Theory of a Local Superconductor in a Magnetic Field

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The Ginzburg–Landau equations, proposed[1] as a phenomenological description of the behavior of a superconductor in a magnetic field, have proved to be strikingly successful in explaining a wide variety of experimental observations.[2] A more fundamental justification of the Ginzburg–Landau theory and a deeper understanding of the reasons for its success has been provided by Gor'kov,[3] on the basis of his Green's function reformulation[4] of the BCS microscopic theory. Gor'kov, however, restricted his attention to the region of temperatures near the critical temperature, $T_c - T \ll T_c$, which entitled him to make a number of simplifying approximations:

(1) that the position-dependent energy gap function $\Delta(\mathbf{R},T)$, proportional to the Ginzburg–Landau order parameter, was small compared to T;

(2) that $\Delta(\mathbf{R},T)$ was slowly varying over distances $\xi(T)$ characterizing the spatial extent of the electron pair correlation;

(3) that the penetration depth $\delta(T)$ was much larger than $\xi(T)$, so that the magnetic field was also slowly varying over a correlation distance;

(4) that the bulk critical field $H_c(T)$ was sufficiently small so that the radius of a cyclotron orbit was large compared with $\xi(T)$.

Using these approximations, that $\Delta, \partial/\partial\mathbf{R}$, and the vector potential $\mathbf{A}$ were small expansion parameters, Gor'kov[3] was not only able to derive the form of the Ginzburg–Landau equations, but to relate the phenomenological parameters to measurable microscopic ones.

There has been considerable recent interest in the possibility of generalizing the Ginzburg–Landau–Gor'kov (GLG) theory to lower temperatures, and several preliminary attempts[5] in this direction have been made. It is reasonable to suppose, for one thing, that Gor'kov's approximation (1) is no more than a mathematical convenience, without real physical necessity. Since the GLG equations are already nonlinear in Δ, removal of the small Δ restriction with an attendant increase in the nonlinearity should not seriously diminish their utility. On the other hand, mathematically, approximations (2)–(4) are vital for converting the nonlocal integral equations, which are the natural description of superconductivity arising from the Green's function formulation, into local differential equations of the GLG character. It is just this local differential character which gives the GLG theory its usefulness and intuitive appeal. Physically, furthermore, approximations (2)–(4) in effect merely state that the correlation distance ξ is the shortest length entering the problem, and hence that a local description is appropriate.[6] Such a local description of superconductivity, however, cannot have as wide a physical relevance at all T as Gor'kov demonstrated it to have near T_c, since a principal feature of the BCS theory of pure bulk superconductors is the intrinsic nonlocality. Nevertheless, there are indications that certain pure transition metals, such as Nb, are in fact local,[7] and it could be argued that the short mean free path of superconducting alloys should make them local as well.

A closely related subject of current interest is the role of the magnetic spin energy in determining critical behavior. It has been pointed out by Chandrasekhar[8] and by Clogston[9] that the difference in spin susceptibilities between normal and superconducting states may be the primary factor limiting the upper critical field, H_{c2}, in negative surface energy superconductors with very high H_{c2}. Evidence for this occurring experimentally has been marshaled by Berlincourt and Hake.[10] The GLG theory, so powerfully brought to bear on the negative surface

[1] V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950).

[2] V. L. Ginzburg, Zh. Eksperim. i Teor. Fiz. **34**, 113 (1958) [English transl.: Soviet Phys.—JETP **7**, 78 (1958)]; D. H. Douglass, Jr., Phys. Rev. **124**, 735 (1961); IBM J. Res. Develop. **6**, 44 (1962).

[3] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **36**, 1918 (1959) [English transl.: Soviet Phys.—JETP **9**, 1364 (1959)]; Zh. Eksperim. i Teor. Fiz. **37**, 1407 (1959) [English transl.: Soviet Phys.—JETP **10**, 998 (1960)].

[4] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **34**, 935 (1958) [English transl.: Soviet Phys.—JETP **7**, 505 (1958)].

[5] M. Tinkham, IBM J. Res. Develop. **6**, 49 (1962); J. Bardeen, Rev. Mod. Phys. **34**, 667 (1962).

[6] It is thus not surprising that a charged boson gas model of superconductivity is described by equations very similar to those of GLG: F. Bloch and H. E. Rorschach, Phys. Rev. **128**, 1697 (1962).

[7] T. F. Stromberg and C. A. Swenson, Phys. Rev. Letters **9**, 370 (1962); S. H. Goedmoed, A. Van der Giessen, D. de Klerk, and C. J. Gorter, Phys. Letters **3**, 250 (1963).

[8] B. S. Chandrasekhar, Appl. Phys. Letters **1**, 7 (1962).

[9] A. M. Clogston, Phys. Rev. Letters **9**, 266 (1962).

[10] T. G. Berlincourt and R. R. Hake, Phys. Rev. **131**, 140 (1963).

energy phase by Abrikosov,[11] nonetheless omits all mention of electron spin.[12]

In a previous paper,[13] we have obtained equations describing a pure local superconductor at arbitrary temperature, by following Gor'kov's derivation,[3] but relaxing approximation (1). We here wish to review the principal results of that work, to present the modifications and extensions due to taking into account a finite mean free path and the magnetic spin energy, and to further discuss the conditions under which such a local description may be valid.

The central result of *I* is to determine a real, gauge invariant, free energy functional, $\mathfrak{F}\{\Delta,\Delta^*,\mathbf{A}\}$, appropriate to a pure local superconductor in a magnetic field. Requiring that the functional be stationary with respect to both Δ (or its complex conjugate Δ^*) and $\mathbf{A}$ leads to coupled differential equations determining these quantities. The former is the extension of the GLG equation for the order parameter or gap function, while the latter is just Maxwell's equation for the induced field with the Meissner currents as a source. When these equations are satisfied, the functional equals the free energy difference between the superconducting and normal phases. The functional is found to be

$$\mathfrak{F} = \int d^3R \left\{ N(0) \left[w(|\Delta(\mathbf{R})|^2) + \frac{v_F^2}{6} \left| \left[\frac{\partial}{\partial \mathbf{R}} - \frac{2ie}{c} \mathbf{A}(\mathbf{R}) \right] \Delta(\mathbf{R}) \right|^2 w''(|\Delta(\mathbf{R})|^2) + \frac{v_F^2}{36} \left(\frac{\partial |\Delta(\mathbf{R})|^2}{\partial \mathbf{R}} \right)^2 w'''(|\Delta(\mathbf{R})|^2) \right] + \frac{[\mathbf{B}(\mathbf{R}) - \mathbf{B}_a]^2}{8\pi} \right\}, \tag{1}$$

with

$$w(x) \equiv -\int_{-\infty}^{\infty} d\epsilon \left[\frac{1}{\beta} \ln \frac{1 + \cosh \beta(\epsilon^2 + x)^{\frac{1}{2}}}{1 + \cosh \beta\epsilon} - x \frac{\tanh \frac{1}{2} \beta_c \epsilon}{2\epsilon} \right]. \tag{2}$$

Here $N(0)$ is the density of states of one spin at the Fermi surface, v_F is the Fermi velocity, $\mathbf{A}$ is the vector potential associated with the total magnetic field $\mathbf{B}$, $\mathbf{B}_a$ is the applied field, β is the inverse temperature with critical value β_c, and primes on w denote differentiation with respect to its argument.

As was pointed out in I, $\mathfrak{F}$ and its stationary conditions reduce to known results in several special cases. For $T \approx T_c$ and $|\Delta| \ll T$, expansion in powers of $|\Delta|^2$ using $w'(0) = -\ln T_c/T$ and $w''(0) = [7\zeta(3)]/[8(\pi T)^2]$ leads back to the usual GLG expressions. On the other hand, ignoring the position dependence of $|\Delta|$ recovers the BCS theory (in the London gauge, Δ real), since $w'(\Delta^2) = 0$ is just the standard BCS energy gap equation, $N(0)w(\Delta^2)$ is the BCS free energy density difference, and $2\Delta^2 w''(\Delta^2)$ is identical to the BCS factor Λ/Λ_T giving the temperature dependence of the London penetration depth. Since $w'' > 0$, inspection of $\mathfrak{F}$ shows that spatial variations of the gap function and magnetic field exclusion increase the superconducting free energy, as expected.

When the energy of interaction of the magnetic field with the electron spins is also taken into account, the functional $\mathfrak{F}$ is modified by the addition of the term

$$\delta\mathfrak{F}_{\text{spin}} = \int d^3R \, \tfrac{1}{2} N(0) (e\mathbf{B}/mc)^2 |\Delta(\mathbf{R})|^2 w''(|\Delta(\mathbf{R})|^2) \, . \tag{3}$$

That this expression is in fact a reasonable one may be seen by again considering Δ independent of $\mathbf{R}$; then $-N(0)(e/mc)^2 \Delta^2 w''(\Delta^2)$ is just the difference in spin susceptibilities between superconducting and normal phases, $\chi_s - \chi_n$, first computed by Yosida.[14] This result, combined with that of the preceding paragraph for the penetration depth, illustrates rather convincingly the point originally made by Bardeen,[15] that, in the local limit, a two-fluid model is valid, and that $2|\Delta(\mathbf{R})|^2 w''(|\Delta(\mathbf{R})|^2)$ is the appropriate position-dependent superfluid density fraction. Again it may be noted that the spin energy increases the superconducting free energy, confirming the remarks of Chandrasekhar[8] and Clogston.[9]

Before comparing the spin energy to the GLG orbital magnetic energy, to determine the relative contribution of the former to the negative surface energy phase, it is also necessary to insert a finite mean free path into Eqs. (1) and (3). A partial answer is obtained very easily by adapting the calculations of Abrikosov and Gor'kov[16] for the one- and two-particle superconducting Green's functions in the presence of dilute random impurity scattering

[11] A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. **32**, 1442 (1957) [English transl.: Soviet Phys.—JETP **5**, 1174 (1957)].

[12] But see N. Kusnezov, Phys. Rev. **130**, 2253 (1963).

[13] N. R. Werthamer, Phys. Rev. **132**, 2440 (1963), henceforth referred to as I. The results obtained here have also been derived independently by L. Tewordt, Phys. Rev. **132**, 595 (1963) and private communication.

[14] K. Yosida, Phys. Rev. **110**, 769 (1958).

[15] J. Bardeen, Phys. Rev. Letters **1**, 399 (1958).

[16] A. A. Abrikosov and L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **35**, 1558 (1958) [English transl.: Soviet Phys.— JETP **8**, 1090 (1959)]; Zh. Eksperim. i Teor. Fiz. **36**, 319 (1959) [English transl.: Soviet Phys.—JETP **9**, 220 (1959)]; Zh. Eksperim. i Teor. Fiz. **39**, 480 (1960). [English transl.: Soviet Phys.—JETP **12**, 337 (1961)].

centers. Their results suffice to show that the w term in $\mathfrak{F}$ is unchanged by the presence of impurities, as is the spin term, whereas w'' is replaced by a new function which may be written as

$$w'' \to w''_\tau \equiv \pi T \sum_{n=0}^{\infty} (\omega_n^2 + |\Delta|^2)^{-1} \times \left[(\omega_n^2 + |\Delta|^2)^{\frac{1}{2}} + \left(\frac{1}{2\tau_{\mathrm{tr}}}\right) \right]^{-1}, \qquad (4)$$

where $\omega_n \equiv (2n+1)\pi T$, τ_{tr} is the transport collision time, and w''_τ reduces to w'' as $\tau_{\mathrm{tr}} \to \infty$. How the w''' term in $\mathfrak{F}$ is modified cannot be determined without additional computation.

A comparison of the spin energy, Eq. (3), with the orbital magnetic energy of Eqs. (1) and (4) may now be made very crudely by approximating A by $B\delta$ (except for films of thickness $d \ll \delta$, in which case $A \sim Bd$). Then, for an ideal pure electron gas model, the orbital energy is seen to dominate by a factor $(p_F\delta)^2$, which might typically be of order 10^5. However, the transition metal compounds with very high T_c and H_{cu}, such as the V_3X compounds or Nb_3Sn, also all have high densities of states at the Fermi surface and very large effective masses, typically $\sim 10^2$. In addition, the electronic mean free path l in these materials as prepared in short compared to the coherence distance ξ_0 of the pure metal. In these more realistic circumstances, the ratio of orbital to spin energies is roughly $(mv_F\delta)^2(l/\xi_0)$, which, in many cases, could be of order unity. Thus a preliminary estimate confirms the importance of the spin energy for high critical field superconductors, and stresses the significance of extending Abrikosov's detailed calculations[11] of the negative surface energy phase to include the spin energy term, Eq. (3).

Finally, it is necessary to point out the limits of applicability of the local theory of superconductivity outlined above. The derivation leading to Eqs. (1)–(4) is an expansion resting crucially on the assumption that the coherence distance is short, and inspection shows that ξ is proportional to w''_τ. However, near a second-order critical point where $|\Delta|$ is small, roughly[17]

$$w''_\tau \backsim (\pi T)^{-1} \left[\pi T + \left(\frac{1}{2\tau_{\mathrm{tr}}}\right) \right]^{-1}.$$

As T tends toward zero, w''_τ approaches infinity and the expansion breaks down, despite the presence of a finite mean free path. Thus even though a superconductor may be local in weak fields, it becomes nonlocal at lower temperatures in fields sufficient to reduce the gap function substantially. Stated differently, as both T and Δ become small (compared to T_c), the coherence distance increases to the point where it no longer can be the shortest characteristic length entering the problem. As an example, the low temperature ($T \ll T_c$) magnetic transitions of a thin film ($d \ll \delta$), predicted to be of first order by Bardeen[5] from a theory resembling ours, must rather be discussed on the basis of the Gor'kov–Shapoval[18] nonlocal integral equations.

[17] Gor'kov (Ref. 3b) gives the exact expression.

[18] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **37**, 833 (1959) [English transl.: Soviet Phys.—JETP **10**, 593 (1960)]; E. A. Shapoval, Zh. Eksperim. i Teor. Fiz. **41**, 877 (1961) [English transl.: Soviet Phys.—JETP **14**, 628 (1962)].

Exact Solution of the Ginzburg–Landau Equations for Slabs in Tangential Magnetic Fields

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I. INTRODUCTION

The Ginzburg–Landau (GL) equations[1] for a slab of superconductor in a tangential external magnetic field may be taken as one dimensional in form, all quantities then being functions only of the transverse coordinate. The equations have usually been solved in the approximation that assumes the order to be constant or nearly constant across the slab,[2,3] which is adequate for films of moderate thickness for small kappa materials. However, the constant order approximation is not only quantitatively poor for thick films and large kappa materials, but completely fails to reveal the important high-field behavior and the

[1] V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950).

[2] V. L. Ginzburg, Zh. Eksperim. i Teor. Fiz. **34**, 113 (1958) [English Transl.: Soviet Phys.—JETP **34**, 78 (1958)].

[3] Paul M. Marcus, in *Proceedings of the Eighth International Conference on Low Temperature Physics, London, 1962* (Butterworths Scientific Publications, Ltd., London, 1962).