Cluster Property of the Correlation Functions of Classical Gases

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INTRODUCTION

It has been shown recently by several authors[1-5] that, under adequate assumptions on the interaction potential, the virial expansion has a finite radius of convergence. Two essentially different methods have been used. In Ref. 1, the absolute integrability of the Ursell functions was established, in Refs. 2 and 3 the boundedness of the correlation functions was used. The aim of this note is to prove a property of absolute integrability similar to that obtained in Ref. 1, but valid for a larger class of potentials. This property is the cluster decomposition property, or cluster property, of the correlation functions. To derive it we will use the ideas and results of Refs. 2 and 3. Furthermore, we will need an algebraic apparatus which we proceed now to describe (see also Ref. 4).

1. THE MAPPING Γ

We will denote by $(x)_n$ the family $(x_1, x_2, \cdots, x_n)$ of vectors x_i in ν-dimensional Euclidean space. Let A be the complex vector space of all sequences ψ

$$\psi = (\psi(x)_n)_{0 \leq n < \infty} \tag{1.1}$$

of classes[6] of bounded (Lebesgue-) measurable functions $\psi(x)_n$. The 0-th component ψ_0 of ψ is thus a complex number. If X is any finite sequence of vectors in ν-dimensional Euclidean space we may write $X = (a_1, \cdots, a_n)$ and define

$$\psi(X) = \psi(a)_n . \tag{1.2}$$

Let now $\psi_1 \in A$, $\psi_2 \in A$, we write

$$\psi(X) = \sum_{Y \subset X} \psi_1(Y)\psi_2(X - Y) , \tag{1.3}$$

where the summation extends over all subsequences Y of X and $X - Y$ is the complement of the subsequence Y in X, i.e., the sequence of the elements of X not contained in Y, in the same order. For the new element $\psi \in A$ defined by (1.3) we will write

$$\psi = \psi_1 * \psi_2 . \tag{1.4}$$

It is easily seen that the product (1.4) is associative and commutative and defines on A a structure of integrity ring with unit element $\mathbf{1}$. We will call A_+ the ideal of A formed by the elements ψ such that $\psi_0 = 0$, then A is the direct sum (as vector space over the complex field) of A_+ and of the subspace generated by $\mathbf{1}$.

Let α be a sequence of functions $\alpha(x)_n$ defined by

$$\alpha_0 = 1 , \quad \alpha(x)_n = \prod_{i=1}^{n} \alpha(x_i) , \tag{1.5}$$

where $\alpha(x)$ is Lebesgue-integrable. To any $\psi \in A$ we may then associate the formal power series

$$\langle \alpha,\psi \rangle(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int d(x)_n \alpha(x)_n \psi(x)_n . \tag{1.6}$$

Lemma. If $\psi_1 \in A$, $\psi_2 \in A$ and $\psi = \psi_1 * \psi_2$ the following identity between formal power series holds:

$$\langle \alpha,\psi \rangle(z) = \langle \alpha,\psi_1 \rangle(z) \langle \alpha,\psi_2 \rangle(z) . \tag{1.7}$$

The verification is immediate (see Ref. 4).

We define now a mapping Γ from A_+ to A as the restriction to A_+ of the exponential

$$\Gamma\phi = \mathbf{1} + \phi + \frac{\phi * \phi}{2} + \frac{\phi * \phi * \phi}{3!} + \cdots . \tag{1.8}$$

Obviously Γ maps A_+ onto $\mathbf{1} + A_+$ and has a unique inverse Γ^{-1}. It is easily seen (see Ref. 4) that $(\Gamma\phi)(x)_n$ is the sum of the products

$$\phi(X_1) \cdots \phi(X_r) \tag{1.9}$$

corresponding to all partitions of $(x)_n$ into subsequences $X_1, \cdots X_r$ for $1 \leqslant r \leqslant n$. From (1.7) and (1.8) we obtain the following identity between formal power series

$$\langle \alpha, \Gamma\phi \rangle(z) = \exp [\langle \alpha,\phi \rangle(z)] \tag{1.10}$$

We define now a derivation D_x in A by the relation[7]

$$(D_x\psi)(x)_n = \psi(x,x_1,x_2,\cdots,x_n) . \tag{1.11}$$

[1] J. Groeneveld, Phys. Letters **3**, 50 (1962).
[2] D. Ruelle, Ann. Phys. **25**, 109 (1963).
[3] O. Penrose, J. Math. Phys. **4**, 1312 (1963).
[4] D. Ruelle, Lecture notes of the Theoretical Physics Institute (1963), University of Colorado, Boulder.
[5] J. L. Lebowitz and O. Penrose (to be published).
[6] Two functions are said to belong to the same class if they differ only on a set of Lebesgue measure zero.
[7] Actually $D_x\psi$ does not make sense at a point x, one should consider the expressions $D_\gamma\psi = \int dx\ \gamma(x)D_x\psi$ defined for Lebesgue-integrable γ.

It is easily seen that D_x is linear and satisfies

$$D_x(\psi_1 * \psi_2) = (D_x\psi_1) * \psi_2 + \psi_1 * (D_x\psi_2) , \tag{1.12}$$

and therefore also

$$D_x(\Gamma\phi) = D_x\phi * \Gamma\phi . \tag{1.13}$$

More generally if X is the sequence $(a_1,\cdots,a_n)$ we write

$$D_X\psi = D_{a_1}\cdots D_{a_n}\psi . \tag{1.14}$$

2. GRAND PARTITION FUNCTION AND CORRELATION FUNCTIONS

We assume that a sequence of real functions

$$U(x)_n = U(x_1,\cdots,x_n) \tag{2.1}$$

is given such that $U_0 = 0$ and

$$U(x)_n \geqslant -nB , \tag{2.2}$$

where B is a nonnegative constant. $U(x)_n$ is supposed to be measurable and may take the value $+\infty$. We write

$$\psi(x)_n = e^{-\beta U(x)n} , \tag{2.3}$$

where β is positive. $U(x)_n$ is interpreted as the potential energy of a system of n particles located at the points $x_1, x_2, \cdots, x_n$ and β as $1/kT$, where T is the absolute temperature. It follows from (2.2) that (1.6) defines an entire function $\langle\alpha,\psi\rangle(z)$ with value 1 for $z = 0$. Its logarithm is thus holomorphic in a neighborhood of the origin and it follows from (1.10) that

$$\Xi = \langle\alpha,\psi\rangle(z) = \exp[\langle\alpha,\Gamma^{-1}\psi\rangle(z)] . \tag{2.4}$$

If $\alpha^\Lambda(x)$ is the characteristic function of a sphere Λ,

$$\Xi = \langle\alpha^\Lambda,\psi\rangle(z) \tag{2.5}$$

is called the grand partition function of Λ. Let

$$(\psi^\Lambda(z))(X) = \Xi^{-1}\langle\alpha^\Lambda,D_X\psi\rangle(z) \tag{2.6}$$

then, the correlation functions are defined by

$$\rho^\Lambda(x)_n = z^n\alpha^\Lambda(x)_n(\psi^\Lambda(z))(x)_n , \quad n > 0 . \tag{2.7}$$

They are meromorphic in z, without pole on the positive real axis. We will by convention put $\rho^\Lambda_0 = 0$ so that $\rho^\Lambda \in A_+$.

Let us define an element $\tilde\phi_X$, $X \neq \varnothing$, of A_+ by

$$\tilde\phi_X = \psi^{-1} * D_X\psi \tag{2.8}$$

We have then the following identity between power series convergent in a neighborhood of the origin

$$\begin{aligned}(\psi^\Lambda(z))(X) &= \Xi^{-1}\langle\alpha^\Lambda,\psi * (\psi^{-1} * D_X\psi)\rangle(z)\\ &= \Xi^{-1}\langle\alpha^\Lambda,\psi\rangle(z)\langle\alpha^\Lambda,\psi^{-1} * D_X\psi\rangle(z)\\ &= \langle\alpha^\Lambda,\tilde\phi_X\rangle(z) .\end{aligned} \tag{2.9}$$

Let $\psi = \Gamma\phi$ and define

$$(\phi^\Lambda(z))(X) = \langle\alpha^\Lambda,D_X\phi\rangle(z) . \tag{2.10}$$

Applying (1.13) repeatedly to (2.6) we find

$$\psi^\Lambda(z) = \Gamma[\phi^\Lambda(z)] . \tag{2.11}$$

If we write

$$\mathbf{1} + \rho^\Lambda = \Gamma(\chi^\Lambda) , \tag{2.12}$$

it follows from (2.7) and (2.11) that

$$\chi^\Lambda(x)_n = z^n\alpha^\Lambda(x)_n(\phi^\Lambda(z))(x)_n . \tag{2.13}$$

Notice finally that because of (1.13) we have

$$D_x\phi = \psi^{-1} * D_x\psi = \tilde\phi_{(x)} . \tag{2.14}$$

3. TWO-BODY POTENTIALS

We assume from now on that the potential energy $U(x)_n$ is given by a two-body interaction

$$U(x)_n = \sum_{1\leq i<j\leq n} \Phi(x_j - x_i) , \tag{3.1}$$

where $\Phi(x)$ is a real measurable function which may take the value $+\infty$ and depends on $|x|$ only,[8] where $|x|$ is the Euclidean norm of x.

Let us impose the following requirements on Φ (see Refs. 2 and 4)

$$(\Phi\,1) \qquad U(x)_n \geqslant -nB , \quad \textit{for some } B \geqslant 0 , \tag{3.2}$$

$$(\Phi\,2) \qquad C(\beta) = \int dx\,|e^{-\beta\Phi(x)} - 1| < +\infty$$

$$\textit{for some } \beta > 0 . \tag{3.3}$$

(Φ 1) is simply a repetition of (2.2); it implies that $\Phi(x)$ is bounded from below and therefore (3.3) holds for any $\beta > 0$.

Define

$$V^x(x)_n = \sum_{i=1}^n \Phi(x_i - x) , \tag{3.4}$$

$$\gamma^x(x)_n = \prod_{i=1}^n (e^{-\beta\Phi(xi-x)} - 1) . \tag{3.5}$$

We have then, with the previous definitions,

$$(D_x\psi)(x)_n = e^{-\beta V^x(x)n}\psi(x)_n \tag{3.6}$$

$$e^{-\beta V^x(x)n} = \sum_{S\subset(x)_n} \gamma^x(S) \tag{3.7}$$

where in (3.7) the sum extends over all subsequences S of $(x)_n$. Let $x \in X$ and $X' = X - (x)$, we have then

$$\tilde\phi_X = \psi^{-1} * D_X\psi = \psi^{-1} * D_xD_{X'}\psi , \tag{3.8}$$

[8] This condition is natural physically but mathematically unnecessary. As pointed out by Penrose (Ref. 3), one might also use a ν-dimensional lattice Z^ν instead of ν-dimensional space R^ν.

where

$$(D_x D_{X'}\psi)(x)_n = e^{-\beta V^x(X')} e^{-\beta V^x(x)_n} \psi(X',(x)_n) . \tag{3.9}$$

Therefore, using (3.7)

$$\begin{aligned}
\tilde{\phi}_X(Y) &= \sum_{Z\subset Y} \psi^{-1}(Y-Z) D_x D_{X'} \psi(Z) \\
&= \sum_{Z\subset Y} \psi^{-1}(Y-Z) e^{-\beta V^x(X')} e^{-\beta V^x(Z)} \psi(X',Z) \\
&= e^{-\beta V^x(X')} \sum_{Z\subset Y} \psi^{-1}(Y-Z) e^{-\beta V^x(Z)} \psi(X',Z) \\
&= e^{-\beta V^x(X')} \sum_{Z\subset Y} \sum_{S\subset Z} \psi^{-1}(Y-Z) \gamma^x(S) \psi(X',Z) \\
&= e^{-\beta V^x(X')} \sum_{S\subset Y} \gamma^x(S) \sum_{S\subset Z\subset Y} \psi^{-1}(Y-Z) \psi(X',Z) .
\end{aligned} \tag{3.10}$$

If we write $T = Z - S$ we obtain finally

$$\begin{aligned}
\tilde{\phi}_X(Y) &= e^{-\beta V^x(X')} \sum_{S\subset Y} \gamma^x(S) \sum_{T\subset Y-S} \psi^{-1}(Y-S-T) \\
&\quad \times \psi(X',S,T) \\
&= e^{-\beta V^x(X')} \sum_{S\subset Y} \gamma^x(S) \tilde{\phi}_{(X',S)}(Y-S)
\end{aligned} \tag{3.11}$$

From (3.11) we will now derive by induction on $m + n$ the inequality

$$\begin{aligned}
\sup_{(x)_m} \int d(y)_n \, |\tilde{\phi}_{(x)_m}(y)_n| &\leqslant n! \xi^{m-1} \\
&\quad \times \{e^{2\beta B} \xi^{-1} \exp [C(\beta)\xi]\}^{m+n-1}
\end{aligned} \tag{3.12}$$

for any positive ξ. Notice first that for $m + n = 1$, i.e., $m = 1$ and $n = 0$, the left and the right-hand sides of (3.12) are equal to 1. Let now $X = (x)_M$, $Y = (y)_N$ and let (3.12) hold for $m + n < M + N$. Then

$$\begin{aligned}
&\int dY \, |\sum_{S\subset Y} \gamma^x(S) \tilde{\phi}_{(X',S)}(Y-S)| \\
&\leqslant \sum_{S\subset Y} \int dS \, |\gamma^x(S)| \int d(Y-S) \, |\tilde{\phi}_{(X',S)}(Y-S)| \\
&\leqslant \sum_{l=0}^{N} \frac{N!}{l!(N-l)!} [C(\beta)]^l (N-l)! \xi^{M+l-2} \\
&\qquad \times \{e^{2\beta B} \xi^{-1} \exp [C(\beta)\xi]\}^{M+N-2} \\
&< e^{-2\beta B} N! \xi^{M-1} \{e^{2\beta B} \xi^{-1} \exp [C(\beta)\xi]\}^{M+N-1} \\
&\qquad \times \exp [-C(\beta)\xi] \sum_{l=0}^{\infty} \frac{[C(\beta)\xi]^l}{l!} .
\end{aligned} \tag{3.13}$$

Therefore, by (3.11), for any $x \in X$

$$\begin{aligned}
\int dY \, |\tilde{\phi}_X(Y)| &\leqslant e^{-\beta V^x(X')} e^{-2\beta B} N! \xi^{M-1} \\
&\quad \times \{e^{2\beta B} \xi^{-1} \exp [C(\beta)\xi]\}^{M+N-1} .
\end{aligned} \tag{3.14}$$

The conclusion follows by taking the geometric mean $e^{-2\beta U(X)/N} \leqslant e^{2\beta B}$ of the factors $e^{-\beta V^x(X')}$ for the different choices of x in X. However, if $M = 1$, then $V^x(X') = 0$ and, from (2.14), we get

$$\begin{aligned}
&\int dx_2 \cdots dx_n |\phi(x_1,x_2,\cdots,x_n)| \\
&\quad \leqslant (n-1)! e^{-2\beta B} \{e^{2\beta B} \xi^{-1} \exp [C(\beta)\xi]\}^{n-1} .
\end{aligned} \tag{3.15}$$

By a more careful majorization we would obtain, following Penrose [see Ref. 3, Eqs. (6.10) and (6.11)], instead of (3.12) and (3.15) the inequalities

$$\begin{aligned}
\sup_{(x)_m} \int d(y)_n \, |\tilde{\phi}_{(x)_m}(y)_n| &\leqslant m(m+n)^{n-1} \\
&\quad \times e^{2(m+n-1)\beta B} [C(\beta)]^n ,
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\int dx_2 \cdots dx_n |\phi(x_1,x_2,\cdots,x_n)| \\
&\quad \leqslant n^{n-2} e^{2(n-2)\beta B} [C(\beta)]^{n-1} .
\end{aligned} \tag{3.17}$$

4. THE CLUSTER PROPERTY

It follows from (3.12) that the power series $\langle \alpha^\Lambda, \tilde{\phi}_X \rangle(z)$ representing $(\psi^\Lambda(z))(X)$ in a neighborhood of the origin converges uniformly provided

$$|z| < \xi e^{-2\beta B} \exp [-C(\beta)\xi] . \tag{4.1}$$

If we choose $\xi = [C(\beta)]^{-1}$, this condition becomes

$$|z| < e^{-2\beta B - 1} [C(\beta)]^{-1} . \tag{4.2}$$

If (4.2) holds, the following series also converges uniformly

$$(\psi(z))(X) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int d(y)_n \tilde{\phi}_X(y)_n \tag{4.3}$$

and provided (4.1) holds, we have

$$|(\psi(z))(x)_m| \leqslant \frac{\{e^{2\beta B} \exp [C(\beta)\xi]\}^{m-1}}{1 - |z| \xi^{-1} e^{2\beta B} \exp [C(\beta)\xi]} . \tag{4.4}$$

It has been shown in Ref. 2 (theorem 1) that when the radius of the sphere Λ, centered at the origin, tends to infinity, the functions $\rho^\Lambda(x)_m$ defined by (2.7) converge uniformly on the compacts to limits $\rho(x)_m$. The same holds therefore also for the coefficients

$$\frac{1}{n!} \alpha^\Lambda(X) \int d(y)_n \, \alpha^\Lambda(y)_n \tilde{\phi}_X(y)_n \tag{4.5}$$

of their expansions in powers of z, which must then have the limits

$$\frac{1}{n!} \int d(y)_n \, \tilde{\phi}_X(y)_n . \tag{4.6}$$

This shows that the correlation functions for infinite volume $\rho(x)_m$ introduced in Ref. 2 are defined by

$$\rho(x)_m = z^m(\psi(z))(x)_m . \tag{4.7}$$

We would now like to prove a "cluster property" (see Ref. 4) of the $\rho(x)_m$, i.e., a property of "decrease at infinity" of the functions $\chi(x)_m$ defined by

$$\mathbf{1} + \rho = \Gamma(\chi) . \tag{4.8}$$

It is equivalent to use the functions $(\phi(z))(x)_m$ defined by

$$\psi(z) = \Gamma(\phi(z)) \tag{4.9}$$

or

$$(\phi(z))(X) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int d(y)_n \, D_X\phi(y)_n \tag{4.10}$$

[compare with (2.11)] because

$$\chi(x)_m = z^m(\phi(z))(x)_m . \tag{4.11}$$

Since $(\phi(z))(X)$ is a finite sum of products of functions $(\psi(z))(Y)$, the series (4.10) converges uniformly in X if (4.2) is satisfied. On the other hand, putting $\xi = [C(\beta)]^{-1}$, we obtain from (3.15)

$$\int dx_2 \cdots dx_m \, |\phi(x)_m| \leqslant (m-1)! e^{-2\beta B}[e^{2\beta B+1}C(\beta)]^{m-1} . \tag{4.12}$$

Therefore $(\phi(z))(x)_m$ is a Lebesgue-integrable function of $x_2, \cdots, x_m$:

$$\int dx_2 \cdots dx_m |(\phi(z))(x)_m|$$

$$\leqslant e^{-2\beta B} \sum_{n=0}^{\infty} \frac{|z|^n}{n!} (m+n-1)![e^{2\beta B+1}C(\beta)]^{m+n-1}$$

$$= \frac{(m-1)! e^{-2\beta B}[e^{2\beta B+1}C(\beta)]^{m-1}}{[1 - |z|e^{2\beta B+1}C(\beta)]^m} , \tag{4.13}$$

where we have used the formula

$$\sum_{n=0}^{\infty} \frac{(m+n-1)!}{n!} t^n = \frac{(m-1)!}{(1-t)^m} , \quad |t| < 1 . \tag{4.14}$$

From (4.11) we obtain then immediately:

Theorem. The functions $\rho(x)_n$ have the following cluster property: If we put $\mathbf{1} + \rho = \Gamma(\chi)$, the functions $\chi(x)_n$ are Lebesgue-integrable as functions of the differences between their arguments x_i for $|z| < e^{-2\beta B-1}[C(\beta)]^{-1}$

$$\int dx_2 \cdots dx_m |\chi(x)_m|$$

$$\leqslant \frac{(m-1)! |z| e^{-2\beta B}[|z|e^{2\beta B+1}C(\beta)]^{m-1}}{[1 - |z|e^{2\beta B+1}C(\beta)]^m} . \tag{4.15}$$

Notice that if Φ has finite range R, $\phi(x_1, \cdots, x_m)$ vanishes for $|x_2 - x_1| > (m-1)R$, hence, for instance

$$\int_{|x_2 - x_1| > mR} dx_2 |\chi(x_1, x_2)| \leqslant |z| e^{-2\beta B} \sum_{n=m}^{\infty} (n+1)t^{n+1}$$

$$= |z| e^{2\beta B}[m(1-t)^{-1} + (1-t)^{-2}]t^{m+1} \tag{4.16}$$

where $t = |z|e^{2\beta B+1}C(\beta) < 1$, an *exponential decrease* in $\lambda = mR$ as one might expect. The implications of this fact for the analyticity of the Fourier transform

$$\hat{\chi}(p) = (2\pi)^{-\frac{1}{2}\nu} \int d(x_2 - x_1) e^{-ip(x_2 - x_1)} \chi(x_1, x_2) \tag{4.17}$$

are obvious.[9]

5. ANALYTICITY PROPERTIES

It is easily seen that the above theorem remains true if Φ is allowed to be complex and satisfies instead of (Φ 1) the condition

(Φ' 1) $$\text{Re } U(x)_n \geqslant -nB , \tag{5.1}$$

while (Φ 2) remains formally unchanged.

We will instead investigate the possibility of taking β complex for a real potential Φ satisfying (Φ 1) and (Φ 2). If Re $\beta > 0$ and $C(\beta)$ is defined by (3.3) then (3.12), (3.15), (3.16), and (3.17) remain unchanged except for the replacement of $e^{2\beta B}$ by exp [2(Re β)B]. Therefore, if the conditions

$$\text{Re } \beta > 0 , \quad |z| < e^{-2(\text{Re}\,\beta)B-1}[C(\beta)]^{-1} \tag{5.2}$$

corresponding to (4.2) are satisfied, the power series defining $\psi(z)$, $\phi(z)$, ρ, and χ converge.

If one notices that $(\partial/\partial\beta)e^{-\beta\Phi} = (\partial/\partial\beta)(e^{-\beta\Phi} - 1)$ is both bounded and integrable one sees easily that $\psi(z)$ and $\phi(z)$ are vector-valued analytic functions of β and z in any domain for the complex variables β and z such that the conditions (5.2) are satisfied. We refrain from a formal definition and proof. Let us however notice that if β is real and (4.2) is satisfied for some z, then (5.2) is also satisfied if β is given

[9] Results in this direction have also been obtained by J. Groeneveld (J. L. Lebowitz, private communication).

a sufficiently small imaginary part, therefore, if (4.2) is satisfied $\psi(z)$ and $\phi(z)$ are analytic functions of both the complex variable z and the real variable β.

This applies in particular to the first component of ϕ, thus to the density and by integration to the pressure. This is the main point of this section: The density and the pressure are analytic in both the complex variable z and the real variable $\beta > 0$ if (4.2) is satisfied.

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Approximation Method in Collision Theory Based on R-Matrix Theory

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We report on numerical calculations using an approximation method for calculating scattering and reaction cross sections. The method is suggested by R-matrix theory and consists in the use of an R matrix calculated with the first Rayleigh–Schrödinger approximation for the resonant energies and reduced widths. Similar to the Born approximation, the method can be applied rather directly to any problem, but it necessitates more calculating than Born's method. The calculations reported on were carried out for a simple potential well. In this case the accurate phase shifts, the Born approximation, and the approximation here considered can all be obtained with relative ease. The approximation method here considered is quite accurate for weak potentials down to zero energy. For larger potentials, which produce several bound states, neither the present, nor the Born approximation is accurate at low energy; all of them work at high energy. The region of validity of the approximation method here considered extends, however, to lower energy.

INTRODUCTION

If the states (i.e., wavefunctions) of the colliding and separating particles are known, the Born approximation provides an expression for the cross section in terms of a single definite integral.[1] This is the great practical advantage of the method. Its drawback is, as that of most approximation methods, that it gives accurate results only for a restricted class of problems. This shortcoming is reflected also in the nonunitary nature of the collision matrix given by the Born approximation. We have endeavored to develop an approximation method which can also be applied with relative ease to any problem and which does yield, at every stage, a unitary (and symmetric) collision matrix. The present article describes the method and a test thereof. Its application is much more complex than that of Born's method, but the fact that the approximate collision matrix which it furnishes is always unitary suggests that it is, perhaps, as a rule more accurate than the Born approximation.[2] The unitary nature of the approximate collision matrix is, on the other hand, not necessarily an advantage: A gross departure of the collision matrix obtained by Born's method from a unitary matrix gives at least a sufficient criterion that it is inaccurate and one can hope, conversely, that if the collision matrix obtained is nearly unitary, it is also reasonably accurate. No similar criterion is known to us with respect to the method to be described.

Another limitation of the method to be described is that it assumes, as does R-matrix theory,[3] that the interactions extend only over a finite range or that, if they have infinite range (such as the electrostatic

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[1] See, for instance, N. F. Mott and H. W. S. Massey, *The Theory of Atomic Collisions* (Oxford University Press, Oxford, England, 1933), or 2nd ed., 1952. An approximation method, similar to the one here proposed, has been considered also in the Master's Thesis of D. F. Hubbard at the University of Virginia. We are much indebted to Professor A. Altman of the University of Maryland for providing us with a copy of this thesis.

[2] See, for instance, the second author's *Review of Collision Theory* in the Solvay Report for 1962 (Interscience Publishers, Inc., New York, 1964).

[3] See, for instance, A. M. Lane and R. G. Thomas, Rev. Mod. Phys. **30**, 257 (1958); or E. Vogt's article in *Nuclear Reactions* (North-Holland Publishing Company, Amsterdam, 1959), p. 215 ff., or G. Breit's article in the *Encyclopedia of Physics* [Springer-Verlag, Berlin, 1959), Vol. 41/1, p. 274 ff.], or, for a short review, L. Eisenbud and E. P. Wigner, *Nuclear Structure* (Princeton University Press, Princeton, New Jersey, 1958), Chap. 9.