Relaxation, Retardation, and Superposition

J. Ross Macdonald and Carl A. Barlow, Jr.
Texas Instruments Incorporated, Dallas 22, Texas

INTRODUCTION

A PHYSICAL process governed by a linear differential equation may exhibit a single retardation or relaxation time or a distribution of such times. The usual distinction between retardation and relaxation systems is based on whether the physical response variable considered primary increases toward a final value upon application of step-function stimulation or decreases and relaxes toward such a final equilibrium or steady state. For example, when a constant mechanical stress is applied at $t = 0$, the noninstantaneous progression of the resulting strain toward a finite or infinite final value is a retardation process. Alternatively, when a constant strain is applied and maintained, the stress relaxes toward a zero or nonzero final value. Clearly, either process may be considered, and the roles of stimulus and response become interchanged on going from one to the other.

The development of mathematical methods of describing the temporal and/or frequency response of linear distributed dielectric and mechanical systems has progressed in a parallel but infrequently tangential fashion for a long time. Many developments have been carried over freely from one field to the other, but this process has been retarded by the multiplicities of different nomenclatures in the two fields. In an effort to bring the notation and methods of the two fields into closer agreement, in the present work we shall first compare equivalent quantities in the two areas using reasonably well standardized notation.[1-3] We shall then show how retardation and relaxation processes can be described by a single set of constitutive relations and shall finally discuss representations of the superposition principle valid when impulse functions occur under the integral sign.

[1] H. Leaderman, Trans. Soc. Rheol. 1, 213 (1957).
[2] J. R. Macdonald and M. K. Brachman, Rev. Mod. Phys. 28, 393 (1956). The following proofing errors should be corrected. First, the ∞ limit in Eq. (1) should be changed to t. Second, the lower limit in the first part of Eq. (2) should be $0-$ not 0.
[3] J. D. Ferry, *Viscoelastic Properties of Polymers* (John Wiley & Sons, Inc., New York, 1961).

COMPARISON OF MECHANICAL AND DIELECTRIC RESPONSE

Table I compares nomenclature for retardation and relaxation systems. We have elected to consider only the mechanical shear situation[1,3]; notation for other types of mechanical deformation, such as volume and longitudinal, appears in the final report of the Committee on Nomenclature of the Society of Rheology.[1] In a mechanical or dielectric retardation system, stress or voltage is applied and the basic response is considered to be the resulting strain or charge. On the other hand, for a mechanical relaxation system, strain is the applied, controlled variable, and the resulting stress the conjugate variable. Since the dielectric relaxation situation where a constant charge is applied and the resulting voltage measured is not used experimentally, it is not further considered herein.

In Table I, C_a and ϵ_a are the capacitance and dielectric constant, respectively, of the dielectric system with air or vacuum replacing the usual material between the plates; their introduction allows simple transformation from capacitance to dielectric constant variables. One difference between corresponding quantities for mechanical and dielectric retardation systems is that the strain $\epsilon(t)$ is an intensive quantity while the measureable charge $q(t)$ is extensive. This difference will be eliminated in later comparison using normalized variables; it can easily be eliminated directly by the use of intensive variables such as current density and electric field strength. It is unfortunate that the same symbol ϵ is used for strain in mechanical systems and for dielectric constant in dielectric ones. The dielectric notation used herein is conventional[2] except for R_{p_0}. This is the low-frequency limiting parallel resistance of the system; it is frequently infinite; and it may be represented by a series RC, in parallel with the rest of the system, in which the series capacitance is infinite, leading to an infinite time constant. Similarly, the C_∞ introduced later is the high-frequency limiting capacitance of a series RC branch, in parallel with the rest of the system, in which the series R is

TABLE I. Comparison of nomenclature.

Retardation				Relaxation	
Mechanical		Dielectric		Mechanical	
Name	Symbol	Name	Symbol	Name	Symbol
Stress	$\sigma(t)$	Voltage	$V(t)$	Stress	$\sigma(t)$
Strain	$\epsilon(t)$	Charge	$q(t)$	Strain	$\epsilon(t)$
Time derivative of strain	$\dot{\epsilon}(t)$	Current	$i(t)$	...	...
Steady-state or equilibrium shear compliance	$J_e = J_g + J_d$	Low-frequency limiting capacitance	$C_s = \left(\frac{C_a}{\epsilon_a}\right)\epsilon_s$	Equilibrium shear modulus	$G_e = G_g - G_d$
Glass shear compliance	$J_g = J_e - J_d$	High-frequency limiting capacitance	$C_\infty = \left(\frac{C_a}{\epsilon_a}\right)\epsilon_\infty$	Glass shear modulus	$G_g = G_e + G_d = J_g^{-1}$
Delayed shear compliance	$J_d = J_e - J_g$	...	$C_s - C_\infty = \left(\frac{C_a}{\epsilon_a}\right) \times \left(\epsilon_s - \epsilon_\infty\right)$	Decay shear modulus	$G_d = G_g - G_e$
Shear viscosity coefficient	η	Low-frequency limiting parallel resistance	R_{p_0}	Shear viscosity coefficient	η

zero; it is frequently termed the geometrical capacitance and leads to a zero time constant or retardation time. The quantity G_e is only different from zero and equal to J_e^{-1} when the viscosity is infinite, as in a cross-linked polymer which admits no plastic flow.

Further quantities may be defined when one considers system response to a forcing step function applied at $t = 0$. We use the conventional unit step function[2,4] $u_0(t)$ and Dirac delta function $\delta(t) = \dot{u}_0(t) \equiv u_1(t)$ centered around $t = 0$. A dot denotes differentiation with respect to the argument throughout. It is convenient to introduce the following symbols for limiting processes:

$$\lim_{\epsilon\to 0}\begin{cases} -\epsilon \equiv 0-, \\ +\epsilon \equiv 0+, \\ t-\epsilon \equiv t-, \\ t+\epsilon \equiv t+, \end{cases} \quad (1)$$

where ϵ is positive. Then $u_0(0-) = 0$, $u_0(0) = \frac{1}{2}$, and $u_0(0+) = 1$.

[4] G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers* (McGraw-Hill Book Company, Inc., New York, 1961), pp. 255–259, 740–745.

Table II shows various step-function responses. Thus, $C(t)$ is the effective time-varying capacitance (or normalized charge) following the application of a step function of voltage at $t = 0$ to an initially uncharged system; the corresponding time-varying effective dielectric constant transient response is $\epsilon(t) = (\epsilon_a/C_a)C(t)$. Note that $J(t)$, $C(t)$, and $G(t)$ are normalized quantities inasmuch as they represent system transient response to a unit-amplitude step-function stimulus. It is usually convenient to separate out that part of the response of the system which does not depend on such quantities as C_∞ and R_{p_0} or J_g and η. The resulting response function, which we denote with a subscript d as shown in Table II, is then associated only with one or more finite retardation or relaxation times or with a continuous distribution thereof. The normalized response functions $\psi(t)$ and $\phi(t)$ of Table II thus describe this part of the system only.

For mechanical systems, $\psi(t)$ is usually termed the creep or retardation function and $\phi(t)$ the relaxation function. It is especially important to emphasize

TABLE II. Step-function response.

Stimulus $f(t)$	Retardation: Mechanical $\sigma(t)=\sigma_0 u_0(t)$ — Name	Symbol	Retardation: Dielectric $V(t)=V_0 u_0(t)$ — Name	Symbol	Relaxation: Mechanical $\epsilon(t)=\epsilon_0 u_0(t)$ — Name	Symbol
Normalized response $r_n(t)$	Shear creep compliance	$J(t)\equiv\epsilon(t)/\sigma_0 = [J_g+t/\eta]u_0(t)+J_d\psi(t)$	Time-varying capacitance	$C(t)\equiv q(t)/V_0 = [C_\infty+t/R_{p_0}]u_0(t)+(C_s-C_\infty)\psi(t)$	Shear relaxation modulus	$G(t)\equiv\sigma(t)/\epsilon_0 = G_e u_0(t)+G_d\phi(t)+\eta\delta(t)$
	Delayed component of shear creep compliance	$J_d(t)\equiv J_d\psi(t)$	Delayed component of time-varying capacitance	$C_d(t)\equiv(C_s-C_\infty)\psi(t)$	Decay component of shear relaxation modulus	$G_d(t)\equiv G_d\phi(t)u_0(t)$
	Retardation shear creep function	$\psi(t)$	Retardation function	$\psi(t)$	Shear relaxation function	$\phi(t)$

that these step-function transient response functions must be zero for $t<0$, i.e., for times before the application of a stimulus. Thus, when expressions for them are written down explicitly, they must involve $u_0(t)$ or higher order impulse functions such as $\delta(t)$. The $u_0(t)$'s present in the $J(t)$, $C(t)$, and $G(t)$ of Table II are usually[1,3,5-7] omitted and $\psi(t)$ and $\phi(t)$ are apparently taken as continuous and involving no impulse functions. This procedure leads to difficulties, as shown later, since it eliminates a possible discontinuity at the origin and implies that the above functions may not be zero for $t<0$.

Table III presents a comparison of important normalized quantities. It is here assumed that the forcing function is applied at $t=0$ and thus involves $u_0(t)$ and/or higher order impulse functions. Table III should be read either from the "General" column to the left or from this column to the right, not completely from left to right. Thus, the general distribution function $F(\tau)$ may be used to represent either $L(\tau)/J_d$, where $L(\tau)$ is usually denoted the spectrum of mechanical retardation times, or $H(\tau)/G_d$, where $H(\tau)$ is the mechanical relaxation time distribution, or spectrum, or probability density. Although $L(\tau)$ and $H(\tau)$ are inter-related[3,5,6] for the same material, as are $\psi(t)$ and $\phi(t)$, the present arrangement is not meant to imply that they are equal but only to show how the same general notation may be used to describe the essential parts of either retardation or relaxation response.

All quantities appearing in Table III below the $f_n(t)$ row pertain to the delayed or decay component of the over-all system response. Therefore, the zero and infinite retardation times arising from, e.g., J_g and η are not included in the $L(\tau)$ retardation spectrum. Thus, in the dielectric case the delayed dielectric response function is $\epsilon_d(t)=\epsilon(t)-[\epsilon_\infty+(\epsilon_a t/C_a R_{p_0}]\ u_0(t)=(\epsilon_s-\epsilon_\infty)\psi(t)$ and the frequency response function $\epsilon_d'(\omega)$ is the usual $[\epsilon'(\omega)-\epsilon_\infty]$ when $R_{p_0}=\infty$.

A number of important relations between quantities in the "General" column are given in the Appendix. These also hold for the specific retardation or relaxation cases but are summarized most conveniently in terms of the present generalized notation. Further relations between the general quantities are given elsewhere.[8] Finally, because of the linearity assumed in the present work, step-function charging from an initial zero charge condition and complete discharging curves, for example, should be of the same form when $R_{p_0}=\infty$. Thus, if the normalized step-function retardation response from $t=0$ is described by $\psi(t)$, then the corresponding discharge quantity, measured after complete charging and with $t=0$ at the instant of discharge, will be $[u_0(t)-\psi(t)]$. Such discharge, or its mechanical equivalent, may be considered a relaxation process (here of charge or strain, not stress), and it is only in this case that the relaxation function $\phi(t)$ equals the corresponding strain function $[u_0(t)-\psi(t)]$ for the same material.

[5] B. Gross, *Mathematical Structure of the Theories of Viscoelasticity* (Hermann & Cie, Paris, France, 1953).
[6] H. Leaderman, *Rheology*, edited by F. R. Eirich (Academic Press Inc., New York, 1958), Vol. 2, pp. 1–61.
[7] A. V. Tobolsky, *Properties and Structure of Polymers* (John Wiley & Sons, Inc., New York, 1961), p. 104.

[8] Reference 2. See Ref. 17 for a discussion of changes needed to bring full consistency between the present work and that of Ref. 2.

TABLE III. Relations between general and specific response functions.

Retardation		General	Relaxation
Mechanical	Dielectric		Mechanical
$\sigma(t)/\sigma_0$	$V(t)/V_0$	$f_n(t)$ (normalized forcing function)	$\epsilon(t)/\epsilon_0$
$\epsilon_d(t)/\sigma_0$	$\epsilon_d(t)/V_0$	$r_n(t)$ (normalized response function)	$\sigma_d(t)/\epsilon_0$
$\psi(t)$	$\psi(t)$	$\xi(t)\begin{cases}\xi(0)=0\\ \xi(\infty)=1\end{cases}$	
$\dot{\psi}(t)$	$A(t)$	$A(t)\equiv\dot{\xi}(t)$	$\delta(t)-\dot{\phi}(t)$
		$u_0(t)-\xi(t)$	$\phi(t)\begin{cases}\phi(0+)=1\\ \phi(\infty)=0\end{cases}$
	$G(\tau)$	$G(\tau)$	
$L(\tau)/J_d$	$F(\tau)$	$F(\tau)\equiv\tau G(\tau)$	$H(\tau)/G_d$
$J_d^*(i\omega)/J_d\equiv[J_d'(\omega) - iJ_d''(\omega)]/J_d$	$Q(i\omega)=\dfrac{\epsilon_d^*(i\omega)}{\epsilon_s-\epsilon_\infty}$	$Q(i\omega)\equiv J(\omega)-iH(\omega)$	
$J_d'(\omega)/J_d$	$J(\omega)=\epsilon_d'(\omega)/(\epsilon_s-\epsilon_\infty)$	$J(\omega)$	
$J_d''(\omega)/J_d$	$H(\omega)=\epsilon_d''(\omega)/(\epsilon_s-\epsilon_\infty)$	$H(\omega)$	
		$\overline{Q}(i\omega)\equiv 1-Q(i\omega)$	$G_d^*(i\omega)/G_d\equiv[G_d'(\omega) + iG''(\omega)]/G_d$
		$1-J(\omega)$	$G_d'(\omega)/G_d$
		$H(\omega)$	$G''(\omega)/G_d=G_d''(\omega)/G_d$

THE SUPERPOSITION PRINCIPLE

The superposition principle for our three present cases is frequently expressed in the form[5,6]

$$\begin{aligned}\sigma(t)&=\int_{-\infty}^{t}\dot{\epsilon}(\tau)G(t-\tau)d\tau\,,\\ \epsilon(t)&=\int_{-\infty}^{t}\dot{\sigma}(\tau)J(t-\tau)d\tau\,,\\ q(t)&=\int_{-\infty}^{t}\dot{V}(\tau)C(t-\tau)d\tau\,,\end{aligned}\tag{2}$$

where the forcing function may be switched on at any specific time. For simplicity, we now change to a more general notation which may represent any of these cases. Then one may write,

$$r(t)=\int_{-\infty}^{t}\dot{f}(\tau)\theta(t-\tau)d\tau=\int_{0}^{\infty}\dot{f}(t-\tau)\theta(\tau)\,d\tau\,,\tag{3}$$

where the second result follows by transformation of variables[6] and $\theta(t)$ may be $G(t)$, $J(t)$, $C(t)$, $\xi(t)$, $\psi(t)$, or $\phi(t)$. We shall not necessarily require $\theta(0+) = 0$, as is the case for $\psi(t)$, but shall take $f(-\infty) = 0$. In general, $f(t)$ and $\theta(t)$ will involve impulse functions, such as $u_0(t)$, $\delta(t)$, etc. To ensure proper spanning of such functions, the t limit in (3) must be replaced by $t+$ or $+\infty$.[9] Then the 0 limit in the second expression of (3) becomes $0-$ or $-\infty$. In the latter case, (3) is transformed to the usual form of the convolution integral.[10] When $f(t)=\sigma(t) = \sigma_0 u_0(t)$, for example, and $\theta(t)=J(t)$, the superposition integral leads to $r(t)=\sigma_0 J(t)\equiv\epsilon(t)$, in agreement with the result in Table II for this case.

The superposition integral is most frequently employed with forcing functions which are applied at $t=0$; therefore, we shall now consider various ways the integral can be expressed in this general case, a case where $f(t)$ and $\theta(t)$ will involve impulse functions centered at the origin. Then transformation

[9] B. Gross, Lineare Systeme, Suppl. Nuovo Cimento **3**, 235 (1956).

[10] I. I. Hirschman and D. V. Widder, *The Convolution Transform* (Princeton University Press, Princeton, New Jersey, 1955).

of variables, differentiation, and integration by parts leads to the following equivalent forms,

$$
\begin{aligned}
r(t) &= \int_{-\infty}^{\infty} \dot{f}(\tau)\theta(t-\tau)d\tau = \int_{-\infty}^{\infty} \dot{f}(t-\tau)\theta(\tau)d\tau\,, \\
&= \frac{d}{dt}\int_{-\infty}^{\infty} f(t-\tau)\theta(\tau)d\tau = \frac{d}{dt}\int_{-\infty}^{\infty} f(\tau)\theta(t-\tau)d\tau\,, \\
&= \int_{-\infty}^{\infty} f(\tau)\dot{\theta}(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)\dot{\theta}(\tau)d\tau\,, \qquad (4)
\end{aligned}
$$

where we have used $f(t) = \theta(t) = 0$ for $t < 0$, and where the integration can be reduced, if desired, to the range $(0-, t+)$ because of the above conditions on $f(t)$ and $\theta(t)$. Further reduction of the range to $(0, t)$ is, in general, not permissible, because impulse-function contributions to the superposition integrals would not then be properly included. It is readily shown that $r(t)$, proportional to current in the dielectric case, is given by equations of the same form as (4) with the function of t under the integral sign replaced by its derivative. In the dielectric case[2] $\ddot{\theta}(t) \equiv B(t) \equiv \dot{A}(t)$, where $B(t)$ measures the current impulse response.

The particular form of the integrands in (4) is required by the principle of temporal antecedence of stimulus with respect to response.[11] Thus, in the case considered, where the stimulus and response were tacitly assumed to be applied and observed at identical points in space, we have required that for all $f(t)$ vanishing for $t < 0$, the step-function response $\theta(t)$ and the general response $r(t)$ must likewise vanish for $t < 0$. This leads to the condition[4] $\theta(t-\tau) = 0$ for $t < \tau$. If one drops the assumption of spatial localization, the relativistic analog becomes that the response vanish at all points lying outside the forward light cone emanating from the stimulus. Analytically, this condition is $\theta(t-\tau) = 0$ for all $t - \tau < \Delta(t)$, where $\Delta(t)$ is defined implicitly by the equation $|\mathbf{r}_{\text{obs}}(t) - \mathbf{r}_{\text{stim}}(t - \Delta(t))| = c\Delta(t)$; in this equation $\mathbf{r}_{\text{obs}}(t)$ defines the "point" of observation of the response and $\mathbf{r}_{\text{stim}}(t-\Delta)$ the "point" of application of the stimulus at the retarded time $t - \Delta$.

The question of impulse functions in the integrands of (4) is the subject of considerable confusion in the literature, especially when the superposition integral is written in some of the forms that are discussed later. An effort to provide clarification has been published by Gross and Güttinger[12] but is insufficiently general in the forms used for $f(t)$ and $\theta(t)$.

First, it is clear that if $f(t)$ is applied at $t = 0$ and is zero for $t < 0$, it must be of the form $f(t) = f_0(t)u_0(t)$ plus possible terms involving higher order impulse functions, where $f_0(t)$ is a continuous function of t, is indefinitely differentiable, and may or may not be zero at $t = 0$. [For simplicity, we omit consideration of the case where $f(t)$ or its derivatives may be discontinuous at points for which $t > 0$.] Thus, even when the higher order impulse terms in $f(t)$ itself are identically zero, it or one of its derivatives must be discontinuous at the origin.

Secondly, since $\theta(t) = 0$ for $t < 0$, as discussed above, $\theta(t)$ must also be of the form $\theta(t) = \theta_0(t)u_0(t)$ plus possible terms involving higher order impulse functions. Again, $\theta(t)$ or its nth derivative must show a discontinuity at the origin. Examples are afforded by $C(t)$ and $G(t)$ of Table II, where the former involves a term $C_\infty u_0(t)$ and the latter involves both $u_0(t)$ and $\delta(t)$ terms.

Several authors have attempted to avoid the difficulties raised by the above discontinuities by specifying that such functions as $f(t)$ and $\theta(t)$ [or $J(t)$] are continuous and that delta functions, when they appear, are to be handled by summations instead of integrations. However, we have seen that $f(t)$ and $\theta(t)$ cannot be continuous functions; they are continuous only when the $u_0(t)$ terms are incorrectly omitted from $f(t)$ and $\theta(t)$. It has apparently been felt by many authors that when $f(t)$ is applied at $t = 0$, setting the lower limit of the superposition integral to zero and using continuous functions in the integrand is an adequate procedure. This is not the case in general if it is correctly required that $r(t)$ be zero for $t < 0$ and differentiable to any order, and is not even correct for $r(t)$ itself in many cases of practical interest. It is hazardous also to omit the $u_0(t)$'s from explicit expressions for $f(t)$ and $\theta(t)$ even when it is stated, instead, that they are zero for $t < 0$. Such a procedure may make it appear that the time drivative of $f(t) = f_0(t)$ [$f(t) = 0$, $t < 0$] is $\dot{f}_0(t)$ [$\dot{f}(t) = 0$, $t < 0$], whereas the correct result is $\dot{f}(t) = f_0(t)\delta(t) + \dot{f}_0(t)u_0(t)$.

The superposition principle for stimulus first applied at $t = 0$ is sometimes written in one of the following forms,

$$r(t) = \int_0^t \dot{f}(\tau)\theta(t-\tau)d\tau\,, \qquad [7] \qquad (5)$$

$$r(t) = f(t)\theta(0) + \int_0^\infty f(t-\tau)\dot{\theta}(\tau)d\tau\,, \qquad [6, 13] \qquad (6)$$

[11] M. Bunge, Am. Scientist **49**, 432 (1961).

[12] B. Gross and W. Güttinger, Appl. Sci. Res. Sec. B: **6**, 189 (1956).

[13] Reference 3, pp. 15, 16.

$$r(t) = f(t)\theta(0) + \int_{0-}^{t} f(\tau)\dot{\theta}(t-\tau)d\tau\,, \quad [14] \qquad (7)$$

$$\dot{r}(t) = f(0)\dot{\theta}(t) + \int_{0}^{t} \dot{f}(\tau)\theta(t-\tau)d\tau\,, \quad [15, 16] \qquad (8)$$

$$\dot{r}(t) = f(0)\dot{\theta}(t) + \int_{0-}^{t} \dot{f}(\tau)\theta(t-\tau)d\tau\,, \quad [17] \qquad (9)$$

$$\dot{r}(t) = f(0-)\dot{\theta}(t) + \int_{0-}^{t} \dot{f}(\tau)\theta(t-\tau)d\tau\,, \quad [14] \qquad (10)$$

where the numbers in square brackets are pertinent references. As is shown subsequently, none of these expressions is sufficiently general to apply without further qualification.

In order to obtain equations to compare with (5) through (10), contributions from impulse functions must be brought out from under the integral sign in equations such as (4). In general, such separation requires care, and there is the possibility of non-commutation of products involving higher order impulse functions. These matters are considered in the general case in a later paper.[18] For the present, however, we may select the simplified case specified by $f(t) = f_0(t)u_0(t)$ and $\theta(t) = \theta_0(t)u_0(t)$, where $f_0(t)$ and $\theta_0(t)$ are "good" functions involving no impulses and are continuously differentiable. The latter condition is necessary to ensure that all resulting equations can be indefinitely differentiated. We obtain

$$r(t) = f_0(0)\theta(t) + u_0(t)\int_{0}^{t} \dot{f}_0(\tau)\theta_0(t-\tau)d\tau\,, \qquad (11)$$

$$r(t) = f(t)\theta_0(0) + u_0(t)\int_{0}^{t} f_0(t-\tau)\dot{\theta}_0(\tau)d\tau\,, \qquad (12)$$

$$\dot{r}(t) = f_0(0)\dot{\theta}(t) + \dot{f}_0(t)\theta_0(0)u_0(t) + u_0(t)\int_{0}^{t} \dot{f}_0(\tau)\dot{\theta}_0(t-\tau)d\tau \qquad (13)$$

$$\dot{r}(t) = f(t)\dot{\theta}_0(0) + \dot{f}(t)\theta_0(0) + u_0(t)\int_{0}^{t} f_0(t-\tau)\ddot{\theta}_0(\tau)d\tau\,, \qquad (14)$$

$$\dot{r}(t) = f_0(0)\dot{\theta}(t) + \dot{f}_0(0)\theta(t) + u_0(t)\int_{0}^{t} \ddot{f}_0(\tau)\theta_0(t-\tau)d\tau\,, \qquad (15)$$

where it will be noted that only "good" functions appear within the integrals and that quantities such as $f(0)$ and $\theta(0)$, which are often left ill-defined, no longer appear. An alternative way of writing Eqs. (11) through (15) is to change the limits to $0+$ and $t-$ and then, if desired, use the complete functions $f(t)$ and $\theta(t)$ rather than the present "good" functions in the integrand. Note that when $\theta(t)$ involves $\delta(t)$, as does $G(t)$ when $\eta \neq 0$, the above equations are insufficiently general and it is simplest to use one of Eqs. (4).

Next, we compare the predictions of Eqs. (5) through (10) with the results (11) through (15). The comparison may be made on the basis of using continuous functions in (5) to (10). On taking $f(t) = f_0(t)$ and $\theta(t) = \theta_0(t)$, one finds that Eqs. (5) and (7) give the correct answer if $r(t)$ is explicitly stated to be zero for $t < 0$ or if the results are multiplied by $u_0(t)$, an equivalent procedure. Equation (6), however, does not yield the proper result because the range of integration remains $(0,\infty)$ and does not become $(0,t)$. Further, $u_0(t)$ needs to be added to each of the results of Eqs. (8), (9), and (10). In addition, however, it is found that the term $\dot{f}_0(t)\theta_0(0)u_0(t)$ is missing from the predictions of these latter equations. More terms will be missing from higher derivatives of $r(t)$ when continuous functions are used.

Secondly, comparison may be made on the basis of the use of the proper functions, $f(t) = f_0(t)u_0(t)$ and $\theta(t) = \theta_0(t)u_0(t)$, in Eqs. (5) to (10). In order properly to span all impulses, the limits of the integrals should all be changed to $(0-, t+)$ or $(-\infty, \infty)$. With this change, Eq. (5) becomes identical with one of Eqs. (4) and is correct. All the other integrals can then be paired with one of the forms in (4) or with the results for $\dot{r}(t)$ following from (4). It thus follows that Eqs. (6) through (10) with their limits properly extended will only give correct answers in the present case provided that $\theta(0) = \theta_0(0)u_0(0)$ and $f(0) = f_0(0)u_0(0)$ are zero. This is only the case, in general, if $u_0(0) = 0$. This require-

[14] J. R. Macdonald, J. Appl. Phys. **32**, 2385 (1961). Quantities such as $\psi(t)$ of this reference are not normalized to unity. When normalization is possible, equivalence with the present work is established by selecting the quantity q such that $\psi(\infty) = 1$ and taking $J_dM = 1$. All explicit expressions for $\psi(t)$ and $A(t)$ in this paper should be multiplied by $u_0(t)$ on the right-hand side of each equation.

[15] J. R. Carson, *Electric Circuit Theory and the Operational Calculus* (Chelsea Publishing Company, New York, 1953), 2nd ed., p. 16.

[16] M. F. Gardner and J. L. Barnes, *Transients in Linear Systems* (John Wiley & Sons, Inc., New York, 1942), p. 234.

[17] Ref. 2, Eq. (2). The $r(t)$ of this reference is the present $\dot{r}(t)$; also Eq. (2) is only equivalent to the present Eq. (9) if the $A(t)$ used in Ref. 2 is identified with the present $\theta(t)$. This requires that the $A(t)$ of Ref. 2 be zero for $t < 0$ and, therefore, that explicit equations for it involve impulse functions. This requirement, unfortunately, has not been entirely followed in Ref. 2. Consistency necessitates that the following changes be made. Replace the t limit by $t+$ in Eqs. (1) and (2) and eliminate the first equation of Eq. (2); eliminate the $A(0)\delta(t)$ term from Eq. (5) and the $A(0)$ from 33; in all explicit results for $r(t)$, $A(t)$, and $B(t)$ multiply all terms not already implicitly or explicitly involving impulse functions by $u_0(t)$—thus, e.g., (27) and (28) would not be changed but (47), (48), and (63) would be; finally, add the condition in Appendix I that if $g(y)$ is the result of a generalized Fourier sine or cosine transform it must be taken zero for $y < 0$.

[18] Work in progress of the present authors.

ment is, however, not consistent with $\delta(t) \equiv \dot{u}_0(t)$ and the use of a delta function centered around $t = 0$. Further, it serves no purpose to put in such terms as $f(t)\theta(0)$ if they are to be eliminated. On the other hand, if the limits of all the integrals in (5) to (10) are changed to $(0+,t-)$ to exclude contributions from discontinuities, difficulties still arise from the consequent omission of $u_0(t)$ multiplying the integrals and the incorrectness of the added terms in the equations.

In conclusion, it appears that the safest procedure for calculating $r(t)$ or any of its derivatives by a superposition integral is to use one of Eqs. (4) together with functions under the integrand that explicitly include impulse-function terms such as $u_0(t)$ or $\delta(t)$ which ensure that the functions are zero for $t < 0$.

ACKNOWLEDGMENT

Numerous valuable discussions with Dr. Robert Stratton are much appreciated.

APPENDIX: SOME GENERAL RELATIONS

$$Q(p) = \int_{0-}^{\infty} A(t)e^{-pt}dt \equiv \mathcal{L}[A(t)] = \int_0^{\infty} \frac{G(\tau)d\tau}{1+p\tau} = \int_{-\infty}^{\infty} \frac{F(\tau)ds}{1+p\tau}, \tag{A1}$$

where $Q(0) = 1$, $\mathcal{L}$ denotes the Laplace transform, $s \equiv \ln(\tau/\tau_0)$, and τ_0 is an arbitrary positive number.

$$\bar{Q}(p) = 1 + \int_{0-}^{\infty} \phi(t)e^{-pt}dt = 1 - \mathcal{L}[A(t)] = 1 - \int_0^{\infty} \frac{G(\tau)d\tau}{1+p\tau} = \int_0^{\infty} \left(\frac{p\tau}{1+p\tau}\right) G(\tau)d\tau = \int_{-\infty}^{\infty} \left(\frac{p\tau}{1+p\tau}\right) F(\tau)ds . \tag{A2}$$

When the complex variable p approaches $i\omega$, then $Q(p) \to Q(i\omega)$ and $\bar{Q}(p) \to \bar{Q}(i\omega)$. It follows that

$$J(\omega) = \int_0^{\infty} \frac{G(\tau)d\tau}{1+(\omega\tau)^2} = \int_{-\infty}^{\infty} \frac{F(\tau)ds}{1+(\omega\tau)^2}, \tag{A3}$$

$$1 - J(\omega) = \int_0^{\infty} \frac{(\omega\tau)^2}{1+(\omega\tau)^2} G(\tau)d\tau = \int_{-\infty}^{\infty} \frac{(\omega\tau)^2}{1+(\omega\tau)^2} F(\tau)ds , \tag{A4}$$

$$H(\omega) = \int_0^{\infty} \frac{(\omega\tau)G(\tau)d\tau}{1+(\omega\tau)^2} = \int_{-\infty}^{\infty} \frac{(\omega\tau)F(\tau)ds}{1+(\omega\tau)^2}, \tag{A5}$$

$$A(t) = \mathcal{L}^{-1}[Q(p)] = u_0(t) \int_0^{\infty} \left[\frac{G(\tau)}{\tau}\right] e^{-t/\tau} d\tau = u_0(t) \int_{-\infty}^{\infty} \left[\frac{F(\tau)}{\tau}\right] e^{-t/\tau} ds , \tag{A6}$$

$$\xi(t) = \int_{0-}^{t} A(x)dx = \left[1 - \int_0^{\infty} G(\tau)e^{-t/\tau}d\tau\right] u_0(t) = \left[1 - \int_{-\infty}^{\infty} F(\tau)e^{-t/\tau}ds\right] u_0(t) . \tag{A7}$$

Finally, $J(\omega)$ and $H(\omega)$ are connected by the Kronig–Kramers relations.[2] Those of the above equations involving τ may conveniently be alternatively expressed[14] in terms of $z \equiv \tau_0/\tau = \exp(-s)$.