served temperature dependence of the critical current is about twice that which has been predicted.[12]

CONCLUSION

The main conclusion of this work is that the magnetization of hysteretic high field superconductors can be well understood in terms of one phenomenological parameter, the macroscopic critical current density. A main problem remaining is the determination of the relationship between this parameter and the microstructure of hard superconductors.

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Hard Superconductivity: Theory of the Motion of Abrikosov Flux Lines

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The central concept in the theory of what one might call the "critical phenomena" of hard superconductors—critical currents, critical fields, decay of persistent currents, "excess" voltages, etc.—is clearly Abrikosov's notion of the quantized flux line.[1] This is made almost obvious by the remark that, because of the now universally accepted validity of the quantization of flux through superconductors, the smallest possible breakdown of superconductivity is the motion of a single quantum of magnetic flux through the wire or ring. Thus, in all cases so far conceived, the lowest activation energy for any critical breakdown is that for the motion—and creation, if necessary—of single Abrikosov flux lines. This statement is independent of whether the mechanism for hard superconductivity is the GLAG one or the Mendelssohn sponge theory, although we assume the former to be valid in most cases. Even the decay of currents in true soft superconductors under α-particle bombardment[2] is probably best explained by the threading of Abrikosov lines through normal holes punched by the α particles.[3]

The purpose of this paper is to see how many of the phenomena of hard superconductivity we can understand qualitatively in terms of the thermally activated motion of Abrikosov lines past pinning centers, without going into unnecessary detail on the nature of the pinning centers—whether they are dislocations, cavities, precipitates, etc.—or the precise internal structure of the superconductor. Our task, then, is to study the process—presumably thermally activated barrier penetration—by which flux lines move.

Let us then suppose that we have a superconductor penetrated by a magnetic field H and carrying a bulk current, for simplicity $\perp H$, $J = c\nabla \times \mathbf{H}/4\pi$. The magnetic field will penetrate in the form of Abrikosov lines; their density is clearly not uniform because of J, and we expect their arrangement is to some extent irregular. The magnetic energy per unit volume is $H^2/8\pi$; we can think of this as a magnetic pressure exerted by the flux lines on each other, and in the absence of pinning centers this pressure would have to be equalized by a rearrangement of the lines, leading to $J = 0$. Examination of Abrikosov's theory shows that actually at all but low fields the internal and external fields are nearly the same, so that we usually assume $B = H$, a minor simplification of which Friedel *et al.* have considered the errors.[4]

In finding the rate of the activation process we need to know two things: the driving force exerted by the magnetic pressure, and the nature of the barriers. The former is more available to us theoret-

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[1] A. A. Abrikosov, Zh. Eksperim i Teor. Fiz. **32**, 1442 (1957) [English transl.: Soviet Phys.—JETP **5**, 1174 (1957)].
[2] P. de Feo and G. Sacerdoti, Phys. Letters **2**, 264 (1962).
[3] N. Cabibbo and S. Doniach, Phys. Letters **4**, 29 (1963). We have proposed a slightly different mechanism.
[4] J. Friedel, P. G. de Gennes, and J. Matricon, Appl. Phys. Letters **2**, 119 (1963).

ically but, on the other hand, more complex, entirely because it is really transmitted only through the interactions between the flux lines themselves. We can only guess at the correct results before more detailed computations of the behavior of nonuniform distributions of flux lines become available.

In Abrikosov's paper, in the appropriate case for hard superconductors ($\kappa \gg 1$, $H_{c2} > H > H_{c1}$) the interaction free energy of the lines is written in two equivalent ways:

$$F_{\text{int}} = (1/8\pi)\,\overline{\mathbf{H}\cdot(\mathbf{H} - \delta^2\nabla^2\mathbf{H})} \tag{1}$$

$$= \frac{H_c^2}{4\kappa^2}\sum_{i,j} K_0\left(\frac{|\mathbf{r}_i - \mathbf{r}_j|}{\delta}\right), \tag{2}$$

κ being the famous Landau–Ginsburg parameter δ_0/ξ_0, δ the penetration depth, and H_c the thermodynamic critical field. The two-dimensional position vectors of the lines are the $\mathbf{r}_i$. In the case most usual in hard superconductors, the distances $|\mathbf{r}_i - \mathbf{r}_j|$ are small compared to δ, $\overline{\nabla^2 H}$ is relatively small, and so the interaction free energy is practically given by the naive expression for the magnetic energy. This would mean that the force per unit volume would be given by $\nabla F = -\mathbf{H} \times \nabla \times \mathbf{H}/4\pi = \mathbf{J} \times \mathbf{H}/c$, the Lorentz force; and the force per flux line is then $\mathbf{J} \times \mathbf{\Phi}_0/c$ per unit length, where the magnitude of $\mathbf{\Phi}_0$ is the flux unit $hc/2e$ and it is directed along the flux line.

It is very useful to notice that (2) is formally the same as the electrostatic interaction between lines of electric charge, except that it is screened away at the penetration depth δ—i.e., K_0 has the logarithmic nature of two-dimensional charge interactions as $r \to 0$, while it is $\sim e^{-r/\delta}$ as $r \to \infty$. This tells us that, because of this relatively long-range interaction, local perturbations of the line density are very unfavorable energetically—for instance, simply putting in locally one extra flux line costs an energy of the order of $H\Phi_0$ per unit length, which is much greater than the energy available from any reasonable pinning centers. Thus the arrangement can be irregular *only* on a scale *greater than* δ—locally the density must be uniform, and any local variation must be only a slight increase in the local density spread out over a region of radius $>\delta$.

This is the idea behind the concept of "flux bundles"[5]—that in fact, while it is probably the individual flux line's internal structure (of size $\sim\xi_0$) which is caught by a pinning center, that line individually cannot jump over the barrier alone, because it would get badly out of equilibrium with the local density in its neighborhood; but rather a whole bundle of lines, of radius $\sim\delta$, must move simultaneously. Therefore, of course, it is the force on the total bundle which acts against the pinning barrier. On the other hand, while the line *density* must be very uniform, the K_0 function is actually a very slowly varying one near $r \to 0$ so that the arrangement of lines need not be crystallographically regular, and the bundles can slide past each other reasonably easliy.

The free energy of a bundle in the region of a barrier, then, can now be written down. The force is about $JH\delta^2 l/c = J\Phi_0 n_b l/c$, where l is the effective length of line over which the force acts, presumably the distance between pinning centers, and n_b is the number of lines in a bundle; as a function of position x of the bundle we have

$$F_{\text{force}} = JH\delta^2 lx/c\,.$$

The size of the barrier is presumably about ξ_0, and the appropriate scale factor for its energy is $(H_c^2/8\pi)\xi_0^3$; suppose a fraction p of this is effective. The total barrier free energy then—realizing that we have both p and l as undetermined parameters so that we need not specify the determinable ones more precisely—is

$$F_b = (pH_c^2\xi_0^3/8\pi) - (JH\delta^2 l\xi_0/c)\,. \tag{3}$$

It is perhaps barely worthwhile to make a guess at the pre-exponential factors in the rate equation, even though almost all the results are controlled entirely by (3). A particular barrier will allow one of the lines in the bundle through at a rate/sec of

$$R = \omega_0 e^{-F_b/kT} \tag{4}$$

ω_0 is a vibration frequency of the bundle, $\sim 10^5 - 10^{10}$/sec. Since the bundles are of width δ, the rate per unit area is obtained by dividing by δ. The equation for diffusion of flux density $|B|$ is given by finding the rate at which flux enters and leaves a small element of volume:

$$d|B|/dt = -\nabla\cdot(\Phi_0\mathbf{R}/\delta)\,, \tag{5}$$

where the gradient is two-dimensional, Φ_0 is the flux unit, and $\mathbf{R}$ is of magnitude (4) and directed in the direction of the gradient of magnetic pressure

$$\alpha = \nabla p = \nabla(H^2/8\pi) = \mathbf{J} \times \mathbf{H}/c\,. \tag{6}$$

We shall often use the notation α for the appropriate combination in the force term even when JH is not correct.

There are two cases in which we would obviously expect the above reasoning to fail: the two extremes of the Abrikosov state, near H_{c2} and H_{c1}, the upper

[5] P. W. Anderson, Phys. Rev. Letters 9, 309 (1962).

and lower critical fields, respectively. Near the lower critical field the lines are more than a distance δ apart; also the differences between B and H, and therefore the complications due to surface currents, etc., are much greater. Qualitatively, we expect that as the number $n_b \rightarrow 1$, the value of the field will matter less and less, and the effective force will reduce to that on a single line, proportional to J alone. This is indeed the qualitative behavior in many materials, e.g., those in which we find the force term as $J(H + B_0)$, with B_0 of the order of H_{c1}.[6] But no justification for this particular form has appeared.

As $H \rightarrow H_{c2}$, the lines will be forced together until the forces between them are no longer the long-range, smooth electromagnetic forces, varying as $\ln(r_i - r_j)$, but are the much more steeply varying forces which ensue when the regions in which $\Psi \neq$ const overlap. One would expect the bundle concept to fail completely, and the "hard core" interaction between lines to lead to new effects. One suggestion one might make would be that the lattice of lines may become *rigid*, so that the bundles can no longer slide independently past each other. In that case the rate might be expected to become much slower, a possible explanation for the "peak effect."[7]

The two immediate conclusions which were drawn from (5) were the critical current curve and the flux creep rate equation.[5] The "critical current" came from supposing that the critical parameters as measured in most cases—notably Kim's experiments[6]—simply represented a point at which the rate R became immeasurably slow. Call this rate R_c. Then we have

$$kT \ln (R_c/\omega_0) = -(F_b)_{\rm crit}$$

or

$$\alpha_{\rm crit} = \frac{(JH)_{\rm crit}}{c} = \frac{pH_c^2}{8\pi}\,\frac{\xi_0^2}{\delta^2 l} - \frac{kT}{\delta^2 l \xi_0} \ln \frac{R_c}{\omega_0} . \qquad (7)$$

A similar expression was found to agree reasonably well with the temperature dependence of $\alpha_{\rm crit}$ in Refs. 5 and 6. This expression is somewhat better qualitatively because one has the most structure-sensitive parameter l in both terms.

The flux creep rate equation was also treated approximately in Ref. 5. Equation (5) may be put in a more useful form by writing it as an equation for the pressure gradient α itself (for simplicity we specialize to a one-dimensional situation as in a tube wall):

$$\frac{\partial \alpha}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{H}{4\pi}\,\frac{\partial H}{\partial x}\right) = \frac{\partial}{\partial x}\,\frac{H\Phi_0}{4\pi\delta}\,\frac{\partial R(\alpha)}{\partial x} .$$

Usually, R will depend exponentially on α. H, of course, will depend only roughly linearly on α. Also, we really do not quite understand the pre-exponential factors in the rate equation anyhow, and an extra factor H could not easily be checked experimentally in most cases. Thus, it is easiest and within errors of the theory to neglect the derivative of H relative to that of R, and we obtain

$$\frac{\partial \alpha}{\partial t} = \frac{H\Phi_0\omega_0}{4\pi\delta} e^{-F_0/kT} \frac{\partial^2}{\partial x^2} e^{\alpha/\alpha_1} = K_0 \frac{\partial^2}{\partial x^2} e^{\alpha/\alpha_1} , \qquad (8)$$

where

$$\alpha_1 = \frac{kT}{\delta^2 l \xi_0} = \frac{\alpha_{\rm crit} - F_0/\delta^2 l \xi_0}{\ln (\omega_0/R_c)} . \qquad (9)$$

α_1 is usually very small, of the order of $10^{-2}\,\alpha_{\rm crit}$, which means that the barriers are indeed high compared to kT. K_0 is defined by this equation and F_0 is the force-free barrier height $F_b(\alpha = 0)$.

A steadily decaying solution of (8) is

$$e^{\alpha/\alpha_1} = (-\alpha_1 x^2 + bx + c)/2K_0 t$$
$$\alpha = \alpha_1\{\ln[(-\alpha_1 x^2 + bx + c)/2K_0] - \ln t\} . \qquad (10)$$

This logarithmic time dependence has been repeatedly observed.[6,8]

Another solution of the nonlinear diffusion equation (8) is the steady-state one, such as one would obtain physically by supplying power from an external source to maintain a current through a tube or plate sample:

$$(\partial/\partial x) e^{\alpha/\alpha_1} = c$$
$$\alpha = \alpha_1(\ln c + \ln x) . \qquad (11)$$

Because α_1 is so small c will be enormously large in all cases, and thus α will be essentially constant throughout the sample, as is physically obvious from the exponential rate dependence on α.

We have made no progress in studying the transient solutions which are relevant when one quickly applies external fields or currents to a hard superconductor. Clearly, the effect will be to create local concentrations of magnetic pressure which could diffuse away but may well not be able to do so stably—i.e., in such a way as to decay continuously into a steady-state solution. We are currently studying this

[6] Y. B. Kim, C. F. Hempstead, and A. R. Strnad, Phys. Rev. Letters 9, 306 (1962).

[7] For instance, S. H. Autler, E. S. Rosenblum, and K. H. Gooen, Phys. Rev. Letters 9, 489 (1962).

[8] Y. B. Kim, C. F. Hempstead, and A. R. Strnad, Phys. Rev. **131**, 2486 (1963).

problem with the idea that possible instabilities in Eq. (8) may well be related to the phenomena of magnet instability.

A whole range of further applications of these ideas are suggested by the observation that the existence of flux creep clearly implies power dissipation in all hard superconductors.[8] That is to say not only that apparently perfectly superconducting samples below the so-called "critical curve" are still dissipating power—and thus offering resistance to current flow—at a finite rate, but that apparently nonsuperconducting, resistive samples far *above* the usually accepted critical conditions are also often truly superconducting in the thermodynamic sense, especially under transient conditions before thermal or flux diffusion instabilities have had time to occur.

A simple way to derive the resistive power dissipation is to start from (5) and Maxwell's equation

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}/c$$

in a simple one-dimensional case where we assume we have B in the z direction, J and E in the y direction, and flux creeping in the x direction. Then,

$$\frac{dE_y}{dx} = -\frac{\dot{B}_z}{c} = \frac{d}{dx}\left[\frac{\Phi_0\omega_0}{\delta c}\exp\left(-\frac{F_0}{kT}+\frac{\alpha}{\alpha_1}\right)\right];$$

i.e., to maintain the flux we have to apply an E field

$$E_y = \frac{\Phi_0}{dc}\exp\left(-\frac{F_0}{kT}+\frac{\alpha}{\alpha_1}\right). \qquad (12)$$

There is, then, clearly a power dissipation

$$P = E_y J_y = \frac{J\Phi_0}{\delta c}\exp\left(\frac{\alpha}{\alpha_1}-\frac{F_0}{kT}\right) \qquad (13)$$

in the material.

This immediately suggests the possibility of severe thermal instabilities in hard superconductors. Let us write down the equation for the heat content of the material:

$$\frac{dQ}{dt} = c_{sp}\frac{dT}{dt} = -\kappa\nabla^2 T + P.$$

κ is the heat conductivity. Suppose a small temperature fluctuation $\delta T(r,t)$ occurs. Whether it grows or decays is determined by the equation

$$c\frac{d\delta T}{dt} = -\kappa\nabla^2(\delta T) + \frac{\partial P}{\partial T}\delta T. \qquad (14)$$

Suppose the fluctuation occurs on a scale of size r. Then for stability we must have

$$\frac{\kappa}{r^2} > \frac{\partial P}{\partial T} = \frac{P}{T}\left(\frac{F_0}{kT}-\frac{\alpha}{\alpha_1}-\frac{1}{kT}\frac{dF_0}{dT}\right).$$

By differentiating Eq. (7) we may obtain an expression for the dimensionless ratio in parentheses:

$$\frac{\kappa}{r^2} > \frac{P}{T}\left|\frac{T}{\alpha_1}\left(\frac{\partial\alpha}{\partial T}\right)_R\right|, \qquad (15)$$

where $(\partial\alpha/\partial T)_R$ means that we fix the rate R. Experimental data tell us that this ratio is of the order 10^2-10^3, which means that thermal instability is a severe problem. That is, the increase in temperature ΔT of the given region over the surroundings caused by the power input P is of the order

$$\Delta T \simeq Pr^2/\kappa$$

so this tells us that

$$\frac{\Delta T}{T} < \left|\frac{T}{\alpha_1}\left(\frac{\partial\alpha}{\partial T}\right)_R\right|^{-1} \simeq 10^{-2}-10^{-3} \qquad (16)$$

is the stability requirement: a very tiny rise in temperature can presage a complete thermal breakdown. In particular, if, because of excessively rapid current or field changes or of "weak spots," the stress α becomes concentrated in a small region, P will be exponentially larger locally while κ/r^2 only increases as the square of the size, so that local thermal instability can be a problem. The obvious practical morals are three: first, that magnet configurations allowing good thermal conduction are vital; second, the rather discouraging remark that so far good hard superconductors are also bad thermal conductors for obvious reasons, so that the better the magnet the worse the stability problem will be; and third, because of the factor $1/r^2$ very big magnets will have extra stability problems.

The final remark we would like to make is that the existence of this effective resistivity caused by flux creep allows us to investigate experimentally the creep rate over a much wider range of α than was possible with the original flux decay measurements, by measuring the resistivity of wire samples. In particular, one can go to far higher stresses—the "resistive state" of superconductors.[8] In this region Kim *et al.* have found that the exponential law $R \propto e^{\alpha/\alpha_1}$ begins to fail and is replaced by a roughly linear relation $R \propto \alpha$. This occurs when the effective barrier $F_0 - (\alpha/\alpha_1)kT$ is no longer large compared to kT; we would then expect a viscous resistance to flux line motion, a process of "flux flow" rather than "flux creep," analogous to the motion of magnetic domain walls above the coercive field. Kim has suggested a very useful semi-empirical formula for the velocity of flux lines:

$$v = [(\text{const}\times\alpha)^{-1} + (\text{const}\times e^{\alpha/\alpha_1})^{-1}]^{-1}. \qquad (17)$$

In summary, the general qualitative ideas of flux creep theory appear to be soundly based and to allow for the qualitative and semiquantitative understanding of a wide range of phenomena. Many problems remain, both for detailed quantitative study and even for better qualitative understanding. To list a few of these:

(1) More detailed understanding of flux line interactions, in particular a sounder basis for the "bundle" concept and an understanding of B_0 and of the peak effect.

(2) The peculiar transient pulses observed by Kim *et al.*[8] These support the creep idea qualitatively, but are too large to be individual lines or bundles and too small to be instabilities. Are they avalanches?

(3) The nonlinear diffusion equation: when is it unstable?

(4) The viscous state. This is a completely new theoretical problem and I know of no obvious way even of approaching it from fundamental theory.

Resistive States of Hard Superconductors

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In earlier reports[1] we have shown that the critical currents in high-field superconductors are limited by the Lorentz force relation

$$\alpha = J(H + B_0) \leqslant \alpha_c\,, \tag{1}$$

where α_c and B_0 are structure-sensitive constants of the material. This relation obtained for J perpendicular to H has been extended to other orientations by Cullen *et al.*[2] Although the relevance of the Lorentz force in hard superconductivity was pointed out first by Gorter,[3] a more definitive formulation of the problem was proposed by Anderson[4] in his flux creep theory. This theory basically assumes the GLAG-type superconductors, but the phenomena involving transport currents are described primarily in terms of the concept of thermally activated motion of flux structures. More recently, our measurements[5] have been extended into what we may call the "resistive state," far above the critical state specified by the relation (1). Here again, we find the prevalence of the Lorentz force parameter α. These results are briefly summarized in this paper.

According to the flux creep theory,[6] the flux structures in a GLAG-type superconductor move with a rate proportional to

$$R = \omega_0 \exp\,(-F_b/kT)\;;$$

$$F_b = \frac{pH_c^2\xi_0^3}{8\pi} - \frac{JH\delta^2 l\xi_0}{c} = F_0 - q\alpha\,. \tag{2}$$

In tube magnetization experiments,[1] for example, the critical state is attained when the above rate falls below a practically observable limit. The exponent in (2) then becomes a constant value and α therein is identified experimentally as α_c. For $\alpha < \alpha_c$, the logarithmic decay of J as predicted by (2) has been verified with a high degree of accuracy. For $\alpha > \alpha_c$, however, the rate (2) is inconveniently large and J is externally supplied to hold α at a desired level. In this situation, the flux creep generates an uncompensated emf proportional to (2).

Figure 1 shows typical voltages observed across a 3Nb–Zr wire sample—plotted as a function of H for different sets of constant J's ($\perp H$). Although the strong dependence of V on J and H is evident, the raw data do not display readily recognizable systematics. If, however, V is plotted as a function of $\alpha = J(H + B_0)$ with $B_0 = 0.5$ kG as determined from the data,[5] $V(H)$'s for different J's all coalesce to a single curve within the scatters shown by the horizontal flags. This indicates that at a given temperature V is a function of α only. According to (2), the slope

$$\partial \ln V/\partial\alpha = q/kT \tag{3}$$

is expected to be constant. As α increases, however, the observed slope decreases until V is almost linear in α. In this region the prevailing process is visualized as "flux flow" rather than "flux creep."[6] Since the

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[3] J. C. Gorter, Phys. Letters **1**, 69 (1962); **2**, 26 (1962).
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[5] Y. B. Kim, C. F. Hempstead, and A. R. Strnad, Phys. Rev. **131**, 2486 (1963).
[6] P. W. Anderson and Y. B. Kim, Rev. Mod. Phys. **36**, 39 (1964).