where γ_1 and γ_2 are orthogonal matrices. But we have already proved by the argument after (D3) that orthogonal transformations

$$\epsilon_1 \rightarrow \tilde{\gamma}_1 \epsilon_1 \gamma_1 ,$$

$$\epsilon_1' \rightarrow \tilde{\gamma}_2 \epsilon_1' \gamma_2 ,$$

do not change volume elements $\Pi\ de$. Thus (E9) is true in general.

(iv) Now in the neighborhood of the point in question, by (59)

$$\Delta_n = (\det Q)(\det \epsilon_1)(-1)^{n-1} + \text{higher orders} .$$

Using (C13) one obtains,

$$\Delta_n = (-1)^n \Delta_4 \det \epsilon_1 + \text{higher orders} .$$

Similarly

$$\Delta_n' = (-1)^n \Delta_4' \det \epsilon_1' + \text{higher orders} .$$

Taking the ratio of these two equations and using (E8) and the fact that at *all* points, by definition,

$$\Delta_n = U^2 \Delta_n' ,$$

one obtains at the point in question

$$\det L = (\Delta_4/\Delta_4')^{\frac{1}{2}}\, U^{-1} . \qquad \text{(E11)}$$

(E9) now yields the desired equation (E2).

Non-Abelian Vector Gauge Fields and the Electromagnetic Field

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INTRODUCTION

Hypothetical vector gauge fields have been introduced in order to give a deeper dynamical foundation for such internal properties as isotopic spin.[1] An essential aspect of isotopic spin is electrical charge, and there is no doubt about the dynamical relation of this property to the electromagnetic field. Do these different types of vector fields simply coexist, or can they be combined to form a more unified theory of vector gauge fields? An integrated formulation can indeed be given, and it is not a trivial one since there are definite dynamical implications with regard to electromagnetic properties and the structure of the non-Abelian transformation group. The unification can encompass all fields that partake in both strong and electromagnetic interactions.[2] This success poses a physical problem, however. As one member of a set of gauge fields, the electromagnetic field is not physically distinguished and fails to perform its physical role of destroying the conservation of isotopic spin. Perhaps it is in this apparent dilemma that we find the clue to the existence in nature of other sets of fields which possess electromagnetic interactions, but no strong interactions. Is it the presence of charged leptonic fields that denies the higher symmetry transformations, relating the electromagnetic field with the non-Abelian fields, and gives to the electromagnetic field its characteristic physical influence?

The inclusion of electromagnetic lepton interactions produces a new difficulty, one of consistency. The gauge invariance of all terms in the Lagrange function save one contradicts the principle of stationary action. Another term that violates gauge invariance must be included. The simplest choice is a mass term in which the mass constant is presumably small, on the scale of strongly interacting particle masses, if a domain of approximate gauge invariance is to exist. And this modification raises again the physical mass problem of gauge fields: Are unit spin particles of small mass implied by the theory?

UNIFIED THEORY

The Lagrange function of a non-Abelian vector gauge field coupled with a spin $\frac{1}{2}$ field is[3]

$$\mathfrak{L} = -\tfrac{1}{2} G^{\mu\nu}[\partial_\mu \phi_\nu - \partial_\nu \phi_\mu + (\phi_\mu i t' \phi_\nu)] + \tfrac{1}{4} G^{\mu\nu} G_{\mu\nu} + \tfrac{1}{2}\, i\psi\alpha^\mu(\partial_\mu - i\, \text{``}T'\phi_\mu\text{''})\, \psi + \tfrac{1}{2}\, i m \psi\beta\psi ,$$

where the matrices t' and T' include coupling con-

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[1] C. N. Yang and R. Mills, Phys. Rev. **96**, 191 (1954).

[2] The problem of compatability has been given a more restricted discussion by R. Arnowitt and S. Deser, to be published.

[3] The notation follows J. Schwinger, Phys. Rev. **125**, 1043 (1962).

stants. We now interpret the internal space to be $(n + 1)$ dimensional and use the notation

$$G_0^{\mu\nu} = F^{\mu\nu}\,, \quad \phi_0^{\mu} = A^{\mu}\,,$$

together with

$$t_0' = eq\,, \quad T_0' = eQ\,,$$
$$t_a' = ft_a\,, \quad T_a' = fT_a\,, \quad a = 1,\cdots n\,.$$

The total antisymmetry of t_{abc}', a, b, $c = 0, 1\cdots n$, should not be overlooked, for

$$(t_a')_{0b} = (t_b')_{a0} = -t_{a0b}' = -(eq)_{ab}\,.$$

The result obtained by writing the Lagrange function in terms of the n-dimensional internal space is given by

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{2}F^{\mu\nu}[\partial_\mu A_\nu - \partial_\nu A_\mu + (\phi_\mu ieq\phi_\nu)] + \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}\\ &-\tfrac{1}{2}G^{\mu\nu}[(\partial_\mu - ieqA_\mu)\phi_\nu - (\partial_\nu - ieqA_\nu)\phi_\mu\\ &\qquad + (\phi_\mu ift\phi_\nu)]\\ &+\tfrac{1}{4}G^{\mu\nu}G_{\mu\nu} + \tfrac{1}{2}i\psi\alpha^\mu(\partial_\mu - ieQA_\mu - if\,``T\,\phi_\mu")\psi\\ &\qquad + \tfrac{1}{2}im\psi\beta\psi\,,\end{aligned}$$

which exhibits the required electromagnetic gauge structure, and also implies an intrinsic magnetic moment for the vector field.[4]

The infinitesimal gauge variations of the unified theory,

$$\begin{aligned}\delta\phi_\mu &= (\partial_\mu - i\,``t'\,\phi_\mu")\,\delta\lambda\\ \delta G_{\mu\nu} &= i\,``t'\,\delta\lambda"\,G_{\mu\nu}\\ \delta\psi &= i\,``T'\,\delta\lambda"\,\psi\,,\end{aligned}$$

are written out as

$$\begin{aligned}\delta A_\mu &= \partial_\mu\,\delta\lambda_0 + (\phi_\mu ieq\,\delta\lambda)\,,\\ \delta\phi_\mu &= ieq\,\delta\lambda_0\,\phi_\mu + (\partial_\mu - ieq\,A_\mu - if\,``t\phi_\mu")\,\delta\lambda\,,\end{aligned}$$

and

$$\begin{aligned}\delta F_{\mu\nu} &= (G_{\mu\nu}ieq\,\delta\lambda)\,,\\ \delta G_{\mu\nu} &= ieq\,\delta\lambda_0\,G_{\mu\nu} - ieq\,\delta\lambda F_{\mu\nu} + if\,``t\,\delta\lambda"\,G_{\mu\nu}\,,\end{aligned}$$

together with

$$\delta\psi = (ieQ\,\delta\lambda_0 + if\,``T\,\delta\lambda")\,\psi\,.$$

The single functional subgroup of electromagnetic gauge transformations is evident.

The transformations generated by $\delta\lambda_a$, $a = 1\cdots n$ do not form a subgroup, unless $e = 0$. Yet, if the neglect of electromagnetic effects is a reasonable approximation to the complete theory, an n-parameter transformation group should be implicit in the latter. We interpret this somewhat vague physical requirement to have the following meaning. The group commutation properties of the $(n + 1)$-dimensional matrices t_a',

$$[t_a', t_b'] = \sum_{c=0}^{n} t_{abc}' t_c'\,,$$

and of the n-dimensional matrices t_a,

$$[t_a, t_b] = \sum_{c=1}^{n} t_{abc} t_c\,,$$

are both valid, with $e \neq 0$. More symmetrically expressed, we require the compatability of

$$\sum_{g=0}^{n}(t_{dag}'t_{gbc}' + t_{dbg}'t_{gca}' + t_{dcg}'t_{gab}') = 0 \qquad a, b, c, d = 0,1,\cdots n$$

with

$$\sum_{g=1}^{n}(t_{dag}t_{gbc} + t_{dbg}t_{gca} + t_{dcg}t_{gab}) = 0 \qquad a, b, c, d = 1,\cdots n\,.$$

If a, b, c, $d \neq 0$ in the $(n + 1)$-dimensional set of structure constant equations, the term in the summation with $g = 0$ must vanish separately, or

$$q_{da}q_{bc} + q_{db}q_{ca} + q_{dc}q_{ab} = 0\,.$$

On setting one of these four variables equal to 0, we learn that

$$[t_a, q] = \sum_{b=1}^{n} q_{ab} t_b\,,$$

and this exhausts the relations implied by the unified theory.

The restrictions imposed on the matrix q can best be appreciated by multiplying the quadratic equation with q_{gd} to form

$$(q^2)_{ga}q_{bc} + (q^2)_{gb}q_{ca} + (q^2)_{gc}q_{ab} = 0\,,$$

and then summing over $g = c$, which yields

$$q^3 = \lambda^2 q\,, \quad \lambda^2 = \tfrac{1}{2}\,\mathrm{tr}q^2\,.$$

With the permissible normalization $\lambda^2 = 1$, the eigenvalues of q, the electric charge matrix of the vector field, can only be 0, ± 1. Furthermore,

$$1 = \tfrac{1}{2}[m(+1) + m(-1)]$$

where $m(\pm 1)$ are the multiplicities of the respective q eigenvalues. On invoking symmetry between positive and negative charge we learn that

$$m(+1) = m(-1) = 1\,.$$

[4] It is interesting to encounter here just the special electromagnetic interaction that I first considered in collaboration with H. C. Corben [Phys. Rev. **58**, 953 (1940)], while we were research fellows at Berkeley. That work was inspired by the cosmic ray investigations of J. R. Oppenheimer, R. Serber, and H. Snyder [Phys. Rev. **57**, 75 (1940)].

Thus the vector gauge field contains one pair of charged components, and all other components are electrically neutral. With an appropriate labeling of field components, the charge matrix q is

$$q = \left(\begin{array}{cc|c} 0 & -i & 0 \\ i & 0 & \\ \hline 0 & & 0 \end{array}\right)$$

and it can be verified that the quadratic conditions on q_{ab} are satisfied.

The structure of the charge matrix asserts that

$$[q,t_1] = it_2\ , \quad [q,t_2] = -it_1\ , \quad [q,t_a] = 0\ , \quad a = 3,\cdots n\ .$$

According to these properties,

$$[t_a,t_b] - \sum_{c=3}^{n} t_{abc}t_c = t_{ab1}t_1 + t_{ab2}t_2 = 0 \qquad a, b = 3,\cdots n\ ,$$

since the left-hand member commutes with q. The implied vanishing of the structure constants t_{ab1}, t_{ab2}, a, $b = 3,\cdots n$, also asserts that

$$[t_1,t_a] = t_{1a2}t_2\ , \quad [t_2,t_a] = t_{2a1}t_1\ , \quad a = 3,\cdots n.$$

By performing a linear transformation of the t_a it can be arranged that

$$t_{12a} = i\delta_{a3}\ .$$

Then

$$[t_3,t_1] = it_2\ , \quad [t_2,t_3] = it_1\ , \quad [t_1,t_2] = it_3\ ,$$

while

$$[t_{1,2,3},t_a] = 0 \qquad a = 4,\cdots n$$

and

$$[t_a,t_b] = \sum_{c=4}^{n} t_{abc}t_c \qquad a, b = 4,\cdots n\ .$$

Thus the group structure is completely factored into the three-dimensional isotopic spin group and a group with $(n - 3)$ parameters. According to the identification

$$q = t_3\ ,$$

only zero and unit isotopic spin representations are contained in the vector field.

It may appear that these structural results refer just to a particular group representation rather than the group, for the general commutation relations of the unified theory

$$[T'_a,T'_b] = \sum_{c=0}^{n} t'_{abc}T'_c$$

imply that

$$[T_a,T_b] = \sum_{c=1}^{n} t_{abc}T_c - (e^2/f^2)q_{ab}Q\ ,$$

together with

$$[T_a,Q] = \sum_{b=1}^{n} q_{ab}T_b\ .$$

In more detail, these are

$$-i[T_1,T_2] = T_3 + (e^2/f^2)Q\ ,$$

$$-i[T_3,T_1] = T_2, \qquad -i[T_2,T_3] = T_1\ ,$$

$$[T_1,Q] = -iT_2\ , \qquad [T_2,Q] = iT_1\ ,$$

$$[Q,T_3] = 0\ ,$$

and

$$[T_{1,2,3},T_a] = 0\ , \quad [T_a,T_b] = \sum_{c=4}^{n} t_{abc}T_c \qquad a, b = 4,\cdots n\ .$$

Thus, the general commutation relations differ by the presence of the term $(e^2/f^2)Q$, added to T_3. But, it suffices to define

$$^*T_3 = \gamma^{-2}[T_3 + (e^2/f^2)Q]\ , \quad ^*T_{1,2} = \gamma^{-1}T_{1,2}\ ,$$

where

$$\gamma = [1 + (e^2/f^2)]^{\frac{1}{2}}\ ,$$

in order to regain the commutator structure of the three-dimensional isotopic spin group for $^*T_{1,2,3}$. It is essential that Q and T_3 have the same commutation properties with $T_{1,2}$. We place

$$Q - T_3 = \gamma^2\, \tfrac{1}{2}\, Y\ , \quad [Y,{}^*T_{1,2,3}] = 0\ ,$$

which yields

$$Q = {}^*T_3 + \tfrac{1}{2}\, Y$$

and identifies Y with the hypercharge. Then

$$T_{1,2} = \gamma^*T_{1,2}\ , \quad T_3 = {}^*T_3 - (\gamma^2 - 1)\, \tfrac{1}{2}\, Y\ .$$

The group commutation relations govern the conserved integral quantities of the theory. In the unified formulation the latter are given by

$$\mathbf{T}'_a = \int (d\mathbf{x})[G^{0k}it'_a\phi_k + \tfrac{1}{2}\,\psi T'_a\psi]\ ,$$

which includes

$$\mathbf{Q} = \int (d\mathbf{x})[G^{0k}iq\phi_k + \tfrac{1}{2}\,\psi Q\psi]$$

and

$$\mathbf{T}_a = \int (d\mathbf{x})[G^{0k}it_a\phi_k + (e/f)(iqG^{0k})_a A_k - (e/f)F^{0k}(iq\phi_k)_a + \tfrac{1}{2}\,\psi T_a\psi]\ .$$

The operators $^*\mathbf{T}_{1,2,3}$ that obey isotopic spin commutation relations then appear as

$$^*\mathbf{T}_3 = \int (d\mathbf{x})[G^{0k}it_3\phi_k + \tfrac{1}{2}\,\psi\,{}^*T_3\psi]$$

and

$$^*\mathbf{T}_{1,2} = \int (d\mathbf{x})[\gamma^{-1}G^{0k}it_{1,2}\phi_k \pm (e/f)\gamma^{-1}(G^{0k}_{2,1}A_k - F^{0k}\phi_{k2,1}) + \tfrac{1}{2}\,\psi\,{}^*T_{1,2}\psi]\,.$$

In the last formula, the lower sign applies to $^*\mathbf{T}_2$.

One can hardly fail now to recognize the variables that make explicit the three-dimensional isotopic spin invariance of the theory. The fields

$$\begin{aligned} ^*\phi_{\mu 3} &= \gamma^{-1}[\phi_{\mu 3} + (e/f)A_\mu]\,, \\ ^*G_3^{\mu\nu} &= \gamma^{-1}[G_3^{\mu\nu} + (e/f)F^{\mu\nu}]\,, \\ ^*A_\mu &= \gamma^{-1}[A_\mu - (e/f)\phi_{\mu 3}]\,, \\ ^*F^{\mu\nu} &= \gamma^{-1}[F^{\mu\nu} - (e/f)G_3^{\mu\nu}]\,, \end{aligned}$$

together with

$$^*\phi_{\mu 1,2} = \phi_{\mu 1,2}\,, \quad ^*G^{\mu\nu}_{1,2} = G^{\mu\nu}_{1,2}\,,$$

are such that

$$^*\mathbf{T}_a = \int (d\mathbf{x})[^*G^{0k}it_a\,{}^*\phi_k + \tfrac{1}{2}\,\psi\,{}^*T_a\psi]\,,$$

$$\mathbf{Q} = \int (d\mathbf{x})[^*G^{0k}iq\,{}^*\phi_k + \tfrac{1}{2}\,\psi Q\psi]\,.$$

For simplicity of presentation, the transformed Lagrange function is written without the fields ϕ_a^μ, $a = 4\cdots n$, which can easily be reinstated, and omitting the star designation on the fields. The result is

$$\begin{aligned} \mathfrak{L} = &-\tfrac{1}{2}\,F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \tfrac{1}{4}\,F^{\mu\nu}F_{\mu\nu} \\ &- \tfrac{1}{2}\,G^{\mu\nu}[\partial_\mu\phi_\nu - \partial_\nu\phi_\mu + (\phi_\mu if\gamma t\phi_\nu)] + \tfrac{1}{4}\,G^{\mu\nu}G_{\mu\nu} \\ &+ \tfrac{1}{2}\,i\psi\alpha^\mu(\partial_\mu - ie\gamma\,\tfrac{1}{2}\,YA_\mu - if\gamma\,\text{“}{}^*T\phi_\mu\text{”})\psi \\ &+ \tfrac{1}{2}\,im\psi\beta\psi\,. \end{aligned}$$

All electromagnetic effects for the gauge field have disappeared, and the field $^*A_\mu$, $^*F_{\mu\nu}$ is identified as a hypercharge field!

It is tempting to assert that $^*A_\mu$, $^*F_{\mu\nu}$ is *the* hypercharge gauge field. The electromagnetic field would then be constructed as a linear combination of the hypercharge gauge field and a component of the isotopic spin gauge field. The alternative possibility is to include in the set ϕ_a^μ, $a = 4,\cdots n$, an Abelian gauge field that is coupled to hypercharge, with coupling constant g. Then two linear combinations of the hypercharge fields exist, one of which has the coupling constant $[g^2 + (\frac{1}{2}\,e\gamma)^2]^{\frac{1}{2}}$, while the other is completely uncoupled.

LEPTON INTERACTIONS

The existence of three-dimensional isotopic spin invariance is conveyed by the unusual transformations in which A_μ, $F_{\mu\nu}$ is combined with ϕ_μ, $G_{\mu\nu}$. A purely electromagnetic interaction with another system destroys this symmetry and gives the electromagnetic field its specific dynamical significance. Let the Lagrange function contain additional terms that are invariant under electromagnetic gauge transformations

$$\mathfrak{L}_e + j_e^\mu A_\mu\,,$$

such as those representing charged lepton fields. The complete Lagrange function responds to an $(n+1)$ parameter infinitesimal gauge transformation as

$$\delta\mathfrak{L} = j_e^\mu(\phi_\mu ieq\,\delta\lambda)\,.$$

Alternatively, we can write

$$A_\mu = \gamma^{-1}[^*A_\mu + (e/f)^*\phi_{\mu 3}]\,,$$

which shows how a preferential direction is introduced in the isotopic space. The response to the three-dimensional isotopic spin gauge transformation

$$\delta\,{}^*\phi_\mu = (\partial_\mu - if\gamma\,\text{“}t\,{}^*\phi_\mu\text{”})\,\delta\,{}^*\lambda$$

is again

$$\delta\mathfrak{L} = j_e^\mu(\phi_\mu ieq\,\delta\lambda)_3\,,$$

since $q = t_3$ and $^*\phi_{\mu 1,2} = \phi_{\mu 1,2}$.

As we have remarked in the Introduction, this result contradicts the principle of stationary action, which demands that $\delta\mathfrak{L}$ be zero, to within a divergence term. Another nongauge invariant expression must be added, such as

$$-\tfrac{1}{2}\,\phi^\mu m_0^2\phi_\mu = -\tfrac{1}{2}\,m_0^2\sum_{a=1}^{3}\phi_a^\mu\phi_{\mu a}\,.$$

Then

$$\delta\mathfrak{L} = j_e^\mu(\phi_\mu ieq\,\delta\lambda) - \phi^\mu m_0^2(\partial_\mu - ieq\,A_\mu)\,\delta\lambda$$

and the stationary action principle asserts that

$$m_0^2(\partial_\mu - ieq\,A_\mu)\phi^\mu = ieq\,\phi_\mu j_e^\mu\,.$$

The field equation that is modified by the m_0 term is

$$(\partial_\nu - ieqA_\nu - if\,\text{“}t\phi_\nu\text{”})G^{\mu\nu} = -ieqF^{\mu\nu}\phi_\nu + k^\mu - m_0^2\phi^\mu\,,$$

$$k^\mu = \tfrac{1}{2}\,\psi\alpha^\mu fT\psi\,.$$

This can be written as

$$\partial_\nu G^{\mu\nu} + m_0^2\phi^\mu = j^\mu\,,$$

where

$$j^\mu = (G^{\mu\nu}ift\phi_\nu) + ieqA_\nu G^{\mu\nu} - ieqF^{\mu\nu}\phi_\nu + k^\mu\,,$$

from which we derive

$$\partial_\mu j^\mu = m_0^2 \partial_\mu \phi^\mu = ieq(j_e^\mu + m_0^2 A_\mu)\phi^\mu \, .$$

It will be noted how the presence of the m_0 term is required to obtain the physically necessary equations of nonconservation. The corresponding integral relations are

$$\partial_0 f\mathbf{T} = \int (d\mathbf{x}) ieq(j_e^\mu + m_0^2 A_\mu)\phi^\mu \, ,$$

which asserts that

$$\partial_0 \, {}^*\mathbf{T}_{1,2} = \pm (e/f)\gamma^{-1} \int (d\mathbf{x}) (j_e^\mu + m_0^2 A_\mu)\phi^\mu_{2,1}$$
$$= -i[{}^*\mathbf{T}_{1,2}, P^0] \, .$$

We do not try to discuss here whether this mechanism implies reasonable magnitudes for the T_3 dependence of mass multiplets.

We make only one comment about the physical mass problem. For an Abelian gauge field coupled to a conserved current, there are sum rules[5] that require the existence of unit spin excitations with mass m less than m_0. The sum rules are different, however, if the current is not conserved,[6] or if the field is non-Abelian. Then, no simple conclusions about the spectrum in relation to m_0 can be drawn.

[5] K. Johnson, Nucl. Phys. **25**, 435 (1961).
[6] D. G. Boulware and W. Gilbert, Phys. Rev. **126**, 1563 (1962).

Generalized Radial Integrals With Hydrogenic Functions

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The calculation of radial quantum hydrogenic integrals is an important part of wave mechanics. The direct procedure of multiplying, term by term, leads to long and involved series. These series can be summed, but the methods employed are too complicated for student application. Textbooks and reference books have studiously avoided the calculation except for small specific values of n and l. I have discovered a simple method for dealing with this summation problem. The procedure does more than bring the method within the scope of a student whose mathematical skill does not go far beyond that of elementary calculus; it leads to the reduction of generalized radial quantum integrals.

The one-electron wave equation,

$$H\psi = -(h^2/8\pi^2 m)\nabla^2\psi + U\psi = E\psi \, , \qquad (1)$$

for a Coulomb field,

$$U = -Z\epsilon^2/r \, , \qquad (2)$$

has the well-known solution[1]

$$\psi = R\Theta\Phi \, , \qquad (3)$$

a product function.

$$\Phi = (2\pi)^{-\frac{1}{2}} e^{im\varphi} \, , \qquad (4)$$

a function of the longitude parameter ϕ.

$$\Theta(l,m) = \frac{(|m| + m)}{2} \left[\frac{2l+1}{2} \, \frac{(l - |m|)!}{(l + |m|)!} \right]^{\frac{1}{2}} \times \sin^{|m|}\theta \, \frac{d^{|m|}}{(d\cos\theta)^{|m|}} P_l(\cos\theta) \, , \qquad (5)$$

[1] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (The Macmillan Company, New York, 1935), pp. 52 and 117.

where θ is the angle measured from the pole along a meridian. Thus

$$\Theta(l,m) = (-1)^m \Theta(l,-m) \, . \qquad (6)$$

This choice of signs (or phase) is important for many purposes. $P_l(\cos\theta)$ is the Legendre polynomial. The radial equation is a function of the radius r,

$$R(n,l) = -\left[\frac{Z(n+l)!}{n^2 \mathrm{a}(n-l-1)!} \right]^{\frac{1}{2}} \frac{e^{-Zr/n\mathrm{a}}}{(2l+1)!} \left(\frac{2Zr}{n\mathrm{a}} \right)^{l+1} \times {}_1F_1(-n+l+1, 2l+2, 2Zr/n\mathrm{a}) \, . \qquad (7)$$

${}_1F_1$ is the confluent hypergeometric function,

$${}_1F_1(-\alpha,\gamma,x) = 1 - \frac{\alpha}{1!\gamma} x + \frac{\alpha(\alpha-1)}{2!\gamma(\gamma+1)} x^2 - \cdots$$
$$= \sum_{j=0}^{\alpha} (-1)^j \frac{\alpha!}{(\alpha-j)!} \, \frac{(\gamma-1)!}{(\gamma-1+j)!} x^j \, . \qquad (8)$$

These functions are normalized to give

$$\int_0^{2\pi} \Phi(m')\Phi(m) \, d\varphi = \delta(m',m) \, , \qquad (9)$$

$$\int_0^{\pi} \Theta(l',m)\Theta(l,m) \sin\theta \, d\theta = \delta(l',l) \, , \qquad (10)$$

$$\int_0^{\infty} R(n',l)R(n,l) \, dr = \delta(n',n) \, , \qquad (11)$$

where δ is the Kronecker δ, equal to unity if $m' = m$ and equal to zero if $m' \neq m$, and likewise for l and n.

In these equations, ϵ and m are the electronic charge and mass, Z is the nuclear charge, and h is Planck's constant. a is the radius of the first Bohr