pendence of η and F_0 on J and H. It should also converge asymptotically to $V = \rho_n J$, the voltage due to the normal resistivity of the material.

As H is varied over a wide range, we find the representation of resistive states in the form of Fig. 1 becomes rather constrained. This trend is readily detectable in Fig. 2, where constant V lines are shown in a logarithmic procession in the log J–log H plane. In this representation, $V(\alpha) = \text{const}$ with $\alpha = J(H + B_0)$ will have a variable slope

$$d \ln J/d \ln H = -(1 + B_0/H)^{-1}, \tag{11}$$

and the flow region ($V \propto \alpha$) will appear as equidistant parallel lines. The usual critical current densities for a Nb_3Sn wire sample are shown in the figure. For the 3Nb–Zr wire sample, constant V lines are shown up to $V = 200\ \mu$V. The region of higher power dissipation cannot be traced because of thermal instability.[5] These data are reducible in the form of $V(\alpha)$[5] and follow the pattern shown in Fig. 1. For the Nb–Ta wire sample, the upper critical field H_{c2} is only 4 kG and the entire region of the mixed state can readily be followed. Near H_{c2}, the resistive behavior is much like that of the 3Nb–Zr wire. As H decreases, however, the slopes of constant V lines rise, or the values of B_0 decrease. This trend becomes much more severe for the annealed Nb–Ta sample[7]—B_0 is now negative for the entire region of the resistive states. Thus, if the Lorentz force parameter is to be maintained in the form $\alpha = J(H + B_0)$, B_0 does not remain constant over a wide range of H variation. As for the dependence on microstructure, B_0 generally decreases as the pinning effects are reduced. In spite of these ramifications, however, Fig. 2 clearly points out the important fact—over 5 decades of $\alpha = JH$ values the resistive states are controlled primarily by the Lorentz force.

[7] The Nb–Ta (R-29) sample has been annealed thoroughly to remove most of the defects. Magnetization measurements showed, however, a considerable hysteresis, indicating that the sample is not defect-free.

Collective Modes of Vortex Lines in Superconductors of the Second Kind

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I. ONE SINGLE VORTEX LINE

We consider an Abrikosov vortex line[1] in a superconductor of the "extreme second kind": the penetration depth λ is much larger than the coherence length ξ. Then the line consists of a small hard core (cylinder of radius ξ) where the order parameter is seriously modified, plus a large "electromagnetic region" (radius $\sim \lambda$) where the distribution of fields and currents is ruled by a simple London equation (Fig. 1). We first recall how the line energy per unit length is computed in this case. To simplify the notation we consider only pure metals, with electrons of effective mass m^*, superfluid concentration n_s, and velocity v_s. (Note that for the superconductors of interest $m^* \sim 10$ to 100 electron masses.) The energy is

$$\mathfrak{J} = \int_{r>\xi} dx\, dy \left(\frac{h^2}{8\pi} + \tfrac{1}{2} n_s m^* v_s^2\right)$$

$$= \int_{r>\xi} dx\, dy\ (1/8\pi)\ (h^2 + \lambda^2 \operatorname{curl}^2 h),$$

$$\lambda^{-2} = 4n_s e^2/m^* c^2. \tag{I.1}$$

The minimization of $\mathfrak{J}$ leads to the condition

$$h + \lambda^2 \operatorname{curl} \operatorname{curl} h = \varphi_0 \delta_2(r), \tag{I.2}$$

where $\varphi_0 = ch/2e$ is the flux quantum and the δ function describes the singularity on the hard core. For $r \ll \lambda$ the solution of (I.2) is of the form

$$h = (\varphi_0/2\pi\lambda^2) \ln(\lambda/r), \tag{I.3}$$

while for $r \gg \lambda h$ decreases exponentially. The energy $\mathfrak{J}$ may be derived from (I.1) by a partial integration

$$\mathfrak{J} = \frac{\lambda^2}{4}[rh|\operatorname{curl} h|]_{r=\xi} = \left(\frac{\varphi_0}{4\pi\lambda}\right)^2 \ln \frac{\lambda}{\xi}. \tag{I.4}$$

The field H_{c1} above which the thermal equilibrium corresponds to a finite density of lines in the super-

[1] A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **32**, 1442 (1957) [English transl.: Soviet Phys.—JETP **5**, 1174 (1957).

conductor is

$$H_{c1} = \frac{4\pi}{\varphi_0}\mathfrak{J} = \frac{\varphi_0}{4\pi\lambda^2}\ln\frac{\lambda}{\xi}\,.$$

Let us now discuss the force acting on a line moving with velocity v along Ox. In a frame $(x'yz)$ tied with the line the superfluid electrons carry a current $j = n_s(-e)(-v)$ and are submitted to a Lorentz force jxh/c. The total force on the electrons is directed along y and equals to $-n_se(v/c)\varphi_0$ (per unit

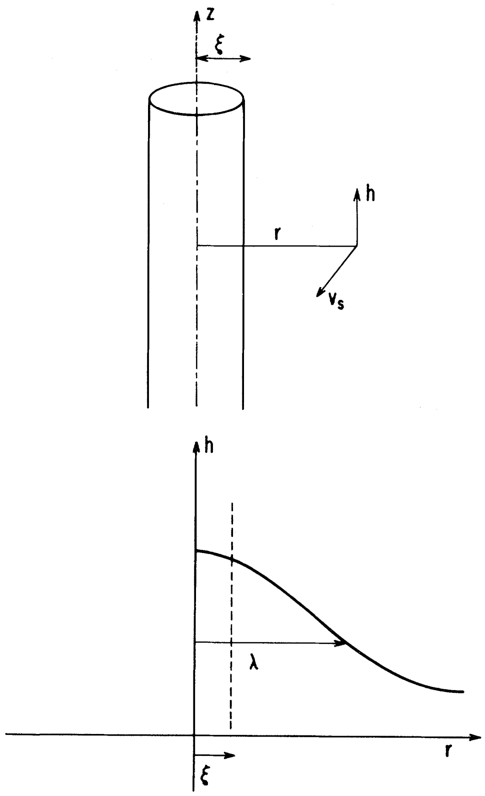

FIG. 1.

length of line). The force f on the line has the opposite sign. Thus,

$$f_y = n_se(v_x/c)\varphi_0\,. \tag{I.5}$$

We can now compute the vibration modes of the line: Balancing the line tension against the Magnus force (I.5) we get

$$-n_se(v_y/c)\varphi_0 = \mathfrak{J}(\partial^2 s_x/\partial z^2)\,, \tag{I.6}$$
$$+n_se(v_x/e)\varphi_0 = \mathfrak{J}(\partial^2 s_y/\partial z^2)\,,$$

where $s(z)$ is the line displacement. Putting $v = i\omega s$ and $s \sim e^{ikz}$ we obtain the dispersion law[2]

$$\omega = (\hbar/4m^*)k^2\ln(\lambda/\xi)\,. \tag{I.7}$$

Typically for $m^* = 50$ electronic masses, $k^{-1} = 10^{-4}$ cm and $\ln(\lambda/\xi) = 4$ (as expected for materials of the Nb_3Sn series) we get $\omega = 2 \times 10^6$. We can also compute the eddy current damping of such a mode from the electric fields associated with the line in motion. The resulting Q factor of the mode is frequency-independent and is of order $Q \cong mc^2/\hbar\sigma$, where σ is the conductivity due to the normal electrons.[3] With $m = 50$ and $\sigma = 5 \times 10^{18}$, $Q \sim 10^3$.

II. INTERACTIONS BETWEEN LINES

Starting from Eq. (I.1) one may show by a detailed calculation that the interaction energy (per unit length) of two parallel lines is

$$U_{12} = (\varphi_0/4\pi)h_{12}\,, \tag{II.1}$$

where h_{12} is the field created at the location of line 2 by the line 1 alone. The energy U_{12} is repulsive, and of range λ (decreasing at large distance r_{12} like $r_{12}^{-\frac{1}{2}}e^{-r_{12}/\lambda}$). It is instructive to write down the force f_{12} on line 2 due to line 1;

$$f_{12x} = -\frac{\varphi_0}{4\pi}\frac{\partial h_{12}}{\partial x_2}\,.$$

Introduce the current $j_1(r)$ which would exist in the presence of line 1 alone; then by Maxwell's equation we get

$$f_{12x} = -(\varphi_0/c)j_{1y} \quad f_y = (\varphi_0/c)j_{1x}\,. \tag{II.2}$$

Thus $\sim\mathbf{f}_{12}$ has the form of a Lorentz force.

Let us now consider an assembly of lines in an external field $H > H_{c1}$. The thermodynamic potential $\mathcal{G}$ (per cm^3) is of the form

$$\mathcal{G} = U(B) - (\mathbf{B}\cdot\mathbf{H}/4\pi)\,. \tag{II.3}$$

The induction $B = n\varphi_0$ is proportional to the line density n. $U(B)$ contains the line tension contribution $n\mathfrak{J}$ and the pair interactions (II.1). The minimum $\mathcal{G}_0$ of (II.3) for fixed H is realized for an induction $B(H)$. At low H (H slightly larger than H_{c1}) the corresponding interline distance d is of order λ. When $H \sim H_{c2}$ (the upper critical field), d is comparable to ξ (and in this region our London-type description breaks down). We focus our attention on the intermediate region $\lambda \gg d \gg \xi$ or $H_{c1} \ll H \ll H_n$ which

[2] J. Friedel, P. G. de Gennes, and J. Matricon, Appl. Phys. Letters **2**, 119 (1963).

[3] In most superconductors of the second kind presently available the mean free paths are short compared with the distance between lines (or with λ) and there are no corrections of the anomalous skin effect type.

is very important in practice for materials of the Nb_3Sn group). The ideal equilibrium state of the lines corresponds to a 2-dimensional lattice.[1] But it is very important to realize that the range of the interactions (II.1) is much longer here than the lattice parameter. Thus,

(1) the lattice structure has only a very weak effect on the energy. In fact, we calculate[2]

$$U(B) = \frac{B^2}{8\pi} + \frac{BH_{c1}}{4\pi} \frac{\ln \alpha d/\xi}{\ln \lambda/\xi}, \quad \text{(II.4)}$$

where α is a constant of order unity for all lattice structures.

(2) the lattice has only a very weak resistance to shear stresses: The shear modulus is $\sim(d/\lambda)^2$ smaller than the bulk modulus. For many effects it is in fact sufficient to describe the lines as an ideal two-dimensional fluid, the stress tensor being reduced to a scalar.

A detailed discussion of the theoretical lattice structures and elastic properties will be given elsewhere.[4] On the experimental side we might hope to detect the "lattice" structure by electron probe methods, but this would display the field distribution only for thin specimens or at the surface of the superconductor. Another method of investigation which is not entirely ruled out is *neutron diffraction:* with neutrons of wavelength $\lambda_n = 6$ Å an array of spacing $d = 360$ Å (corresponding to an induction $B = \varphi_0/d^2 \sim 20\,000$ G) would give a Bragg reflection at a scattering angle $\lambda n/d$ of order 1°. The interaction energy is $\mu_n h(r)$, where μ_n is the neutron moment and the scattering amplitude for a wave vector τ is easily deduced from the Fourier transform of Eq. (I.2). The amplitude per atom is

$$a = \mu_n \varphi_0 \frac{M}{2\pi h^2} \frac{n v_0}{1 + \lambda^2 \tau^2} \cong \frac{1.91}{4} \frac{n v_0}{\lambda^2 \tau^2}, \quad \text{(II.5)}$$

where M is the neutron mass and v_0 the atomic volume of the superconductor. Taking $\tau = 2\pi/d$ and $n = 1/d^2$ (square lattice of lines) we get $a = (1.91/16\pi^2)(v_0/\lambda^2)$ or, using the definition of λ,

$$a = \frac{1.91\, n_s v_0}{4\pi} \frac{e^2}{m^* c^2}. \quad \text{(II.6)}$$

If $n_s v_0 = 4$ and $m^* = 20$ (m electron), this corresponds to a coherent cross section $4\pi a^2$ in the millibarn range. Of course the main interest of the neutrons is that they would explore the interior of the sample.

[4] P. G. de Gennes and J. Matricon (to be published).

III. COLLECTIVE MODES

A. Dispersion Relations

Consider an array of lines in a field $H(\|Oz)$. Each line is displaced by a variable amount $s_x(r)$, $s_y(r)$. In an isotropic approximation the potential may be expanded as follows:

$$\begin{aligned} \mathcal{G} = \mathcal{G}_0 &+ \tfrac{1}{2} K_1 \left[\left(\frac{\partial s_x}{\partial x}\right)^2 + \left(\frac{\partial s_y}{\partial y}\right)^2 \right] + K_2 \left[\left(\frac{\partial s_x}{\partial x}\right)^2 \right. \\ &\left. + \left(\frac{\partial s_y}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial s_x}{\partial y} + \frac{\partial s_y}{\partial x}\right)^2 \right] \\ &+ \tfrac{1}{2} K_3 \left(\frac{\partial s_x}{\partial z}\right)^2 + \left(\frac{\partial s_y}{\partial z}\right)^2, \end{aligned} \quad \text{(III.1)}$$

K_1 and K_2 are the Lame coefficients of the two-dimensional "line lattice." We may derive some relations for the K's by the general formula

$$\mathcal{G} = \mathcal{G}_0 + \frac{1}{8\pi}\left[\frac{(B_z - B(H))^2}{\mu_z} + \frac{B_x^2 + B_y^2}{\mu_\perp}\right], \quad \text{(III.2)}$$

where $\mu_z = dB(H)/dH$ and $\mu_\perp = B(H)/H$ are the reversible permeabilities—both nearly equal to 1 in the region $\lambda \gg d$.

$$\begin{aligned} \tfrac{1}{2}K_1 + K_2 &= B^2(H)/4\pi\mu_z \cong B^2/4\pi, \\ K_3 &= HB(H)/4\pi. \end{aligned} \quad \text{(III.3)}$$

The shear modulus K_2 is small $K_2 \sim K_1(d/\lambda)^2$. From (III.1) we derive the forces acting on each filament in a nonhomogeneous distorted state, balance them against (I.5), and look for eigenmodes $s = s_0 e^{i(kr-\omega t)}$. We get the dispersion relations

$$\omega = \frac{eH}{m^* c} k^2 \lambda^2 \qquad (k \text{ along } Oz), \quad \text{(III.4)}$$

$$\omega \cong \frac{eB}{m^* c} k^2 \lambda d \qquad (k \text{ normal to } Oz). \quad \text{(III.5)}$$

The mode (III.4) is circularly polarized and in the low density limit ($H \to H_{c1}$) it coincides with (I.7). The mode (III.5) is elliptically polarized (if k is along Ox, $s_x/s_y \sim d/\lambda \ll 1$). In the calculation of (III.5) it is not possible to neglect K_2.

The above results apply to an ideal array of lines. If, in fact, the lines are pinned to defects in the lattice structure, the potential $\mathcal{G} - \mathcal{G}_0$ will contain terms in s_x^2, s_y^2. For long wavelengths there still exist well-defined plane-wave modes, but with a gap in the frequency spectrum ($\omega = \omega_0 + Dk^2$). This is important in some applications.

B. Effects on Physical Properties

We shall first discuss the *surface impedance* of a superconductor of the second kind, the static field H

being applied normal to the surface. At frequencies well below the fermion energy gap, the dynamic permeability $\mu_\perp(k\omega)$ (with k along Oz) is controlled by the modes (III.4) and given by

$$\mu_{\perp(\pm)} = Bk^2/(\pm[4\pi n_s e\omega/c] + k^2 H) , \quad \text{(III.6)}$$

where the $\pm$ sign refers to the two states of circular polarization. Inserting this value of $\mu_\perp$ in the Maxwell equations

$$\operatorname{curl} E = -(1/c)\partial B/\partial t ,$$
$$\operatorname{curl} H = 4\pi\sigma E/c ,$$

we obtain the surface inpedance

$$Z_\pm = \pm i4\pi c(E/H)_{z=0} = -i(c^2/\sigma)k_\pm ,$$

$$k_\pm^2 = \mp\frac{\omega}{(eH/m^*c)\lambda^2} + i\frac{4\pi\sigma\omega}{c^2} \ (\operatorname{Im} k < 0) . \quad \text{(III.7)}$$

One of the modes (+) penetrates only to a depth $\sim\lambda(\omega_H/\omega)^{\frac{1}{2}}$ (where $\omega_H = eH/m^*c$). The penetration depth of the other mode is larger $\sim\lambda(\omega_H/\omega)^{\frac{1}{2}}(2/\rho)$ where

$$\rho = \frac{4\pi\sigma\lambda^2\omega_H}{c^2} = \frac{n_n}{n_s}\,\omega_H\tau \ll 1 , \quad \text{(III.8)}$$

n_n being the density of normal electrons and τ the collision time. This difference in behavior leads to a rotation of the polarization plane (of a linearly polarized incoming radiation) by reflection, of angle

$$\beta \cong (c/4\pi\sigma\lambda)(\omega/\omega_H)^{\frac{1}{2}} . \quad \text{(III.9)}$$

At low temperatures σ may be small: taking[5] $\sigma = 10^{17}$ (resistivity 10 $\mu\Omega$-cm), $\lambda = 10^{-5}$, $H = 10^4$ G, $m^* = 40$ electronic masses, and $\omega/2\pi = 10^9$ we get $\beta \sim 1°$. Thus a measurement of β in clean samples (where the pinning frequency ω_0 is much smaller than ω) would give direct information on the collective modes. A more complete discussion, including the effects of "surface pinning" of the lines, is given in Ref. 3. The influence of the collective modes on the *nuclear resonance* behavior does not seem very easy to separate:

(1) There is a contribution to the nuclear relaxation by thermal motion of the lines. However this term is small, extremely sensitive to the pinning frequency ω_0, and may be hidden by many effects related to the Fermi-type excitations.

(2) There is an indirect interaction between nuclei via the collective modes. We have also calculated this effect, and again find that it depends strongly on the pinning of the vortex lines.

Similar conclusions apply to the experiments on *ultrasonic attenuation*. In an ideal material, the dispersion relation for the collective modes (III.4), (III.5) does not intersect the phonon dispersion relation (in the range of frequencies which are physically allowed). On the other hand, if there is a pinning frequency ω_0, then we find a crossover at a phonon frequency close to ω_0. It is not entirely impossible that the very strange peaks observed in attenuation vs temperature by Clairborne and Einspruch in Nb–Zr alloys[6] might be explained by such crossovers (provided that some trapped flux existed in the samples).

ACKNOWLEDGMENTS

We would like to thank J. Friedel and J. M. Winter for discussions on these and related topics.

[5] Note that the field dependence of β is not simple, since σ probably depends on H.

[6] L. T. Clairborne, and N. G. Einspruch, Phys. Rev. Letters **10**, 49 (1963).

Discussion 2

W. F. VINEN, *University of Birmingham:* I wonder if I might ask Dr. Anderson if he has considered the consequences, or at least the possible consequences, of the fact that these flux lines are vortices. The reason I ask this is that the dynamical behavior of a vortex is not always quite what one might think it is, as we have in fact just heard from Dr. de Gennes. For example, if you apply a force to it, it moves not in the direction of the force, but at right angles to the force. My impression on thinking about this very crudely is that Dr. Anderson is right in the limit where the effective force of friction between the moving flux lines and the lattice is very large. But if it is not all that large, as in a rather perfect crystal, then you might get a more complicated effect in which the array of lines does not move in quite the direction you would expect it to move.

P. W. ANDERSON, *Bell Telephone Laboratories:* I think everything you say is absolutely right, but I think that, in all the cases we are talking about experimentally, the lines in fact are moving very slowly. Certainly where thermal activation is involved we are just interested in the free energies of flux lines at two different points. We don't care whether they get from one to the other sideways or any other way. In the flow region I think the motion is still very slow, but your suggestion may nonetheless be relevant.

VINEN: Presumably it would be interesting to produce a crystal in which the force between the lines and the lattice is small; you could then get these effects which seem to be characteristic of vortices, and so get good evidence that vortices are really relevant.

D. J. VAN OOIJEN, *Philips Research Laboratories:* I want to comment briefly on the experimental dynamics of the mixed state of the type-II superconductor. The jumping of flux lines under the influence of a changing applied magnetic field can be demonstrated experimentally by recording the noise on a pickup coil surrounding the superconductor. The noise was found to start at the onset of the mixed state and then to decrease rather rapidly with increasing fields. On decreasing the fields, the noise persisted down to zero field thus showing a hysteresis of the magnetization.

ANDERSON: What was the order of the magnitude of the noise?

VAN OOIJEN: Well, I did not measure it, but it is, I think, 10 or 100 times smaller than the well-known Barkhausen noise that occurs in the magnetization of a ferromagnetic. The effect is rather similar to this Barkhausen noise.

C. P. BEAN, *General Electric Research Laboratory:* In the Pb–Vycor system that I showed there is also noise which is rather flat across the audio spectrum. It is a function of the driving and applied fields and is of the order of millivolts for a coil of 100 turns. It is rather large—very similar to Barkhausen noise.

ANDERSON: I wondered if Kim wanted to comment. He has of course observed pulses coming through his tubes in some of these flux decay experiments.

Y. B. KIM, *Bell Telephone Laboratories:* We have seen this type of pulse. Recently we have been making a systematic analysis of these pulses observed particularly in NbZr tubes. Here we can really correlate flux creep with the frequency of the pulses we observe. At the moment it seems that the size of the pulses are very sensitive to the microstructure. For example, we observe large pulses in NbZr, but in Nb_3Sn the pulses we see are rather small. We also tried this on a sintered niobium powder sample, which is one of the best examples showing the Lorentz-force criteria, and saw nothing. The pulses presumably exist but are smaller than the noise level; our detection limit right now is about 20 flux quanta. Therefore, if pulses exist due to flux motion in niobium powder samples, the flux bundles must be smaller than 20 flux quanta. This is the region in which Anderson anticipates the theory to work. Therefore, I believe that the pulses we observe in the NbZr may be of a different kind.

B. S. CHANDRASEKHAR, *Western Reserve University:* I regret that Dr. Kim did not have time to go into the flux creep and flux flow regions of his curves, but if I understand him right, the appearance of resistance just before the final transition to the normal state he explains ultimately because of the Lorentz force. I believe similar curves are observed when the transition is measured in a field parallel to the current direction. Would you care to comment on how one interprets the shape of the transition curves in this case?

KIM: I believe this will be covered by the paper of Cullen, Cody, and McEvoy. They actually applied this Lorentz-force model to the case where the field is not perpendicular to the current direction. It seems the relation holds out very well.

GORTER: Perhaps I may make a point in connection with the problems of the layer models discussed this morning. The main difference between the results of the two kinds of models, as far as these phenomena are concerned, would concern the creep of small flux bundles. For the case of the large flux bundles (flux flow), similar results could be obtained. For small flux bundles one should find a strong temperature dependence at very low temperatures. It might be worthwhile for the experimentalists to investigate the temperature dependence.

VINEN: I am slightly worried about the result (of de Gennes) insofar as it concerns the term proportional to k_z^2 in the dispersion relation. I stand open to correction, but I think I am right in saying that this term is in fact simply the helicon dispersion relation, and it appears therefore that this term has nothing to do directly with superconductivity with vortices. I am worried also because I tried to work this problem out myself and got a result which differed from the one given here essentially in that there was an extra term proportional to k_z^2 which involved the energy per unit length of one of these vortices. If my result were right it would be rather nice because this would in principle give us a good method for measuring the energy per unit length of one of these vortices, which would enable us to have a rather good check against the microscopic theory.

GORTER: In connection with the temperature dependence of the specific heat—what I said this morning in answer to Professor Pippard was not quite correct. For a mixed state one should expect, in Van Beelen's computation of the layer model, a term in the specific heat proportional to T. I wish to mention it in view of your (de Gennes) suggestion of this interesting $T^{\frac{3}{2}}$ dependence.

RALPH BENAROYA, *Argonne National Laboratory:* I have some experimental evidence in support of Bean's field-dependent penetration-depth theory. The experiment, in part, consists of scanning the field between two rectangular, $\frac{1}{4}$-in. thick, Nb_3Sn plates while an external orthogonal field up to 20 kG is applied to them. The scanned curves, when compared to calculated field distributions corresponding to various current-loop models, were found to match different ones depending on the applied field. Thus at low fields, the current loops which are found to be concentrated on the outer surfaces of the plates tend to migrate inwards with the application of higher fields. Even more spectacularly, the current loops, which at low fields flow in the peripheries of the plates, seem to fill in until continuous current loops covering the whole plate are established at higher fields. This latter effect might be attributed to the intensity of the Lorentz forces as the field difference between the inside and outside of the plates increases.

PIPPARD: Question to Dr. Bean. It seems to me that when one is in the irreversible phase of his model, one is feeding in or taking out of the sample a stream of fluxoids, and I want to know where they are when they go in. His meshes are about 10^{-12} cm^2 (I think) in area, which means that, if they were just little regions containing flux alone, there would be 200 000 G contained in each one of these. That is to say, the flux quantization is obviously a matter of extreme importance in such a fine mesh. And what I want to know is, has he tried to work out which long-range interactions between the meshes are caused by this flux quantization and, if so, does it have any observable effect on the question of what are the critical currents, for example, in his mesh units?

BEAN: I have not made calculations of the type Professor Pippard proposes, although I have had this in mind, and I think it is a very interesting idea.