and longitudinal momenta of the secondaries are supposed uncorrelated, with the remarkable result that the energy variations of longitudinal momentum distribution and multiplicity are found to be severely limited. The limitations found are compatible with the experimental evidence from cosmic rays (Sec. 5).

Most of the material reported in Secs. 2 to 5 summarizes and extends already published work. We mention in a brief closing section some experimental questions regarding inelastic collisions which come up naturally in the present type of analysis.

2. APPROXIMATE ENERGY INDEPENDENCE OF CROSS SECTIONS AND IMAGINARY CHARACTER OF THE SCATTERING AMPLITUDE

Let $T(st)$ be the scattering amplitude for the elastic scattering $A + B \rightarrow A + B$, s being the square of the center-of-mass (c.m.) energy and $-t$ the square of the c.m. momentum transfer from A to B. Our problem is to show under plausible assumptions that, at very high energy, approximate energy independence of total cross-section σ_t and differential elastic cross-section $d\sigma_{el}/dt$ requires T to be almost purely imaginary. This has been done in two recent notes[2,3] for the case where σ_t and $d\sigma_{el}/dt$ tend to nonvanishing limits as $s \to +\infty$, analyticity of $T(st)$ in the upper half s plane being exploited through the dispersion relation at constant t in the fashion first shown by Pomeranchuk when he derived his famous theorem on total cross sections of crossed reactions.[4] Under slightly modified assumptions, use of the dispersion relation in establishing theorems of the Pomeranchuk type can be replaced by a very elegant application of the Phragmén–Lindelöf theorem,[5] a method which allows us also to discuss the case where the cross sections do not become constant at high energy.[6] The investigations just referred to dealt mainly with the problem of the relation between $A + B \to A + B$ and the crossed reaction $\overline{A} + B \to \overline{A} + B$. Our purpose here being different, we present the argument anew. Spin, which is no essential complication for these questions,[6] is neglected.

Our first and main assumption (*assumption a*) is that the contributions to $A + B \to A + B$ of exchange of the charge conjugation quantum numbers $C = \pm 1$ in the t channel ($A + \overline{A} \to B + \overline{B}$) have different asymptotic dependences on s, such that one is negligible compared to the other for $s \to +\infty$ and t in a given interval $t_0 \leqslant t \leqslant 0$. It is an assumption of nondegeneracy between the asymptotic behaviors of $C = 1$ and $C = -1$ contributions in the t channel: They for example should not go with the same power of s or $\ln s$ as $s \to +\infty$.[3] Denoting by $\overline{T}(st)$ the scattering amplitude for the crossed process $\overline{A} + B \to \overline{A} + B$, this assumption means

$$\lim_{s\to+\infty} \overline{T}(st)/T(st) = \pm 1\,, \quad t_0 \leqslant t \leqslant 0 \qquad (2.1)$$

the $\pm$ sign corresponding to dominance of $C = \pm 1$ exchange. We now remember that by the unitarity of the S matrix (optical theorem) $\mathrm{Im}\, T(s0) > 0$ unless $\sigma_t = 0$, and similarly for $\overline{T}$. As a consequence, unless σ_t vanishes identically above a certain finite energy, we can only have the $+$ sign in (2.1) and the dominant exchange in the t channel is therefore $C = 1$. We thus have[7]

$$\lim_{s\to+\infty} \overline{T}(st)/T(st) = 1\,, \quad t_0 \leqslant t \leqslant 0\,. \qquad (2.2)$$

Crossing symmetry expresses $\overline{T}$ in terms of the analytic continuation of T in the upper half s plane

$$\overline{T}(st) = T^*(ut)\,, \quad u = 2m_A^2 + 2m_B^2 - s - t\,, \quad s \geqslant (m_A + m_B)^2 \qquad (2.3)$$

so that assumption (a) gives rise to the following equation:

$$\lim_{s\to+\infty} T^*(2m_A^2 + 2m_B^2 - s - t,t)/T(st) = 1\,. \qquad (2.4)$$

Here, and further on in the present section, t is in the interval $t_0 \leqslant t \leqslant 0$.

The usual analyticity properties postulated for T, analyticity in s and majorization by a polynomial in s in the upper half s plane, are obviously insufficient to derive from (2.4) what is the phase of T as $s \to +\infty$. We need an assumption on the high-energy behavior of T (assumption b). Its detailed contents depend to a certain degree on the method followed to exploit it. Following Meiman[5] and Logunov *et al.*,[6] we use as the main tool the Phragmén–Lindelöf theorem. We then choose to formulate assumption (b) as follows:

$$\lim_{s\to+\infty} s^{-\alpha(t)}(\ln s)^{-\beta(t)}T(st) = c(t) \qquad (2.5)$$

with $\alpha(t)$ and $\beta(t)$ real, and $c(t)$ complex nonvanishing. It is a rather general *ansatz* for the high-energy behavior of T at fixed t (for further generalization see below). We rewrite it as

$$\lim_{s\to+\infty} T(st)/\varphi(st) = 1 \qquad (2.6)$$

with φ a function which can be chosen to be

$$\varphi(st) = c(t)(s+i)^{\alpha(t)}[\ln(s+i)]^{\beta(t)}\,. \qquad (2.7)$$

We take the cut of φ to be $s = -ix$, $x \geqslant 1$. From (2.4) one derives

$$\lim_{s\to-\infty} T(st)/\varphi^*(-s,t) = 1\,. \qquad (2.8)$$

But from (2.7)

$$\lim_{s\to-\infty} \varphi^*(-s,t)/\varphi(s,t) = [c^*(t)/c(t)]\exp[-i\pi\alpha(t)] \equiv \gamma(t)\,. \qquad (2.9)$$

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[2] L. Van Hove, Phys. Letters **5**, 252 (1963).
[3] L. Van Hove, Phys. Letters **7**, 76 (1963).
[4] I. Ia. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. **34**, 725 (1958) [English transl.: Soviet Phys.—JETP **7**, 499 (1958)].
[5] N. N. Meiman, Zh. Eksperim. i Teor. Fiz. **43**, 2277 (1962) [English transl.: Soviet Phys.—JETP **16**, 1609 (1963)].
[6] A. A. Logunov, N. van Hieu, I. T. Todorov, and O. A. Khrustalev, Phys. Letters **7**, 69 (1963).

[7] A slightly different proof is given in Ref. 3. Theorems on possible dominance of the exchange of certain quantum numbers in elastic scattering have been given for pion–pion scattering by D. Amati, A. Stanghellini, and S. Fubini [Nuovo Cimento **26**, 896 (1962), Eq. (9.20) and thereafter]; for general isospin values by L. L. Foldy and R. F. Peierls [Phys. Rev. **130**, 1585 (1963)]; and for general groups by D. Amati, L. L. Foldy, A. Stanghellini, and L. Van Hove (to be published).

Hence,

$$\lim_{s\to-\infty} T(st)/\varphi(st) = \gamma(t) . \qquad (2.10)$$

The function T/φ is analytic in the upper half s plane and $|T/\varphi|$ increases more slowly than an exponential $\exp(L|s|^\epsilon)$ with $L > 0$, $\epsilon < 1$. According to the Phragmén–Lindelöf theorem applied to T/φ,[8] Eqs. (2.6) and (2.10) then imply $\gamma(t) = 1$. This determines the phase of $c(t)$, and hence of $T(st)$, to be $-\frac{1}{2}\pi\alpha(t) + n\pi$, $n = 0, 1$.

But the energy variation of cross sections at high s is given by

$$\sigma_t \propto s^{-1} \operatorname{Im} T(s0) , \quad d\sigma_{el}/dt \propto s^{-2}|T(st)|^2 . \qquad (2.11)$$

Clearly, slowly varying cross sections at high energy correspond in (2.5) to $\alpha(t)$ close to 1, and hence to almost purely imaginary scattering amplitude $T(st)$. It is remarkable that this conclusion, being independent of $\beta(t)$, is unaffected by logarithmic variations of the cross sections at large s.

The above reasoning could be extended to more general forms of the asymptotic behavior of T as $s \to +\infty$. One can repeat the argument for any function φ such that T/φ has the properties mentioned immediately after (2.10), and such that

$$\varphi^*(2m_A^2 + 2m_B^2 - s - t,t)/\varphi(st) \qquad (2.12)$$

has a finite limit $\gamma(t)$ for $s \to +\infty$. The value of $\gamma(t)$ is then 1 by virtue of the Phragmén–Lindelöf theorem, and this uniquely fixes the phase factor of $\varphi(st)$ for every t.

Instead of relying entirely on the Phragmén–Lindelöf theorem one could have used it only to prove that T/φ is bounded in the upper half s plane, and then proceed by writing down a once subtracted dispersion relation in s expressing Im (T/φ) in terms of Re (T/φ). By the original method of Pomeranchuk one then obtains $\gamma(t) = 1$, but finds in addition a sum rule for $\operatorname{Re}(T/\varphi)$, which is an extension to arbitrary particles of the sum rule $d(t) = 0$ obtained in Ref. 2 for pion–nucleon scattering (sum rule of Goldberger, Miyazawa, and Oehme).

3. A CLASS OF UNCORRELATED JET MODELS

The present Section is devoted to a class of uncorrelated jet models and prepares the calculation of the corresponding shadow scattering. We recall first the formulas needed for deducing the shadow scattering; they follow from unitarity of the S matrix and are derived in some detail in an earlier paper on this subject.[9] The partial wave amplitude for angular momentum l is denoted by $i\eta_l$.[10] The quantity η_l, which is real under our assumption of a purely imaginary scattering amplitude, is related to the inelastic collisions by the set of equations

$$\eta_l - \tfrac{1}{2}\eta_l^2 = f_l , \qquad (3.1)$$

$$\sum_{l=0}^{\infty} (2l+1) f_l P_l(\cos\theta) = F(k\theta) , \qquad (3.2)$$

$$F(k\theta) = 2\pi k\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2)^{-1} \times \langle f(\mathbf{k}')|\prod_0^3 \delta(\sigma P^\mu - p_0^\mu)|f(\mathbf{k})\rangle , \qquad (3.3)$$

$$\prod_0^3 \delta(P^\mu - p_0^\mu)|f(\mathbf{k})\rangle = (1-Q)S|\mathbf{k},-\mathbf{k}\rangle . \qquad (3.4)$$

All symbols refer to the c.m. system. $|\mathbf{k}, -\mathbf{k}\rangle$ denotes the incident state in which particles A and B have momenta $\mathbf{k}$ and $-\mathbf{k}$, respectively, and energies

$$\epsilon_1 = (m_A^2 + k^2)^{\frac{1}{2}}, \quad \epsilon_2 = (m_B^2 + k^2)^{\frac{1}{2}}, \quad k = |\mathbf{k}| . \qquad (3.5)$$

The corresponding final state $S|\mathbf{k}, -\mathbf{k}\rangle$ has an elastic component $QS|\mathbf{k}, -\mathbf{k}\rangle$ in the channel where A and B but no other particle are present (Q denotes the projection operator on this component). The sum of all inelastic components, to be called *inelastic final state*, is the vector (3.4) where the four delta functions express energy–momentum conservation. P^μ ($\mu = 0,1,2,3$) is the total energy–momentum operator, p_0^μ its value in the incident state

$$p_0^0 = s^{\frac{1}{2}} = \epsilon_1 + \epsilon_2 , \quad \mathbf{p}_0 = 0 . \qquad (3.6)$$

$f(\mathbf{k}')$ in (3.3) relates to the inelastic final state of a collision with incident state $|\mathbf{k}', -\mathbf{k}'\rangle$, where $\mathbf{k}'$ has the same length as $\mathbf{k}$ and forms an angle θ with it. F has the significance of an *overlap function* between inelastic final states of two collisions with the same total energy and momentum. Unitarity requires the equivalent inequalities

$$|1 - \eta_l| \leqslant 1 , \qquad 0 \leqslant f_l \leqslant \tfrac{1}{2} . \qquad (3.7)$$

In high-energy collisions the differential elastic cross section

$$\frac{d\sigma_{el}}{dt} = \frac{\pi}{k^2}\frac{d\sigma_{el}}{d\Omega} = \frac{\pi}{4k^4}\left|\sum_l (2l+1)\eta_l P_l(\cos\theta)\right|^2 , \quad t = 2k^2(\cos\theta - 1) \qquad (3.8)$$

[8] For a convenient formulation of the Phragmén–Lindelöf theorem for our purpose, see N. N. Maimann, loc. cit., Sec. 2, A and B. As mentioned there, the theorem also applies when the function becomes unbounded on a finite interval of the real axis. It has, however, to be analytic on the real axis outside of such an interval. Another useful reference for the theorem is E. C. Titchmarsh, *The Theory of Functions* (Oxford University Press, London, 1939), 2nd ed., Sec. 5.61 to 5.64.

[9] L. Van Hove, Nuovo Cimento **28**, 798 (1963), hereafter referred to as I. See also L. Van Hove, "Strong interactions at very high energy," course given at the Cargèse Summer School, Corsica, 1963.

[10] The corresponding expectation value of the S matrix is $1 - \eta_l$.

is only appreciable at small angles. The main region of experimental interest for our considerations is $|t|^{\frac{1}{2}} \simeq k\theta \lesssim 1$ GeV/c, in which the differential cross section is approximately energy-independent and has the shape (1.1), (1.2) with[11] $a \simeq 10(\text{GeV}/c)^{-2}$. Hence similar properties have to hold for $F(k,\theta)$ (see Sec. 4 for more details), and both η_l and f_l are smooth functions of l approaching zero for $l \gg k/m_p$. This requires η_l to be the smallest of the two roots of (3.1)

$$\eta_l = 1 - (1 - 2f_l)^{\frac{1}{2}} \tag{3.9}$$

We also write down in our notation the total, total elastic, and total inelastic cross sections

$$\sigma_t = (2\pi/k^2) \sum (2l+1)\eta_l\,, \tag{3.10}$$

$$\sigma_{el} = (\pi/k^2) \sum (2l+1)\eta_l^2\,, \tag{3.11}$$

$$\sigma_{inel} = (2\pi/k^2) \sum (2l+1)f_l = (2\pi/k^2)F(k0)\,. \tag{3.12}$$

Selecting a *jet model* means selecting a model wavefunction for the inelastic final state (3.4). A most interesting example based on field theoretical considerations is the multiperipheral model of Amati, Stanghellini, and Fubini.[12] Its shadow scattering, calculated by Amati, Cini, and Stanghellini,[13] gives a differential cross section satisfying (1.1) and (1.2). The diffraction peak, however, is found to shrink indefinitely for increasing energies ($a \to +\infty$ for $s \to +\infty$) whereas experiment suggests constancy of a to be the most common asymptotic behavior.[11] In an attempt to study more generally how qualitative properties of jets reflect themselves in the behavior of shadow scattering, we have considered in I[9] two classes of jet models which contain arbitrary functions. In one of them, $|f(\mathbf{k})\rangle$, is assumed to be a product wavefunction of N identical bosons, all in the same single particle state characterized by its wavefunction in momentum space $\varphi(\mathbf{q})$. The second one is much more realistic, its main assumption is lack of correlation between secondaries in different intervals of longitudinal momentum. As its earlier discussion was very sketchy, we shall give a more complete treatment presently. Our aim is to show that at high energy and small angles the overlap function has the following angular dependence

$$F(k\theta) = F(k0) \exp(-A\theta^2)\,, \tag{3.13}$$

or, in terms of $t \simeq -(k\theta)^2$,

$$F(k\theta) = F(k0) \exp (At/k^2)\,. \tag{3.14}$$

A is a large positive quantity ($A \gg 1$) which we shall calculate in terms of the inelastic final state. The simple angular dependence (3.13), which has been found also in the two other jet models mentioned above, is a manifestation of the lack of correlation between secondaries and is therefore expected to hold quite generally for weakly correlated jets of appreciable multiplicity. It is shown in the next section to imply for the elastic cross section the form (1.1), (1.2), with a simple relation between a, b, and A. This will provide experimental information on A, especially on its variation with incident energy, which will then be discussed in Sec. 5.

Our assumption concerning the inelastic final state (3.4) can be formulated as follows. We call longitudinal momentum q_l of a secondary the component of its momentum $\mathbf{q}$ along the incident direction, which we take to be the direction of $\mathbf{k}$. The interval of possible values of q_l, which is $-k \leqslant q_l \leqslant k$ at high energy, is supposed to be divided in $n \gg 1$ subintervals

$$q^{(j-1)} \leqslant q_l < q^{(j)}\,, \quad j = 1,\cdots,n \tag{3.15}$$

$$q^{(0)} = -k < q^{(1)} < \cdots q^{(n)} = k\,, \tag{3.16}$$

and $|f(\mathbf{k})\rangle$ is supposed to have the product form

$$|f(\mathbf{k})\rangle = \prod_{j=i}^{n} g_j(\mathbf{k})|0\rangle\,, \tag{3.17}$$

where $|0\rangle$ is the vacuum state and $g_j(\mathbf{k})$ is a polynomial in creation operators of particles with longitudinal momenta in the interval (3.15). Inserting into (3.4) we see that the only correlations between secondaries belonging to different longitudinal momentum intervals are the ones imposed by energy–momentum conservation.[14] From (3.17) we have

$$|f(\mathbf{k}')\rangle = \prod_{j=1}^{n} g_j(\mathbf{k}')|0\rangle\,, \tag{3.18}$$

where $g_j(\mathbf{k}')$ is obtained from $g_j(\mathbf{k})$ by the rotation of angle θ which brings $\mathbf{k}$ on $\mathbf{k}'$.

We must study for large n the angular dependence of

$$F(k\theta) = c_k\langle 0| \prod_{j'} g_{j'}^*(\mathbf{k}') \prod_{\mu} \delta(P^\mu - p_0^\mu) \prod_{j} g_j(\mathbf{k})|0\rangle\,, \tag{3.19}$$

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[11] For a recent review of the experimental situation, see A. M. Wetherell's report on Experimental GeV Physics, International Conference on Elementary Particles, Sienna (October, 1963).

[12] D. Amati, A. Stanghellini, and S. Fubini, Nuovo Cimento **26**, 896 (1962).

[13] D. Amati, M. Cini, and A. Stanghellini, Phys. Letters **4**, 270 (1961); Nuovo Cimento **30**, 193 (1963).

[14] Compatibility of cosmic ray data with an uncorrelated jet model of the type here considered has been established in O. Czyzewski and A. Krzywicki, Nuovo Cimento **30**, 603 (1963), and in A. Krzywicki, "A possible interpretation of ultrahigh energy experimental data," CERN preprint (October, 1963). Our treatment does not include the excited isobars produced by incident nucleons. They can be inserted simply as additional factors in the product (3.17).

with c_k a proportionality factor independent of θ. The only difficulty involved is caused by the delta functions for energy–momentum conservation. It is an experimental fact that the secondaries of inelastic collisions have transverse momenta

$$\mathbf{q}_t = \mathbf{q} - q_l \mathbf{k}/k\,, \tag{3.20}$$

which do not increase with increasing incident energy. But, as mentioned before, the relevant values of the angle θ decrease as k^{-1}. This allows us to make the approximation that $g_j(\mathbf{k}')$ only creates particles within the same momentum region (3.15) as $g_j(\mathbf{k})$. As a consequence (3.19) can be written as

$$F(k\theta) = c_k \int \prod_\mu \delta\left(\sum_j p_j^\mu - p_0^\mu\right) \prod_j f_j(p_j^\mu,\theta)\, d_4p_j^\mu\,, \tag{3.21}$$

$$f_j(p^\mu,\theta) = \langle 0|g_j^*(\mathbf{k}') \textstyle\prod_\mu \delta(P^\mu - p^\mu) g_j(\mathbf{k})|0\rangle\,, \quad j = 1,\cdots,n\,. \tag{3.22}$$

In these equations the dependence of f_j on $\mathbf{k}$ and $\mathbf{k}'$ has been condensed in a dependence on θ only (see below).

Equation (3.21) is very similar to Eq. (3.3) of I. The effect of energy–momentum conservation has there been handled under the simplifying assumption that all functions involved are real and positive; it was then sufficient to use the central limit theorem of probability theory. For $\theta \neq 0$, however, the functions f_j defined above are, in general, complex, and this complication must therefore be taken into account. We follow a method inspired on the standard proof of the central limit theorem. By Fourier transformation (3.21) becomes

$$F(k\theta) = (2\pi)^{-4} c_k \int d_4u \exp\left(-i \sum_\mu u_\mu p_0^\mu\right) \prod_j \tilde{f}_j(u_\mu,\theta), \tag{3.23}$$

$$\tilde{f}_j(u_\mu,\theta) = \int \exp\left(i \sum_\mu u_\mu p^\mu\right) f_j(p^\mu,\theta)\, d_4p\,. \tag{3.24}$$

We need an asymptotic estimate of F for large n, i.e., for the case where the product in (3.23) contains many factors. The procedure to be followed is to determine the values $\tilde{u}_\mu$ of u_μ which make $\prod_j|\tilde{f}_j(u_\mu,\theta)|$ maximum ($\tilde{u}_\mu$ depends on θ), and to expand each $\ln \tilde{f}_j$ in powers of $u_\mu - \tilde{u}_\mu$ up to second order. For the expansion coefficients we write

$$\alpha_j^\mu + i\beta_j^\mu = \partial(\ln \tilde{f}_j)/\partial u_\mu\,, \qquad \gamma_j^{\mu\nu} + i\delta_j^{\mu\nu} = \partial^2(\ln \tilde{f}_j)/\,\partial u_\mu\, \partial u_\nu \tag{3.25}$$

for $u_\mu = \tilde{u}_\mu$, where all α, β, γ, and δ are real functions of θ. The condition of maximum for $\prod_j|\tilde{f}_j|$ determines the four numbers $\tilde{u}_\mu$ to be roots of the equations

$$\textstyle\sum_j \alpha_j^\mu = 0\,, \quad \mu = 0,1,2,3\,. \tag{3.26}$$

For $\theta = 0$, all $f_j{}^\mu(p^\mu 0)$ are real positive and $\tilde{u}_\mu = 0$ gives the maximum. For $\theta \neq 0$, we shall have in general $\tilde{u}_\mu \neq 0$ (in I we had treated the case $\tilde{u}_\mu = 0$ for all θ). If (3.26) has more than one set of roots, the correct choice of $\tilde{u}_\mu$ can be made by continuity. Inserting the power expansion of $\ln \tilde{f}_j$ in (3.23) one obtains by integration the asymptotic estimate

$$F(k\theta) = (\tfrac{1}{2}\pi)^2 c_k |D|^{-\frac{1}{2}} [\textstyle\prod_j \tilde{f}_j(\tilde{u}_\mu,\theta)] \exp[-i \sum_\mu \tilde{u}_\mu p_0^\mu + \tfrac{1}{2}\sum_{\mu\nu}(\Gamma_{\mu\nu} + i\Delta_{\mu\nu})(p_0^\mu - \beta^\mu) \times (p_0^\nu - \beta^\nu)]\,. \tag{3.27}$$

D is the determinant of the 4×4 matrix $\sum_j(\gamma_j^{\mu\nu} + i\delta_j^{\mu\nu})$, the inverse of this matrix is written $\Gamma_{\mu\nu} + i\Delta_{\mu\nu}$($\Gamma$ and Δ real), and β^μ stands for $\sum_j \beta_j^\mu$.

The cylindrical symmetry of the inelastic final state (3.4) around $\mathbf{k}$ implies symmetry relations for the functions f_j and their Fourier transforms $\tilde{f}_j$. To formulate them simply we select special orientations of $\mathbf{k}$ and $\mathbf{k}'$ in momentum space, corresponding to the coordinates

$$k^1 = k'^1 = k\cos(\tfrac{1}{2}\theta)\,, \qquad k^2 = -k'^2 = k\sin(\tfrac{1}{2}\theta)\,, \qquad k^3 = k'^3 = 0\,. \tag{3.28}$$

It is on the basis of this choice that f_j and $\tilde{f}_j$ can be regarded as functions of θ rather than $\mathbf{k}$, $\mathbf{k}'$. The angle θ is now allowed to take negative as well as positive values, and the definition of $F(k\theta)$ is extended to negative θ by taking this function to be even in θ. The symmetry relations for f_j are

$$\begin{aligned} f_j(p^0,p^1,-p^2,p^3,\theta) &= f_j(p^0,p^1,p^2,p^3,-\theta) \\ &= f_j^*(p^0,p^1,p^2,p^3,\theta)\,, \\ f_j(p^0,p^1,p^2,-p^3,\theta) &= f_j(p^0,p^1,p^2,p^3,\theta)\,, \end{aligned} \tag{3.29}$$

and similarly for $\tilde{f}_j$

$$\begin{aligned} \tilde{f}_j(u_0,u_1,-u_2,u_3,\theta) &= \tilde{f}_j(u_0,u_1,u_2,u_3,-\theta) \\ &= \tilde{f}_j^*(-u_0,-u_1,-u_2,u_3,\theta)\,, \\ \tilde{f}_j(u_0,u_1,u_2,-u_3,\theta) &= \tilde{f}_j(u_0,u_1,u_2,u_3,\theta)\,. \end{aligned} \tag{3.30}$$

As a consequence it can then be seen that $\tilde{u}_\mu$ verifies

$$\tilde{u}_0 = \tilde{u}_1 = \tilde{u}_3 = 0\,. \tag{3.31}$$

Only $\tilde{u}_2$ is, in general, nonvanishing and it is to be found from the $\mu = 2$ equation of the set (3.26), the other equations of the set being automatically satisfied. One can see that $\tilde{u}_2$ is an odd function of θ, which implies that $\tilde{f}_j(\tilde{u}_\mu,\theta)$ is real and even in θ.

Further consequences of the symmetry relations are

$$\beta_j^2 = \beta_j^3 = 0\,,\quad \beta^2 = \beta^3 = 0\,;$$
$$\delta_j^{\mu\nu} = 0 \text{ for } \mu\,,\quad \nu = 0{,}1\,,\quad \delta_j^{22} = \delta_j^{33} = 0\,;$$
$$\gamma_j^{\mu 2} = \gamma_j^{2\mu} = 0 \text{ for } \mu = 0{,}1\,,$$
$$\gamma_j^{\mu 3} + i\delta_j^{\mu 3} = \gamma_j^{3\mu} + i\delta_j^{3\mu} = 0 \text{ for } \mu = 0{,}1{,}2\,;$$
$$\Delta_{\mu\nu} = 0 \text{ for } \mu,\nu = 0{,}1\,,\quad \Delta_{22} = \Delta_{33} = 0\,;$$
$$\Gamma_{\mu 2} = \Gamma_{2\mu} = 0 \text{ for } \mu = 0{,}1\,,$$
$$\Gamma_{\mu 3} + i\Delta_{\mu 3} = \Gamma_{3\mu} + i\Delta_{3\mu} = 0 \text{ for } \mu = 0{,}1{,}2\,.$$

One also verifies that all nonvanishing β, γ, and Γ are even functions of θ, whereas the nonvanishing δ and Δ are odd. The determinant D is then real and even in θ. Using (3.6) and all consequences of the symmetry relations we find for (3.27) the simplified expression

$$F(k\theta) = (\tfrac{1}{2}\pi)^2 c_k |D|^{-\frac{1}{2}} [\textstyle\prod_j \tilde{f}_j(\bar{u}_\mu,\theta)] \times \exp\left[\tfrac{1}{2}\sum_{\rho,\sigma=0,1} \Gamma_{\rho\sigma}(p_0^\rho - \beta^\rho)(p_0^\sigma - \beta^\sigma)\right]. \tag{3.32}$$

We can now show that the angular dependence of F is in agreement with (3.13). To this end we consider $\tilde{f}_j(\bar{u}_\mu,\theta)$ to lowest nonvanishing order in θ

$$\tilde{f}_j(\bar{u}_\mu,\theta) = \tilde{f}_j(0,0)(1 - a_j\theta^2)\,. \tag{3.33}$$

From the Schwarz inequality a_j is easily seen to be non-negative. Since n is large we can write

$$\textstyle\prod_j \tilde{f}_j(\bar{u}_\mu,\theta) = [\prod_j \tilde{f}_j(0,0)] \exp(-A\theta^2)\,,\quad A = \sum_j a_j\,. \tag{3.34}$$

We shall calculate A explicitly below and verify that it is positive. The approximation (3.34) is valid in the following range of variation of the exponent

$$A\theta^2 \ll 2(\textstyle\sum a_j)^2/\sum a_j^2 = 2n\bar{a}^2/\overline{a^2}\,,\quad \overline{a^m} = n^{-1}\sum_j a_j^m\,. \tag{3.35}$$

As will be seen later on, we expect at very high energy the asymptotic behavior $A \propto k^2$, and we shall be able to select $n \to \infty$ in such a way that $n\,\bar{a}^2/\,\overline{a^2}$ tends to infinity with k. The approximation (3.34) thus becomes increasingly accurate. The relevant range of θ varies as $\theta \backsim A^{-\frac{1}{2}} \propto k^{-1}$.

These facts allow to neglect the θ dependence of the other factors in (3.32). For $\theta = 0$ one easily calculates from (3.22) and (3.25)

$$\beta_j^\mu = G_j\langle 0|g_j^*(\mathbf{k})P^\mu g_j(\mathbf{k})|0\rangle\,, \tag{3.36}$$

$$\gamma_j^{\mu\nu} = -G_j\langle 0|g_j^*(\mathbf{k})(P^\mu - \beta_j^\mu)(P^\nu - \beta_j^\nu)g_j(\mathbf{k})|0\rangle\,, \tag{3.37}$$

$$G_j^{-1} = \langle 0|g_j^*(\mathbf{k})g_j(\mathbf{k})|0\rangle\,. \tag{3.38}$$

From the latter equations it is natural to expect the γ to remain finite at high energies. Hence we take Γ of order n^{-1}. From (3.36) and (3.37) one finds that for $\theta = 0$

$$\beta^\mu = \textstyle\sum_j \beta_j^\mu = \langle f(\mathbf{k})|P^\mu|f(\mathbf{k})\rangle/\langle f(\mathbf{k})|f(\mathbf{k})\rangle\,. \tag{3.39}$$

In the spirit of the *ansatz* (3.4) and (3.17) for the inelastic final state one can naturally assume (3.39) to be the energy–momentum value p_0^μ of the collision. Consequently β^μ reduces to $p_0^\mu + b^\mu\theta^2$ for small θ and the exponent in (3.32) becomes

$$\tfrac{1}{2}\,\theta^4 \textstyle\sum_{\rho\sigma} \Gamma_{\rho\sigma} b^\rho b^\sigma\,.$$

From (3.37) the quadratic form is negative definite. We make the following order of magnitude estimates: $\Gamma \propto n^{-1}$, $b \propto k$, and therefore obtain for the exponent $n^{-1}k^2\theta^4 \propto n^{-1}k^{-2}$ in the relevant angular range $\theta \propto k^{-1}$. This quantity is obviously negligible. The last factor of (3.32) which depends on θ is the determinant D. One can estimate $\partial \ln D/\partial(\theta^2)$ at $\theta = 0$. It turns out to be of order $\bar{a}$, while θ^2 is of order $(n\,\bar{a})^{-1}$, so that we can replace D by its value at $\theta = 0$. We are left with the very simple result

$$F(k,\theta) = F(k,0)\exp(-A\theta^2)\,, \tag{3.40}$$

$$A = -\textstyle\sum_j [\partial \ln f_j(\bar{u}_\mu,\theta)/\partial(\theta^2)] \quad \text{for } \theta = 0\,. \tag{3.41}$$

We finally calculate the explicit value of A. From the symmetry properties of $\tilde{f}_j$ one immediately gets to second order

$$f_j(00\bar{u}_\mu 0\theta) = \tilde{f}_j + \tfrac{1}{2}[\bar{u}_2^2(\partial^2\tilde{f}_j/\partial u_2^2) + 2\bar{u}_2\theta(\partial^2\tilde{f}_j/\partial u_2\partial\theta) + \theta^2(\partial^2\tilde{f}_j/\partial\theta^2)] \tag{3.42}$$

where in the right-hand side $\tilde{f}_j$ and its derivatives must be taken at $u_\mu = \theta = 0$. We recall that $\bar{u}_2$ is obtained from the equation

$$\textstyle\sum_j [\partial \ln \tilde{f}_j(00\bar{u}_2 0\theta)/\partial\bar{u}_2] = 0\,. \tag{3.43}$$

It gives to first order, by means of (3.42),

$$\bar{u}_2[\textstyle\sum_j \tilde{f}_j^{-1}(\partial^2\tilde{f}_j/\partial u_2^2)] + \theta[\sum_j \tilde{f}_j^{-1}(\partial^2\tilde{f}_j/\partial u_2\partial\theta)] = 0\,. \tag{3.44}$$

Substituting back in (3.42) we obtain

$$2A = -[\textstyle\sum_j \tilde{f}_j^{-1}(\partial^2\tilde{f}_j/\partial\theta^2)] + [\sum_j \tilde{f}_j^{-1}(\partial^2\tilde{f}_j/\partial u_2\partial\theta)]^2[\sum_j \tilde{f}_j^{-1}(\partial^2\tilde{f}_j/\partial u_2^2)]^{-1}\,. \tag{3.45}$$

Here again $\tilde{f}_j$ and its derivatives are to be taken at $u_\mu = \theta = 0$. To pursue the calculation further one must go back to the explicit expression of $\tilde{f}$, as given by (3.22), (3.24). We shall limit ourselves to the 0 and 1 particle components of $g_j(\mathbf{k})$, many particle components being treated in identical fashion,

$$g_j(\mathbf{k}) = g_j^{(0)} + \int g_j^{(1)}(\mathbf{q}) A_{\mathbf{q}}^* d_3\mathbf{q} + \cdots . \quad (3.46)$$

$A_{\mathbf{q}}^*$ is the creation operator normalized to a delta function in the three-momentum $\mathbf{q}$. The function $g^{(1)}$ is cylindrically symmetric around $\mathbf{k}$ and vanishes for the longitudinal component q_l of $\mathbf{q}$ outside of the interval (3.15). Denoting by q_2 and q_3 Cartesian coordinates in the transverse plane, one finds after some calculation

$$A = A_1 + A_2 , \quad (3.47)$$

$$2A_1 = \sum_j \int (\mathrm{Re}\, \phi_j)^2 \rho_j \, d_3\mathbf{q} , \quad (3.48)$$

$$2A_2 = \left[\sum_j \int (\mathrm{Im}\, \phi_j)^2 \rho_j \, d_3\mathbf{q} \right] - \left[\sum_j \int \mathrm{Im}\, \phi_j q_2 \rho_j \, d_3\mathbf{q} \right]^2 \left[\sum_j \int q_2^2 \rho_j \, d_3\mathbf{q} \right]^{-1} \quad (3.49)$$

$$\rho_j = |g_j^{(1)}(\mathbf{q})|^2 \Big/ \left\{ |g_j^{(0)}|^2 + \int |g_j^{(1)}(\mathbf{q}')|^2 \, d_3\mathbf{q}' \right\} \quad (3.50)$$

$$\phi_j = q_l[\partial \ln g_j^{(1)}(\mathbf{q})/\partial q_2] - q_2[\partial \ln g_j^{(1)}(\mathbf{q})/\partial q_l] . \quad (3.51)$$

While A_1 is entirely independent of the phases of the wavefunctions $g_j^{(1)}$, these phases are responsible for A_2. Both quantities are non-negative and A_1 is easily seen to be positive for any reasonable choice of $g_j^{(1)}$.

4. SHADOW SCATTERING OF UNCORRELATED JETS

In the previous section we have obtained for the overlap function the angular dependence

$$F(k\theta) = F(k0) \exp(-A\theta^2) , \qquad A \gg 1 . \quad (4.1)$$

We believe it to have a validity much more general than the jet models considered to derive it. We expect in fact that it should hold for all weakly correlated jets of appreciable multiplicity. We now proceed to discuss the implications of this simple angular dependence for elastic scattering, assuming the latter to be shadow scattering (i.e., to have a purely imaginary amplitude) in line with the arguments of Sec. 2.[15]

Because of $A \gg 1$, the partial wave expansion (3.2) of F is calculated using the familiar equations

$$\lim_{l \to +\infty} P_l(1 - x^2/2l^2) = J_0(x) , \quad (4.2)$$

$$\int_0^\infty \exp(-\tfrac{1}{2} x^2) J_0(x\xi) x \, dx = \exp(-\tfrac{1}{2} \xi^2) . \quad (4.3)$$

[15] The subsequent calculation is of course quite analogous to the shadow scattering computation for the multiperipheral model which also obeys (4.1), see Ref. 13.

One finds

$$f_0 = F(k,0)/4A , \quad (4.4)$$

$$f_l = f_0 \exp(-l^2/4A) . \quad (4.5)$$

The unitarity requirements (3.7) are satisfied if

$$0 \leqslant f_0 \leqslant \tfrac{1}{2} . \quad (4.6)$$

The partial wave amplitude $i\eta_l$ for elastic scattering is given by (3.9). We expand it in powers of f_l[16]

$$\eta_l = \sum_{m=1}^{\infty} c_m f_l^m \quad (4.7)$$

$$c_1 = 1 , \quad c_m = 1 \cdot 3 \cdots (2m - 3)/m ! \quad \text{for} \quad m > 1 . \quad (4.8)$$

Inserting (4.5) and using (4.2), (4.3) we obtain for the elastic scattering amplitude

$$T(st) = i \sum_l (2l + 1) \eta_l P_l(\cos \theta) = 4A \sum_1^\infty m^{-1} c_m f_0^m \exp(At/mk^2) \quad (4.9)$$

with, as usual,

$$t = 2k^2(\cos \theta - 1) \simeq -(k\theta)^2 .$$

The total cross section is obtained from (3.10)

$$\sigma_t = 8\pi A k^{-2} \sum_1^\infty m^{-1} c_m f_0^m = 8\pi A k^{-2} \int_0^{f_0} [1 - (1 - 2f)^{\frac{1}{2}}] f^{-1} \, df = 16\pi A k^{-2} \{ \ln [\tfrac{1}{2} + \tfrac{1}{2} (1 - 2f_0)^{\frac{1}{2}}] - (1 - 2f_0)^{\frac{1}{2}} + 1 \} . \quad (4.10)$$

(3.12) and (4.4) give the total inelastic cross section

$$\sigma_{\mathrm{inel}} = 8\pi A k^{-2} f_0 \quad (4.11)$$

from which σ_{el} is obtained as $\sigma_t - \sigma_{\mathrm{inel}}$. The ratio $\sigma_{\mathrm{el}}/\sigma_t$ is a function of the dimensionless parameter f_0. As f_0 increases from 0 to its maximum value $\frac{1}{2}$, $\sigma_{\mathrm{el}}/\sigma_t$ increases from 0 to a maximum, which is found to be

$$\max(\sigma_{\mathrm{el}}/\sigma_t) = (\sigma_{\mathrm{el}}/\sigma_t)_{f_0 = \frac{1}{2}} = 1 - (4 - 4 \ln 2)^{-1} = 0.213 . \quad (4.12)$$

This is quite a low value, and it is remarkable that measured values of $\sigma_{\mathrm{el}}/\sigma_t$ are close to it or somewhat smaller. We thus expect actual values of f_0 to be rather close to $\frac{1}{2}$. This will be confirmed later.

[16] All series to be considered are convergent when f_0 is within the limits (4.6).

To discuss the differential elastic cross section we first write (4.9) in the form

$$T(st) = T(s0)\exp[a_1 x + a_2 x^2 + \cdots], \qquad x = At/k^2 \tag{4.13}$$

the coefficients a_i are functions of f_0 which are given by

$$a_1 = \alpha_1/\alpha_0, \quad a_2 = (\alpha_2/2\alpha_0) - \tfrac{1}{2}a_1^2, \cdots \tag{4.14}$$

with

$$\alpha_i = \sum_{m=1}^{\infty} m^{-i-1} c_m f_0^m; \quad i = 0,1,2\cdots. \tag{4.15}$$

For all allowed values of f_0, a_1 is close to 1 and $a_2, \cdots$ close to 0. For example,

$$f_0 = \tfrac{1}{2}, \quad a_1 = 0.86, \quad a_2 = 0.03, \quad a_3 = -0.005, \quad a_4 = 0.0003; \tag{4.16}$$

$$f_0 = 0, \quad a_1 = 1, \quad a_2 = a_3 = \cdots = 0. \tag{4.17}$$

The differential cross section becomes

$$d\sigma_{el}/dt = (\sigma_t^2/16\pi)\exp[2a_1 x + 2a_2 x^2 + \cdots]. \tag{4.18}$$

Its value in the forward direction follows from the optical theorem under our assumption of purely imaginary scattering amplitude. We rewrite (4.18) in the form (1.1) suggested by experiment

$$d\sigma_{el}/dt = (\sigma_t^2/16\pi)\exp(at + bt^2) \tag{4.19}$$

and find

$$a = 2a_1 A/k^2, \tag{4.20}$$

$$b/a^2 = a_2/2a_1^2. \tag{4.21}$$

High-energy experiments tell us that both σ_t and a remain constant or vary at most very slowly with energy. Going back to equations (4.10) and (4.20), which determine the basic parameters A and f_0 of our theoretical description, we conclude that the dimensionless quantity f_0 and the area A/k^2 are essentially energy independent. Experiment gives $a \simeq 10(\text{GeV}/c)^{-2} = 4$ mb both for pp and $\pi^{\pm}p$ scattering.[11] Combining (4.10) and (4.20) we have

$$\sigma_t = 8\pi\alpha_0 A k^{-2} = 4\pi(\alpha_0^2/\alpha_1)a. \tag{4.22}$$

From $\sigma_t \simeq 40$ mb and 25 mb we obtain $\alpha_0^2/\alpha_1 \simeq 0.8$ (pp) and $0.5(\pi p)$. The ratio α_0^2/α_1, which is f_0 at small f_0, takes the value 0.74 at $f_0 = \frac{1}{2}$. While the situation is quite comfortable for πp, suggesting a value of f_0 close to 0.4, we find that σ_t/a is too large by about 6% for pp scattering. If we remark that it is precisely for pp scattering that experiment revealed a slow shrinking of the diffraction peak, corresponding to a slow increase of a, while σ_t was found to remain quite constant, it is tempting to speculate that this tendency will maintain toward somewhat higher energies, giving rise to a further slight increase of a and a corresponding decrease of σ_t/a. A possible reason for the different behaviour of πp and pp scattering in this respect may be that, because of the abundant excitation of nucleon isobars in inelastic collisions, correlations in jets remain important in pp collisions at higher energies than in πp collisions.

We finally discuss the result (4.21) for the t^2 term in $\ln d\sigma_{el}/dt$. The important point is that (4.21) is small for all allowed values of f_0. Its value at $f_0 = \frac{1}{2}$ is found to be 0.02. Although experimental results for b are still rather uncertain, most measurements seem to give $b \backsim 2$ to $3\ (\text{GeV}/c)^{-4}$, so that $b/a^2 \backsim 0.02$ to 0.03 in good agreement with our value. It may be of interest to contrast this number with the value of b/a^2 obtained for an optical model assuming a totally absorbing sphere; it turns out to be $-\frac{1}{6} = -0.17$, wrong both in order of magnitude and sign (of course the black sphere value of $\sigma_{el}/\sigma_t = \frac{1}{2}$ is also far too large). The very fact that b/a^2 is so small will require more accurate experimental work, especially at very small values of $|t|$, to go further with the above considerations. One might hope to obtain at the same time information on the real part of the scattering amplitude which may well play a much bigger role than the 2% effect just discussed.

5. MULTIPLICITY AND MOMENTUM DISTRIBUTION OF UNCORRELATED JETS

The analysis of elastic scattering just made has revealed that the quantity Ak^{-2} is required by experiment to be essentially constant at high energy. This quantity has been calculated in Sec. 3 in terms of the model wavefunction adopted for the jet, see Eqs. (3.47) to (3.51). We now investigate what restrictions the constancy of Ak^{-2} imposes on the properties of the jet. Since we know very little on the detailed structure of jets our discussion will not go beyond the exploration of what seem to be promising indications. We shall reach them through some simplifying assumptions.

We first determine the leading contribution to A using the fact that for most secondaries the longitudinal momentum q_l is much larger than the transverse one $q_t = (q_1^2 + q_2^2)^{\frac{1}{2}}$. We approximate (3.51) by

$$\phi_j = q_l[\partial \ln g_j^{(1)}(\mathbf{q})/\partial q_2] \tag{5.1}$$

and we neglect the second term in A_2 on the ground that a large compensation between positive and negative q_l must occur in the quantity

$$\sum_j \int \mathrm{Im}\, \phi_j q_2 \rho_j \, d_3\mathbf{q} \, . \qquad (5.2)$$

Then

$$A = \sum_j \int q_l^2 \psi_j \rho_j \, d_3\mathbf{q} \, , \qquad (5.3)$$

$$\psi_j = |\partial \ln g_j^{(1)}(\mathbf{q})/\partial q_2|^2 \, . \qquad (5.4)$$

From (3.46) the density distribution of secondaries in momentum space, i.e., the average number of secondaries per unit volume in $\mathbf{q}$ space, is seen to be

$$N(\mathbf{q}) = \rho_j(\mathbf{q}) \qquad (5.5)$$

where $q^{(j-1)} \leqslant q_l < q^{(j)}$. Introducing the average multiplicity of the inelastic collision

$$N = \int N(\mathbf{q}) \, d_3\mathbf{q} = \sum_j \int \rho_j(\mathbf{q}) \, d_3\mathbf{q} \, , \qquad (5.6)$$

we can write (5.3) in the form

$$A = N\langle q_l^2 \psi \rangle \qquad (5.7)$$

where $\psi = \psi(\mathbf{q})$ is the function defined by (5.4) in each longitudinal momentum interval and the average is taken over all secondaries. Let us also write down the condition (3.39) for $\mu = 0$, which states that the average energy in $|f(\mathbf{k})\rangle$ is equal to the incident energy $\epsilon_1 + \epsilon_2 \simeq 2k$. It is

$$2k = \sum_j \int q_0 \rho_j \, d_3\mathbf{q} = N\langle q_0 \rangle \, , \qquad (5.8)$$

q_0 denoting the energy of a secondary of momentum $\mathbf{q}$. From the preceding equations we find a simple expression for Ak^{-2}

$$Ak^{-2} = (4/N)(\langle q_l^2 \psi \rangle / \langle q_0 \rangle^2) \, ,$$

or, since $q_l \simeq q_0$,

$$Ak^{-2} = (4/N)(\langle q_0^2 \psi \rangle / \langle q_0 \rangle^2) \, . \qquad (5.9)$$

This is the quantity which has to be constant at high energy. The multiplicity N is of course known to increase with energy, and we conclude that $\langle q_0^2 \psi \rangle / \langle q_0 \rangle^2$ must increase at the same rate as N.

With the exception of ψ, Eq. (5.9) only contains simple quantities. From (5.4) ψ is clearly related to the logarithmic slope of the final state wavefunction in the transverse direction. We shall continue our discussion under the assumption that ψ, or at least its average over transverse momenta at fixed longitudinal momentum q_l, is independent of q_l as well as of incident energy. Roughly speaking, this would be true if the transverse momentum distribution is independent of incident energy and if there are no correlations between longitudinal and transverse momenta (except the over-all correlation imposed by energy–momentum conservation). Would experimental work reveal that such correlations exist, the following considerations will have to be revised. Under our new assumption ψ is treated as a constant in (5.9) and we have

$$Ak^{-2} = (4\psi/N)(\langle q_0^2 \rangle / \langle q_0 \rangle^2) \, . \qquad (5.10)$$

Since we have assumed ψ to be independent of k, the dependence of Ak^{-2} on incident energy is now entirely determined by the k dependence of the energy distribution $w(q_0)$ of secondaries. We indeed know from (5.8) that the multiplicity N is entirely fixed in terms of $w(q_0)$.

Energy and momentum conservation implies that no secondary can have an energy larger than k. We thus can put

$$w(q_0) = 0 \quad \text{for} \quad q_0 \geqslant k \, . \qquad (5.11)$$

Hence, using (5.8),

$$\langle q_0^2 \rangle = \int q_0^2 w(q_0) \, dq_0 \leqslant k \int q_0 w(q_0) \, dq_0 = k \langle q_0 \rangle = \tfrac{1}{2} N \langle q_0 \rangle^2 \, . \qquad (5.12)$$

This means that in our model Ak^{-2} can certainly not increase with k. In other words, shrinking of the diffraction peak is impossible. Furthermore, when the energy variation of $w(q_0)$ is such as to give $N \to \infty$, the corresponding value for Ak^{-2} will in general tend to zero as $k \to \infty$. Only special types of k dependence for $w(q_0)$ will give at the same time constant Ak^{-2} and increasing N. What is required is to have $\langle q_0 \rangle / k$ approach zero while $\langle q_0^2 \rangle / k \langle q_0 \rangle$ tends to a nonvanishing limit. The ratio $\langle q_0^2 \rangle / \langle q_0 \rangle^2$ must increase, so that the distribution must grow flatter toward large q_0. We have tried the following *ansatz*[9]

$$w(q_0) = c(k) q_0^{-\alpha} \quad \text{for} \quad q_0 \leqslant k \, ,$$
$$= 0 \quad \text{for} \quad q_0 > k \, . \qquad (5.13)$$

$c(k)$ is determined by the normalization condition

$$\int w(q_0) \, dq_0 = 1 \, .$$

The exponent α, which might in principle vary with k, is limited to the interval $1 \leqslant \alpha \leqslant 2$ by the requirement that $\langle q_0^2 \rangle / \langle q_0 \rangle^2 \to \infty$. We assume for simplicity that it becomes constant at large k. One then finds (A' = positive constant)

$$Ak^{-2} \to A' \, , \quad N \propto \ln k \quad \text{for} \quad \alpha = 1 \, ,$$
$$Ak^{-2} \to A' \, , \quad N \propto k^{\alpha-1} \quad \text{for} \quad 1 < \alpha < 2 \, ,$$
$$Ak^{-2} \propto (\ln k)^{-1} \, , \quad N \propto k (\ln k)^{-1} \quad \text{for} \quad \alpha = 2 \, . \qquad (5.14)$$

It is remarkable that constancy of Ak^{-2} is achieved for all α satisfying

$$1 \leqslant \alpha < 2\,, \tag{5.15}$$

an interval compatible with cosmic ray evidence concerning the increase of N (which seems to favor $1 \leqslant \alpha \lesssim \frac{3}{2}$). Also the corresponding distribution (5.13) for secondaries is compatible with cosmic ray data.[14]

The last point we have to check is whether the distributions (5.13) and (5.15) allow the exponential approximation (3.34) to be made. We have to show that the quantity $n\,\bar{a}^2/\overline{a^2}$, defined in (3.35), tends to infinity for some reasonable law of increase for n. Actually n cannot increase faster than k if the length of the longitudinal momentum intervals (3.15) is to remain finite. Under the assumptions made in the present section a_j is of order of $\rho_j[q^{(j)}]^2$, hence

$$\overline{a^m} = n^{-1} \sum a_j^m \propto \int q_0^{2m} w(q_0)\, dq_0\,.$$

We then readily calculate

$$\overline{a^m} \propto k^{2m}/\ln k \qquad \text{for} \quad \alpha = 1\,,$$
$$\overline{a^m} \propto k^{2m+1-\alpha} \qquad \text{for} \quad 1 < \alpha < 2\,;$$
$$\left(\sum a_j\right)^m / \sum a_j^m = n^{m-1}\bar{a}_j^m/\overline{a_j^m}$$
$$\propto (n/\ln k)^{m-1} \quad \text{for} \quad \alpha = 1\,,$$
$$\propto (n/k^{\alpha-1})^{m-1} \quad \text{for} \quad 1 < \alpha < 2\,.$$

This tends to infinity for all $m \geqslant 2$ if we take $n \propto k^{\epsilon}$ with $\alpha - 1 < \epsilon < 1$, a rate of increase which is entirely reasonable.

6. CONCLUDING REMARKS

As was stressed in the introduction, our theoretical analysis of high-energy collisions is strongly phenomenological in character, in the sense that we relied heavily on experimental indications. We therefore conclude by mentioning a few questions on which further experimental work is highly desirable. In the course of our discussion we established logical connections between approximate constancy of cross sections, imaginary character of the elastic amplitude, linearity of $\ln\,(d\sigma_{\rm el}/dt)$ in t and various jet properties such as lack of correlations, multiplicity increase, and energy distribution of secondaries. One aspect of jet structure which played a central part in our discussion is the assumed uncorrelated emission of secondaries, which was found to account for the linearity of $\ln\,(d\sigma_{\rm el}/dt)$ in t. It is on this randomness in the emission process that the need for even crude experimental data is felt most severely.

Except for the requirement of energy–momentum conservation, we assumed lack of correlation between secondaries emitted in different intervals of longitudinal momentum. This is a natural assumption which could be tested experimentally (actual tests have been proposed at the end of I). If correlations are found, two possibilities are still to be considered. Either jets are uncorrelated when analyzed in some other grouping of the secondaries, or the correlations are present in any set of variables. The latter would, for example, be the case if a strict fire-ball model would hold, in which a small number of fire balls with uniquely defined mass, charge, and spin would be formed in all or most inelastic collisions. The former possibility would occur in a modified fire-ball model where the fire balls would be no more than clusters of weakly correlated secondaries.

The considerations of Sec. 2 show that, under plausible assumptions, slow energy variation of total and elastic cross sections implies an almost imaginary elastic amplitude. Even weaker assumptions are known to imply that total and elastic cross sections become equal for the processes $A + B$ and $\bar{A} + B$.[2,6] These features of high energy collisions are qualitatively supported by experiment. They raise the question of the relation between inelastic collisions of an incident particle A and its antiparticle $\bar{A}$ on the same target particle B. One wonders how the jets formed by $A + B$ compare with those formed by $\bar{A} + B$. The most natural assumption is that the nature of the incident particle A or $\bar{A}$ reflects itself only among those secondaries in the jet which have their longitudinal momentum closest to the one of A or $\bar{A}$. This is realized in the multiperipheral model[12] but could hold true in many other models if they ensure retention of charge, baryon number, and strangeness of the incident particle A (or B) by the group of secondaries with longitudinal momenta closest to the one of A (or B). There is already clear experimental evidence in favor of such a behavior, but much more detailed work is possible and desirable. Also from the theoretical standpoint, one should include this effect into the discussion of shadow scattering given above.

In addition to these questions, which all belong to jet physics, continuation of the very interesting work done up to now on elastic collisions will of course be most useful. One can hope to develop gradually in this way a rather definite theoretical description of high-energy collisions, which is probably a necessary prerequisite for a deeper understanding of strong interactions at very high energies.