Singular Potentials and Peratization.* I.

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I. INTRODUCTION

Recently, several explorations have been made in the domain of unrenormalizable field theories. Examples are spin-1 electrodynamics and various weak-interaction field theories. One does not know whether such theories actually exist in any sense. If it is assumed that they do, then one may hope that a way of obtaining meaningful finite answers is to be by a rearrangement of the perturbation expansion. In order to be able to even recognize the nature of the terms in this expansion it is necessary to introduce an (invariant) cutoff of some kind or other. The program is first to isolate the leading singular part of each term in the expansion, then to sum these parts, and next to see if one can give a finite meaning to this sum as the cut off tends to infinity. One then takes the next to leading singular part of the perturbation terms and proceeds likewise, till ultimately only finite parts of the perturbation terms survive.

For vector-meson electrodynamics one can in certain cases isolate the leading term in this new expansion as coming exclusively from a logarithmically divergent lowest radiative correction.[1] That is, in this case it is sufficient to assume the existence of the limit of certain infinite series of the kind mentioned above. If the program makes sense, then to leading order it is not necessary to calculate such limits. For the case of weak interactions such limits must be evaluated from the start, and to leading and next to leading order this has been done for an infinite subset of graphs.[2–4]

We do not review here any further the many problems which arise in this "peratization" program[5] which is in its early stages of development. Broadly speaking, it is a characteristic of these attempts that one applies somewhat unfamiliar techniques to a problem for which the suitability of the method is uncertain. In this respect, the situation is not dissimilar to one met earlier in another application of series of divergent terms. This is the quantum mechanical treatment of the virial expansion for a hard-sphere gas by the binary collision method.[6] While a hard-sphere gas is vastly less obscure than a relativistic interaction, it is nevertheless not known whether the binary collision expansion exists—though it seems quite plausible that the leading terms obtained in this way are correct. At any rate, it would seem helpful for the understanding of the methods under discussion to apply them to a problem in which there is no doubt about existence questions, and where, in fact, one knows the answers in explicit form from the outset.

A very convenient instance of this situation is provided by the quantum theory of repulsive singular potentials. This paper is devoted to the study of some examples of this kind.

We denote the potential by $gV(r)$, where g is a coupling constant. We call $V(r)$ singular if

$$\int_0^b r|V(r)|dr \qquad \text{is divergent} \qquad (1.1)$$

for any fixed $b > 0$. We restrict ourselves to such $V(r)$ that

$$\int_c^\infty r^2|V(r)|dr \qquad \text{exists} \qquad (1.2)$$

for any $c > 0$, in order not to be bothered by complications (of no interest for our purpose) arising from a too slow falling off at infinity. For a repulsive potential satisfying (1.1) and (1.2) there exists, of course, a solution to the scattering integral equation

$$\Psi(\mathbf{x}) = e^{i\mathbf{kx}} - \frac{g}{4\pi}\int d\mathbf{y}\,\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\,V(y)\Psi(\mathbf{y}), \quad (1.3)$$

even though this integral equation is "singular," in

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[1] The general theory of vector-meson electrodynamics is given by T. D. Lee and C. N. Yang, Phys. Rev. **128**, 885 (1962). For applications of the "logarithmic singularity method" see T. D. Lee, Phys. Rev. **128**, 899 (1962); J. Bernstein and T. D. Lee, Phys. Rev. Letters **11**, 512 (1963).

[2] G. Feinberg and A. Pais, Phys. Rev. **131**, 2724 (1963).

[3] G. Feinberg and A. Pais, Phys. Rev. **133**, 477B (1964).

[4] Y. Pwu and T. T. Wu, Phys. Rev., to be published.

[5] For such a review see A. Pais, "Methods and Problems in the Dynamics of Weak Interactions," in *Proceedings of the 1963 Sienna Conference*, to be published.

[6] T. D. Lee and C. N. Yang, Phys. Rev. **117**, 12 (1960).

the specific sense that every term of the Born series is divergent. For the purpose of illustrating the peratization idea this is just what we need: a divergent series that one wishes to sum to an answer known to exist.

The method we use bears resemblance to the treatment of the Bethe–Salpeter equation given earlier.[3] There we made the kernel nonsingular by regularization. Likewise we regulate here the potential itself by studying Eq. (1.3) for

$$V(r) \to V(r,\alpha)\,, \tag{1.4}$$

where

$$\int_0^b r|V(r,\alpha)|dr \qquad \text{exists}\,, \tag{1.5}$$

$$\int_c^\infty r^2|V(r,\alpha)|dr \qquad \text{exists}\,. \tag{1.6}$$

α is a parameter characterizing the regulated potential $V(r,\alpha)$.

As has been emphasized,[5] it is highly desirable to show for the field theoretical problems that, if finite results can be attained at all by resuming techniques, these should be independent of the way the cut off is introduced. This is largely an open problem. In the present case, however, we are able to give such a proof of independence of regularization. As is shown in Sec. II for a class of singular potentials, the conditions (1.5, 6) are *sufficient* for getting the desired result, we need never specify $V(r,\alpha)$ any further.

The specific examples to be discussed in Sec. II are the repulsive "power potentials" for which

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} - \frac{g}{r^m}\right]\psi(k,l,r) = 0\,,$$

$$\Psi(\mathbf{x}) = \frac{1}{r}\sum\nolimits_l \psi(k,l,r)Y_{l0}(\theta)\,, \tag{1.7}$$

and where $g > 0$. We have[7]

$$m > 3\,, \tag{1.8}$$

in order to satisfy Eq. (1.2). In the field theory case,[2] leading singularities could be associated with zero external momenta. The situation here turns out to be likewise, so we first focus attention on $k = 0$. In this case, the regular solution is,[8] for all l,

$$\psi(0,l,r) = r^{\frac{1}{2}}K_{(2l+1)/(m-2)}\left[\frac{2g^{\frac{1}{2}}}{m-2}\,r^{1-\frac{1}{2}m}\right]. \tag{1.9}$$

[7] The case $m < 3$ is discussed in L. Landau and E. Lifshitz, *Quantum Mechanics* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1958), pp. 404–405.

[8] K is the Bessel function for imaginary argument as defined in G. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, England, 1944), 2nd ed., p. 78.

For potentials as singular as the present ones, the zero-energy scattering is nevertheless pure S wave,[9] so we need only consider

$$\psi(0,0,r) \equiv \psi(r)\,. \tag{1.10}$$

At large distances,

$$\psi(r) = \text{const}\,(r + a)\,, \tag{1.11}$$

where a is the zero-energy scattering amplitude given by

$$a = -(\nu g^{\frac{1}{2}})^{2\nu}\{\Gamma(1-\nu)/\Gamma(1+\nu)\}\,,$$
$$\nu = 1/(m-2)\,. \tag{1.12}$$

In replacing $V(r)$ by $V(r,\alpha)$ we can imagine having chosen a g sufficiently small such that the Born expansion for the zero-energy integral equation converges.[10] (This equation is given in Sec. II.) This means, in particular, that the wavefunction corresponding to the regulated potential is analytic in g for small g. Nevertheless, in the limit $V(r,\alpha) \to V(r)$, we get a wavefunction that has a branch point in g at $g = 0$ [see Eq. (1.9)]. It has sometimes been thought[11] that this analyticity situation constitutes proof that peratization can not be applied in this instance. As we see in the next section this conclusion is not correct. There we provide an example [see Eq. (2.25)] which explicitly shows how the limit $\alpha = 0$ is reached. We take a specific form for $V(r,\alpha)$ for which we give both the wavefunction and the zero-energy amplitude in closed form. One can check directly in that case that for $\alpha \neq 0$ the amplitude is analytic in g near $g = 0$, and for $\alpha = 0$ it has an essential singularity in g at the origin.

It should be noted that for a singular potential there does not exist a Lippman–Schwinger equation.[12] In fact there does not even exist a Born term. This situation has in a sense already been encountered in the field theory of weak interactions, where it was nevertheless possible to give meaning to the solution after resuming a series with cutoff.[13] Likewise we can write down a Lippman–Schwinger equation for $V(r,\alpha)$. We show in Sec. III that the power count on that equation gives the correct dependence of the zero-energy scattering amplitude on g. In that section we also briefly discuss the case $k \neq 0$ and the case of singular potentials which are not of the power type. There we also give an instance of a closed form for the off-the-shell zero-energy scattering

[9] See Ref. 7, p. 405.

[10] R. Jost and A.Pais, Phys. Rev. **82**, 840 (1951).

[11] R. F. Sawyer, "Non-renormalizability and the short range force in some field-theoretic models," preprint.

[12] B. Lippman and J. Schwinger, Phys. Rev. **79**, 469 (1950).

[13] The "forbidden" leptonic processes fall in this category, see Refs. 2, 3.

amplitude which depends on one momentum variable q and which for $q \neq 0$ develops a logarithmic dependence on g [see Eq. (3.8)]. (Such situations have also been conjectured as happening in field theory, see Ref. 3, Appendix B).

It should be emphasized that there is an essential difference between the potential and the field theoretical problem. After all, a regulated potential is still a bona fide potential, while a regulated-field theory does not satisfy the same postulates as an unregulated one. Nevertheless, we believe that the present examples are illuminating in regard to the method of summing series of divergent terms. In particular, the choice of order of limits: first integrate over small coordinate distances, then let a cut off tend to zero, can apparently be made with impunity. We are encouraged to think that perhaps also in field theory the corresponding prescribed order of operations may not be so arbitrary, and that the rules of peratization may be part of a more general mathematical discipline.

II. PERATIZATION OF THE ZERO-ENERGY INTEGRAL EQUATION

Our starting point is the zero-energy scattering integral equation for regulated repulsive power potentials. Clearly, there are infinitely many ways of defining functions $V(r,\alpha)$ which satisfy (1.5, 6) and the limit

$$\lim_{\alpha\to 0} V(r,\alpha) = r^{-m}, \qquad m > 3\,. \tag{2.1}$$

To identify these different ways of regulating the potentials g/r^m we write

$$V(y,\alpha) = \alpha^{-m}U(y/\alpha)\,. \tag{2.2}$$

The conditions (1.5) and (1.6) imply

$$\int_0^a \zeta U(\zeta)d\zeta < \infty\ ;\quad \int_b^\infty \zeta^2 U(\zeta)d\zeta < \infty\,. \tag{2.3}$$

Also, if (2.1) is to hold, $U(\zeta)$ must vanish asymptotically as ζ^{-m} as $\zeta \to \infty$. We write $V(y,\alpha)$ in the form (2.2) to emphasize the dependence of regularization on a parameter α and a structural form U. Equation (2.2) is also useful for power counting in momentum space, see Sec. III.

According to Eq. (1.3) the zero-energy scattering integral equation for the potentials (2.2) is

$$\Psi(\mathbf{x};\alpha,U) = 1 - \frac{g}{4\pi}\int \frac{V(y,\alpha)\Psi(\mathbf{y};\alpha,U)}{|\mathbf{x}-\mathbf{y}|}\,d^3y\,. \tag{2.4}$$

The wavefunction, of course, depends on our choice of the functional form U of the regulated potential, as well as on the cutoff parameter α.

For $\alpha \neq 0$ Eq. (2.4) has a unique and convergent iterative solution for each choice of U, provided g is taken small enough to make the Born series converge. Furthermore, the solution of (2.4) only depends on $|\mathbf{x}|$. We have

$$\Psi(x;\alpha,U) = 1 - \frac{g}{x}\int_0^x y^2dyV(y,\alpha)\Psi(y;\alpha,U) - g\int_x^\infty ydyV(y,\alpha)\Psi(y;\alpha,U)\,. \tag{2.5}$$

In the limit as $\alpha \to 0$ both Eqs. (2.4) and (2.5) are singular in the sense that each term in the Born series for (2.5) blows up as $\alpha \to 0$, and one is led to a situation very similar to that faced in studying the Bethe–Salpeter equation for singular field theories. Working directly with (2.5) we show that not only does the limit of the solution exist as $\alpha \to 0$, but also that this limit is independent of the choice of U and is identical with the correct answer given in (1.9). We next give a prescription which seems well suited for dealing with such highly singular equations.

The method of attack is closely related to that used in Ref. 3. The idea is to separate Eq. (2.5) into two equations, one of which is singular for $\alpha \to 0$, and one which is regular in that limit. We write

$$\Psi(x;\alpha,U) = \Psi_1(x;\alpha,U) + \Psi_2(x;\alpha,U)\,, \tag{2.6}$$

where now Ψ_1 and Ψ_2 satisfy the following integral equations,

$$\Psi_1(x;\alpha,U) = -\frac{g}{x}\int_0^\infty y^2dyV(y,\alpha) \times [\Psi_1(y;\alpha,U) + \Psi_2(y;\alpha,U)]\,, \tag{2.7}$$

$$\Psi_2(x;\alpha,U) = 1 - \frac{g}{x}\int_x^\infty ydy(x-y)V(y,\alpha)\Psi_1(y;\alpha,U) - \frac{g}{x}\int_x^\infty ydy(x-y)V(y,\alpha)\Psi_2(y;\alpha,U)\,. \tag{2.8}$$

Equations (2.7) and (2.8) are identical with (2.5). For $\alpha \neq 0$ a solution always exists and from (2.7) Ψ_1 is given by

$$\Psi_1(x;\alpha,U) = a(\alpha,U)/x\,. \tag{2.9}$$

The constant $a(\alpha,U)$ is upon closer inspection seen to be nothing but the zero-energy scattering amplitude. Substituting, (2.9) in (2.8) we obtain a simple Volterra-type equation for Ψ_2,

$$\Psi_2(x;\alpha,U) = 1 - ga(\alpha,U)f(x;\alpha,U) - \frac{g}{x}\int_x^\infty ydy(x-y)V(y,\alpha)\Psi_2(y;\alpha,U)\,, \tag{2.10}$$

where

$$f(x;\alpha,U) = \frac{1}{x}\int_x^\infty dy V(y,\alpha)(x-y)\,. \quad (2.11)$$

The solution of (2.10) is of the form

$$\Psi_2(x;\alpha,U) = \Psi_2^{(1)}(x;\alpha,U) + a(\alpha,U)\Psi_2^{(2)}(x;\alpha,U)\,, \quad (2.12)$$

where the dependence on a has been explicitly factored out.

Let us for the moment assume that the limit of $a(\alpha,U)$ as $\alpha\to 0$ exists. (We show below that this is the case.) Then the remarkable thing about (2.10) is that because of its Volterra form it is nonsingular as $\alpha\to 0$. In the limit as $\alpha\to 0$ (2.10) becomes

$$\Psi_2(x;0,U) = \left[1+\frac{ga(0,U)}{(m-1)(m-2)x^{m-1}}\right] - \frac{g}{x}\int_x^\infty dy\left[\frac{x-y}{y^{m-1}}\right]\Psi_2(y;0,U)\,. \quad (2.13)$$

This last equation depends on U only through the constant $a(0,U)$. Furthermore, for any $m>3$, Eq. (2.13) can be solved exactly, and gives[14] [using (2.12)]

$$\Psi_2^{(1)}(x;0,U) = (\nu g^{\frac{1}{2}})^{\nu}\Gamma(1-\nu)x^{-\frac{1}{2}}I_{-\nu}(z)\,,$$
$$\Psi_2^{(2)}(x;0,U) = (\nu g^{\frac{1}{2}})^{-\nu}\Gamma(1+\nu)x^{-\frac{1}{2}}I_{\nu}(z) - 1/x\,, \quad (2.14)$$

where

$$\nu = 1/(m-2)$$
$$z = 2\nu g^{\frac{1}{2}}x^{-\frac{1}{2}\nu}\,. \quad (2.15)$$

One can easily check that both expressions in (2.14) are analytic in g and have no essential singularities for $g=0$. The constants $a(0,U)$ however also depend on g and, as we see in a moment, they have an essential singularity at $g=0$.

Let us now compute $a(0,U)$. From (2.7) and (2.9) we have

$$a(\alpha,U) = \lim_{\sigma\to 0} -g\int_\sigma^\infty y^2 dy V(y,\alpha) \times[\Psi_1(y;\alpha,U)+\Psi_2(y;\alpha,U)]\,. \quad (2.16)$$

At this stage, with $\alpha\neq 0$, the σ-limit is superfluous; however we need it later on. Substituting for Ψ_1 the expression (2.9) and for Ψ_2 (2.12) we obtain

$$a(\alpha,U) = \lim_{\sigma\to 0}\frac{-g\int_\sigma^\infty y^2 dy V(y,\alpha)\Psi_2^{(1)}(y;\alpha,U)}{1+g\int_\sigma^\infty y^2 dy V(y,\alpha)[\Psi_2^{(2)}(y;\alpha,U)+1/y]}\,. \quad (2.17)$$

[14] $I_\nu(z)$ is as defined in Ref. 8, p. 77.

Now let $\alpha\to 0$ and interchange the order of the limits; we get from the known limits in (2.14)

$$a(0,U) = \lim_{\sigma\to 0}\frac{-g\int_\sigma^\infty y^{\frac{3}{2}-m}dy I_{-\nu}(z)(\nu g^{\frac{1}{2}})^{\nu}\Gamma(1-\nu)}{1+g\int_\sigma^\infty y^{\frac{3}{2}-m}dy I_{\nu}(z)(\nu g^{\frac{1}{2}})^{-\nu}\Gamma(1+\nu)}\,. \quad (2.18)$$

The integrals in both numerator and denominator can be done explicitly. They are of course divergent as $\sigma\to 0$; however they diverge in exactly the same way and the limit gives

$$a(0,U) = -(\nu g^{\frac{1}{2}})^{2\nu}\{\Gamma(1-\nu)/\Gamma(1+\nu)\}\,. \quad (2.19)$$

This last expression, which is also the zero-energy scattering amplitude for the potential gr^{-m}, is clearly independent of the choice of U. One also notes that it is the only possible expression for $a(0,U)$ which when substituted in (2.12) and (2.6) gives a solution $\Psi(x;0,U)$ which is not divergent for $x\to 0$. Indeed one could have obtained $a(0,U)$ from (2.12) by imposing this condition. It is evident that (2.19) coupled with (2.14) and (2.6) gives the correct answer (1.9) and (1.10) for the zero-energy wavefunction, as[15]

$$K_\nu(z) = \tfrac{1}{2}\pi(1/\sin\pi\nu)[I_{-\nu}(z) - I_\nu(z)]\,. \quad (2.20)$$

For certain choices of the regulator U one can solve (2.10) exactly in closed form for $\alpha\neq 0$, and directly check the limits we have taken. The simplest such choice we can think of is to take

$$U(\zeta) = U_1(\zeta) = \frac{1}{(1+\zeta)^m}\,, \quad (2.21)$$

which corresponds to $V(r,\alpha) = (r+\alpha)^{-m}$. With this potential we can by a simple change of variables reduce (2.10) to a form very similar (2.13). One obtains

$$\begin{aligned}\Psi_2(x;\alpha,U_1) &= \frac{x+\alpha}{x}\{(\nu g^{\frac{1}{2}})^{\nu}I_{-\nu}(z_+)\Gamma(1-\nu)(x+\alpha)^{-\frac{1}{2}}\} \\ &+ a(\alpha,U)\frac{(x+\alpha)}{x}\left\{(\nu g^{\frac{1}{2}})^{-\nu}I_\nu(z_+)\times\Gamma(1+\nu)(x+\alpha)^{-\frac{1}{2}} - \frac{1}{(x+\alpha)}\right\} \\ &- \alpha\frac{(x+\alpha)}{x}\left\{(\nu g^{\frac{1}{2}})^{-\nu}I_\nu(z_+)\Gamma(1+\nu)(x+\alpha)^{-\frac{1}{2}}\right\}\,,\end{aligned} \quad (2.22)$$

where

$$z_+ = 2\nu g^{\frac{1}{2}}(x+\alpha)^{-\frac{1}{2}\nu}\,. \quad (2.23)$$

[15] Reference 8, p. 77.

On the other hand, Ψ_1 is still given by (2.9) and the full wavefunction is

$$\Psi(x;\alpha,U_1) = [a(\alpha,U_1)/x] + \Psi_2(x;\alpha,U_1)\,, \quad (2.24)$$

with Ψ_2 given by (2.22). Using (2.17) we can now obtain the explicit expression for the zero-energy scattering amplitude $a(\alpha,U_1)$.

$$a(\alpha,U_1) = -\alpha\left[\frac{(\frac{1}{2}Z_\alpha)^\nu I_{-\nu}(Z_\alpha)\Gamma(1-\nu)}{(\frac{1}{2}Z_\alpha)^{-\nu} I_\nu(Z_\alpha)\Gamma(1+\nu)} - 1\right], \quad (2.25)$$

where

$$Z_\alpha = 2\nu\, g^{\frac{1}{2}}\alpha^{-\frac{1}{2}\nu}\,. \quad (2.26)$$

One checks that in the limit $\alpha \to 0$ (2.25) goes into (2.19) and that (2.22) goes into (2.14). Furthermore one sees from the definition[14] of the I_ν functions that $a(\alpha,U_1)$ as given in (2.26) is analytic in g. More precisely, it is a ratio of two entire functions in g. However [see (2.19)], in the limit as $\alpha \to 0$, $a(0,U)$ has an essential singularity at $g = 0$ (a branch point of order $1/\nu$). Thus (2.25) gives a clear counterexample to the claims made by Sawyer.[11] A function analytic in g but depending on a parameter α can easily develop an essential singularity in g in the limit $\alpha = 0$ even if it had been analytic at that value of g for all $\alpha \neq 0$. The function $a(\alpha,U_1)$ is meromorphic in g and has a power-series expansion in g with a nonzero radius of convergence. Equation (2.25) could of course have been obtained by summing a Born series instead of using our method.

Finally, as a function of α, the amplitude $a(\alpha,U_1)$ has some curious analytic properties. Both numerator and denominator in (2.25) have essential singularities in α at $\alpha = 0$. However, the limit of $I_\nu(x)/I_{-\nu}(x)$ as $x \to \infty$ (along the real axis) is unity.[14] At this point it is clear that the reality of α and of $V(r,\alpha)$ is important!

III. CONCLUDING REMARKS

We first ask what one can learn by applying power counting to the Lippmann–Schwinger integral equation in momentum space. For convenience we write the scattering amplitude, $F(k_f,k_i;\alpha)$, in the form

$$F(\mathbf{k}_f,\mathbf{k}_i;\alpha) = f(\alpha\mathbf{k}_f,\alpha\mathbf{k}_i;\alpha)\,. \quad (3.1)$$

The Lippmann–Schwinger equation for f now reads

$$f(\alpha\mathbf{k}_f,\alpha\mathbf{k}_i;\alpha) = \frac{-g}{4\pi}\frac{\tilde U(\alpha|\mathbf{k}_f-\mathbf{k}_i|)}{\alpha^{m-3}} - \frac{g}{(2\pi)^3\alpha^{m-2}}\int d^3p\,\frac{\tilde U(|\alpha\mathbf{k}_f-\mathbf{p}|)\,f(\mathbf{p},\alpha\mathbf{k}_i,\alpha)}{p^2-\alpha^2k^2-i\epsilon}\,, \quad (3.2)$$

Here $\tilde U$ is the three-dimensional Fourier transform of U, and the variable $\mathbf{p}$ is dimensionless. Iterating (3.2) and taking leading singular terms in each order we get

$$F(\mathbf{k}_f,\mathbf{k}_i;\alpha) = \alpha\sum_{n=1}^{\infty}\left(\frac{g}{\alpha^{m-2}}\right)^n a_n + \text{less singular terms}\,. \quad (3.3)$$

The a_n's are not dependent on k and are essentially the Born terms of order n evaluated with $\alpha k = 0$. It is obvious, therefore, that the series of leading singular terms in (3.3) is identical with the exact series for the zero-energy amplitude $F(0,0;\alpha)$. Put

$$\lambda = g/\alpha^{m-2}\,. \quad (3.4)$$

Then for small k we get to a good approximation

$$F(\mathbf{k}_f,\mathbf{k}_i;\alpha) \approx g^{1/(m-2)}\frac{\sum_n \lambda^n a_n}{1/\lambda^{(m-2)}} \approx g^{1/(m-2)}W(\lambda)\,, \quad (3.5)$$

and as $\lambda \to \infty$

$$F(\mathbf{k}_f,\mathbf{k}_i;0) \approx g^{1/(m-2)} \times \text{const}\,. \quad (3.6)$$

The constant is of course the same as that of the zero-energy amplitude and we get for $F(\mathbf{k}_f,\mathbf{k}_i,0)$ the expression (2.19) apart from k-dependent terms which are of higher order in g.

The question arises whether there exists a k domain for which the zero-energy scattering amplitude a is still a good approximation to the full amplitude. At this point for the first time the question of the magnitude of g enters, as the k-region in question is given by

$$kg^\nu \ll 1\,. \quad (3.7)$$

It follows immediately from power counting on (3.2) that this combination is decisive.

Equation (3.7) is not meant to imply that the amplitude $F(\mathbf{k}_f,\mathbf{k}_i;0)$ is analytic in k near $k = 0$. It probably is not as the following result indicates. Take the case $m = 4$ for which the zero-energy off-shell amplitude $F(q,0;0)$ takes a particularly simply form:

$$F(q,0;0) = -gq^{-1}\int_0^\infty \sin qr\, r^{-1}\exp(-g^{\frac{1}{2}}/r)dr \quad (3.8)$$
$$= g^{\frac{1}{2}}[K_2(\beta) + K_2(\beta^*)]$$

with

$$\beta = 2g^{\frac{1}{4}}q^{\frac{1}{2}}e^{\frac{1}{4}i\pi}\,.$$

This equation shows first that the zero-energy off-shell amplitude is of order $g^{\frac{1}{2}}$ for $g^{\frac{1}{2}}q \ll 1$ in accordance with Eq. (3.7). However, $F(q,0)$ has a logarithmic branch point at $q = 0$.

Equation (3.7) is the analog of the relation $gk/m \ll 1$ which in W-field theory[2] defines the "low-energy regime" (g = meson lepton coupling constant, $m = W$ mass), and $F(q,0)$ is in fact the analog of the zero-energy Bethe–Salpeter amplitude discussed in reference 3. While we know little about the high-energy regime in field theory ($gk \gg m$), it is interesting to note the high-energy exponential damping exhibited in the potential problem by Eq. (3.8).

Unlike the W-field theory, the amplitude for a power potential has only a leading singular series for $k = 0$, no subsequent summations are called for at fixed energy. This is because we only have one constant g in the potential case, while there are two (g and m) in field theory. We can also study potentials with more constants. Take for example singular short-range potentials like

$$V_\mu(r) = -g(e^{-\mu r}/r^m) . \qquad (3.9)$$

By power counting one readily verifies that the zero-energy scattering amplitude for $V_0(r)$ is a good approximation to the one for $V_\mu(r)$ as long as g and μ satisfy

$$\mu g^{1/(m-2)} \ll 1 . \qquad (3.10)$$

In conclusion we wish to emphasize that for singular power potentials the peratization program is completed once the case $k = 0$ has been understood. This is so because the knowledge of the zero-energy wavefunction makes it possible to reduce the integral equation (1.3) for $k \neq 0$ to one which now is "regular" in the sense that the Neumann series does exist.

We show this for the case $m = 4$ and limit ourselves to the S-wave case. Once the zero-energy solutions are known one can obtain a regular Volterra integral equation for the nonzero energy case.[16] The zero-energy solutions of (1.7) in the present case are $xe^{-g^{\frac{1}{2}}/x}$ and $xe^{+g^{\frac{1}{2}}/x}$. If we now write

$$\psi(k,0,r) = re^{-g^{\frac{1}{2}}/r} f(k,r) ,$$

we get for f the integral equation

$$f(k,x) = 1 - \frac{k^2}{2g^{\frac{1}{2}}} \int_0^x y^2 dy f(k,y) + \frac{k^2}{2g^{\frac{1}{2}}} \int_0^x y^2 dy f(y) e^{2g^{\frac{1}{2}}/x - 2g^{\frac{1}{2}}/y} . \qquad (3.11)$$

This is an iterable Volterra equation and leads to a function analytic in k^2.

[16] E. Predazzi and T. Regge, Nuovo Cimento **24**, 518 (1962).

Physical Regions in Invariant Variables for n Particles and the Phase-Space Volume Element

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In considering the reactions

$$A \to 1 + 2 + \cdots + n \qquad (1)$$

or

$$A + \bar{1} \to 2 + 3 + \cdots + n , \qquad (2)$$

it is oftentimes natural to express probabilities, (i.e., square of matrix elements) in terms of the scalar products of the 4-momenta of the particles involved. (For particles with spin, a similar situation obtains after averaging over the spins of the particles involved.) Denoting by $\{i,j\}$ the scalar product of the 4-momenta of particles i and j, one may ask whether it is possible and convenient to pursue subsequent calculations (such as for the total probability) entirely in terms of these quantities $\{i,j\}$. These considerations[1] led to the investigations described in the present paper.

We discuss the following three questions.

(a) What are the kinematically allowed values of the variables $\{i,j\}$ for reactions (1) or (2)?

[1] For somewhat related discussions, see V. E. Asribekov, Zh. Eksperim. i Teor. Fiz. **42**, 565 (1962) [English Transl. Soviet Phys.–JETP **15**, 394 (1962)]; J. Tarski, J. Math. Phys. **1**, 149 (1960); B. Jacobsohn, Bull. Am. Phys. Soc. **7**, 503 (1962); and D. Hall and A. S. Wightman, Mat. Fys. Medd. Dan. Vid. Selsk **31**, No. 5 (1957).