independently by two different groups.[64,65] Their results are in substantial agreement:

$$A^+(Na^{24}): +0.104 \pm 0.026 \ (63) ;$$
$$+0.091 \pm 0.017 \ (64) ,$$
$$A^-(Al^{24}): -0.089 \pm 0.057 \ (63) ;$$
$$-0.086 \pm 0.054 \ (64) .$$

Therefore, it implies that β decay in complex nuclei is consistent with CVC theory well within experimental error. However, it is not necessary to imply that it is in disagreement with the old Fermi theory since the Fermi matrix element, due to mesonic effect, could be very small. On the other hand, the isotopic spin purity seems to be definitely better than theoretical estimate by the j–j coupling model.

[64] E. L. Hasse, H. A. Hill, and D. B. Knudson, Phys. Letters **4**, 338 (1963).
[65] L. G. Mann, S. O. Bloom, A. Scott, R. Polichas, and J. R. Richardson, in Proceedings of the Manchester Conf. on Low- and Medium-Energy Physics, 1963; and also Phys. Rev. (to be published).

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Electromagnetic Interactions of a Yang–Mills Field

R. J. FINKELSTEIN
Department of Physics, University of California, Los Angeles, California

1. INTRODUCTION

Since it is possible that the weak interactions are mediated by vector bosons, associated with a conserved vector current, it is of some interest to investigate a possible genetic relationship between these hypothetical particles and the photon. In particular it has been conjectured that the photon is the neutral member of a multiplet which embraces the charged W-mesons.[1] Since one is then faced with a parity violating generalization of quantum electrodynamics, it is perhaps also natural in such speculations to examine the possibility of magnetic poles which, like the W-mesons, may also violate parity[2] and may also be produced only at very high energies. In a theory of this kind it is clear that the existence, or the nonexistence, of magnetic poles would have important implications for the structure of the weak interactions.

The Maxwell equations may be written

$$\partial_\mu F^{\alpha\mu} = J^\alpha , \tag{A}$$

$$\partial_\mu \mathbf{F}^{\alpha\mu} = 0 , \tag{B}$$

where $\mathbf{F}^{\alpha\mu}$ is the dual field:

$$\mathbf{F}^{\alpha\beta} = \tfrac{1}{2} \epsilon^{\alpha\beta\mu\nu} F_{\mu\nu} . \tag{1.1}$$

The conservation of electric current follows from the first equation since $F^{\alpha\beta}$ is antisymmetric. Equation (B) directly expresses the nonexistence of magnetic poles and permits the usual representation of the six-vector field $\mathrm{F}_{\alpha\beta}$ in terms of the four-vector A_α.

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha . \tag{1.2}$$

This representation is invariant under gauge transformations:

$$A_\alpha' = A_\alpha + \partial_\alpha \Lambda . \tag{1.3}$$

In this way the nonexistence of magnetic poles leads to the gauge group. If one generalizes either by allowing magnetic poles or by expanding the gauge group, the single 4-vector potential must be replaced by a field with more degrees of freedom.

2. GEOMETRICAL BASIS

Since the electromagnetic field is so important for determining our present ideas about physical space time, one would expect any generalization of the former to have important geometrical implications; or turning the argument around, one might seek a geometrical basis for generalizing Maxwell's equations. In the context of general relativity this has often been done, but until recently without reference to the weak interactions.

In this paper we postulate a local gauge group which is non-Abelian, compact, and contains the electromagnetic gauge transformation. Therefore a general vector field may be designated by $\mathrm{A}_{\mu M}$ where

[1] A. Salam and J. Ward, Nuovo Cimento **11**, 568 (1959).
[2] N. Cabibbo and E. Ferrari, Nuovo Cimento **23**, 1147 (1962).

μ is the usual vector space–time index and M is the gauge index labeling some irreducible unitary representation of the gauge group. The operator $A_{\mu M}$, assigned to a pair of indices (μM), then depends on both the coordinate system and the local gauge and transforms as follows:

$$A'_{\mu M} = A_{\sigma M}(\partial x^{\sigma}/\partial x^{\mu'}) , \qquad (2.1)$$

$$A'_{\mu M} = A_{\mu N}V_{NM} . \qquad (2.2)$$

The gauge indices are suppressed in the rest of this paper.

Because both coordinate and gauge transformations are position dependent, it is not possible to assert the equality of fields at remote points in an invariant manner. In order to define equivalence at nearby points one introduces the concept of parallel transfer

$$\delta A^{\mu} = -A^{\alpha}L^{\mu}_{\alpha\beta}\delta x^{\beta} , \qquad (2.3)$$

where $L^{\mu}_{\alpha\beta}$ is the displacement field or the connection of the space.

We assume that the geometry is entirely characterized by the connection $L^{\mu}_{\alpha\beta}$ and $g^{\alpha\beta}$, which defines the light cone and generalizes the metric tensor. Both of these objects are gauge variant, as follows:

$$L^{\mu'}_{\alpha\beta} = V^{-1}L^{\mu}_{\alpha\beta}V - \delta^{\mu}_{\alpha}V^{-1}\partial_{\beta}V , \qquad (2.4)$$

$$g^{\alpha\beta'} = V^{-1}g^{\alpha\beta}V . \qquad (2.5)$$

The transformation law (2.4) is required in order that the fundamental equivalence (2.3) be gauge invariant. The corresponding relation (2.5) is postulated for $g^{\alpha\beta}$ in order that it transform like the bilinear combination $(A^{\alpha})^{\dagger}A^{\beta}$ (where the dagger indicates adjoint and we assume a unitary representation of the gauge group).

Instead of $L^{\mu}_{\alpha\beta}$ we may use the connection density $U^{\mu}_{\alpha\beta}$ defined as follows:

$$U^{\mu}_{\alpha\beta} = L^{\mu}_{\alpha\beta} - \delta^{\mu}_{\beta}L^{\sigma}_{\alpha\sigma} . \qquad (2.6)$$

In terms of $U^{\mu}_{\alpha\beta}$, the Ricci tensor is

$$L_{\alpha\beta} = \partial_{\mu}U^{\mu}_{\alpha\beta} - U^{\rho}_{\alpha\sigma}U^{\sigma}_{\rho\beta} + \tfrac{1}{3}\, U^{\rho}_{\alpha\rho}U^{\sigma}_{\sigma\beta} \qquad (2.7)$$

with the gauge transformation law

$$L'_{\alpha\beta} = V^{-1}L_{\alpha\beta}V . \qquad (2.8)$$

The following traces[3]

$$I_1 = \mathrm{Tr}\, L_{\alpha\beta}g^{\alpha\beta} , \qquad (2.9)$$

$$I_2 = \mathrm{Tr}\, g_{\alpha\beta}g^{\alpha\beta} , \qquad (2.10)$$

are invariant under gauge as well as space–time transformations.

[3] Here the notation Tr means a normalized trace so that Tr 1 = 1.

We consider theories with the Lagrangian density

$$L = \mathrm{Tr}\,[L_{\alpha\beta}g^{\alpha\beta} - \lambda g_{\alpha\beta}g^{\alpha\beta}] . \qquad (2.11)$$

3. GENERAL FORMULATION OF VECTOR THEORIES

Since the basic field variables $g^{\alpha\beta}$ and $U^{\mu}_{\alpha\beta}$ are neither symmetric nor Hermitian the resulting theory is very general. In order to arrive at a more familiar situation we now make certain special assumptions. In the first place we describe a generalization of the symmetry condition, which is similar to transposition invariance,[4] and will be called conjugation invariance.[5]

Denote conjugate field variables by (+) and (−). They are related by definition as follows

$$g_{\alpha\beta}\,(-) = [g_{\beta\alpha}\,(+)]^{\dagger} , \qquad (3.1)$$

$$U^{\mu}_{\alpha\beta}\,(-) = [U^{\mu}_{\beta\alpha}\,(+)]^{\dagger} , \qquad (3.2)$$

where the symbol † means Hermitian adjoint. We now impose the conditions

$$U^{\mu}_{\alpha\beta}\,(-) = U^{\mu}_{\alpha\beta}\,(+) = U^{\mu}_{\alpha\beta} , \qquad (3.3)$$

$$g_{\alpha\beta}\,(-) = g_{\alpha\beta}\,(+) = g_{\alpha\beta} . \qquad (3.4)$$

According to these assumptions the symmetric parts are Hermitian while the antisymmetric parts are anti-Hermitian.

As usual we interpret the symmetric structure in terms of the gravitational field. We also provisionally assume that the gravitational variables are gauge invariant:

$$(g_{(\alpha\beta)},g_{\lambda\mu}) = (g_{(\alpha\beta)},U^{\sigma}_{\lambda\mu}) = 0 , \qquad (3.5a)$$

$$(U^{\mu}_{(\alpha\beta)},g_{\lambda\rho}) = (U^{\mu}_{(\alpha\beta)},U^{\sigma}_{\lambda\rho}) = 0 . \qquad (3.5b)$$

We assume that the antisymmetric part of the field describes, after quantization, particles of unit spin:

$$U^{\mu}_{[\alpha\beta]} = \delta^{\mu}_{\alpha}D_{\beta} - \delta^{\mu}_{\beta}D_{\alpha} . \qquad (3.6)$$

After these special assumptions I_1 simplifies as follows:

$$I_1 = \mathrm{Tr}\, g^{\alpha\beta}L_{\alpha\beta} = g^{(\alpha\beta)}R_{(\alpha\beta)} + \mathrm{Tr}\, g^{[\alpha\beta]}R_{[\alpha\beta]} , \qquad (3.7)$$

where

$$R_{(\alpha\beta)} = \partial_{\mu}U^{\mu}_{(\alpha\beta)} - U^{\sigma}_{(\alpha\rho)}U^{\rho}_{(\sigma\beta)} + \tfrac{1}{3}\, U^{\sigma}_{(\alpha\sigma)}U^{\rho}_{(\rho\beta)} \qquad (3.8)$$

and

$$R_{[\alpha\beta]} = \partial_{\alpha}D_{\beta} - \partial_{\beta}D_{\alpha} + (D_{\alpha},D_{\beta}) . \qquad (3.9)$$

The Lagrangian (2.11) becomes

$$L = \mathrm{Tr}\,[g^{(\alpha\beta)}R_{(\alpha\beta)} + \lambda_1 g^{[\alpha\beta]}R_{[\alpha\beta]} - \lambda_2 g^{[\alpha\beta]}g_{[\alpha\beta]} - \lambda_3 g^{(\alpha\beta)}g_{(\alpha\beta)}] , \qquad (3.10)$$

[4] A. Einstein and B. Kaufmann, Ann. Math. **62**, 128 (1955).
[5] R. Finkelstein and W. Ramsay, Ann. Phys. **21**, 408 (1963).

where

$$\lambda_1 = 1\,, \quad \lambda_2 = \lambda_3 = \lambda\,. \tag{3.10a}$$

Since the four parts of (3.10) are irreducible, it is also possible to assume arbitrary values of λ_1, λ_2, and λ_3. However, we shall here assume (3.10a) in agreement with (2.11).

4. EQUATIONS OF MOTION

The independent variables are $g^{(\alpha\beta)}$ and $U^{\mu}_{(\alpha\beta)}$ on the one hand, and $g^{[\alpha\beta]}$ and D^α on the other; and the corresponding equations of motion following from (3.10) are

$$\delta\mathfrak{L}/\delta g^{(\alpha\beta)} = 0 \quad \text{or} \quad [\partial/\partial g^{(\alpha\beta)} - \tfrac{1}{2}\, g_{(\alpha\beta)}]L = 0\,, \tag{4.1}$$

$$\delta\mathfrak{L}/\delta U^{\mu}_{(\alpha\beta)} = 0 \quad \text{or} \quad g^{(\alpha\beta)}_{|\gamma} = g^{(\alpha\beta)}_{;\gamma} = 0\,, \tag{4.2}$$

$$\delta\mathfrak{L}/\delta g^{[\alpha\beta]} = 0 \quad \text{or} \quad g_{[\alpha\beta]} = \lambda^{-1} R_{[\alpha\beta]}\,, \tag{4.3}$$

$$\delta\mathfrak{L}/\delta D_\alpha = 0 \quad \text{or} \quad g^{[\alpha\beta]}_{|\beta} = 0\,, \tag{4.4}$$

where we have denoted the covariant derivatives with respect to the symmetric connection by $g_{\alpha\beta;\gamma}$ and with respect to the total connection by $g_{\alpha\beta|\gamma}$. These two kinds of derivatives are by definition the following:

$$g_{\alpha\beta|\mu} = \partial_\mu g_{\alpha\beta} - L^{\sigma}_{\alpha\mu} g_{\sigma\beta} - g_{\alpha\sigma}(L^{\sigma}_{\beta\mu})^{\dagger}\,, \tag{4.5}$$

$$g_{\alpha\beta;\mu} = \partial_\mu g_{\alpha\beta} - C^{\sigma}_{\alpha\mu} g_{\sigma\beta} - g_{\alpha\sigma} C^{\sigma}_{\beta\mu}\,. \tag{4.6}$$

Here $g_{\alpha\beta|\mu}$ is constructed in such a way that it transforms properly under changes of gauge as well as changes of coordinate system. That is

$$g'_{\alpha\beta|\mu} = V^{-1} g_{\alpha\beta|\mu} V\,.$$

In Eq. (4.6), $C^{\mu}_{\alpha\beta}$ is the symmetric Hermitian part of $L^{\mu}_{\alpha\beta}$.

The special form of $L^{\mu}_{\alpha\beta}$ following from our assumptions about $U^{\mu}_{\alpha\beta}$ in paragraph 3 is

$$L^{\mu}_{\alpha\beta} = C^{\mu}_{\alpha\beta} + \delta^{\mu}_{\alpha} D_\beta\,, \tag{4.7}$$

and therefore by (4.5)

$$g_{\alpha\beta|\mu} = g_{\alpha\beta;\mu} + (g_{\alpha\beta}, D_\mu)\,. \tag{4.8}$$

For the symmetric part of $g_{\alpha\beta}$ we have

$$(g_{(\alpha\beta)}, D_\mu) = 0\,,$$

so that the two covariant derivatives of $g_{(\alpha\beta)}$ are equal.

Equation (4.2) may be solved for $C^{\mu}_{(\alpha\beta)}$ in the usual way and one finds

$$C^{\mu}_{(\alpha\beta)} = \left\{{}^{\mu}_{\alpha\beta}\right\}\,, \tag{4.9}$$

where $\left\{{}^{\mu}_{\alpha\beta}\right\}$ is the Christoffel connection.

Equation (4.1) may be written as

$$R_{(\alpha\beta)} - \tfrac{1}{2}\, R g_{(\alpha\beta)} = -KT_{\alpha\beta}\,, \tag{4.1$'$}$$

where

$$KT_{(\alpha\beta)} = [\partial/\partial g^{(\alpha\beta)} - \tfrac{1}{2}\, g_{(\alpha\beta)}]\, \mathrm{Tr}\, g^{[\alpha\beta]}[R_{[\alpha\beta]} - \lambda g_{[\alpha\beta]}]\,. \tag{4.10}$$

Therefore Eqs. (4.1) and (4.2) are the usual equations of gravitational theory, where $T_{\alpha\beta}$ is the energy momentum tensor of the antisymmetric field. Carrying out the operation indicated in (4.10) and using the equations of motion (4.3), one finds

$$KT_{\alpha\beta} = 6\lambda\, \mathrm{Tr}\, [g_{[\alpha\lambda]} g^{(\lambda\mu)} g_{[\mu\beta]} + \tfrac{1}{4}\, g_{(\alpha\beta)} g_{[\lambda\mu]} g^{[\lambda\mu]}]\,. \tag{4.11}$$

The antisymmetric field is determined by the vector potential D^μ. According to (4.3), the field strengths are represented by either $g_{[\alpha\beta]}$ or $R_{[\alpha\beta]}$, which differ only by a scale factor. Except for gauge structure (4.11) is of course just the usual Maxwell energy momentum tensor.

We have just seen how (4.1) may be written in a form such that the right-hand side represents the source of the $g_{(\alpha\beta)}$ field. In a similar way (4.4) may be written so that the right-hand side represents the source of the $g_{[\alpha\beta]}$ field, as follows:

$$g^{[\alpha\mu]}_{;\mu} = S^\alpha\,, \tag{4.4$'$}$$

where the current is

$$S^\alpha = -(g^{[\alpha\mu]}, D_\mu)\,. \tag{4.12}$$

While (4.1)$'$ is a single scalar equation, (4.4)$'$ is a matrix equation and the current S^α has as many independent components as there are independent parameters of the gauge group.

The covariant divergence of the left-hand side of (4.1)$'$ and (4.4)$'$ vanishes identically and therefore in the limit of flat space

$$\partial_\beta T^{\alpha\beta} = 0\,, \quad \partial_\alpha S^\alpha = 0\,.$$

These equations give the conservation of energy–momentum and of charge–current for the D-field.

5. THE DUAL FIELD AND MAGNETIC CURRENTS

Except for gauge properties all the preceding equations are the same as one would write for the Maxwell field. In particular, the currents (4.12) represent a generalization of the electric current.

We now consider the source of the dual field, which provides the analogue of the magnetic currents. The dual field is[6]

$$\mathbf{g}^{[\alpha\beta]} = \tfrac{1}{2}\, \epsilon^{\alpha\beta\lambda\mu} g_{[\lambda\mu]} \tag{5.1}$$

[6] Throughout this paper the dual field will be denoted by boldface symbols.

Its covariant divergence is

$$\mathbf{g}^{[\alpha\beta]}_{|\beta} = (1/3!)\,\epsilon^{\alpha\lambda\mu\nu}S_{\lambda\mu\nu}\,, \qquad (5.2a)$$

where

$$S_{\lambda\mu\nu} = g_{[\lambda\mu]|\nu} + g_{[\mu\nu]|\lambda} + g_{[\nu\lambda]|\mu} = \sum{}' g_{[\lambda\mu]|\nu} \qquad (5.2b)$$

and $\sum'$ means the cyclic sum.

The general form of the covariant derivative is

$$g_{\lambda\mu|\nu} = g_{\lambda\mu;\nu} + (g_{\lambda\mu},D_\nu)\,, \qquad (5.3)$$

but

$$\sum{}' g_{[\lambda\mu];\nu} = \sum{}' \partial_\nu g_{[\lambda\mu]}\,. \qquad (5.4)$$

Therefore,

$$\lambda \sum{}' g_{[\lambda\mu]|\nu} = \sum{}' \partial_\nu\{\partial_\lambda D_\mu - \partial_\mu D_\lambda - (D_\lambda,D_\mu)\} + \sum{}' (\{\partial_\lambda D_\mu - \partial_\mu D_\lambda - (D_\lambda,D_\mu)\},D_\nu)\,,$$

by Eqs. (4.3) and (3.9) or

$$\lambda \sum{}' g_{[\lambda\mu]|\nu} = -\sum [\partial_\nu(D_\lambda,D_\mu) + (\partial_\lambda D_\mu - \partial_\mu D_\lambda,D_\nu)]$$

with the aid of Jacobi's identity. We conclude

$$\sum{}' g_{[\lambda\mu]|\nu} = 0 \qquad (5.5a)$$

or

$$\mathbf{g}^{[\alpha\sigma]}_{|\sigma} = 0\,. \qquad (5.5)$$

The expression (5.5) is of the same form as (4.4). The corresponding magnetic currents are

$$u^\alpha = -(\mathbf{g}^{[\alpha\mu]},D_\mu)\,. \qquad (5.6)$$

In the case of the Maxwell field the gauge group is Abelian and both S^α and u^α vanish.

An important difference between (5.5) and (4.4) is that (5.5) vanishes identically, while (4.4) vanishes only if the D-field is free: If the D-field is coupled to a fermion field then (4.4) becomes

$$g^{[\alpha\sigma]}_{|\sigma} = j^\alpha\,, \qquad (5.7)$$

but (5.5) remains unchanged.

6. CONSISTENCY OF CONSERVATION LAWS

The divergence of the left-hand sides of (4.1)′ and (4.4)′ vanishes identically. Let us verify that the divergence of the right-hand sides also vanishes. Consider (4.4)′ first. The consistency condition is then

$$\partial_\alpha s^\alpha = \partial_\alpha(g^{[\mu\alpha]},D_\mu) = 0\,, \qquad (6.1)$$

or

$$2(\partial_\alpha g^{[\mu\alpha]},D_\mu) + (g^{[\mu\alpha]},\partial_\alpha D_\mu - \partial_\mu D_\alpha) = 0\,,$$
$$2((g^{[\mu\alpha]},D_\alpha),D_\mu) + (g^{[\mu\alpha]},(D_\mu,D_\alpha)) = 0\,,$$

or

$$((g^{[\mu\alpha]},D_\alpha),D_\mu) + ((D_\alpha,D_\mu),g^{[\mu\alpha]}) + ((D_\mu,g^{[\mu\alpha]}),D_\alpha) = 0\,, \qquad (6.2)$$

which vanishes by the Jacobi identity.

In the Maxwell case we know that the divergence of the energy momentum tensor gives rise to the Lorentz force working on the external source (5.7). If the original Lagrangian is complete then j^α vanishes, there is no Lorentz force, and the divergence of $T_{\mu\nu}$ vanishes. In the general case when j^α does not vanish, consider the following matrix whose trace is the energy momentum tensor:

$$\Theta_{\alpha\beta} = g_{[\alpha\lambda]}g^{(\lambda\mu)}g_{[\mu\beta]} + \tfrac{1}{4}\,g_{(\alpha\beta)}g_{[\lambda\sigma]}g^{[\lambda\sigma]}\,, \qquad (6.3)$$

where

$$T_{\alpha\beta} = \mathrm{Tr}\,\Theta_{\alpha\beta}\,. \qquad (6.3a)$$

Then

$$\Theta^{\alpha\beta}_{|\beta} = g^{[\alpha\lambda]}g_{(\lambda\mu)}g^{[\mu\beta]}_{|\beta} + g^{[\alpha\lambda]}_{|\beta}g_{(\lambda\mu)}g^{[\mu\beta]} + \tfrac{1}{4}\,g^{(\alpha\beta)}[g_{[\lambda\sigma]|\beta}g^{[\lambda\sigma]} + g_{[\lambda\sigma]}g^{[\lambda\sigma]}_{|\beta}]\,,$$

$$\mathrm{Tr}\,\Theta^{\alpha\beta}_{|\beta} = \tfrac{1}{2}\,\mathrm{Tr}\,[g_{[\alpha\lambda]|\sigma} + g_{[\sigma\alpha]|\lambda} + g_{[\lambda\sigma]|\alpha}]g^{[\lambda\nu]} + \mathrm{Tr}\,g^{[\alpha\mu]}j_\mu\,,$$

where j_μ is given by (5.7). Therefore

$$\mathrm{Tr}\,\Theta^{\alpha\beta}_{|\beta} = \mathrm{Tr}\,g^{[\alpha\beta]}j_\beta \qquad (6.4)$$

by the identity (5.5).

Since

$$\mathrm{Tr}\,\Theta^{\alpha\beta}_{|\beta} = \mathrm{Tr}\,[\partial_\beta\Theta^{\cdot\beta}_{\alpha} + (\Theta^{\cdot\beta}_{\alpha},D_\beta)] = \partial_\beta\,\mathrm{Tr}\,\Theta^{\cdot\beta}_{\alpha}\,,$$

we have the result by (6.3a)

$$\partial_\beta T^{\alpha\beta} = \mathrm{Tr}\,g^{[\alpha\beta]}j_\beta\,, \qquad (6.5)$$

where the right-hand side is the generalization of the Lorentz force and vanishes if $j_\beta = 0$.

7. FERMION FIELDS

We consider a fermion field which is a gauge multiplet and adopt a representation where local spacetime rotations work on the left and gauge transformations on the right:

$$\Psi' = S\Psi \quad \text{local ennuple rotation}\,, \qquad (7.1)$$

$$\Psi' = \Psi V \quad \text{local gauge transformation}\,. \qquad (7.2)$$

Ψ is of course invariant under global coordinate transformations.

Let the covariant derivative of Ψ be

$$\Psi_{|\alpha} = \partial_\alpha\Psi + \Gamma_\alpha\Psi + \Psi D_\alpha\,, \qquad (7.3)$$

where Γ_α is the spin connection and D_α is the gauge connection which we have been discussing. Then we have

$$\Psi' = S\Psi\,, \quad \Gamma'_\alpha = S\Gamma_\alpha S^{-1} + S(\partial_\alpha S^{-1})\,,$$
$$D'_\alpha = D_\alpha\,, \quad \Psi'_{|\alpha} = S\Psi_{|\alpha}\,, \qquad (7.4)$$

under local rotations, where S is the spin representation of the rotation. Similarly,

$$\Psi' = \Psi V\ , \quad \Gamma'_\alpha = \Gamma_\alpha\ ,$$
$$D'_\alpha = V^{-1}D_\alpha V - V^{-1}\partial_\alpha V\ , \quad \Psi'|_\alpha = \Psi|_\alpha V\ , \qquad (7.5)$$

under local gauge transformations.

The simplest choice of fermion Lagrangian is

$$\mathrm{Tr}\ \bar\Psi\gamma^\mu\Psi|_\mu\ . \qquad (7.6)$$

The complete Lagrangian density describing fermions, vector bosons, and the gravitational field is

$$L = \mathrm{Tr}\ [g^{(\alpha\beta)}R_{(\alpha\beta)} + \lambda_1 g^{[\alpha\beta]}R_{[\alpha\beta]} + \lambda_2 g^{[\alpha\beta]}g_{[\alpha\beta]} + \lambda g^{(\alpha\beta)}g_{(\alpha\beta)} + \epsilon\bar\Psi\gamma^\mu\Psi|_\mu]\ . \qquad (7.7)$$

We then find (5.7) instead of (4.4) with

$$j^\mu = \epsilon\bar\Psi\gamma^\mu\Psi\ . \qquad (7.8)$$

The equation of motion for the fermion field itself is

$$\gamma^\mu\Psi|_\mu = 0\ , \qquad (7.9)$$

or

$$\gamma^\mu(\partial_\mu\Psi + \Gamma_\mu\Psi + \Psi D_\mu) = 0\ . \qquad (7.9a)$$

The Γ_μ couplings are gravitational and the D_μ couplings represent the gauge generalization of the usual minimal electromagnetic couplings.

8. THE ELECTROMAGNETIC FIELD

Since the antisymmetric field equations are the same as for the electromagnetic field it is possible to assume that the electromagnetic field is already included as the neutral member of the gauge multiplet.

As the simplest example let the gauge group be SU_2 and consider the three component B-field whose neutral component is the electromagnetic potential. We introduce dimensional fields B_μ and $B_{\mu\nu}$ as follows:

$$D_\mu = (ie/\hbar c)(\mathbf{B}_\mu)\ , \qquad (8.1a)$$
$$g_{[\mu\nu]} = (ie/\hbar c)a^2(\mathbf{B}_{\mu\nu})\ , \qquad (8.1b)$$

where a is a length.

Here

$$(\mathbf{B}_\mu) = \begin{pmatrix} \frac{1}{2}A_\mu & \frac{1}{\sqrt2}B_\mu \\ \frac{1}{\sqrt2}B^*_\mu & -\frac{1}{2}A_\mu \end{pmatrix} \qquad (8.2a)$$

and

$$(\mathbf{B}_{\mu\nu}) = \begin{pmatrix} \frac{1}{2}f_{\mu\nu} & \frac{1}{\sqrt2}B_{\mu\nu} \\ \frac{1}{\sqrt2}B^*_{\mu\nu} & -\frac{1}{2}f_{\mu\nu} \end{pmatrix}\ . \qquad (8.2b)$$

Then by (3.9) and (4.3)

$$g_{[\mu\nu]} = a^2[\partial_\mu D_\nu - \partial_\nu D_\mu + (D_\mu,D_\nu)]\ , \qquad (8.3)$$

where we have put $\lambda = a^{-2}$. Then

$$f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - (ie/\hbar c)(B_\mu B^*_\nu - B_\nu B^*_\mu)\ , \qquad (8.4a)$$
$$B_{\mu\nu} = \nabla_\mu B_\nu - \nabla_\nu B_\mu\ , \qquad (8.4b)$$
$$B^*_{\mu\nu} = \nabla^*_\mu B^*_\nu - \nabla^*_\nu B^*_\mu\ , \qquad (8.4c)$$

where

$$\nabla_\mu = \partial_\mu - (ie/\hbar c)A_\mu\ ,$$
$$\nabla^*_\mu = \partial_\mu + (ie/\hbar c)A_\mu\ .$$

Consider the gauge transformation

$$V = \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}\ . \qquad (8.5)$$

Then

$$D'_\mu = V^{-1}D_\mu V - V^{-1}\partial_\mu V\ , \qquad (8.6)$$

or

$$A'_\mu = A_\mu + \partial_\mu\varphi\ , \qquad (8.6a)$$
$$B'_\mu = B_\mu e^{-i\varphi}\ , \qquad (8.6b)$$
$$B^{*\prime}_\mu = B^*_\mu e^{i\varphi}\ . \qquad (8.6c)$$

Therefore V has the significance of an electromagnetic gauge transformation which combines gauge transformations of the first and second kind. It is also clear that $f_{\mu\nu}$ and $B_{\mu\nu}$ are invariant under this transformation.

The charged B-particles carry electrical currents which may be calculated from (4.12) and (8.2)

$$S^\alpha = -\frac{ie}{\hbar c}\begin{pmatrix} \frac{1}{2}[B^{\alpha\mu}(B_\mu)^* - (B^{\alpha\mu})^*B_\mu] & \frac{1}{\sqrt2}[f^{\alpha\mu}B_\mu - B^{\alpha\mu}A_\mu] \\ \frac{1}{\sqrt2}[(B^{\alpha\mu})^*A_\mu - f^{\alpha\mu}(B_\mu)^*] & \frac{1}{2}[(B^{\alpha\mu})^*B_\mu - (B^{\alpha\mu})(B_\mu)^*] \end{pmatrix}\ . \qquad (8.7)$$

The neutral component of (4.4)′ reads

$$\partial_\mu f^{\alpha\mu} = S^\alpha_3\ ,$$

where S^α_3 is the neutral component of the currents:

$$S^\alpha_3 = -(ie/\hbar c)[B^{\alpha\mu}(B_\mu)^* - (B^{\alpha\mu})^*B_\mu]\ , \qquad (8.8)$$

which is just the usual expression for the electric current carried by vector particles of charge e (and $-e$).

One may now ask about the possible existence of magnetic currents. One finds according to (5.6)

$$\partial_\sigma \mathbf{f}^{\alpha\sigma} = u^\alpha_3\ , \qquad (8.9)$$

where

$$u^\alpha = -(ie/\hbar c)(\mathbf{B}^{\alpha\mu},B_\mu)\ . \qquad (8.9a)$$

The magnetic current is

$$u^\alpha_3 = -(ie/\hbar c)\ \mathrm{Tr}\ (\mathbf{B}^{\alpha\mu},B_\mu)\tau_3$$
$$= (ie/\hbar c)[\mathbf{B}^{\alpha\mu}(B_\mu)^* - (\mathbf{B}^{\alpha\mu})^*B_\mu]\ . \qquad (8.10)$$

In particular

$$\operatorname{div}\mathbf{H} = -(ie/\hbar c)[\mathbf{B}^{0\kappa}B^*_\kappa - (\mathbf{B}^{0\kappa})^*B_\kappa]\,, \tag{8.11}$$

corresponding to

$$\operatorname{div}\mathbf{E} = (ie/\hbar c)[B^{0\kappa}B^*_\kappa - B^{0\kappa}B_\kappa]\,.$$

But

$$\mathbf{B}^{0\kappa} = \tfrac{1}{2}\,\epsilon^{\kappa ij}B_{ij} = \epsilon^{\kappa ij}\nabla_i B_j\,,$$

and

$$(B^{0\kappa})(B^*_\kappa) = \epsilon^{\kappa ij}(B_\kappa)^*(\nabla_i B_j)\,.$$

Therefore

$$\operatorname{div}\mathbf{H} = (ie/\hbar c)[(\boldsymbol{\nabla}\times\mathbf{B})\cdot\mathbf{B}^* - (\boldsymbol{\nabla}\times\mathbf{B})^*\cdot\mathbf{B}]\,. \tag{8.11a}$$

It has been pointed out that the electrical charge density in Eq. (8.12) is just what one would compute for charged vector mesons. Since this expression involves conjugate momenta ($B^{0\kappa}$), it is quantized.

In contrast the effective magnetic charge density given by (8.11) or (8.11a) does not involve conjugate momenta and therefore has a continuous spectrum. In fact it follows from (8.11a) that

$$\operatorname{div}\mathbf{B} = 0\,, \tag{8.12}$$

where

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I}\,, \tag{8.12a}$$

$$4\pi\mathbf{I} = -(ie/\hbar c)(\mathbf{B}\times\mathbf{B}^*)\,. \tag{8.12b}$$

Therefore the source of div **H** represents magnetic moment and does not arise from magnetic poles.

The asymmetry between the electric and magnetic charges results as usual from the fact that the magnetic equations (5.5) are identities while the corresponding electric eqs (5.7) are true equations of motion.

9. MAGNETIC MOMENT OF VECTOR MESONS

In order to check the interpretation of **I** as magnetic moment consider the equations of motion of the complete D_μ-field, as determined by (4.4)′ and (8.7)

$$\partial_\mu\begin{pmatrix}\tfrac{1}{2}f^{\alpha\mu} & \tfrac{1}{\sqrt{2}}B^{\alpha\mu}\\ \tfrac{1}{\sqrt{2}}(B^{\alpha\mu})^* & -\tfrac{1}{2}f^{\alpha\mu}\end{pmatrix} = -\frac{ie}{\hbar c}\begin{pmatrix}\tfrac{1}{2}[B^{\alpha\mu}B^*_\mu - (B^{\alpha\mu})^*B_\mu & \tfrac{1}{\sqrt{2}}[f^{\alpha\mu}B_\mu - B^{\alpha\mu}A_\mu]\\ \tfrac{1}{\sqrt{2}}[(B^{\alpha\mu})^*A_\mu - (f^{\alpha\mu})B^*_\mu] & \tfrac{1}{2}[(B^{\alpha\mu})^*B_\mu - (B^{\alpha\mu})B^*_\mu]\end{pmatrix},$$

or

$$(\partial_\mu - (ie/\hbar c)A_\mu)B^{\alpha\mu} + (ie/\hbar c)f^{\alpha\mu}B_\mu = 0\,, \tag{9.1}$$

$$(\partial_\mu + (ie/\hbar c)A_\mu)(B^{\alpha\mu})^* - (ie/\hbar c)f^{\alpha\mu}B^*_\mu = 0\,; \tag{9.2}$$

in addition to the earlier equation

$$\partial_\mu f^{\alpha\mu} = -(ie/\hbar c)[B^{\alpha\mu}B^*_\mu - (B^{\alpha\mu})^*B_\mu]\,. \tag{9.3}$$

These equations of motion follow from the Lagrangian

$$\begin{aligned}L = &-\tfrac{1}{2}(B^{\alpha\beta})^*(B_{\alpha\beta}) + \tfrac{1}{2}(B^{\alpha\beta})^*(\nabla_\alpha B_\beta - \nabla_\beta B_\alpha)\\ &+ \tfrac{1}{2}(B^{\alpha\beta})(\nabla_\alpha B_\beta - \nabla_\beta B_\alpha)^*\\ &+ \tfrac{1}{2}(ie/\hbar c)f^{\alpha\beta}(B^*_\alpha B_\beta - B_\alpha B^*_\beta)\,.\end{aligned} \tag{9.4}$$

The preceding expression may be compared with our initial Lagrangian where the dimensions are made explicit by the introduction of the charge e and the length a:

$$L = -e^2a\,\mathrm{Tr}\,[L_{[\alpha\beta]}g^{[\alpha\beta]} - \tfrac{1}{2}a^{-2}g_{[\alpha\beta]}g^{[\alpha\beta]}]\,,$$

$$L_{[\alpha\beta]} = \partial_\alpha D_\beta - \partial_\beta D_\alpha + (D_\alpha, D_\beta)\,, \tag{9.5}$$

which gives

$$\begin{aligned}L = (e^2/\hbar c)^2a^3\{&-\tfrac{1}{4}f_{\alpha\beta}f^{\alpha\beta} + \tfrac{1}{2}[\partial_\alpha A_\beta - \partial_\beta A_\alpha]f^{\alpha\beta}\\ &- \tfrac{1}{2}B_{\alpha\beta}(B^{\alpha\beta})^* + \tfrac{1}{2}[(\nabla_\alpha B_\beta - \nabla_\beta B_\alpha)(B^{\alpha\beta})^*\\ &+ (\nabla_\alpha B_\beta - \nabla_\beta B_\alpha)^*B^{\alpha\beta}]\\ &- (ie/\hbar c)(B_\alpha B^*_\beta - B_\beta B^*_\alpha)f^{\alpha\beta}\}\,.\end{aligned} \tag{9.5a}$$

If Eq. (9.1) is replaced by

$$(\partial_\mu - (ie/\hbar c)A_\mu)B^{\alpha\mu} + (ie/\hbar c)Kf^{\alpha\mu}B_\mu = 0\,, \tag{9.6}$$

then this new equation describes a particle of gyromagnetic ratio $1 + K$. The particular theory we are describing is therefore characterized by gyromagnetic ratio 2.

Although the natural choice of the neutral field component is $f_{\alpha\beta}$, it is also possible to write the final equations in terms of

$$\mathring{f}_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha\,.$$

Then

$$f_{\alpha\beta} = \mathring{f}_{\alpha\beta} - (ie/\hbar c)[B_\alpha B^*_\beta - B^*_\alpha B_\beta]$$

and

$$\begin{aligned}\nabla_\mu B^{\alpha\mu} + (ie/\hbar c)\mathring{f}^{\alpha\mu}B_\mu - (ie/\hbar c)^2[B^\alpha(B^\mu)^*\\ - (B^\alpha)^*B^\mu]B_\mu = 0\end{aligned}$$

$$\begin{aligned}\partial_\mu\mathring{f}^{\alpha\mu} = &-(ie/\hbar c)[B^{\alpha\mu}B^*_\mu - (B^{\alpha\mu})^*B_\mu]\\ &+ (ie/\hbar c)\partial_\mu[B^\alpha(B^\mu)^* - (B^\alpha)^*B^\mu]\,.\end{aligned}$$

Thus, if $\mathring{f}^{\alpha\beta}$ is used instead of $f^{\alpha\beta}$, it is then necessary to modify the current by a perfect divergence and to add direct pair interactions.

10. MODIFICATION OF LORENTZ FORCE

According to (6.5)

$$\partial_\beta T^{\alpha\beta} = 0 \tag{10.1}$$

if the D_μ-field is free. The energy momentum tensor in (10.1) is by (6.3)

$$T_{\alpha\beta} = T^{(0)}_{\alpha\beta} + T^{(\text{charged})}_{\alpha\beta}, \tag{10.2}$$

where

$$T^{(\text{charged})}_{\alpha\beta} = B_{\alpha\lambda}(B^{\lambda}_{\cdot\beta})^* + \tfrac{1}{4}(B_{\lambda\sigma})(B^{\lambda\sigma})^* g_{\alpha\beta} + (B_{\alpha\lambda})^*(B^{\lambda}_{\cdot\beta}) + \tfrac{1}{4}(B_{\lambda\sigma})^*(B^{\lambda\sigma})g_{\alpha\beta} \tag{10.2a}$$

and

$$T^{(0)}_{\alpha\beta} = f_{\alpha\lambda}f^{\lambda}_{\cdot\beta} + \tfrac{1}{4}f_{\lambda\sigma}f^{\lambda\sigma}g_{\alpha\beta}\,. \tag{10.2b}$$

The charged component $T^{(\text{charged})}_{\alpha\beta}$ is obtained by making the substitution $\partial_\alpha B_\beta - \partial_\beta B_\alpha \rightarrow \nabla_\alpha B_\beta - \nabla_\beta B_\alpha$ in the usual energy momentum tensor of an uncharged vector field. According to (10.1)

$$\partial_\beta(T^{\alpha\beta})^{(\text{charged})} = -\partial_\beta(T^{\alpha\beta})^0\,. \tag{10.3}$$

We now calculate

$$\partial_\beta(T^{\alpha\beta})^0 = f_{\alpha\lambda}s^\lambda + \tfrac{1}{2}[\partial_\beta f_{\alpha\lambda} + \partial_\lambda f_{\beta\alpha} + \partial_\alpha f_{\lambda\beta}]f^{\lambda\beta}\,. \tag{10.4}$$

In the Maxwell case, the bracket vanishes and the right-hand side gives the Lorentz force. Here there is an additional force, namely:

$$\begin{aligned}\tfrac{1}{2}[\partial_\alpha f_{\lambda\beta} + \partial_\lambda f_{\beta\alpha} + \partial_\beta f_{\alpha\lambda}]f^{\lambda\beta} &= (ie/2\hbar c)[\partial_\alpha(B_\beta B^*_\lambda - B^*_\beta B_\lambda) \\ &\quad + \partial_\lambda(B_\beta B^*_\alpha - B^*_\beta B_\alpha) + \partial_\beta(B_\alpha B^*_\lambda - B^*_\alpha B_\lambda)]f^{\lambda\beta} \\ &= (ie/\hbar c)[f^{\nu\mu}\delta^\sigma_\alpha + f^{\mu\sigma}\delta^\nu_\alpha - f^{\nu\sigma}\delta^\mu_\alpha]\partial_\sigma(B^*_\nu B_\mu)\,.\end{aligned} \tag{10.5}$$

The additional term may be rewritten to modify the energy momentum tensor and the current.

11. NONLOCAL REPRESENTATION OF OPERATORS

With the aid of a nonlocal formalism similar to that proposed by Mandelstam for electrodynamics[7] it is possible to eliminate the connection from the equations of motion. Define

$$U = \lim U(s_n)\cdots U(s_1)\,, \tag{11.1a}$$

where

$$U(s_\kappa) = e^{\eta(s\kappa)}\,, \tag{11.1b}$$

$$\eta(s_\kappa) = D_\mu(s_\kappa)\Delta x^\mu\,, \tag{11.1c}$$

or

$$U(x,P) = \exp\left[\int_0^x D_\mu \quad dx^\mu\right], \tag{11.1}$$

where the integration is along a space-like path P, and the ordering is with respect to geodesic distance from the origin. As paths we allow only curves for which the geodesic distance increases as the path parameter increases. Then

$$U^{-1} = \exp\left[\int_x^0 D_\mu \quad dx^\mu\right]. \tag{11.2}$$

Let ∂_μ indicate differentiation with respect to end point. Then

$$\partial_\mu U = D_\mu U\,, \tag{11.3}$$

$$\partial_\mu U^{-1} = -U^{-1}D_\mu\,. \tag{11.4}$$

Define new fermion operators

$$\Psi(x,P) = \psi(x)U(x,P)\,. \tag{11.5}$$

Then

$$\begin{aligned}\partial_\mu\Psi(x,P) &= (\partial_\mu\psi)U(x,P) + \psi\partial_\mu U(x,P) \\ &= [\partial_\mu\psi + \psi D_\mu]U(x,P)\end{aligned}$$

or

$$\partial_\mu\Psi(x,P) = \psi|_\mu(x)U(x,P)\,. \tag{11.6}$$

In this way a correspondence is set up between the ordinary partial derivative of the path-dependent field and the covariant derivative of the path-independent field.

The corresponding construction for the field strengths is

$$\mathcal{B}_{\mu\nu}(x,P) = U^{-1}(x,P)B_{\mu\nu}(x)U(x,P)\,, \tag{11.7}$$

$$\partial_\lambda\mathcal{B}_{\mu\nu} = U^{-1}B_{\mu\nu|\lambda}U\,, \tag{11.8}$$

where

$$B_{\mu\nu|\lambda} = \partial_\lambda B_{\mu\nu} + (B_{\mu\nu},D_\lambda)\,. \tag{11.8a}$$

The equations of motion then become

$$\partial_\lambda\mathcal{B}_{\mu\lambda} = \epsilon\bar{\Psi}\gamma_\mu\Psi\,, \tag{11.9}$$

$$\partial_\lambda\mathcal{B}_{\mu\lambda} = 0\,, \tag{11.10}$$

$$\gamma^\mu\partial_\mu\Psi = 0\,, \tag{11.11}$$

where the covariant derivatives of the path-independent variables have been replaced by ordinary partial derivatives of path-dependent quantities.

12. GAUGE TRANSFORMATIONS

The nonlocal representation just given may be shown to be gauge invariant as follows. By (11.1c)

$$\eta(\kappa) = D_\mu(s_\kappa)\Delta x^\mu\,.$$

Then under a gauge transformation one has

$$\eta(\kappa)' = V_\kappa^{-1}\eta(\kappa)V_\kappa - V_\kappa^{-1}(V_\kappa - V_{\kappa-1})\,,$$

where

$$V_\kappa = V(s_\kappa)\,.$$

[7] S. Mandelstam, Ann. Phys. (N. Y.) 19, 1 (1962).

Then

$$e^{\eta(\kappa)'} = \exp\,[V_\kappa^{-1}\eta(\kappa)V_\kappa]\ \exp\,[-V_\kappa^{-1}(V_\kappa - V_{\kappa-1})]$$
$$= \exp\,[V_\kappa^{-1}\eta(\kappa)V_\kappa]\,[1 - V_\kappa^{-1}(V_\kappa - V_{\kappa-1})]$$
$$= (V_\kappa^{-1}e^{\eta(\kappa)}V_\kappa)(V_\kappa^{-1}V_{\kappa-1})$$

to terms of lowest order, or

$$e^{\eta(\kappa)'} = V_\kappa^{-1}e^{\eta(\kappa)}V_{\kappa-1}\,.$$

Therefore

$$e^{\eta(N)'\cdots\eta(1)'} = V_N^{-1}[e^{\eta(N)}\cdots e^{\eta(1)}]V_0$$

and

$$U' = V_N^{-1}UV_0\,, \qquad (12.1\text{a})$$
$$(U^{-1})' = V_0^{-1}U^{-1}V_N\,. \qquad (12.1\text{b})$$

Similarly

$$\Psi(x,P) = \psi(x)U(x,P)\,,$$
$$\Psi(x,P)' = [\psi(x)V(x)][V^{-1}(x)UV(0)]$$
$$= \Psi(x,P)V(0)\,, \qquad (12.2)$$

and

$$B_{\mu\nu}(x,P) = U^{-1}(x,P)B_{\mu\nu}(x)U(x,P)\,,$$
$$B_{\mu\nu}(x,P)' = [V^{-1}(0)U^{-1}V(x)]B'_{\mu\nu}(x)[V^{-1}(x)UV(0)]$$
$$= V^{-1}(0)[U^{-1}B_{\mu\nu}(x)U]V(0)$$
$$= V^{-1}(0)B_{\mu\nu}(x,P)V(0)\,. \qquad (12.3)$$

It is possible to normalize the gauge transformation by choosing

$$V(0) = 1\,. \qquad (12.4)$$

Then

$$\Psi(x,P)' = \Psi(x,P)\,, \qquad (12.5)$$
$$B_{\mu\nu}(x,P)' = B_{\mu\nu}(x,P)\,. \qquad (12.6)$$

13. PATH DEPENDENCE

The present description is gauge-independent but not path-independent. In fact

$$U(x,P') = U(x,P)\exp\left[\int R_{[\mu\nu]}\quad dS^{\mu\nu}\right] \qquad (13.1)$$

when the integration is carried out over any surface bounded by the space-like paths P and P'.

The curve P' may be obtained from P by repeated small deformations. Each such deformation will enclose an element of surface for which (13.1) may be proved as follows. Let:

$$U = U_a(1)U_b(1)U_a^{-1}(2)U_b^{-1}(2) \qquad (13.2)$$

where

$$U_a(1) = 1 + a(1)\,,\quad \text{etc.}\,,$$

and where $a(1)$, $b(1)$, $a(2)$, $b(2)$ are constructed for the sides of an infinitesimal parallelogram as follows:

$$a(1) = D_x(1)a\,,\qquad b(1) = D_y(1)b\,,$$
$$a(2) = D_x(2)a\,,\qquad b(2) = D_y(2)b\,.$$

Here a and b are the dimensions of the parallelogram in the x and y directions. Then

$$U = 1 + [\partial_x D_y - \partial_y D_x + (D_x,D_y)](ab) \qquad (13.3)$$

and Eq. (13.1) follows.

According to (11.5), a change in path induces the transformation

$$\Psi(x,P') = \Psi(x,P)\exp\left[\int R_{[\mu\nu]}\quad dS^{\mu\nu}\right] \qquad (13.4)$$

and a corresponding equation for $B_{\mu\nu}$.

For consistency we require

$$\exp\left[\int R_{[\mu\nu]}\quad dS^{\mu\nu}\right] = I \qquad (13.5)$$

where I is the unit matrix if the integration is over a closed surface. If the gauge group is SU1, then

$$\int R_{[\mu\nu]}\quad dS^{\mu\nu} = 2n\pi i\,, \qquad (13.6)$$

or

$$\frac{e}{\hbar c}\int f_{[\mu\nu]}\quad dS = 2n\pi\,.$$

This integral represents magnetic flux over a closed surface and therefore measures the enclosed magnetic charge, if any. If the magnetic charge is Q_m, then[2]

$$eQ_m/\hbar c = 2\pi n\,. \qquad (13.7)$$

The minimum possible value of Q_m therefore satisfies the Dirac condition

$$eQ_m/\hbar c = 2\pi\,. \qquad (13.7\text{a})$$

In the case of a general gauge group, we may expand the magnetic flux matrix as follows:

$$\int R_{[\mu\nu]}\quad dS^{\mu\nu} = i\sum \lambda_s e_s \qquad (13.8)$$

where the e_s are the idempotents:

$$e_s e_t = e_s\delta_{st} \qquad (13.9)$$

and

$$\exp\int R_{[\mu\nu]}\quad dS^{\mu\nu} = \sum\,[\exp i\lambda_s]e_s \qquad (13.10)$$

In order to satisfy (13.5) one must choose

$$\lambda_s = 2n_s\pi\,, \qquad (13.11)$$

and therefore

$$\frac{e}{\hbar c}\int B_{[\mu\nu]} \quad dS^{\mu\nu} = 2\pi \sum n_s e_s \,.$$

Again the integral represents magnetic charge, which is now the matrix

$$Q_m = (\hbar c/e)2\pi \sum n_s e_s \,.$$

The eigenvalues of Q_m are

$$Q'_m = (\hbar c/e)2\pi \sum n_s e'_s \qquad (13.12)$$

where $e'_s = 1,0$. Equation (13.12) is the generalization of the Dirac condition to the case of an arbitrary gauge group.[8]

14. MAGNETIC POLES

We have seen that the Yang–Mills field whose neutral component is the electromagnetic field does not admit magnetic poles. However there is also a theory in which the neutral component is dual to the usual electromagnetic field. In the one case the charged bosons carry electric charge, magnetic moment, and no magnetic charge. In the dual theory they carry magnetic charge, electric moment, and no electric charge. In the latter case the fermion sources are also magnetic monopoles, and the magnetic charge is always given by (13.12). If both classes of particles exist the observed electric (magnetic) fields are due to electric (magnetic) monopoles at rest and magnetic (electric) monopoles in motion.

[8] S. Mandelstam, Ann. Phys. (N. Y.) 19, 1 (1962).

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The Vacuum Trajectory in Conventional Field Theory*

M. GELL-MANN
California Institute of Technology, Pasadena, California
M. L. GOLDBERGER
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
AND
F. E. LOW
Department of Physics and Laboratory of Nuclear Science, Massachusetts Institute of Technology, Cambridge, Mass.

1. INTRODUCTION

It has recently been shown that the conventional field theory of vector bosons (γ) interacting through a conserved current with spin one-half fermions (N) (i.e. quantum electrodynamics with massive photons) possesses several remarkable and hitherto unexpected properties. First, it appears to have finite nonperturbative solutions, as shown by Johnson, Baker, and Willey.[1] Second, as shown by Marx, Zachariasen, and the present authors,[2] the spin $\frac{1}{2}$ particle, which in second-order perturbation theory appears as a fixed singularity in the angular momentum, is, as a result of radiative corrections, found to lie on a Regge trajectory

$$l = J - \tfrac{1}{2} = \alpha(W)\,,$$

where W is the total energy and $\alpha(m) = 0$, with m the fermion mass. The function $\alpha(W)$ has a power series expansion in the coupling constant, called γ, such that $\alpha(W) \sim \gamma^2$ as $\gamma \to 0$.

In this paper we investigate the generation of a Pomeranchuk-like (or P) trajectory in the same theory. This differs from the previously discussed fermion trajectory problem in several ways. The P trajectory in no way corresponds to an elementary particle of field theory but is more analogous to the well understood trajectories of potential theory (or ladder approximations in field theory) with the difference that it approaches $J = 1$ as $\gamma^2 \to 0$ rather than $J = -1$ as in potential theory. This is due to the spin of the particles involved. Another difference from the fermion trajectory is, as we shall see, that $J - 1 \equiv \Delta(W^2) \sim \gamma^4$ as $\gamma \to 0$.

The processes whose asymptotic behavior at large z is determined by the P trajectory are $\gamma + \gamma \to \gamma + \gamma$, $N + \overline{N} \to N + \overline{N}$ and $\gamma + \gamma \to N + \overline{N}$. The latter amplitude can be calculated in order γ^2, the two former ones in γ^4. The generation of a Regge

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