with Eq. (3.7). However, $F(q,0)$ has a logarithmic branch point at $q = 0$.

Equation (3.7) is the analog of the relation $gk/m \ll 1$ which in W-field theory[2] defines the "low-energy regime" (g = meson lepton coupling constant, $m = W$ mass), and $F(q,0)$ is in fact the analog of the zero-energy Bethe–Salpeter amplitude discussed in reference 3. While we know little about the high-energy regime in field theory ($gk \gg m$), it is interesting to note the high-energy exponential damping exhibited in the potential problem by Eq. (3.8).

Unlike the W-field theory, the amplitude for a power potential has only a leading singular series for $k = 0$, no subsequent summations are called for at fixed energy. This is because we only have one constant g in the potential case, while there are two (g and m) in field theory. We can also study potentials with more constants. Take for example singular short-range potentials like

$$V_\mu(r) = -g(e^{-\mu r}/r^m) . \qquad (3.9)$$

By power counting one readily verifies that the zero-energy scattering amplitude for $V_0(r)$ is a good approximation to the one for $V_\mu(r)$ as long as g and μ satisfy

$$\mu g^{1/(m-2)} \ll 1 . \qquad (3.10)$$

In conclusion we wish to emphasize that for singular power potentials the peratization program is completed once the case $k = 0$ has been understood. This is so because the knowledge of the zero-energy wavefunction makes it possible to reduce the integral equation (1.3) for $k \neq 0$ to one which now is "regular" in the sense that the Neumann series does exist.

We show this for the case $m = 4$ and limit ourselves to the S-wave case. Once the zero-energy solutions are known one can obtain a regular Volterra integral equation for the nonzero energy case.[16] The zero-energy solutions of (1.7) in the present case are $xe^{-g^{\frac{1}{2}}/x}$ and $xe^{+g^{\frac{1}{2}}/x}$. If we now write

$$\psi(k,0,r) = re^{-g^{\frac{1}{2}}/r} f(k,r) ,$$

we get for f the integral equation

$$f(k,x) = 1 - \frac{k^2}{2g^{\frac{1}{2}}} \int_0^x y^2 dy f(k,y) + \frac{k^2}{2g^{\frac{1}{2}}} \int_0^x y^2 dy f(y) e^{2g^{\frac{1}{2}}/x - 2g^{\frac{1}{2}}/y} . \qquad (3.11)$$

This is an iterable Volterra equation and leads to a function analytic in k^2.

[16] E. Predazzi and T. Regge, Nuovo Cimento **24**, 518 (1962).

Physical Regions in Invariant Variables for n Particles and the Phase-Space Volume Element

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In considering the reactions

$$A \rightarrow 1 + 2 + \cdots + n \qquad (1)$$

or

$$A + \bar{1} \rightarrow 2 + 3 + \cdots + n , \qquad (2)$$

it is oftentimes natural to express probabilities, (i.e., square of matrix elements) in terms of the scalar products of the 4-momenta of the particles involved. (For particles with spin, a similar situation obtains after averaging over the spins of the particles involved.) Denoting by $\{i,j\}$ the scalar product of the 4-momenta of particles i and j, one may ask whether it is possible and convenient to pursue subsequent calculations (such as for the total probability) entirely in terms of these quantities $\{i,j\}$. These considerations[1] led to the investigations described in the present paper.

We discuss the following three questions.

(a) What are the kinematically allowed values of the variables $\{i,j\}$ for reactions (1) or (2)?

[1] For somewhat related discussions, see V. E. Asribekov, Zh. Eksperim. i Teor. Fiz. **42**, 565 (1962) [English Transl. Soviet Phys.-JETP **15**, 394 (1962)]; J. Tarski, J. Math. Phys. **1**, 149 (1960); B. Jacobsohn, Bull. Am. Phys. Soc. **7**, 503 (1962); and D. Hall and A. S. Wightman, Mat. Fys. Medd. Dan. Vid. Selsk **31**, No. 5 (1957).

(b) For a given assignment of values for $\{i,j\}$ which is kinematically allowed, how many Lorentz-transformation-inequivalent kinematic configurations describing reactions (1) or (2) are there that realize this set $\{i,j\}$?

(c) In terms of the variables $\{i,j\}$, what is the volume element

$$d\tau_{n,A} = (\prod d^4p_i)[\prod \delta(p_i^2 - m_i^2)]\delta^4[(\sum p_i) - p_A] \tag{3}$$

for reaction (1), and what is the similar expression for reaction (2)?

The quantities $\{i,j\}$ are clearly not all independent. In discussing questions (a) and (c) defined above, one can choose to regard some of them as independent, and the others as dependent. We follow in this paper, however, the principle of seeking for symmetrical expressions in terms of the indices i and j. It will be seen that the resulting volume element $d\tau$ assumes a remarkably simple form. The derivation we give in the present paper, starting from the symmetrical expression (3) and ending in the symmetrical expression (79), goes through a route utilizing intermediate steps in which symmetry is not maintained. This is perhaps not desirable, but we have not yet found a proof that maintains symmetry throughout. [Since the invariant volume element is a *measure* of the kinematic configuration, it is perhaps not very surprising that the resultant expression (79) is so simple. But we have no nonpedestrian understanding of (79).]

It is proved that for large n the allowed values for $\{i,j\}$ are superconical singularities, of an algebraic surface. The geometry of this surface is closely related to a mapping of its points to the kinematics of reactions (1) and (2). It is also proved that in terms of the geometry of *one universal* algebraic surface the problem for all nonvanishing masses m_i for the n particles can be formulated.

In the main part of the paper we discuss reaction (1), returning to reaction (2) in Sec. 9.

To study the geometry and the mapping mentioned above it is convenient to study not only the physical 4-dimensional space, but the general space[2] of dimension D_{ab}. Thus in Sec. 1 we discuss problem (a) in general dimensions with arbitrary mass m_A. Starting from Sec. 2 the discussions are limited mostly to dimensions D_{1b}, i.e., one time component only.

All quantities in this paper are real.

[2] We use the notation dimension D_{ab} or D_{ab} dimension to denote a vector space with a diagonal metric consisting of $+1$ a times and -1 b times. Physical space is D_{13}.

I.

We assume throughout this paper that the particles $1,2,\cdots n$ have fixed nonvanishing masses $m_1,m_2,\cdots m_n$. Consider the real symmetrical matrix

$$M_n = \left\|\begin{matrix} m_1^2\{1,2\}\cdots\{1,n\} \\ \cdots \\ \{n,1\}\cdots\cdots m_n^2 \end{matrix}\right\| . \tag{4}$$

By the space $\mathfrak{M}_n$ we mean the Cartesian space

$$\{i,j\} , \qquad \text{where} \quad 1 \leqq i < j \leqq n .$$

The matrix M_n has real eigenvalues. A point in space $\mathfrak{M}_n$ and its corresponding matrix M_n are said to belong to the region r_{ab} if M_n has a positive nonvanishing eigenvalues, b negative nonvanishing eigenvalues, and $n - a - b$ eigenvalues which are zero.

We show now that the division of the space $\mathfrak{M}_n$ into regions r_{ab} is closely related to the realizability[3] of the matrix M_n in terms of vectors. From such a relation we obtain in Theorem 5 information about how many disconnected parts each of the regions r_{ab} possesses.

Theorem 1. If a point of $\mathfrak{M}_n$ is in the region r_{ab}, it is realizable in D_{ab} dimension with n vectors of which $a + b$ are linearly independent. If it is also realizable in dimension $D_{a'b'}$, then

$$a' \geqq a , \qquad b' \geqq b .$$

Proof. The first part of this theorem is obvious by diagonalizing the matrix M_n by a real orthogonal transformation. The second part is easily proved with the aid of Lemma 1 of Appendix A.

By the second part of this theorem, since the momenta vectors of the particles $1,\cdots n$ in (1) are realized in dimension D_{13}, the corresponding matrix M_n must be in a region r_{ab} with $a \leqq 1$, $b \leqq 3$. Since

$$\text{Trace}\, M_n = \sum m^2 > 0 ,$$

at least one eigenvalue of M_n must be > 0. Thus $a = 1$. Hence *the physically realizable M_n for reaction (1) is within* r_{13}, r_{12}, r_{11}, and r_{10}. The latter regions represent, respectively, cases where the 4-momenta vectors span 3, 2, 1, or zero space-like dimensions.

Theorem 2. Consider n vectors in dimension D_{ab} of lengths $m_1,m_2,\cdots m_n$ for which $a + b$ are linearly

[3] Realization in terms of, or realization by, n vectors means the n vectors give scalar products that correspond to the point in $\mathfrak{M}_n$, their lengths being given by the diagonal elements of (4). Realization in D_{ab} dimensions means that the n vectors are in the dimension D_{ab} defined in Ref. 2. We sometimes refer to the n vectors as the realization.

independent. The matrix M_n constructed from these vectors by Eq. (4) is in the region r_{ab}.

Proof. Since $a+b$ of the vectors are linearly independent, the rank of M_n as defined by (4) is clearly $a+b$. Now let M_n be in the region $r_{a'b'}$. An orthogonal transformation leaves the rank of a matrix invariant. Thus the rank of M_n is $a'+b'$. Thus

$$a+b=a'+b' .$$

But by the second part of Theorem 1,

$$a \geqq a' , \qquad b \geqq b' .$$

Hence $a=a'$, $b=b'$ which proves this theorem.

Theorem 3. Consider a matrix M_n in the region r_{ab}. All realizations of M_n by n vectors in D_{ab} dimension are Lorentz transformable[4] into each other.

Proof. (a) By Theorem 1 there exists a realization R_0 with n vectors in D_{ab} dimension. Let the first $a+b$ of these be linearly independent. Consider the first diagonal minor M_{a+b} of dimension $(a+b) \times (a+b)$ of M_n. Any realization R_1 of M_n in D_{ab} dimension gives also a realization of M_{a+b} by its first $a+b$ vectors in the same dimension. Focussing attention only on the realization of M_{a+b} it is readily proved that the first $a+b$ vectors of R_0 and of R_1 are transformable into each other by a Lorentz transformation.

(b) As for the other $n-a-b$ vectors, they are linearly dependent on the first $a+b$, and their projection on the first $a+b$ are fixed by the elements of M_n. So the last $n-a-b$ vectors of R_0 and of R_1 are transformed into each other by the same Lorentz transformation.

Remark. (i) If one wants to generalize this theorem to realizations in higher dimensions than D_{ab}, one has to be careful about situations as illustrated in the following simple example:

$$M_2=\begin{Vmatrix} 1 & 1 \\ 1 & 1 \end{Vmatrix}$$

is realizable by

R_0: the vectors (1), (1) in dimension D_{10},

or R_1: the vectors (100), (1, 1, 1) in dimension D_{21},

with metric

$$\begin{Vmatrix} 1 & & \\ & 1 & \\ & & -1 \end{Vmatrix} .$$

[4] A Lorentz transformation in D_{ab} is a transformation described by a matrix L so that $Lg\tilde{L}=g$, where g is the metric of the space D_{ab}.

To apply a Lorentz transformation to R_1 to reduce it to R_0 with two redundant dimensions requires a Lorentz transformation with the velocity 1 which exists only as a limit.

(ii) The above mentioned complication does not arise, however, if we confine ourselves to dimensions $D_{ab'}$ only. In fact we have

Theorem 4. Consider a matrix M_n in the region r_{ab}. Any realization R_1 of it by n vectors in $D_{ab'}$ dimension $[b'>b]$ can be transformed by a Lorentz transformation in $D_{ab'}$ dimension to n vectors occupying only D_{ab} dimension with components zero in the remaining $b'-b$ dimensions.

Proof. (a) This part is entirely similar to part (a) of the proof of Theorem 3. One concludes that there exists a Lorentz transformation L in $D_{ab'}$ dimension that transforms the first $a+b$ vectors of R_1 into the first $a+b$ vectors of the R_0 of Theorem 3 with zeros added as components of these vectors in the redundant $b'-b$ dimensions.

(b) Apply L to all the n vectors of R_1. The components of the first $a+b$ resultant vectors are [according to (a)], written in $a+b$ rows,

$$\|A \quad 0\| , \tag{5}$$

where the size of A is $(a+b) \times (a+b)$ and that of 0 is $(a+b) \times (b'-b)$. Let the components of the remaining $(n-a-b)$ resultant vectors be

$$\|B, C\| . \tag{6}$$

The corresponding components of the vectors of R_0 are

$$\|A\| , \tag{5'}$$

and

$$\|B'\| . \tag{6'}$$

Evidently

$$\det \quad A \neq 0 .$$

R_0 and the L-transformed R_1 share the same M_n. Identifying the scalar products between the vectors of (5) and those of (6) with the corresponding scalar products between the vectors of (5′) and (6′) one obtains

$$B=B' .$$

Now the components in C of (6) all have negative signature (i.e., are space-like). For the vectors (6) to have the same lengths as those of (6′) one must have $C=0$. This completes the proof.

Theorem 5. For $a>1$, the region r_{ab} of $\mathfrak{M}_n$ is a connected region. For $a=1$, the region r_{1b} of $\mathfrak{M}_n$

consists of 2^{n-1} disconnected parts, each being a connected subregion.

Proof. Theorems 1 and 2 establish the correspondence of the points in the region r_{ab} of $\mathfrak{M}_n$ with a set of n vectors in D_{ab} dimension, of which $a+b$ are linearly independent. This correspondence is a continuous one. Starting from any point in the region r_{ab} of $\mathfrak{M}_n$, one could reach another point in the same region if, and only if, the corresponding set of n vectors can be continuously deformed from the one to the other, subject to the conditions (1) $a+b$ of them are linearly independent, and (2) the length of each vector remains unchanged. If $a>1$, such deformation is always possible. For $a=1$, a time-like vector could be forward time-like or backward time-like. Since the masses are >0, a forward time-like vector cannot be continuously changed into a backward time-like vector. Thus, the space of n vectors in D_{1b} dimension is divided into 2^n disconnected regions according to the sign of the time component of the vectors. But, a reversal of the signs of all components of all vectors does not change the matrix M_n. We thus have Theorem 5.

This proof of Theorem 5 also shows that for $a=1$ the region r_{1b} is divided into 2^{n-1} disconnected subregions[5] according to the signs of the scalar products $\{1,i\}$, $i=2\cdots n$. This follows from the fact that the scalar product of two forward time-like vectors is always positive (masses being >0).

In Sec. 4 the case $n=3$ is discussed in detail, illustrating the above theorems.

II.

In this section we specialize to the case of one time-like dimension, i.e., $a=1$. Also, since only reaction (1) is considered, all the n vectors are forward time-like. We thus confine ourselves to subregions discussed in Theorem 5 which are denoted by r^0_{1b}, i.e., for which[5]

$$\{i,j\} > 0\,. \tag{7}$$

We are interested in orthochronous proper Lorentz transformations only. Theorems 3 and 4 appear now as

Theorem 6. Consider a matrix M_n in the subregion r^0_{1b}. Consider any realization R_1 of M_n by n forward time-like vectors in D_{1b} dimension. These n vectors are simultaneously transformable by an orthochronous proper Lorentz transformation into either of two specific realizations R_0 and R_0'. The realizations R_0 and R_0' are related to each other by a space reflection.

[5] We use the notation r^0_{1b} to denote that subregion of r_{1b} in which $\{i,j\}>0$. r^0_{1b} is a connected subregion, by the proof of Theorem 5.

Theorem 7. Consider a matrix M_n in the subregion r^0_{1b}. Consider any realization R_1 of M_n by n forward time-like vectors in $D_{1b'}$ dimension, $b'>b$. These n vectors can be simultaneously transformed by an orthochronous proper Lorentz transformation in $D_{1b'}$ dimension to the realization R_0 described in Theorem 6 occupying only D_{1b} dimension with components 0 in the remaining $b'-b$ dimensions. It could also be transformed by an orthochronous proper Lorentz transformation in $D_{1b'}$ dimension to the realization R_0' described in Theorem 6 occupying only D_{1b} dimension with components 0 in the remaining $b'-b$ dimensions.

These two theorems can be proved in a manner completely similar to the proof of Theorems 3 and 4.

The physical Lorentz transformations are in D_{13} space. We thus have the following conclusions:

The n momenta of (1) *realize a matrix* M_n *in the region* r^0_{13}, r^0_{12}, r^0_{11}, *or* r^0_{10}.

Each matrix M_n *in* r^0_{13} *is realized by two kinematic configurations describing the n momenta of* (1) *which are space reflections of each other. All other realizations can be obtained from these two by orthochronous proper Lorentz transformations.*

Each matrix M_n *in* r^0_{12}, r^0_{11}, *or* r^0_{10} *is realized by one kinematic configuration describing the n momenta of* (1). *All other realizations can be obtained from this by orthochronous proper Lorentz transformations.*

These results completely answer questions (a) and (b) in the Introduction.

III.

In this section we establish a theorem which gives the algebraic description of the region r_{1s} in terms of polynomial invariants constructed out of the matrix M_n.

We define the invariants Δ_l by

$$\Delta_l = (-1)^{l-1} \sum \text{det of all } (l\times l) \text{ diagonal minors of } M_n\,. \tag{8}$$

Thus

$$\Delta_1 = \sum_1^n m_i^2$$
$$\Delta_n = (-1)^{n-1} \det M_n\,.$$

Theorem 8. A necessary and sufficient condition for a point of $\mathfrak{M}_n$ to be in r_{1s} is

$$\Delta_1, \Delta_2, \cdots \Delta_{s+1} \quad \text{all} \quad >0$$
$$\Delta_{s+2} = \Delta_{s+3} = \cdots = \Delta_n = 0\,. \tag{9}$$

Proof. If (9) is satisfied, the eigenvalue equation of M_n is

$$\lambda^n - \Delta_1\lambda^{n-1} - \cdots - \Delta_{s+1}\lambda^{n-s-1} = 0\,. \quad (10)$$

This equation has exactly $n - s - 1$ eigenvalues 0. To study the other eigenvalues, write $\lambda = 1/\mu$, then

$$\Delta_{s+1}\mu^{s+1} + \Delta_s\mu^s + \cdots + \Delta_1\mu - 1 = 0\,.$$

This equation has exactly one positive solution for μ because of condition (9). We have thus established the sufficiency of (9).

To prove its necessity we use the first part of Theorem 1. If M_n is in r_{1s}, M_n is realized by n vectors R in D_{1s} dimension of which $1 + s$ are linearly independent. Thus all its diagonal minors of size larger than $(1 + s) \times (1 + s)$ have det = 0. Hence $\Delta_{s+2} = \Delta_{s+3} = \cdots = \Delta_n = 0$.

Now take any diagonal minor M_l of size $l \times l$ of M_n, with $l \leqq s + 1$. It is realized in dimension D_{1s} by a subset of R. By the second part of Theorem 1 the minor M_l is in the region r_{ab} of $\mathfrak{M}_l$ where

$$a \leqq 1\,, \qquad b \leqq s\,.$$

Clearly $a \neq 0$, since Trace $M_l > 0$. Thus $a = 1$. Thus

$$(-1)^{l-1} \det M_l \geqq 0\,.$$

We need then only prove that for each $l \leqq s + 1$ some diagonal minor M_l has a determinant $\neq 0$. Now since $1 + s$ of the vectors in R are linearly independent, one can always find l of them that are linearly independent. Let the corresponding M_l be in the region r_{1b} of $\mathfrak{M}_l$, $1 + b \leqq l$. Applying Theorem 4 one concludes that linear independence of the l vectors requires $1 + b = l$. Thus det $M_l \neq 0$ and we complete the proof of Theorem 8.

The points of r_{1s} satisfy

$$\Delta_{s+2} = \Delta_{s+3} = \cdots = \Delta_n = 0\,. \quad (11)$$

But not all points on (11) are in r_{1s}, since the inequalities of (9) may be violated. The boundary of r_{1s} on the surface (11) must evidently be points at which at least one additional eigenvalue of M_n vanishes. Thus the boundary of r_{1s} satisfies

$$\Delta_{s+1} = 0\,.$$

Specializing to $s = 3$ and combining with previous results, we have the following theorem:

Theorem 9. The n momenta of (1) realize a matrix M_n which is in r_{13}^0 or on its boundary. The region r_{13}^0 is connected and is on the surface

$$\Delta_5 = \Delta_6 = \cdots = \Delta_n = 0\,.$$

r_{13}^0 are those points on this surface satisfying

$$\Delta_1 > 0\,,\quad \Delta_2 > 0\,,\quad \Delta_3 > 0\,,\quad \Delta_4 > 0\,,$$

and

$$\{i,j\} > 0\,.$$

The boundary of r_{13}^0 satisfies

$$\Delta_4 = 0\,.$$

IV.

In this section we consider in detail the geometry of the case $n = 3$ which is the well-known Dalitz–Fabri problem. Writing the matrix M_3 in the form

$$M_3 = \begin{Vmatrix} m_1^2 & z & y \\ z & m_2^2 & x \\ y & x & m_3^2 \end{Vmatrix} \quad (12)$$

one obtains the surface $\Delta_3 = 0$ as a cubic surface with four conical points at

$$C_0: (x,y,z) = (m_2m_3, m_3m_1, m_1m_2)\,, \quad (13A)$$

$$C_1: (x,y,z) = (m_2m_3, -m_3m_1, -m_1m_2)\,, \quad (13B)$$

$$C_2: (x,y,z) = (-m_2m_3, m_3m_1, -m_1m_2)\,, \quad (13C)$$

$$C_3: (x,y,z) = (-m_2,m_3, -m_3m_1, m_1m_2)\,. \quad (13D)$$

Call the surface S. It consists of five pieces. One central piece S_0 is a closed surface contained wholly in the rectangular parallelpepiped whose 8 vertices are

$$(\pm\, m_2m_3,\ \pm\, m_3m_1,\ \pm\, m_1m_2)\,. \quad (14)$$

This central portion contains the 4 points C and, in fact, contains the 6 straight line segments connecting these four points which form a tetrahedron.

In addition to this central piece S_0, S has four horns H_0, H_1, H_2, H_3 growing out of the four points C. The horn H_0 grows out of C_0 and has its x,y,z coordinates all $\geqq$ that of C_0. H_1 grows out of C_1, and has its x, $-y$, $-z$ coordinates all $\geqq$ that of C_1, etc. The five pieces S_0, H_0, H_1, H_2, and H_3 do not intersect except at the C's which are conical points. The division of the x,y,z space by these pieces of the surface $\Delta_3 = 0$ is schematically illustrated in Fig. 1.

The regions of $\mathfrak{M}_3$ are:

r_{30}: The connected region inside S_0.

r_{21}: The region outside of S_0 and outside of all horns. It is connected.

r_{12}: The inside of the four horns.[6]

r_{20}: The surface S_0 excluding the four points C.

r_{11}: The surfaces[6] H_0, H_1, H_2, H_3 excluding the points C. $H_0 = r_{11}^0$ (both exclude C_0).

r_{10}: The four[6] points C. $C_0 = r_{10}^0$.

[6] Its division into four disconnected subregions is in accordance with Theorem 5.

The physically realizable regions for reaction (1) are r_{12}^0, r_{11}^0 and r_{10}^0, which are the inside, the surface, and the vertex of the horn H_0. For fixed mass m_A for reaction (1), however, one has the additional condition

$$m_A^2 = \sum_i m_i^2 + 2 \sum_{i<j} \{i,j\} , \qquad (15)$$

or

$$x + y + z = \tfrac{1}{2} [m_A^2 - m_1^2 - m_2^2 - m_3^2] . \qquad (16)$$

(16) is a (111) plane. Its intersection with the inside of the horn H_0 is the Dalitz region. Its intersection with the horn H_0 is the boundary of the region. If it

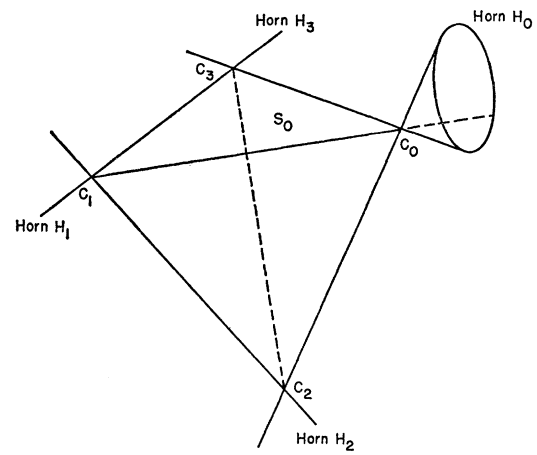

FIG. 1. Schematic diagram of cubic surface det $M_3 = 0$. The surface consists of S_0 and the four horns. S_0 contains on its surface the six straight line segments forming the tetrahedron $C_0C_1C_2C_3$. These four vertices are the only singular points on the surface. They are conical points. The different regions r_{ab} in relation to this surface are described in full in the text. The intersection of H_0 with *any* plane gives a curve which is the boundary of a Dalitz region with appropriate masses. Conversely *all* Dalitz region boundaries with nonvanishing masses can be obtained in this way, from this single horn H_0. See Sec. VIII for this discussion. See Fig. 2 for photographs of a model of horn H_0.

does not intersect H_0, the mass m_A is too small to allow for reaction (1).

Plane (16) also intersects the inside of horn H_1. The intersection may represent reactions

$$\bar{1} + A \to 2 + 3$$

or

$$\bar{1} \to 2 + 3 + \bar{A} .$$

V.

Much of the geometry discussed above can be generalized to cases $n > 3$. In particular, the surface det $M_3 = 0$ consists of regular points which are r_{20} and r_{11}, and conical singularities C which are r_{10}. This fact is generalizable.

Theorem 10. Consider the Cartesian space $\mathfrak{M}_4$ of six dimensions. Consider the surface

$$\det M_4 = 0 . \qquad (17)$$

(i) It consists of regular points which form the regions r_{30} (connected), r_{21} (connected), and r_{12} (8 separate subregions, each connected);

(ii) conical[7] points which form the regions r_{20} (connected) and r_{11} (eight separate subregions, each connected); and

(iii) superconical[7] points of order 3 which form the region r_{10} (eight separate points).

Proof. The surface (17) represents the points where one or more eigenvalues of M_4 is zero. It consists evidently of the regions r_{30}, r_{21}, r_{12}; r_{20}, r_{11}, and r_{10}.

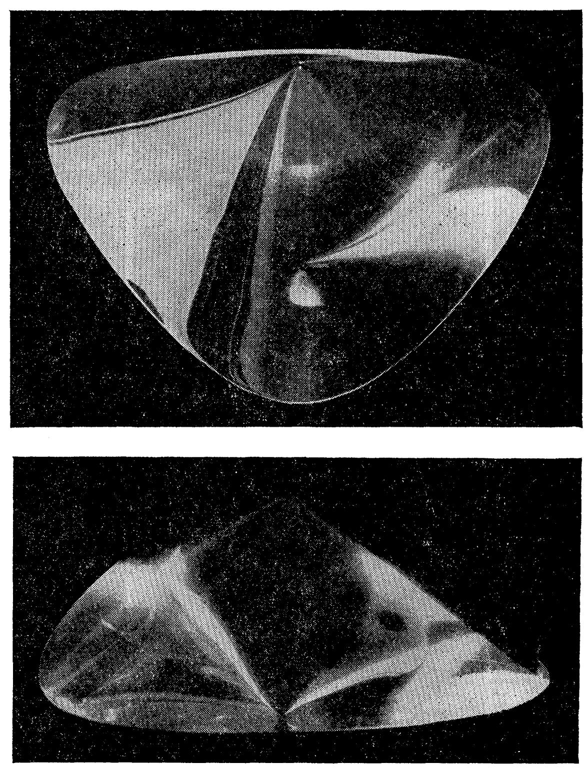

FIG. 2. Two views of a model of horn H_0 of universal surface (80) for decay into three particles. Surface (80) is a specialization of the one schematically represented in Fig. 1 to the case $m_1 = m_2 = m_3 = 1$. The base of the model is the plane $x + y + z = 63$, which is the Dalitz region for $m_1 = m_2 = m_3 = 1$, $m_A = (129)^{\frac{1}{2}}$. The model was made by J. W. Benoit, A. Lemonick, and Theodore Forseman of Princeton University.

To prove the *local* property of (17) in each of these regions the procedure is similar. We illustrate by considering a point $(M_4)_0$ in r_{11}. By Theorem 1 it is

[7] The exact definition of conical and superconical points of various orders appears later in the proof.

realized in D_{11} by four 2-vectors. In other words

$$(M_4)_0 = Vg\tilde{V}, \quad V = 4 \times 2 \text{ matrix}, \tag{18}$$

where

$$g = \text{metric of } D_{11}. \tag{19}$$

Consider the 2 columns of V as vectors in Euclidean space, (i.e., D_{40}). There obviously exist two orthogonal normalized vectors ξ and η in this space orthogonal to these two vectors. Let

$$\Gamma_1 = ||\xi\eta|| = 4 \times 2 \text{ matrix}, \tag{20}$$

and Γ_2 be a 4×2 matrix such that

$$\Gamma = ||\Gamma_1\Gamma_2|| \tag{21}$$

is proper and orthogonal. (That is, $\det \Gamma = 1$, $\tilde{\Gamma}\Gamma = 1$.) Then

$$\tilde{\Gamma}V = ||{}^0_W||. \quad (W = 2 \times 2). \tag{22}$$

By Theorem 1, V is of rank 2. Thus $\det W \neq 0$. Now define

$$\epsilon = \tilde{\Gamma}[M_4 - (M_4)_0]\Gamma \tag{23}$$

[where Γ is defined above for the fixed point $(M_4)_0$]. The elements of ϵ are homogeneous and linear in the deviations in space $\mathfrak{M}_4$ from the point $(M_4)_0$. To study *local* properties we only calculate quantities to the lowest nonvanishing order of the elements of ϵ.

By (22)

$$\tilde{\Gamma}(M_4)_0\Gamma = \begin{Vmatrix} 0 & 0 \\ 0 & Q \end{Vmatrix}, \tag{24}$$

where

$$q = \det Q \neq 0. \tag{25}$$

Thus (23) gives

$$\tilde{\Gamma}M_4\Gamma = \begin{Vmatrix} \epsilon_1 & \epsilon_2 \\ \tilde{\epsilon}_2 & \epsilon_3 + Q \end{Vmatrix}. \tag{26}$$

Hence

$$\det M_4 = q(\det \epsilon_1) + \text{higher order in } \epsilon. \tag{27}$$

If the elements of ϵ_1 are

$$\epsilon_1 = \begin{Vmatrix} e_{11} & e_{12} \\ e_{12} & e_{22} \end{Vmatrix}, \tag{28}$$

we have, near $(M_4)_0$,

$$\det M_4 = q(e_{11}e_{22} - e_{12}^2) + 0(\epsilon^3). \tag{29}$$

Thus (17) becomes, to lowest order,

$$\begin{vmatrix} e_{11} & e_{12} \\ e_{12} & e_{22} \end{vmatrix} = 0. \tag{30}$$

In Appendix B it is proved that the quantities e_{11}, e_{12}, and e_{22} are linearly independent. The surface (30) is a cone.

The connectivity of the regions were derived already in Theorem 5.

For the other regions the reasoning is entirely similar. In the neighborhood of a point in r_{10}, e.g., the surface (17) is, to the lowest order, of the form

$$\begin{vmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{vmatrix} = 0. \tag{31}$$

[These e's are however not the same as those of (28).] Such an equation with linearly independent variables e (to be shown in Appendix B) is what is called a supercone of order 3. Its vertex is called a superconical point of order 3 in the statement of the present theorem. Thus we complete the proof of Theorem 10.

The generalizations of the statement of this theorem to cases with $n > 4$, of its proof, and of Appendix B are straightforward. We see that, e.g., for $n = 5$, the physical region r_{13}^0 consists of regular points of $\det M_5 = 0$. For $n = 6$, the physical points r_{13}^0 are conical points of $\det M_6 = 0$. For $n > 6$, the physical points r_{13}^0 are superconical points of order $n - 4$ of the surface $\det M_n = 0$.

VI.

We now describe the calculation of the volume element (3) for reaction (1). It is to be expressed in terms of the variables $\{i,j\}$, subject to the condition

$$m_A^2 = \sum_i m_i^2 + 2 \sum_{i<j} \{i,j\},$$

or

$$\sum \{i,j\} = \tfrac{1}{2} m_A^2 - \tfrac{1}{2} \sum m_i^2 \equiv K. \tag{32}$$

Thus we seek to find the factor $F_{n,4}$ in

$$d\tau_{n,4} = F_{n,4}\left[\prod_{i<j} d\{i,j\}\right]\delta\left[\sum_{i<j} \{i,j\} - K\right]M_A^{-2}. \tag{33}$$

The factor M_A^{-2} is included for convenience.

For $n = 3$, the standard Dalitz–Fabri uniform density gives

$$F_{3,4} = 8\pi^2. \tag{34}$$

For $n = 4$, we have, using the coordinate system in which A is at rest

$$d\tau_{4,4} = d^3\mathbf{p}_1 d^3\mathbf{p}_2 d^3\mathbf{p}_3 \delta(E_A - E_1 - E_2 - E_3 - E_4) \times (E_1E_2E_3E_4)^{-1}, \tag{35}$$

where $\mathbf{p}_i$ and E_i are the 3-momenta and energy of the ith particle.

Define spherical coordinates for $\mathbf{p}_3$:

$$\theta(\mathbf{p}_3;\mathbf{p}_1+\mathbf{p}_2) = \text{angle between } \mathbf{p}_3 \text{ and } \mathbf{p}_1+\mathbf{p}_2\,, \quad (36)$$

and

$$\phi(\mathbf{p}_3;\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_1) = \text{azimuthal angle of } \mathbf{p}_3 \text{ with } z \text{ axis parallel to } \mathbf{p}_1+\mathbf{p}_2 \text{ and } \mathbf{p}_1 \text{ in } z\text{–}x \text{ plane.} \quad (37)$$

The $d\theta$ integration cancels the δ function of (35), yielding

$$d^3\mathbf{p}_3\delta(E_A-E_1-E_2-E_3-E_4)(E_3E_4)^{-1} = [dE_3d\phi(\mathbf{p}_3;\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_1)]|\mathbf{p}_1+\mathbf{p}_2|^{-1}\,. \quad (38)$$

To express $d\phi$ in more convenient coordinates, we keep[8] $\mathbf{p}_1,\mathbf{p}_2,|\mathbf{p}_3|$ and $\theta(\mathbf{p}_3,\mathbf{p}_1+\mathbf{p}_2)$ fixed. Then

$$d\{1,3\} = |\mathbf{p}_1|\cdot|\mathbf{p}_3|\cdot\sin\theta(\mathbf{p}_1,\mathbf{p}_1+\mathbf{p}_2) \times \sin\theta(\mathbf{p}_3,\mathbf{p}_1+\mathbf{p}_2)\sin\phi\, d\phi\,. \quad (39)$$

Now an explicit evaluation gives[9]

$$\det|p_1+p_2,p_3+p_4,p_1,p_3| = E_A|\mathbf{p}_1+\mathbf{p}_2|\cdot|\mathbf{p}_1| \times|\mathbf{p}_3|\sin\theta(\mathbf{p}_1,\mathbf{p}_1+\mathbf{p}_2)\cdot\sin\theta(\mathbf{p}_3,\mathbf{p}_1+\mathbf{p}_2)\sin\phi\,. \quad (40)$$

Equations (38), (39), and (40) give

$$d^3\mathbf{p}_3\delta(E_A-E_1-E_2-E_3-E_4)(E_3E_4)^{-1} = E_AdE_3d\{1,3\}[\det|p_1,p_2,p_3,p_4|]^{-1}\,. \quad (41)$$

We now evaluate

$$d\tau = (d^3\mathbf{p}_1/E_1)(d^3\mathbf{p}_2/E_2)\,\{\text{Eq. (41)}\}$$

by first fixing $\mathbf{p}_1$, $\{1,3\}$, E_3, and integrating over the azimuth of $\mathbf{p}_2$ around $\mathbf{p}_1$

$$E_2^{-1}d^3\mathbf{p}_2 = 2\pi|\mathbf{p}_1|^{-1}dE_2d\{1,2\}\,. \quad (42)$$

Fixing E_1, E_2, $\{1,2\}$, $\{1,3\}$, and E_3 we can integrate over the direction of $\mathbf{p}_1$ in space

$$E_1^{-1}d^3\mathbf{p}_1 = 4\pi|\mathbf{p}_1|dE_1\,. \quad (43)$$

Multiplying all factors and using

$$E_A = M_A$$
$$M_AE_i = \sum_j\{i,j\}\,, \quad (44)$$

we obtain[10]

$$d\tau_{4,4} = \frac{16\pi^2\Big[\prod_{i<j}d\{i,j\}\Big]\delta(\sum\{i,j\}-K)}{M_A^2\det|1,2,3,4|}\,, \quad (45)$$

or equivalently

$$F_{4,4} = 16\pi^2(\Delta_4)^{-\frac{1}{2}}\,. \quad (46)$$

In (45) and (46) we have included an extra factor of 2 to account for the 1 to 2 relationship between M_4 and kinematics discussed in Sec. 2. The same factor 2 is included in all subsequent calculations for $d\tau_{n,r}$ for which $n \geqq r$.

For $d\tau_{n,4}$ with $n \geqq 5$, the evaluation of $d\tau$ goes through the following steps:

(1) We have in the center of mass system of A

$$d\tau_{n,4} = d^3\mathbf{p}_nd^3\mathbf{p}_{n-1}\cdots d^3\mathbf{p}_2\delta(E_A-E_n-E_{n-1}\cdots E_1) \times(E_n\cdots E_1)^{-1}\,. \quad (47)$$

Define

$$\zeta_i = \sum_{j=i}^n p_j\,,\qquad \eta_i = \sum_{j=1}^i p_j\,. \quad (48)$$

Then

$$p_A = \eta_1+p_2+p_3+\zeta_4\,. \quad (49)$$

We treat η_1, p_2, p_3, and ζ_4 in the same manner that we treated p_4, p_3, p_1, and p_2 in the discussions of (36)–(41), obtaining

$$d^3\mathbf{p}_2\delta(E_A-E_n-\cdots-E_1)(E_2E_1)^{-1} = E_AdE_2d\{2,3\}[\det|\eta_1,2,3,\zeta_4|]^{-1}\,. \quad (50)$$

Now write

$$p_A = \eta_2+p_3+p_4+\zeta_5\,. \quad (51)$$

Choose spherical coordinates for $\mathbf{p}_3$ around $\mathbf{p}_4+\zeta_5$:

$$E_3^{-1}d^3\mathbf{p}_3 = |\mathbf{p}_3|dE_3d\cos\theta(\mathbf{p}_3,\mathbf{p}_4+\zeta_5)d\phi(\mathbf{p}_3;\mathbf{p}_4+\zeta_5,\mathbf{p}_4)\,.$$

The evaluation of this $d\phi$ follows the same method as described in (39) and (40). $d\theta$ is easily expressed in terms of $d\{3,\zeta_4\}$ giving

$$E_3^{-1}d^3\mathbf{p}_3 = E_A[\det|\eta_2,3,4,\zeta_5|]^{-1}dE_3d\{3,4\}d\{3,\zeta_4\}\,. \quad (52)$$

Similarly we can evaluate $E_4^{-1}d^3\mathbf{p}_4$, etc., up to $E_{n-2}^{-1}d\mathbf{p}_{n-2}$. The last two 3-momenta integration $d^3\mathbf{p}_{n-1}$ and $d^3\mathbf{p}_n$ are similar to (42) and (43). The final result is, for $n \geqq 5$,

$$d\tau_{n,4} = \frac{16\pi^2\Big[\prod_2^n dE_i\Big]\Big[\prod_2^{n-1}d\{i,i+1\}\Big]\Big[\prod_3^{n-2}d\{i,\zeta_{i+1}\}\Big](M_A)^{n-3}}{\prod_1^{n-3}\det|\eta_i,i+1,i+2,\zeta_{i+3}|}\,. \quad (53)$$

(2) Using (44) one can reduce all the differentials in (53) into sums of $d\{i,j\}$. The result is expressed in

[8] It is obvious that fixed $\mathbf{p}_1$, $\mathbf{p}_2$, $|\mathbf{p}_3|$ (by energy conservation) implies fixed E_4, and hence fixed $\theta(\mathbf{p}_3,\mathbf{p}_1+\mathbf{p}_2)$.

[9] det here means the determinant formed by the 16 components of the four 4-vectors in the argument.

[10] Where there is no confusion possible, we write i for p_i.

scalar products, but is not symmetrical with respect to permutations of the labeling $1 \rightarrow n$.

To obtain a symmetrical expression for the volume element we consider in $\mathfrak{M}_n$ the neighborhood of the point $(M_n)_0$ realized by the n 4-vectors $p_1 \cdots p_n$ in question. Assume[11] $(M_n)_0$ to be in r_{13}^0. The arguments (18) to (27) can be generalized in a straightforward way. The matrices Q and ϵ_1 are now of the following dimensions

$$Q = 4 \times 4$$
$$\epsilon_1 = (n-4) \times (n-4) .$$

ϵ_1 is a real symmetrical matrix. Its elements $e_{\alpha\beta}$, $[\alpha \leqq \beta; \alpha,\beta = 1 \rightarrow (n-4)]$ are, according to Appendix B, linear in $\{i,j\}$ and are independent.

Now consider at $(M_n)_0$ on the one hand the differential volume

$$\partial_1 = \prod_{i<j} d\{i,j\} , \qquad (54)$$

and on the other hand

$$\partial_2 = [\text{the diffl. vol. in RHS of (53)}]$$
$$\times\, d\left(\sum_{i<j} \{i,j\} - K\right) \prod_{\alpha\leqq\beta} de_{\alpha\beta} . \qquad (55)$$

It is shown in Appendix C that the Jacobian at $(M_n)_0$ is

$$J \equiv \frac{\partial_2}{\partial_1} = \frac{1}{M_A^{n-1}} \frac{2^{n-4}}{(\Delta_4)^{(n-3)/2}} \prod_1^{n-3} \det |\eta_i, i+1, i+2, \zeta_{i+3}| . \qquad (56)$$

Now

$$\text{diffl. vol. in RHS of (53)}$$
$$= \delta\left(\sum_{i<j} \{i,j\} - K\right) \prod_{\alpha\leqq\beta} \delta(e_{\alpha\beta}) \partial_2 .$$

Writing $\partial_2 = J\partial_1$ and substituting into (53) one obtains

$$d\tau_{n,4} = \frac{2^n \pi^2}{M_A^2 (\Delta_4)^{(n-3)/2}} \delta\left(\sum_{i<j} \{i,j\} - K\right) \prod_{\alpha\leqq\beta} \delta(e_{\alpha\beta}) \partial_1 . \qquad (57)$$

Or, comparing with (33), one obtains

$$F_{n,4} = 2^n \pi^2 (\Delta_4)^{-(n-3)/2} \prod_{\alpha\leqq\beta} \delta(e_{\alpha\beta}) . \qquad (58)$$

(3) We now express the factor $\prod \delta(e_{\alpha\beta})$ in (58) in terms of

$$\delta_l \equiv \delta(\Delta_l) .$$

By the generalization of (26),

$$\Delta_n = qT_{n-4} + \text{higher order in } \epsilon$$
$$\Delta_{n-1} = qT_{n-3} + \text{higher order in } \epsilon$$
$$\cdots$$
$$\Delta_5 = qT_1 + \text{higher order in } \epsilon$$
$$\Delta_4 = -q + \text{higher order in } \epsilon , \qquad (59)$$

where

$$T_l = (-1)^{l-1} \sum \text{det. of all } (l \times l) \text{ diagonal minors of } \epsilon_1 . \qquad (60)$$

Now consider the relationship between T and the elements $e_{\alpha\beta}$. If all $T_1, \cdots, T_{n-4}$ are zero, the eigenvalues of ϵ_1 are all zero, so that all $e_{\alpha\beta} = 0$. This fact suggests Theorem 12 below. To obtain this theorem we need first

Theorem 11. Consider a real symmetrical matrix $\epsilon_1 = ||e_{\alpha\beta}||$ of dimension $\nu \times \nu$, with distinct eigenvalues $\lambda_1, \cdots, \lambda_\nu$. There exists a real proper orthogonal matrix Λ, determined up to some $\pm$ signs, so that

$$\epsilon_1 = \Lambda \begin{Vmatrix} \lambda_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \lambda_\nu \end{Vmatrix} \Lambda^{-1} . \qquad (61)$$

Then (a)

$$\prod_{\alpha\leqq\beta} de_{\alpha\beta} = dT_1 \cdots dT_\nu d\Lambda ,$$

where T is defined in (60) and $d\Lambda$ is the group volume element of the orthogonal group formed by Λ, normalized so that near the identity element

$$d\Lambda = \prod d \text{ (all independent off diagonal elements)}.$$

(b) The correspondence $\epsilon_1 \rightleftharpoons \Lambda$ is $1 \rightleftharpoons 2^{\nu-1}$.

This theorem is quite well known in the literature.[12]

Theorem 12. Consider a real symmetrical matrix $\epsilon_1 = ||e_{\alpha\beta}||$ of dimension $\nu \times \nu$. Let T_l be defined by (60). Then[13]

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[11] This represents no loss of generality since the regions r_{12}^0, r_{11}^0, and r_{10}^0 are of smaller dimensions than r_{13}^0 and are irrelevant to the volume element calculation.

[12] See C. E. Porter and N. Rosenzweig, Suomalaisen Tiedeakatemian Toimituksia AVI, No. 44 (1960); N. Rosenzweig and C. E. Porter, Phys. Rev. **120**, 1698 (1960); F. J. Dyson, J. Math. Phys. **3**, 140 (1962).

[13] Notice that the number of δ functions on the two sides of (62) are different. The exact definition of the right-hand side of (62) is

$$\lim \prod_{l=1}^{\nu} \delta(T_l - t_l) ,$$

where $t_l \rightarrow 0$ along any path so that the equation

$$\lambda^\nu - t_1\lambda^{\nu-1} - \cdots - t_\nu = 0$$

always has ν real solutions. [E.g. for $\nu = 2$, $t_1 \rightarrow 0$, $t_2 \rightarrow 0$ such that $t_1^2 + 4t_2 \geqq 0$.]

The meaning of (62) is as follows: Let its left-hand side be denoted by LHS. Let $F(e_{\alpha\beta})$ be a function of all $e_{\alpha\beta}$ which is continuous in the neighborhood of the origin $e_{\alpha\beta} = 0$. Then

$$\int (\text{LHS}) F \prod_{\alpha\leqq\beta} de_{\alpha\beta} = \lim_{t\to 0} \int F \prod_{l=1}^{\nu} \delta(T_l - t_l) \prod_{\alpha\leqq\beta} de_{\alpha\beta} , \qquad (62')$$

where $t \rightarrow 0$ along a path subject to the condition above.

$$2^{1-\nu}G_\nu \prod_{\alpha \leqq \beta} \delta(e_{\alpha\beta}) = \prod_{l=1} \delta(T_l) , \tag{62}$$

where

$$G_\nu = \text{total volume of the group of proper orthogonal real } \nu \times \nu \text{ matrices} . \tag{63}$$

The explicit expression[14] for G_ν is

$$G_1 = 1 ,$$

$$G_\nu = \prod_2^\nu S_i , \tag{64}$$

$$S_\nu = \text{surface area of sphere } x_1^2 + \cdots + x_\nu^2 = 1$$
$$= 2(\pi)^{\nu/2}/\Gamma(\tfrac{1}{2}\nu) . \tag{65}$$

These two theorems are proved in Appendix D. Applying *Theorem* 12 and Eqs. (59), one obtains

$$\prod_{\alpha \leqq \beta} \delta(e_{\alpha\beta}) = (2^{n-5}/G_{n-4})(\Delta_4)^{n-4}\delta_5\delta_6\cdots\delta_n . \tag{66}$$

So we have finally by (66) and (58), for $n \geqq 5$,

$$F_{n,4} = 2^{2n-5}\pi^2(\Delta_4)^{(n-5)/2}[G_{n-4}]^{-1}\delta_5\delta_6\cdots\delta_n . \tag{67}$$

[The region where the volume element $d\tau_{n,4}$ is applied to was stated in Theorem 9.] We see that aside from the δ functions which define the surface of the physical region in $\mathfrak{M}_n$, there is also a density factor proportional to $(\Delta_4)^{(n-5)/2}$. At the boundary of the physical region

$$\text{density} = 0 \quad , \quad \text{for } n > 5$$
$$\text{density} = \infty \quad , \quad \text{for } n = 4 .$$

For $n = 5$ the density factor is a numerical constant throughout the physical region.

VII.

It is obvious that the above can be generalized in a straightforward manner to any space with one time and $r - 1$ space coordinates. For reaction (1) one then writes, in place of (3) and (33)

$$d\tau_{n,r} = (\prod d^r p_i)[\prod \delta(p_i^2 - m_i^2)]\delta^r[(\sum p_i) - p_A] , \tag{68}$$

and

$$d\tau_{n,r} = F_{n,r}\left[\prod_{i<j} d\{i,j\}\right]\delta\left[\sum_{i<j}\{i,j\} - K\right]M_A^{2-r} . \tag{69}$$

It is easy to prove that for $n \geqq 3$, $r \geqq 2$,

$$n > r , \quad F_{n,r} = f_{n,r}(\Delta_r)^{(n-r-1)/2}(G_{n-r})^{-1}\delta_{r+1}\delta_{r+2}\cdots\delta_n , \tag{70}$$

$$n = r , \quad F_{n,n} = f_{n,n}\Delta_r^{-\frac{1}{2}} , \tag{71}$$

$$n < r , \quad F_{n,r} = f_{n,r}\Delta_n^{(r-n-1)/2} , \tag{72}$$

where $f_{n,r}$ are numerical factors.

(72) is interesting in the following way. Consider, e.g., case $n = 3$, the Dalitz–Fabri case. The allowed region of $\mathfrak{M}_3$ for all $r \geqq 3$ is the *same* as that for $r = 3$: namely r_{12}^0, r_{11}^0, and r_{10}^0. This fact merely reflects the possibility of embedding any kinematic configuration for $n = 3$, $r \geqq 3$ in the space $r = 3$, as proved in Theorem 4. But the phase-space density in the Dalitz region is, by (72), only uniform for $r - n - 1 = 0$. That is, $r = 4$. *For higher and higher dimensions, the points with small* Δ_3, i.e., *points near the Dalitz boundary* (*or points representing almost collinear momenta*) *are weighted less and less.*

VIII.

The diagonal elements of M_n are the square of the masses. By the transformation

$$\{i,j\} = m_i m_j \{i,j\}' \tag{73}$$

we can go into primed quantities, which are physically scalar products of velocity 4-vectors. We define other primed quantities starting from $\{i,j\}'$. Thus

$$M_n' = \left\| \begin{matrix} 1 & \{1,2\}'\cdots\{1,n\}' \\ & \cdots \\ \{n,1\}' & 1 \end{matrix} \right\| \tag{74}$$

$\Delta_l' = (-1)^{l-1} \sum$ det. of all $(l \times l)$ diagonal minors of M_n', etc.

It is clear that

$$\Delta_n = U^2\Delta_n' ,$$
$$U = \prod m . \tag{75}$$

The spaces $\mathfrak{M}_n$ and $\mathfrak{M}_n'$ are related to each other by a simple scale transformation. Since $\Delta_n = 0$ is equivalent to $\Delta_n' = 0$, Theorem 10 and its generalizations have entirely similar counterparts for the surface $\det M_n' = 0$ in $\mathfrak{M}_n'$. Since M_n' does not depend on the masses of the particles, the surface

$$\det M_n' = 0 \tag{76}$$

is *universal*, and can be used for any nonvanishing masses.

Thus the physical region is represented by regular points or conical points or superconical points of the universal surface (76). We now show that the volume element $d\tau$ can also be simply expressed in the primed space even though Δ_l'/Δ_l is not in general simple. Write

$$d\tau_{n,4} = F_{n,4}'\left[\prod_{i<j} d\{i,j\}'\right]\delta\left[\sum_{i<j}\{i,j\}'m_i m_j - K\right]M_A^{-2} \tag{77}$$

[14] See, e.g., F. J. Dyson, J. Math. Phys. 3, 140 (1962), Eq. (108).

so that by (75), $F'_{n,4} = F_{n,4}U^{n-1}$.
(34) and (46) lead to

$$F'_{3,4} = 8\pi^2(\prod m)^2 ,$$
$$F'_{4,4} = 16\pi^2(\Delta'_4)^{-\frac{1}{2}}(\prod m)^2 . \qquad (78)$$

For higher n values we have

Theorem 13. For $n \geqq 5$, the volume element $d\tau$ is given by (77) where

$$F'_{n,4} = 2^{2n-5}\pi^2(\Delta'_4)^{(n-5)/2} \times [G_{n-4}]^{-1}\delta(\Delta'_5)\delta(\Delta'_6)\cdots\delta(\Delta'_n)(\prod m)^2 , \qquad (79)$$

and G was defined in (63)–(65).

This theorem is proved in Appendix E. Notice that what corresponds to Theorem 9 for $\mathfrak{M}'_n$ states that the physically allowed region is a connected region $r^{0\prime}_{13}$ (in $\mathfrak{M}'_n$) and its boundary. $r^{0\prime}_{13}$ are those points on the surface

$$\Delta'_5 = \Delta'_6 = \cdots = \Delta'_n = 0$$

satisfying

$$\Delta'_1 > 0 , \quad \Delta'_2 > 0 , \quad \Delta'_3 > 0 , \quad \Delta'_4 > 0$$

and

$$\{i,j\}' > 0 .$$

Its boundary satisfies $\Delta'_4 = 0$.

For the special case of $n = 3$, the surface

$$\begin{vmatrix} 1 & z & y \\ z & 1 & x \\ y & x & 1 \end{vmatrix} = 0 \qquad (80)$$

is the *universal* surface. For a given set of m_1, m_2, m_3, all > 0, the Dalitz boundary is the intersection of Horn H_0 of (80) with the plane

$$m_1m_2z + m_2m_3x + m_3m_1y = \tfrac{1}{2}[m_A^2 - m_1^2 - m_2^2 - m_3^2] .$$

The Dalitz region is within this boundary on the plane. (Cf. Fig. 2.)

IX.

For reaction (2) one still has $p_A = \sum p_i$, but the four-vector p_1 is now backward time-like. These n vectors therefore realize a point M_n which is in a different subregion (i.e., horn) of r_{1b} than r^0_{1b} (see Theorem 5), $b = 3,2,1,0$. Reaction

$$\bar{1} \to \bar{A} + 2 + 3 + \cdots n \qquad (81)$$

is also realized in the same subregion. Also in the same subregion there are points realizing (2) [or (81)] with a 4-vector momentum for A which is space-like and, therefore, unphysical.

The volume element for (2)

$$d\tau^{(2)}_{n,4} = \left(\prod_2^n d^4p_i\right)\left[\prod_2^n \delta(p_i^2 - m_i^2)\right]\delta^4(\sum p - p_A)$$

can be simply obtained from the expression (3) by not including the last integration $(d^3\mathbf{p}_1)/E_1$ which contributed the factor (43). Thus

$$d\tau^{(2)}_{n,4} = [\delta(E_1 + c_1)/4\pi|\mathbf{p}_1|] \times \text{Eq. (77)} , \qquad (82)$$

where c_1 and $|\mathbf{p}_1|$ are the energy and momentum of $\bar{1}$ in reaction (2) in the Lorentz system in which A is at rest.

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APPENDIX A

Lemma 1. Let A and B be two real symmetrical matrices and $TA\tilde{T} = B$, where T is a real rectangular or square matrix. Let a_A, a_B be the number of positive nonvanishing eigenvalues of A and B; b_A, b_B the number of negative nonvanishing eigenvalues of A and B. Then

$$a_A \geqq a_B ,$$
$$b_A \geqq b_B . \qquad (A1)$$

If T is square and det $T \neq 0$, then the equalities in (A1) hold.

The lemma is well known and is easily proved.

APPENDIX B

This appendix proves the linear independence of the elements of ϵ_1. We focus on the example discussed in the proof of Theorem 10.

It is obviously possible to choose the last column of Γ_2 of (21) so that none of its elements are zero. Such a choice does not effect Γ_1, hence it leaves ϵ_1 unchanged. The elements e of ϵ in (23) are linear in $\{i,j\}$ with coefficients quadratic in the elements of Γ:

$$\partial e_{\alpha\beta}/\partial\{i,j\} = \Gamma_{i\alpha}\Gamma_{j\beta} + \Gamma_{i\beta}\Gamma_{j\alpha} . \qquad (B1)$$

The number of $\{i,j\}$, $i < j$, is $\frac{1}{2}n(n-1)$. Consider the collection $\langle e\rangle$ of $e_{\alpha\beta}$, $1 \leqq \alpha \leqq \beta \leqq n-1$. The number of e's in this collection is $\frac{1}{2}n(n-1)$. The collection includes all elements of ϵ_1. We can apply Lemma 2, below, to this problem ($\theta = \phi = \Gamma$) and calculate the Jacobian. The factors on the right-hand side of (B3) are, by Lemma 3, equal in magnitude to the elements $\Gamma_{1n}, \Gamma_{2n}, \cdots, \Gamma_{nn}$, all of which are nonvanishing. Thus $J \neq 0$ and all elements of ϵ_1 are linearly independent.

Lemma 2. Consider quantities $e_{\alpha\beta}$ and $\{i,j\}$ with

$$1 \leqq \alpha \leqq \beta \leqq n-1$$
$$1 \leqq i < j \leqq n\,.$$

$e_{\alpha\beta}$ depend on $\{i,j\}$ so that

$$\partial e_{\alpha\beta}/\partial\{i,j\} = \theta_{i\alpha}\phi_{j\beta} + \theta_{i\beta}\phi_{j\alpha}\,. \tag{B2}$$

The Jacobian is given by

$$J = \partial(\text{all } e)/\partial(\text{all } \{i,j\}) = 2^{n-1}[1,2,\cdots n-1] \times [1,2,\cdots(n-2),\bar{n}] \times [1,2,\cdots(n-3),\overline{(n-1)},\bar{n}] \times \cdots[\bar{2},\bar{3},\cdots\bar{n}]\,, \tag{B3}$$

where the [ ]'s are defined so that, e.g.,

$$[a,b,\cdots\bar{d},\bar{e}] = \begin{vmatrix} \theta_{a1}\theta_{a2}\cdots\ \theta_{a(n-1)} \\ \theta_{b1}\theta_{b2}\cdots\ \theta_{b(n-1)} \\ \cdots \\ \phi_{d1}\phi_{d2}\cdots\phi_{d(n-1)} \\ \phi_{e1}\phi_{e2}\cdots\phi_{e(n-1)} \end{vmatrix}$$

(unbarred indices refer to θ, barred to ϕ).

Proof. (i) Write the Jacobian as a determinant of a matrix $\mathcal{J}$, with rows labeled by $\{i,j\}$, columns by $e_{\alpha\beta}$. Consider any $(n-1) \times 1$ column matrix ψ.

$$\psi = \begin{Vmatrix} \psi_1 \\ \vdots \\ \psi_{n-1} \end{Vmatrix}\,. \tag{B4}$$

Define F_ψ to be a $\frac{1}{2}n(n-1) \times 1$ column matrix with elements $\frac{1}{2}\psi_1^2, \frac{1}{2}\psi_2^2, \cdots \frac{1}{2}\psi_n^2, (\psi_1\psi_2), \cdots (\psi_{n-2}\psi_{n-1})$. Then the element in the row $\{i,j\}$ of $\mathcal{J}F_\psi$ is

$$\left(\textstyle\sum_\alpha \theta_{i\alpha}\psi_\alpha\right)\left(\textstyle\sum_\alpha \phi_{j\alpha}\psi_\alpha\right)\,. \tag{B5}$$

(ii) Consider J as a polynomial in the elements of θ and ϕ. If $[1,2,\cdots n-1] = 0$, there exists a column vector ψ so that

$$\psi \neq 0\,, \quad \textstyle\sum_\alpha \theta_{i\alpha}\psi_\alpha = 0\,.$$

Thus $\mathcal{J}F_\psi = 0$. Hence $J = 0$. Thus J contains as a factor $[1,2,\cdots,n-1]$.

(iii) We can prove similarly that J contains the other factors displayed in (B3). To determine the coefficient, we take

$$\theta_{i\alpha} = \delta_{i\alpha}\,, \quad \phi_{j\beta} = \delta_{(j-1)\beta}\,,$$

and evaluate J and the factors, obtaining (B3).

Lemma 3. If Γ is real orthogonal, complementary minors of Γ have determinants that are equal in value but may have opposite signs.

This lemma is easy to prove.

APPENDIX C[15,16]

To prove (56) we take, e.g., the case of $n = 7$. The general case is entirely similar.

(i) We relabel the indices $1 \to 7$ by $7 \to 1$ in (55):

$$\partial_2 = d^5(23,34,45,56,12)d^3(13+23,14+24+34, \times 15+25+35)d^4(E_2,E_3,E_4,E_5) \times d^2(E_1,E_6)d\left(\sum_{i<j} ij - K\right)\prod_{\alpha\leqq\beta} de_{\alpha\beta}\,. \tag{C1}$$

Define

$$\partial_3 = d^{15}(\text{all } ij \text{ except } 24,25,26,35,36,46) \times d^6(\overline{24},\overline{25},\overline{26},\overline{35},\overline{36},\overline{46}) \tag{C2}$$

where

$$\overline{ab} = ab - 1b - a7 + 17\,. \tag{C3}$$

Now write J as the determinant of a matrix $\mathcal{J}$ with the rows labeled by ij, $i<j$, and columns by the differentials of (C1) in the order exhibited. Let $J_1 = \partial_3/\partial_1$ and let $\mathcal{J}_1$ be the corresponding matrix, with the rows labeled by ij, and columns by the differentials of (C2) in the order exhibited. Thus

$$\mathcal{J}_1 = \begin{Vmatrix} 1 & X \\ 0 & 1 \end{Vmatrix}\,,$$

where the upper 1 is of dimension 15×15, the lower 6×6.

(ii) Now it is straightforward to show that in a corresponding division

$$\tilde{\mathcal{J}}_1\mathcal{J} = \begin{Vmatrix} A_1 & A_3 \\ 0 & A_2 \end{Vmatrix}\,.$$

It is further easy to evaluate det A_1 by successive expansion according to its first, second, $\cdots$ columns:

$$\det A_1 = M_A^{-6}\,.$$

Thus

$$J = \det \tilde{\mathcal{J}}_1\mathcal{J} = M_A^{-6} \det A_2\,. \tag{C4}$$

(iii) Now the elements $e_{\alpha\beta}$ of ϵ were defined in (23). It is clear that A_2 is quadratic in the elements of Γ.

---

[15] In Appendix C we write ij for $\{i,j\}$.

[16] In this appendix we refer to Eq. (18) to (28), meaning generalized versions of these equations, which have the same form.

That is

$$\begin{aligned}\langle \overline{24}|A_2|e_{\alpha\beta}\rangle &= \Gamma_{2\alpha}\Gamma_{4\beta} + \Gamma_{4\alpha}\Gamma_{2\beta} \\ &\quad - (\text{same with } 24 \to 14) \\ &\quad - (\text{same with } 24 \to 27) \\ &\quad + (\text{same with } 24 \to 17) \\ &= (\Gamma_{2\alpha} - \Gamma_{1\alpha})(\Gamma_{4\beta} - \Gamma_{7\beta}) \\ &\quad + (\Gamma_{2\beta} - \Gamma_{1\beta})(\Gamma_{4\alpha} - \Gamma_{7\alpha}) \,, \qquad \text{(C5)}\end{aligned}$$

etc.

We can thus directly apply Lemma 2 for the evaluation of det A_2, with the correspondences:

$$n = 4 \,,$$

$$e_{\alpha\beta} \to e_{\alpha\beta} \,,$$

$$24, 25, 26, 35, 36, 46 \text{ here} \to \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\} \text{ of Lemma 2} \,,$$

$$(\Gamma_{a\alpha} - \Gamma_{1\alpha}) \text{ here} \to \theta_{i\alpha} \text{ of Lemma 2} \,,$$

$$(\Gamma_{b\beta} - \Gamma_{7\beta}) \text{ here} \to \phi_{j\beta} \text{ of Lemma 2} \,,$$

$$\text{(C5)} \to \text{(B2)} \,.$$

Thus

$$\det A_2 = 8 \times \text{four determinants each being } 3 \times 3 \,. \qquad \text{(C6)}$$

(iv) A typical example of the determinants in (C6) is

$$\begin{vmatrix} \Gamma_{21} - \Gamma_{11} & \Gamma_{22} - \Gamma_{12} & \Gamma_{23} - \Gamma_{13} \\ \Gamma_{31} - \Gamma_{11} & \Gamma_{32} - \Gamma_{12} & \Gamma_{33} - \Gamma_{13} \\ \Gamma_{41} - \Gamma_{11} & \Gamma_{42} - \Gamma_{12} & \Gamma_{43} - \Gamma_{13} \end{vmatrix} \,. \qquad \text{(C7)}$$

The transposed of (C7) is equal to

$$\langle 234\rangle - \langle 231\rangle - \langle 134\rangle - \langle 214\rangle \qquad \text{(C8)}$$

where

$$\langle abc\rangle = \begin{vmatrix} \Gamma_{a1} & \Gamma_{b1} & \Gamma_{c1} \\ \Gamma_{a2} & \Gamma_{b2} & \Gamma_{c2} \\ \Gamma_{a3} & \Gamma_{b3} & \Gamma_{c3} \end{vmatrix} \,. \qquad \text{(C9)}$$

Using Lemma 4, below, one can reduce (C8) to

$$(1/\Delta_4^{\frac{1}{2}}) \det |1 + 2 + 3 + 4{,}5{,}6{,}7| \,.$$

The other determinants in (C6) can be processed in a similar manner. Using (C6) and (C4) one arrives at (56).

To state Lemma 4 consider any permutation *abcdefg* of $1 \to 7$. Then

Lemma 4.

$$(\Delta_4)^{\frac{1}{2}} \langle abc\rangle = \pm \det |defg|$$

with $+(-)$ sign for even (odd) permutations.

Proof. By Lemma 3,

$$\langle abc\rangle = \pm \begin{vmatrix} \Gamma_{d4} & \Gamma_{d5} & \Gamma_{d6} & \Gamma_{d7} \\ \Gamma_{e4} & \Gamma_{e5} & \Gamma_{e6} & \Gamma_{e7} \\ \Gamma_{f4} & \Gamma_{f5} & \Gamma_{f6} & \Gamma_{f7} \\ \Gamma_{g4} & \Gamma_{g5} & \Gamma_{g6} & \Gamma_{g7} \end{vmatrix} \,. \qquad \text{(C10)}$$

Now (22) gives

$$V = \Gamma \left\| \begin{matrix} 0 \\ W \end{matrix} \right\| = \Gamma_2 W \,.$$

Taking the d,e,f,g rows of this and evaluating the determinant, one obtains, using (C10),

$$\det |d,e,f,g| = \pm\langle abc\rangle \det W \,. \qquad \text{(C11)}$$

Now the Q of (24) is, by definition,

$$Q = WGW \,. \qquad \text{(C12)}$$

Thus

$$\det Q = -(\det W)^2 \,.$$

But (24) shows that

$$\Delta_4 = -\det Q \,. \qquad \text{(C13)}$$

Collecting factors we obtain Lemma 4. The sign choice was inherited from (C10) and Lemma 3. Since $\det \Gamma = 1$, one proves the sign choice quite readily.

APPENDIX D

Proof of Theorem 11. (i) By direct evaluation one obtains

$$\prod_l dT_l = \prod_{\alpha<\beta} (\lambda_\alpha - \lambda_\beta) \prod_\alpha d\lambda_\alpha \,. \qquad \text{(D1)}$$

(ii) To prove part (a) of the theorem we evaluate the Jacobian

$$J = d(\text{all } e_{\alpha\beta})/(\prod_\alpha d\lambda_\alpha) d\Lambda \,. \qquad \text{(D2)}$$

Let us evaluate J at

$$(\epsilon_1)_0 = \Lambda_0 \left\| \begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_\nu \end{matrix} \right\| \Lambda_0^{-1}$$

Consider the transformation

$$\eta = \Lambda_0^{-1} \epsilon_1 \Lambda_0 \,. \qquad \text{(D3)}$$

From the equation

$$\text{Trace } \eta^2 = \text{Trace } \epsilon_1^2 \,,$$

one concludes that the transformation from

$$e_{11}, \cdots e_{\nu\nu}, \sqrt{2} e_{12}, \sqrt{2} e_{12}, \cdots \sqrt{2} e_{(\nu-1)\nu} \,, \qquad \text{(D4)}$$

to the corresponding elements of η is an orthogonal transformation. Let $\eta = ||\eta_{\alpha\beta}||$. Thus

$$J = d(\text{all}\, \eta_{\alpha\beta})/(\prod d\lambda_\alpha)d\Lambda \,. \tag{D5}$$

To first order the two types of variations $d\lambda$ and $d\Lambda$ induce, respectively, variations in only the diagonal and only the off-diagonal elements of η. For the variations $d\lambda$ this is obvious. For the variations $d\Lambda$ this follows from (iii) below.

(iii) Consider any parameter ξ of Λ. At $\Lambda = \Lambda_0$,

$$\frac{\partial}{\partial\xi}\eta = \Lambda_0^{-1}\left[\frac{\partial}{\partial\xi}\Lambda\begin{Vmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_\nu\end{Vmatrix}\Lambda^{-1}\right]_{\Lambda_0}\Lambda_0$$
$$= \left(\Lambda^{-1}\frac{\partial\Lambda}{\partial\xi}\right)_0\begin{Vmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_\nu\end{Vmatrix} - \begin{Vmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_\nu\end{Vmatrix}\left(\Lambda^{-1}\frac{\partial\Lambda}{\partial\xi}\right)_0 .$$

Now Λ is orthogonal, so Λ^{-1} $(\partial\Lambda/\partial\xi)$ is antisymmetrical. Thus at $\Lambda = \Lambda_0$

$$\left(\frac{\partial\eta}{\partial\xi}\right)_{\alpha\beta} = \left[\left(\Lambda^{-1}\frac{\partial\Lambda}{\partial\xi}\right)_0\right]_{\alpha\beta}(\lambda_\beta - \lambda_\alpha) \,.$$

(iv) Consider the off-diagonal elements of $(d\xi)\Lambda_0^{-1} \times (\partial\Lambda/\partial\xi)_0$ as the components of a vector. The collection of such vectors for different ξ form a parallelepiped whose volume is by definition the group volume element at Λ_0. This leads to a direct evaluation of (D5):

$$J = \prod_{\alpha<\beta}(\lambda_\alpha - \lambda_\beta) \,.$$

(D1) and (D2) now give part (a) of the theorem.

(v) To prove part (b), one uses the fact that Λ can be transformed by

$$\Lambda \to \Lambda\begin{Vmatrix}\pm1 & & & \\ & \pm1 & & \\ & & \ddots & \\ & & & \pm1\end{Vmatrix}$$

without changing (61).

Proof of Theorem 12. We need to prove (62′) of footnote 13. Its left-hand side is

$$2^{1-\nu}G_\nu F(0) \,.$$

To evaluate its right-hand side we use Theorem 11. For infinitesimal $T_l = t_l$, $e_{\alpha\beta} \cong 0$ and the right-hand side becomes

$$F(0) \int d\Lambda \,.$$

But the same ϵ space is covered, by Theorem 11, $2^{\nu-1}$ times when we allow Λ to go through *all* proper $(\nu \times \nu)$ orthogonal matrices. Thus we have Theorem 12.

APPENDIX E

(i) To prove (79) we need only prove that

$$\delta(\Delta_5')\delta(\Delta_6')\cdots\delta(\Delta_n') = U^{n-3}(\Delta_4/\Delta_4')^{(n-5)/2}\delta(\Delta_5)\cdots\delta(\Delta_n) \,, \tag{E1}$$

where

$$U = \prod m \,.$$

Apply (66) to the left side of (E1) and a corresponding (66′) to the right side. (E1) becomes

$$\prod_{\alpha\leqq\beta}\delta(e'_{\alpha\beta}) = U^{n-3}(\Delta_4/\Delta_4')^{-(n-3)/2}\prod_{\alpha\leqq\beta}\delta(e_{\alpha\beta}) \,. \tag{E2}$$

(ii) To prove this we first study the relationship between $e'_{\alpha\beta}$ and $e_{\alpha\beta}$. By[16] (22),

$$\tilde{\Gamma}_1 V = 0 \,. \quad [\tilde{\Gamma}_1 \text{ is } (n-4)\times n] \,. \tag{E3}$$

Now

$$V' = (1/m)V \,, \tag{E4}$$

where

$$m = \begin{Vmatrix}m_1 & & \\ & \ddots & \\ & & m_n\end{Vmatrix} . \tag{E5}$$

The matrix $\tilde{\Gamma}_1 m^2 \Gamma_1$ is positive definite. Thus there exists a matrix L of dimension $(n-4)\times(n-4)$ so that

$$\tilde{L}\tilde{\Gamma}_1 m^2 \Gamma_1 L = 1 \,, \qquad \det L \neq 0 \,. \tag{E6}$$

The matrix $\tilde{L}\tilde{\Gamma}_1 m$ is real orthogonal, and can be taken as Γ_1':

$$\Gamma_1' = \tilde{L}\tilde{\Gamma}_1 m \,, \tag{E7}$$

since it satisfies

$$\tilde{\Gamma}_1' V' = 0 \,.$$

Thus the definition (23) of ϵ gives

$$\epsilon_1' = \tilde{\Gamma}_1'[M_n' - (M_n')_0]\Gamma_1' = \tilde{L}\epsilon_1 L \,, \tag{E8}$$

since

$$mM_n'm = M_n \,.$$

(iii) We now prove that

$$\prod_{\alpha\leqq\beta} de'_{\alpha\beta} = [\det L]^{n-3}\prod_{\alpha\leqq\beta} de_{\alpha\beta} \,. \tag{E9}$$

This is obvious by (E8) if L is diagonal. If L is not diagonal, it can be transformed into a diagonal from L_d:

$$L = \gamma_1 L_d \tilde{\gamma}_2 \,,$$

where γ_1 and γ_2 are orthogonal matrices. But we have already proved by the argument after (D3) that orthogonal transformations

$$\epsilon_1 \to \tilde{\gamma}_1 \epsilon_1 \gamma_1 \,,$$
$$\epsilon_1' \to \tilde{\gamma}_2 \epsilon_1' \gamma_2 \,,$$

do not change volume elements $\Pi\, de$. Thus (E9) is true in general.

(iv) Now in the neighborhood of the point in question, by (59)

$$\Delta_n = (\det Q)(\det \epsilon_1)(-1)^{n-1} + \text{higher orders}\,.$$

Using (C13) one obtains,

$$\Delta_n = (-1)^n \Delta_4 \det \epsilon_1 + \text{higher orders}\,.$$

Similarly

$$\Delta_n' = (-1)^n \Delta_4' \det \epsilon_1' + \text{higher orders}\,.$$

Taking the ratio of these two equations and using (E8) and the fact that at *all* points, by definition,

$$\Delta_n = U^2 \Delta_n' \,,$$

one obtains at the point in question

$$\det L = (\Delta_4/\Delta_4')^{\frac{1}{2}}\, U^{-1}\,. \tag{E11}$$

(E9) now yields the desired equation (E2).

Non-Abelian Vector Gauge Fields and the Electromagnetic Field

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INTRODUCTION

Hypothetical vector gauge fields have been introduced in order to give a deeper dynamical foundation for such internal properties as isotopic spin.[1] An essential aspect of isotopic spin is electrical charge, and there is no doubt about the dynamical relation of this property to the electromagnetic field. Do these different types of vector fields simply coexist, or can they be combined to form a more unified theory of vector gauge fields? An integrated formulation can indeed be given, and it is not a trivial one since there are definite dynamical implications with regard to electromagnetic properties and the structure of the non-Abelian transformation group. The unification can encompass all fields that partake in both strong and electromagnetic interactions.[2] This success poses a physical problem, however. As one member of a set of gauge fields, the electromagnetic field is not physically distinguished and fails to perform its physical role of destroying the conservation of isotopic spin. Perhaps it is in this apparent dilemma that we find the clue to the existence in nature of other sets of fields which possess electromagnetic interactions, but no strong interactions. Is it the presence of charged leptonic fields that denies the higher symmetry transformations, relating the electromagnetic field with the non-Abelian fields, and gives to the electromagnetic field its characteristic physical influence?

The inclusion of electromagnetic lepton interactions produces a new difficulty, one of consistency. The gauge invariance of all terms in the Lagrange function save one contradicts the principle of stationary action. Another term that violates gauge invariance must be included. The simplest choice is a mass term in which the mass constant is presumably small, on the scale of strongly interacting particle masses, if a domain of approximate gauge invariance is to exist. And this modification raises again the physical mass problem of gauge fields: Are unit spin particles of small mass implied by the theory?

UNIFIED THEORY

The Lagrange function of a non-Abelian vector gauge field coupled with a spin $\frac{1}{2}$ field is[3]

$$\mathfrak{L} = -\tfrac{1}{2} G^{\mu\nu}[\partial_\mu \phi_\nu - \partial_\nu \phi_\mu + (\phi_\mu i t' \phi_\nu)] + \tfrac{1}{4} G^{\mu\nu} G_{\mu\nu} + \tfrac{1}{2}\, i\psi \alpha^\mu (\partial_\mu - i\,``T'\phi_\mu")\,\psi + \tfrac{1}{2}\, i m \psi \beta \psi\,,$$

where the matrices t' and T' include coupling con-

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[1] C. N. Yang and R. Mills, Phys. Rev. **96**, 191 (1954).

[2] The problem of compatability has been given a more restricted discussion by R. Arnowitt and S. Deser, to be published.

[3] The notation follows J. Schwinger, Phys. Rev. **125**, 1043 (1962).

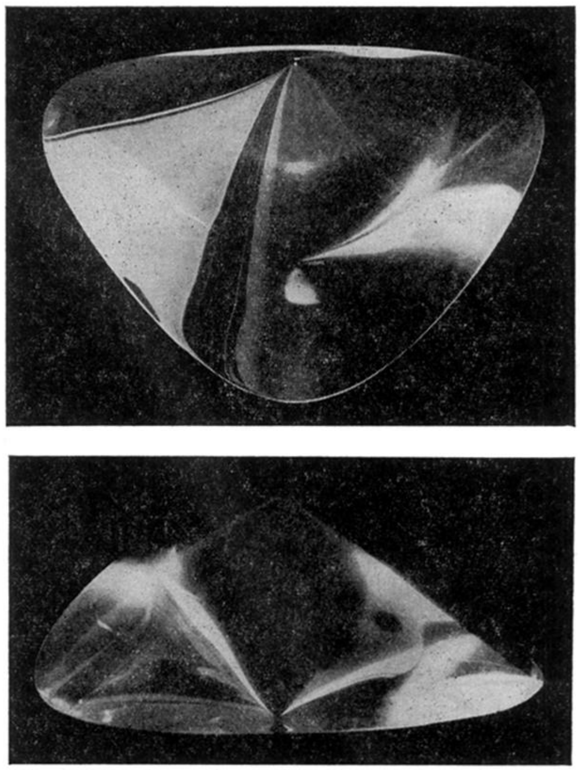

Fig. 2. Two views of a model of horn H_0 of universal surface (80) for decay into three particles. Surface (80) is a specialization of the one schematically represented in Fig. 1 to the case $m_1 = m_2 = m_3 = 1$. The base of the model is the plane $x + y + z = 63$, which is the Dalitz region for $m_1 = m_2 = m_3 = 1$, $m_A = (129)^{\frac{1}{2}}$. The model was made by J. W. Benoit, A. Lemonick, and Theodore Forseman of Princeton University.