This can be done by restricting the range of integrations over the Feynman parameters in a reasonably straightforward manner. The diagram 3c has only a negative z cut and is uninteresting. Another difficulty is that although the predicted large z dependence of the amplitude $f_1^{\dagger}{}_{-1;1\,-1}$ is $\sim 1/z$, the natural order of the various graphs is $\ln(-z)$ and consequently one is not involved with just the computation of the dominant contribution from the various graphs. It is unfortunately the case that we have not yet been given sufficient "running time" to have completed the measurement of the requisite $\gamma\gamma$ scattering amplitude. Consequently we cannot report the evaluation of the trajectory.

It is not profitable to speculate about the outcome of the calculation at any great length. We have seen that the idea of having a vacuum trajectory generated by the exchange of two massive vector bosons is consistent with elastic unitarity. One of the more interesting questions is to locate the place where $\Delta = \alpha - 1$ goes through zero. We can, of course, calculate Im Δ from unitarity as we have shown, but it is precisely the unknown subtraction question that forces our $\gamma\gamma$-scattering experiment. Perhaps the nicest result would be for Δ to be zero at $W^2 = 0$, as the fabled Pomeranchuk trajectory is supposed to behave. We have verified that as the boson mass goes to zero this is indeed the case. It could of course also be that Δ goes to zero at $W^2 = 0$ only when the coupling gets strong, or perhaps for some special value of the mass ratio, λ/m, other than the zero value we have mentioned.

ACKNOWLEDGMENTS

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High-Energy Proton–Proton Scattering

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The optical model for high-energy proton–proton scattering which has been proposed in an earlier paper[1] predicts that, for large momentum transfers, the dependence of the elastic cross section on the square of the momentum transfer is approximately an inverse sixth power law. This prediction is borne out very well by the new measurements of Cocconi *et al.*[2] for their highest proton energies, near 30 BeV. The prediction has been checked down to cross sections as small as 2×10^{-12} of the forward scattering cross section and to center-of-mass scattering angles as large as 82°. For lower energies the measured cross sections deviate from the theoretical curve, becoming larger as the center-of-mass scattering angle approaches 90°.

The cross section for large momentum transfers depends on the behavior of the absorptive potential near $r = 0$, while that for small momentum transfers depends on the behavior for large r. A potential can be constructed to fit the observations for the entire range of momentum transfers. This is of Yukawa form for $r < 0.33 \times 10^{-13}$ cm, and of Gaussian form for $r > 1.1 \times 10^{-13}$ cm. The range of the Yukawa potential is determined by the width of the diffraction curve for large momentum transfer, the range of the Gaussian by the width for momentum transfer near zero.

The general features of high-energy elastic proton–proton scattering with large momentum transfer have been explained[1] in terms of a simple optical model. The change in wave number in the region of interaction was described by an absorptive potential,

$$k' - k = iV(r) \tag{1}$$

and V was supposed to be of Yukawa form,

$$V(r) = \eta e^{-\Lambda r}/r\,. \tag{2}$$

This leads to a cross section formula

$$(1/k^2)d\sigma/d\Omega = (1/\Lambda^4)F(t/\Lambda^2)^2\,, \tag{3}$$

with t the square of the momentum transfer. Numeri-

* This work was supported in part by the United States Atomic Energy Commission.

[1] R. Serber, Phys. Rev. Letters 10, 357 (1963).

[2] G. Cocconi, V. T. Cocconi, A. D. Krisch, J. Orear, R. Rubinstein, D. B. Scarl, W. F. Baker, E. W. Jenkins, and A. L. Read, Phys. Rev. Letters 11, 499 (1963). I am indebted to these authors for making available some additional information before publication.

cal values of F, for $\eta = 1$, are given in Table I for a greater range of the argument than was covered in Ref. 1. For $t/\Lambda^2 > 100$, the asymptotic expansion

$$F(z) = (25.378/z^3)\{1 + (9/z)[\ln z - \tfrac{5}{3}]\}$$

is in error by less than 2.5%. For the particular value $\eta = 1$ it was shown that (3) gives a good qualitative account of the observed cross section over a range of t for which $d\sigma/d\Omega$ varies by a factor of 10^7.

TABLE I. $F(z)$.

z	$F(z)$
0	1.419
0.0625	1.330
0.25	1.053
1	0.5300
4	0.1082
9	2.389×10^{-2}
16	6.264×10^{-3}
25	1.934×10^{-3}
36	6.666×10^{-4}
49	2.806×10^{-4}
64	1.237×10^{-4}
100	3.135×10^{-5}
200	3.691×10^{-6}
350	6.557×10^{-7}

For this qualitative comparison, the remaining parameter Λ was chosen to fit the observed total cross section. While (3) then reproduced the empirical t^{-5} law fairly well, the width of the predicted scattering curve was too great, as shown in Fig. 1 of Ref. 1. To get a more quantitative description of the large momentum transfer measurements, one should choose the scale factor in (3) to fit the observed width. The scaling law given by (3) can be described as follows. Make a log–log plot of the measured values of $(4\pi/k\sigma_{\text{tot}})^2 d\sigma/d\Omega$ vs t, as in Fig. 1 of this paper. On tracing paper, make a similar log–log plot of $[F(t/\Lambda^2)/F(0)]^2$ vs t, using the value $\Lambda^2 = 0.1750$ $(\text{BeV}/c)^2$ of Ref. 1. If the tracing paper is superposed on the plot of experimental points, we get the comparison used in Ref. 1. To change the scale factor Λ^2, shift the tracing paper on a 45° line upwards and to the left, or downwards and to the right.

The result of such a rescaling is shown in Fig. 1. For values of the ordinate between 10^{-2} and 10^{-4} the experimental points of Diddens *et al.*[3] show no significant dependence on the laboratory proton momentum p_0, for p_0 between 18 and 26 BeV/c. The calculated curve was therefore adjusted to fit smoothly to the measured points in this interval. It is represented in Fig. 1 by the solid curve for ordinates less than 10^{-4}, and the dashed curve for ordinates greater than 10^{-4}. The shift was such that $t = 1$ on the tracing paper fell on $t = 0.4$ on the experimental plot, meaning that the proper scale factor is

$$\Lambda^2 = 0.4 \times 0.175(\text{BeV}/c)^2 = 0.070(\text{BeV}/c)^2,$$

$$\Lambda = 1.341(10^{-13}\,\text{cm})^{-1}.$$

In the following, lengths are always measured in units of 10^{-13} cm. For ordinates less than 10^{-4}, the theoretical curve is roughly a t^{-6} law.

The CERN points at highest energy follow the theoretical curve quite well down to ordinates of 10^{-6}. Our calculations were made prior to the recent Brookhaven experiment of Cocconi *et al.*,[2] so that for ordinates below 10^{-6} the curve can fairly be described as a prediction of the theory. For laboratory momenta greater than 26 BeV/c, the new experimental results follow the curve down to an ordinate of 2×10^{-12}. The measured point of highest t represents $p_0 = 31$ BeV/c scattering at $82\tfrac{1}{2}°$ center-of-mass angle, and exhausts the capabilities of presently existing accelerators. For lower energies the measured points deviate[4] above the curve, the deviation becoming quite large as the center-of-mass scattering angle approaches 90°. In Ref. 1 it was suggested that the optical model, which contains in itself no prediction of any dependence on energy, gives the limiting behavior for very high energy. This idea is borne out, to the extent that the higher the proton energy, the further down the scattering follows the predicted curve before the deviation sets in.

A word must now be said about the meaning of the scale factor Λ^2, determined as described above. The Yukawa form of absorptive potential is only nominal; the scattering for large momentum transfers is determined only by the behavior of the potential near $r = 0$. If (2) is expanded in a power series,

$$V = \eta[(1/r) - \Lambda + \tfrac{1}{2}\Lambda^2 r + \cdots],$$

the essential parameters are the coefficient of the $1/r$

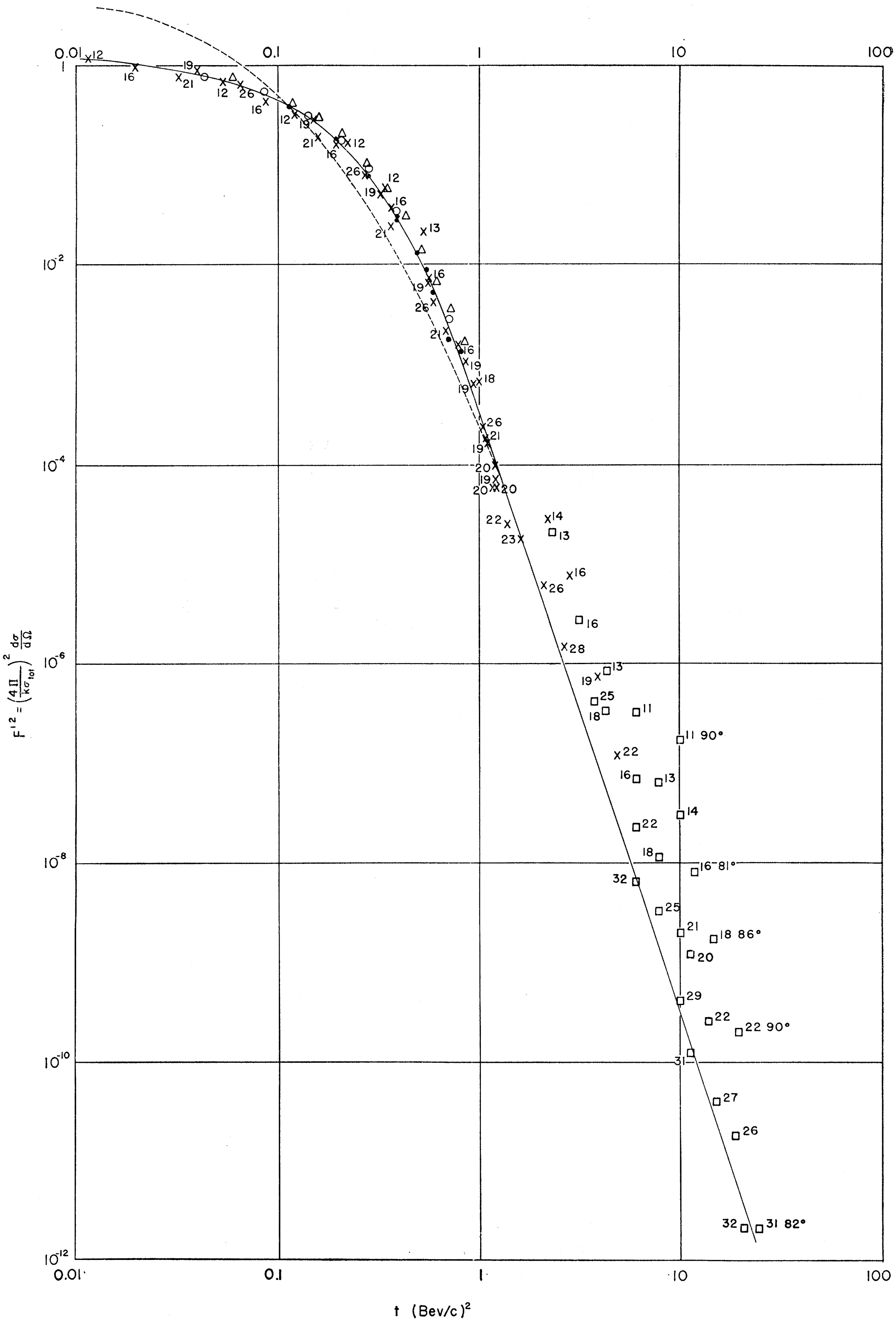
[3] A. N. Diddens, E. Lillethun, G. Manning, A. E. Taylor, T. G. Walker, and A. M. Wetherell, in *Proceedings of the International Conference on High Energy Nuclear Physics, Geneva, 1962* (CERN Scientific Information Service, Geneva, Switzerland, 1962), p. 576.

[4] For scattering near 90°, i.e., for $t/2k^2$ near 1, the cross section should be taken in symmetrized form. Denoting the ordinate of Fig. 1 by $F'(t)^2$, this amounts to replacing $F'(t)^2$ by

$$F'(t)^2 + F'(4k^2 - t)^2 - F'(t)F(4k^2 - t).$$

This leaves the 90° scattering cross section unchanged.

FIG. 1. Log–log plot of $F'^2 = (4\pi/k\sigma_{\text{tot}})^2 d\sigma/d\Omega$ vs t. The experimental points are labeled as follows: Foley *et al.*,[7] $p_0 = 19.6$ BeV/c ●, $p_0 = 16.7$ BeV/c ○, $p_0 = 10.8$ BeV/c △; Diddens *et al.*[3] ×; Cocconi *et al.*[2] □. The numbers next to the points give p_0 in BeV/c, and in a few cases the center-of-mass scattering angles.



0.01
0.1
1
10
100
1
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
$F'^2 = \left(\frac{4\Pi}{k\sigma_{tot}}\right)^2 \frac{d\sigma}{d\Omega}$
t (Bev/c)^2

term which must be taken to be $\eta = 1$, and the ratio of the term linear in r to the $1/r$ term, which determines the scale factor. A more precise statement of the relationships involved can be obtained by considering an absorptive potential of the form

$$V(r) = \int \eta(\Lambda)(e^{-\Lambda r}/r)d\Lambda\,. \qquad (4)$$

Let us define the moments

$$\eta = \int \eta(\Lambda)d\Lambda\,,$$

$$\eta \ln \Lambda_l = \int \eta(\Lambda) \ln \Lambda d\Lambda\,,$$

$$\eta\Lambda^2 = \int \eta(\Lambda)\Lambda^2 d\Lambda\,.$$

As before, η and $\eta\Lambda^2$ determine the $1/r$ and the r term in the power series expansion of $V(r)$. The logarithmic moment arises in the evaluation of the phase shift as a function of the impact parameter $\rho, \delta(\rho) = i\chi(\rho)$,

$$\chi(\rho) = \int_\rho^\infty \frac{rV(r)}{(r^2-\rho^2)^{\frac{1}{2}}}\,dr\,. \qquad (5)$$

For small ρ,

$$\chi(\rho) = -\eta(1+\tfrac{1}{4}\Lambda^2\rho^2)\ln(\tfrac{1}{2}\gamma\Lambda_l\rho) \qquad (6)$$

plus terms of order $\Lambda^2\rho^2$. Here $\ln \gamma = 0.557\cdots$ is Euler's constant. These logarithmically singular terms in $\chi(\rho)$ in turn determine the leading terms in the asymptotic form of the scattering amplitude for large t [see (18) of Ref. 1],

$$\begin{aligned} F(t) = \frac{2}{\pi}\frac{\Lambda^2}{t}\left(\frac{\gamma^2\Lambda_l^2}{t}\right)^\eta \Bigg\{ &\Gamma(1+\eta)^2 \sin \pi\eta \\ &\times\left[1+(1+\eta)^2\frac{\Lambda^2}{t}\ln\frac{t}{\Lambda^2}+O\left(\frac{\Lambda^2}{t}\right)\right] \\ &-\Gamma(2+\eta)^2\pi\eta\cos\pi\eta\,\frac{\Lambda^2}{t}\Bigg[1+(2+\eta)^2 \\ &\times\frac{\Lambda^2}{t}\ln\frac{t}{\Lambda^2}+O\left(\frac{\Lambda^2}{t}\right)\Bigg]\Bigg\}. \end{aligned} \qquad (7)$$

For $\eta = 1$, (7) reduces to

$$F(t) = 8\gamma^2\frac{\Lambda_l^2}{\Lambda^2}\left(\frac{\Lambda^2}{t}\right)^3\left[1+9\frac{\Lambda^2}{t}\ln\frac{t}{\Lambda^2}+O\left(\frac{\Lambda^2}{t}\right)\right]. \qquad (8)$$

The occurrence of the factor (Λ_l^2/Λ^2) in (8) shows that we have more freedom in our scaling law than was indicated in our earlier discussion. In fact, we may change the scales of ordinate and abscissa independently, thus determining both Λ^2 and Λ_l^2. If only the leading term of (8) were considered the two parameters would not be uniquely defined, since one could always make a displacement parallel to the straight line representing the t^{-6} law. It is for this reason that we have included the next largest term in the asymptotic expansion, to show that no new moments of the distribution occur, and that the determination of the parameters is, at least in principle, possible. In adjusting the optical model curve to the experimental results we have not, however, taken advantage of this extra freedom.

In comparing the optical model calculation with the measurements, we have identified the variable t which appears in formulas such as (3) and (8) with the square of the invariant momentum transfer. Some question may be raised as to whether this is the correct comparison when the scattering angle in the center-of-mass system is not small. The Cocconi–Orear experiments extend to 90° scattering in the center-of-mass system. The choice we have made corresponds to taking the scattering amplitude in the center-of-mass system to be

$$\frac{f}{ik} = \int_0^\infty (1-e^{-2\chi(\rho)})J_0(2k\rho\sin\tfrac{1}{2}\theta)\rho d\rho\,, \qquad (9)$$

whereas in Ref. 1 the argument of the Bessel function was written in the small-angle approximation as $k\rho\theta$. Equation (9) is obtained from the expression

$$\frac{f}{ik} = \left(\frac{1}{2k^2}\right)\sum_{l=0}^\infty (2l+1)(1-\exp(-2\chi_l))(P_l(\cos\theta) \qquad (10)$$

by replacing the sum by an integral, using the connection $l + \frac{1}{2} = k\rho$, and by making the approximation

$$P_l(\cos\theta) = J_0((l+\tfrac{1}{2})2\sin\tfrac{1}{2}\theta)\,. \qquad (11)$$

One argument that this is a judicious choice of variables is that (9) then gives the proper Born approximation result for $\chi_l \ll 1$.[5]

For the Yukawa potential (2),

$$\chi_l = K_0((l+\tfrac{1}{2})\Lambda/k)\,, \quad \text{or} \quad \chi(\rho) = K_0(\Lambda\rho)\,. \qquad (12)$$

With this choice of χ, we have re-examined the evaluation of the asymptotic form of f for large momentum transfers, including the higher terms in Macdonald's expansion, of which (11) is the leading term.[6] This investigation indicates that (7) and (8) are correct to within a factor $[1 + O(\sin^2\frac{1}{2}\theta)]$.

As a further check, both of (11) and of replacing the sum by an integral, we have calculated (10)

[5] See also the discussion by R. J. Glauber, *High Energy Collision Theory, Lectures in Theoretical Physics* (Interscience Publishers, Inc., New York, 1959), p. 345.

[6] See G. N. Watson, *Theory of Bessel Functions*, (Cambridge University Press, New York, 1962), p. 158.

numerically, using $\Lambda/k = 0.12$, which corresponds to a proton of momentum $p_0 = 11.26$ BeV/c. In this calculation we kept terms up to $l = 110$. The result for $\theta = 90°$ was $F'(t) = 2.01 \times 10^{-5}$, whereas the evaluation of (9) gave $F'(t) = 1.99 \times 10^{-5}$ [$F'(t)$ is defined by (15)].

Another question about the optical model when large angle scattering is considered concerns the integration of the phase shift along a straight line ray. If we suppose our scattering problem to be the solution of a Klein–Gordon equation, i.e., (20) of Ref. 1, this is equivalent to keeping the first term in the expansion of the WKB expression for the phase shift in powers of V, or η. A question arises because the leading term in (7) vanishes for $\eta = 1$, so that even a small correction to the effective value of η would be multiplied, in comparison to the second term, by a factor $t/4\Lambda^2$. However it is not difficult to evaluate the WKB phase shift exactly for small ρ, and show that the leading term in the phase shift, the $-\eta \ln \rho$ term in (6), has no corrections to any order in η. An interesting point is that it is essential to keep the V^2 term in the Klein–Gordon equation to get this result.

We turn now to a discussion of the observed scattering for small momentum transfers. As a result of the scale change needed to fit the data for large t the range of the Yukawa potential has been increased by a factor 2.5. This would lead to a total cross section 2.5 times the correct value, and to an increase in the forward scattering cross section, which is proportional to σ_{tot}^2, by a factor 6.25. It is easy to see what change in the model must be made to bring its predictions back into agreement with the facts. The high t scattering is determined by the behavior of $V(r)$ near the origin; the contributions to the total cross section, on the other hand, come mostly from large impact parameters. The Yukawa potential has too much long-range tail; by chopping off the tail we can come back into agreement with the observed total and forward scattering cross sections, without affecting the cross section for large t. In addition, since the integrand in the expression for σ_{tot} is proportional to $(1 - e^{-2\chi(\rho)})$, while that for σ_{el} is proportional to $(1 - e^{-2\chi(\rho)})^2$, reducing $\chi(\rho)$ for large ρ, where $\chi(\rho) \ll 1$, reduces σ_{tot} more than σ_{el}. Cutting off the tail will also serve to raise $\sigma_{\text{el}}/\sigma_{\text{tot}}$ from the value 0.185 given by the Yukawa model, to the value $\sigma_{\text{el}}/\sigma_{\text{tot}} = 0.244 \pm 0.012$ observed by Lindenbaum, Yuan, *et al.*[7] for $p_0 = 19.6$ BeV/c.

It is not difficult to determine the form of $V(r)$ needed to fit the small t measurements, if it is supposed that the scattering amplitude is purely imaginary. For then $f(t)$ is determined directly by the experimental results, $f(t) = i(d\sigma/d\Omega)^{\frac{1}{2}}$, and (9) can be inverted to determine $\chi(\rho)$. This method has been used by Krisch,[8] who considered 16 BeV/c scattering, and showed that $\chi(\rho)$ agrees with (12) for small ρ, but falls off more rapidly as ρ increases. The inversion formula for (9) is

$$1 - e^{-2\chi(\rho)} = \int_0^\infty \frac{f(y^2)}{ik} J_0(\rho y) y \, dy \,, \qquad (13)$$

where $y = t^{\frac{1}{2}} = 2k \sin \frac{1}{2}\theta$. We can rewrite (13) as

$$1 - e^{-2\chi(\rho)} = \frac{\sigma_{\text{tot}}}{4\pi} \int_0^\infty F'(y^2) J_0(\rho y) y \, dy \,, \qquad (14)$$

with[9] $F'(y^2) = 4\pi f(y^2)/ik\sigma_{\text{tot}}$, or

$$F'(t) = [(4\pi/k\sigma_{\text{tot}})^2 d\sigma/d\Omega]^{\frac{1}{2}} \,. \qquad (15)$$

The form (15) of $F'(t)$ is the one used in making the connection with the experimental results. It is just the square root of the ordinate of Fig. 1, or the ordinate itself if the spacing between the horizontal lines is read as one decade rather than two. In passing from (13) to (14) and (15) we have not used the optical theorem for the forward scattering amplitude, but simply introduced σ_{tot} to obtain a convenient measure for $d\sigma/d\Omega$.

For $\rho = 0$, (14) gives[10]

$$1 - e^{-2\chi(0)} = \frac{\sigma_{\text{tot}}}{8\pi} \int_0^\infty F'(t) dt \,. \qquad (16)$$

Equation (16) expresses the opacity for a central ray directly in terms of experimentally measured quantities. Since the integral on the right is largely determined by the scattering for small t, it provides a critical test of our model, that is, of our supposition that the large t behavior of the scattering requires a $1/r$ absorption, which gives $\chi(0) = \infty$, and

$$1 = \frac{\sigma_{\text{tot}}}{8\pi} \int_0^\infty F'(t) dt \,. \qquad (17)$$

Some uncertainty is introduced, however, by the assumption that the scattering amplitude is purely imaginary. Lindenbaum and Yuan[7] find, for their highest momentum, $p_0 = 19.6$ BeV/c, $F'(0)^2 = 1.20$

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[7] K. J. Foley, S. J. Lindenbaum, W. A. Love, S. Ozaki, J. J. Russell, and L. C. L. Yuan, Phys. Rev. Letters 11, 425 (1963).

[8] A. D. Krisch, Phys. Rev. Letters 11, 217 (1963).

[9] The function $F'(t)$ differs from the $F(t/\Lambda^2)$ of (3) by a factor,

$$F'(t) = (4\pi/\sigma_{\text{tot}}\Lambda^2) F(t/\Lambda^2) \,.$$

With $\Lambda = 1.341$, $\sigma_{\text{tot}} = 39.5$ mb,

$$F'(t) = 1.769\, F(t/\Lambda^2) \,.$$

[10] Note that the scaling law of (3) leaves the right-hand side of (16) invariant.

$\pm$ 0.20, the error including the uncertainties in the absolute scale of the scattering cross section, in the extrapolation of the cross section to $t = 0$, and in the total cross section. If the amplitude were purely imaginary we should have $F'(0)^2 = 1$. There is some indication in the results of Lindenbaum and Yuan that the deviation from the optical theorem is still larger for smaller values of p_0, and thus that f has a real part which decreases as p_0 increases.

If $f(t) = f_r(t) + if_i(t)$, (16) should be replaced by

$$\mathrm{Re}\,(1 - e^{-2\chi(0)}) = \frac{\sigma_{\mathrm{tot}}}{8\pi}\int_0^\infty F'(t)\left(1 + \frac{f_r(t)^2}{f_i(t)^2}\right)^{-\frac{1}{2}} dt\,. \tag{18}$$

If $f_r(t)/f_i(t)$ were independent of t, (18) would say that $F'(t)$ should be renormalized to $F'(0) = 1$ before making the comparison (17). On the other hand, if $f_r(t)/f_i(t)$ decreases rapidly with t, (17) would be unaltered. The latter situation corresponds to the phase shift having a real part only for large impact parameters. In this circumstance it is easy to see that the scattering amplitude coming from small impact parameters would be modified only by multiplication by a constant phase factor, and the scattering cross section for large t would be unchanged.

In Fig. 1, the solid curve for ordinate greater than 10^{-4} represents the parametric fit given by Lindenbaum and Yuan[6] for their $p_0 = 19.6$ BeV/c data,

$$d\sigma/dt = Ae^{-bt+ct^2}, \tag{19}$$

with $A = 96.5$ mb/(BeV/c)^2, $b = 10.48$(BeV/c)$^{-2}$, $c = 2.25$(BeV/c)$^{-4}$. The optical limit value of A would be $A = 79.7$ mb/(BeV/c)^2, taking $\sigma_{\mathrm{tot}} = 39.5$ mb. On carrying out the integration in (17), using (19) for $F'^2 > 10^{-4}$ and the optical model curve for $F'^2 < 10^{-4}$, we find

$$\frac{\sigma_{\mathrm{tot}}}{8\pi}\int_0^\infty F'(t)dt = 0.96 \pm 0.08\,. \tag{20}$$

The error has been estimated on the basis of Lindenbaum and Yuan's statement of the errors in their experiment. The agreement between (20) and (17) assures us that we can determine a $\chi(\rho)$ from (14) which will reproduce the experimental scattering curve well within the experimental errors. Given $\chi(\rho)$, the corresponding absorptive potential is readily found, since (5) can be inverted to give[11]

$$\int_r^\infty rV(r)dr = \int_0^\infty \chi([r^2 + s^2]^{\frac{1}{2}})ds\,. \tag{21}$$

[11] See Ref. 5, p. 386.

Notice the one-sidedness of this expression, $V(r)$ depends on the values of $\chi(\rho)$ only for $\rho \geqslant r$.

For ρ near zero we have already determined $\chi(\rho)$ by fitting the large t scattering. To terms of order $\Lambda^2\rho^2$,

$$\chi(\rho) = -(1 + \tfrac{1}{4}\Lambda^2\rho^2)\ln\tfrac{1}{2}\gamma\Lambda\rho + \tfrac{1}{4}\Lambda^2\rho^2\,. \tag{22}$$

Figure 2 shows how the curve of $\chi(\rho)$ calculated

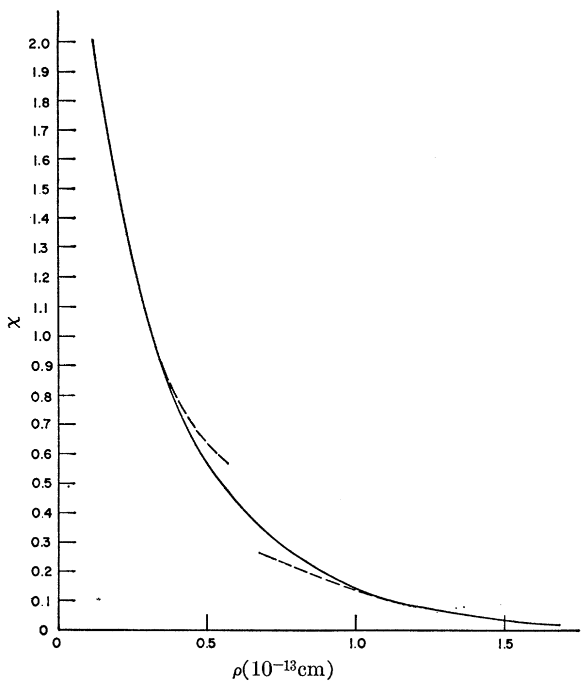

Fig. 2. χ as a function of ρ. The dashed curves give the continuation of (22) to larger ρ, and the continuation of the Gaussian, (23), to smaller ρ.

numerically from (14) for larger ρ joins smoothly to that given by (22). The dotted portion for $\rho > 0.33$ is the continuation of (22). For $\rho > 1.1$, the numerical results for $\chi(\rho)$ can be represented very well by a Gaussian dependence

$$\chi(\rho) = Ae^{-\lambda^2\rho^2} \tag{23}$$

with $A = 0.454$, $\lambda^2 = 1.224$. The continuation of (23) to smaller ρ is also indicated by a dotted line in Fig. 2.

Using (21) and (23) we find

$$\begin{aligned} V(r) &= (2/\pi^{\frac{1}{2}})A\lambda e^{-\lambda^2 r^2} \\ &= 0.567\,e^{-1.224r^2}\,,\ \rho > 1.1\,. \end{aligned} \tag{24}$$

Over the region for which (23) holds, $\chi(\rho) < 0.1$.

Thus the Born approximation is valid, which suggests an immediate connection between λ^2 and Lindenbaum and Yuan's parameter b in (19). For small t, $d\sigma/dt = A \exp(-bt)$, which is the Born approximation scattering from a Gaussian potential of range

$$\lambda^2 = 1/2b\,. \tag{25}$$

Expressed in $(10^{-13}\text{ cm})^2$, rather than $(\text{BeV}/c)^{-2}$, $b = 0.4080$, and (25) gives $\lambda^2 = 1.225$, in excellent agreement with the value obtained from the numerical integration.

For small r,

$$V(r) = e^{-1.341r}/r\,, \quad r < 0.33\,. \tag{26}$$

For $0.33 < r < 1.1$, $rV(r)$ was calculated by numerical integration of the right side of (21), followed by differentiation to give $rV(r)$. The results are shown in Fig. 3.

In summary, we have shown that it is possible to find an absorptive potential which will represent the data well for both large and small t. For small r, $V(r)$ behaves like a Yukawa potential with a range determined by the width of the scattering curve for large t, while for large r it behaves like a Gaussian with a range determined by the initial rate of decrease of the scattering for small t.

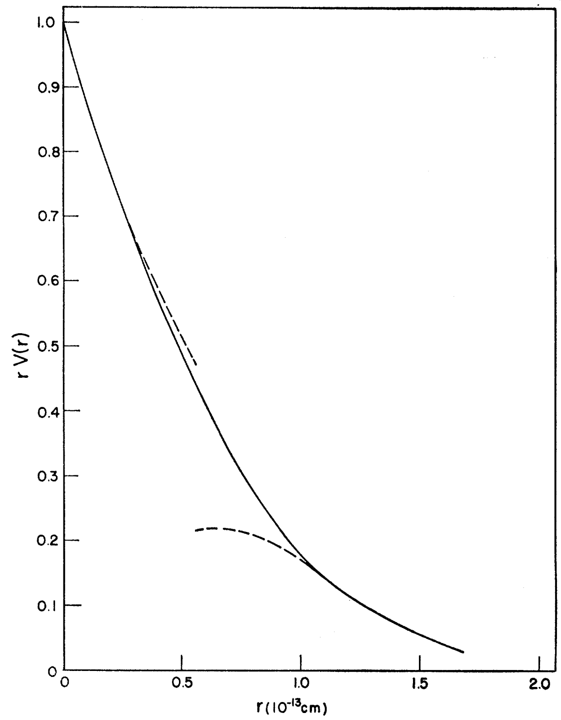

FIG. 3. $rV(r)$ as a function of r. The dashed curves give the continuation of the Yukawa potential (26) (times r) to larger r, and the continuation of the Gaussian potential (24) (times r) to smaller r.

High-Energy Collisions of Strongly Interacting Particles

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1. INTRODUCTION

We are concerned in the present paper with the theoretical analysis of strongly interacting particle collisions in the multi-GeV energy range (laboratory energy above 5 GeV). While one has expected for a long time that the gross features of such collisions would be qualitatively very simple, progress in their detailed experimental and theoretical study has been extremely slow. Reliable and accurate accelerator experiments have started quite recently and have mainly elucidated some of the basic properties of small angle elastic scattering up to 30 GeV. The work on inelastic collisions is only at its most timid beginnings. The systematic study of the most common type of inelastic collisions, often referred to as jets, has hardly started at accelerator energies and is strongly limited by the short-comings of most present detectors (what is required is identification and momentum determination of all pions in the three charge states, for inelastic collisions in pure hydrogen). Regarding our theoretical understanding, fundamental theory has essentially failed to make any progress despite many attempts. It is striking in this context to read over again the early and most interesting paper devoted in 1948 to multiple meson production by Lewis, Oppenheimer, and Wouthuysen,[1] and to compare it with recent attempts at applying field theory or dispersion techniques to this subject. The difficulty to explain even the most predominant features of high-energy collisions remains as great as it was sixteen years ago.

It is in our opinion unavoidable that progress in theoretical understanding of these phenomena will

[1] H. W. Lewis, J. R. Oppenheimer, and S. A. Wouthuysen, Phys. Rev. **73**, 127 (1948).