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The Octet Model and its Clebsch-Gordan Coefficients

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1. INTRODUCTION

In trying to understand the structure of the strong interactions, several higher symmetry schemes have been proposed.[1,2] These higher symmetries should conserve the isospin I and the hypercharge Y. Especially interesting in this respect is the octet model (unitary symmetry) proposed independently by Gell-Mann[3] and Ne'eman.[4] In this model one assumes the strongest interactions to be invariant under transformations belonging to $SU(3)$, i.e., under unimodular unitary transformations in some three-dimensional complex linear vector space ("unitary spin space"). The symmetry of these strong interactions is broken by some unknown weaker mechanism, but in such a way that the isospin and the hypercharge are still conserved. A still weaker interaction, the electromagnetic interaction, breaks this lower symmetry in such a way that only the hypercharge and the third component of isospin are conserved. In this unitary symmetry model one assigns groups of strongly interacting particles with the same quantum numbers (not the same are I, Y, I_3, and directly related ones as strangeness, charge, G parity, etc.), to irreducible representations (IR's) of the group $SU(3)$. The lowest nontrivial IR in the octet model, which is physically possible (i.e., has integer quantum numbers for the hypercharge), is the IR $\{8\}$. The eight well-known baryons N,Λ,Σ, and Ξ, as well as the eight pseudoscalar mesons, K,η,π, and $\bar{K}$, are assigned to IR's $\{8\}$. One assumes, moreover, the existence of eight vector mesons which belong to such a representation. Perhaps the mesons ρ,ω,K^*, and $\bar{K}^*$ constitute this octet. A difficulty here is which K^* to take. There seem to be two $(K\pi)$ resonances, one[5] at 730 MeV and the other[6] at 888 MeV. One favors the 888-MeV resonance because it seems to have all the correct quantum numbers. The next higher IR can contain 10 particles. It is suggested[7] that the familiar (3,3) pion-nucleon resonance, the Y_1^* (1385 MeV), the recently discovered[8,9] $I = \frac{1}{2}$, $\Xi\pi$ resonance at 1532 MeV and a still unknown baryon $\Omega^-(Y = -2,\ I = 0,\ \pm 1685$ MeV) belong to this IR $\{10\}$. A discovery of this Ω^- would be a great triumph for this octet model. Okubo[10] has derived a mass formula for the different members belonging to the same IR. For the octets (IR $\{8\}$), this formula reduces to a mass relation between the different members. This mass relation is very well satisfied for the baryons and for the pseudoscalar mesons. However, for the vector mesons, neither the 888-MeV nor the 730-MeV $(K\pi)$ resonance fulfills this relation. For the IR $\{10\}$ this mass formula is again very well satisfied. Coleman and Glashow[11] have given a relation connecting the electromagnetic mass differences within the baryon octet. This relation is also very well satisfied.

The main purpose of this paper is to derive the

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[1] A very nice survey of the different higher symmetry schemes in strong interactions is given by R. E. Behrends, J. Dreitlein, C. Fronsdal, and B. W. Lee, Rev. Mod. Phys. **34**, 1 (1962). The reader is referred there to the large existing literature about this subject.

[2] D. R. Speiser and J. Tarski, Math. Phys. **4**, 588 (1963).

[3] M. Gell-Mann, California Institute of Technology, Report CTSL–20, March, 1961 (unpublished); Phys. Rev. **125**, 1067 (1962).

[4] Y. Ne'eman, Nucl. Phys. **26**, 222 (1961).

[5] G. Alexander, G. R. Kalbfleisch, D. H. Miller, and G. A. Smith, Phys. Rev. Letters **8**, 447 (1962).

[6] For extensive references, see, *Proceedings of the 1962 Annual International Conference on High-Energy Physics, at CERN* (CERN, Geneva, 1962), p. 781.

[7] M. Gell-Mann, *Proceedings of the 1962 Annual International Conference on High-Energy Physics, at CERN* (CERN, Geneva, 1962), p. 805.

[8] G. M. Pjerrou, D. J. Prowse, P. Schlein, W. E. Slater, D. H. Stork, and H. K. Ticho, *Proceedings of the 1962 Annual International Conference on High-Energy Physics, at CERN* (CERN, Geneva, 1962), p. 289.

[9] L. Bertanza, V. Brisson, P. L. Connolly, E. L. Hart, I. S. Mittra, G. C. Moneti, R. R. Rau, N. P. Samios, S. S. Yamamoto, M. Goldberg, L. Gray, J. Leitner, S. Lichtman, and J. Westgard, *Proceedings of the 1962 Annual International Conference on High-Energy Physics, at CERN* (CERN, Geneva, 1962), p. 279.

[10] S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).

[11] S. Coleman and S. L. Glashow, Phys. Rev. Letters **6**, 423 (1961).

Clebsch–Gordan coefficients[12–14] (CG coefficients) of $SU(3)$ for the products of the most important irreducible representations (Secs. 10, 11, and 18). Special care is taken to define properly all the relevant phase factors (Secs. 7, 10). Some useful symmetry relations for the CG coefficients are derived (Sec. 14). The Wigner–Eckart theorem[15] for this group is given (Sec. 15) and applied to derive a general mass formula for the octets (Sec. 16). A special case gives the Gell-Mann–Okubo mass relation (16.3). Another special case, however, gives a mass relation (16.15) for the octets which is very well satisfied by the vector mesons if one takes as the K^* the 730-MeV $(K-\pi)$ resonance. For completeness and to demonstrate how to handle some special phase assignments, we have considered in Sec. 17 the Yukawa couplings between the baryons and the mesons. To be able to show clearly how the results in the later sections are derived, some additional sections were necessary to define properly the different symbols and concepts used. This leads us to the alternate purpose of this paper; only slight extensions of the existing sections and a few additional ones were necessary to give a rather complete insight in the mathematical framework of this special model for the strong interactions. The treatment is as much as possible "physical"[16] and tries not to rely too heavily on results obtained by purely abstract group theoretic methods. However, where necessary, results only easily obtained (to the author's knowledge) by such methods are stated and used. A very good example is Sec. 12 where the beautifully simple method of Speiser[17] for reducing the direct product of two IR's is explained, but not proved.

2. TENSORS[18]

The group $SU(3)$ consists of all the unitary unimodular transformations in the three-dimensional vector space C_3 over the complex numbers. Let us denote a vector in this space by x^i and its complex conjugate by x_i; thus $x_i = (x^i)^*$. Under a transformation of the group, the vectors x^i and x_i get transformed into the vectors $\bar{x}^i$ and $\bar{x}_i$ according to[19]

$$\bar{x}^i = \alpha_{ij}x^j \,, \tag{2.1a}$$

$$\bar{x}_j = \alpha_{ij}^* x_j = \alpha_{ji}^{-1} x_j \,, \tag{2.1b}$$

because unitarity of α implies[20]

$$\alpha^+ = \alpha^{-1}\,, \quad \text{or} \quad \alpha_{ij}^* = \alpha_{ji}^{-1}\,.$$

In this vector space C_3 we can define mixed tensors $A_{ij\cdots k}^{\alpha\beta\cdots\gamma}$ which transform according to

$$\bar{A}_{ij\cdots k}^{\alpha\beta\cdots\gamma} = \alpha_{\alpha\lambda}\alpha_{\beta\mu}\cdots\alpha_{\gamma\nu}\alpha_{li}^{-1}\alpha_{mj}^{-1}\cdots\alpha_{nk}^{-1}A_{lm\cdots n}^{\lambda\mu\cdots\nu}\,. \tag{2.2}$$

Very special tensors are δ_j^i, ϵ^{ijk}, and ϵ_{ijk}; they are unchanged under a transformation of the group. We have

$$\bar{\delta}_j^i = \alpha_{ik}\alpha_{lj}^{-1}\delta_l^k = \alpha_{ik}\alpha_{kj}^{-1} = \delta_j^i\,,$$

and

$$\bar{\epsilon}^{ijk} = \alpha_{il}\alpha_{jm}\alpha_{kn}\epsilon^{lmn} = \det\alpha\,\epsilon^{ijk} = \epsilon^{ijk}\,,$$

because of the restriction to unimodular transformations ($\det\alpha = 1$).

The monomials $M(p,q)$

$$x^\alpha y^\beta \cdots z^\gamma u_i v_j \cdots w_k$$

with p upper indices and q lower indices are transformed into each other by transformations of the group. These monomials could, therefore, conveniently be used as a basis to construct representations of the group. These representations will, in general, be reducible, as will be shown below, because of the existence of the tensors δ_j^i, ϵ^{ijk}, and ϵ_{ijk}.

With the help of these special tensors, we can construct, from the general mixed tensor $A_{ij\cdots l}^{\alpha\beta\cdots\delta}$ with p upper and q lower indices, the mixed tensors B, C, and D. Where

$$B_{j\cdots l}^{\beta\cdots\delta} = \delta_\alpha^i A_{ij\cdots l}^{\alpha\beta\cdots\delta}$$

is a tensor with $(p-1)$ upper indices and $(q-1)$ lower indices,

$$C_{\mu ij\cdots l}^{\gamma\cdots\delta} = \epsilon_{\mu\alpha\beta}A_{ij\cdots l}^{\alpha\beta\gamma\cdots\delta}$$

is a tensor with $(p-2)$ upper and $(q+1)$ lower indices, and

$$D_{k\cdots l}^{m\alpha\beta\cdots\delta} = \epsilon^{mij}A_{ijk\cdots l}^{\alpha\beta\cdots\delta}$$

is a tensor with $(p+1)$ upper and $(q-2)$ lower indices. The tensors B, C, and D are linear combina-

[12] In several other papers [e.g., Refs. 13 and 14] tables of CG coefficients can be found. Special care has to be taken in using these tables in combination with some of the theorems of this paper. The phase definitions for these CG coefficients are not the same and mostly not stated.

[13] A. R. Edmonds, Proc. Roy. Soc. (London) **A268**, 567 (1962).

[14] M. A. Rashid, Nuovo Cimento **26**, 118 (1962).

[15] C. Eckart, Rev. Mod. Phys. **2**, 302 (1930); E. P. Wigner, Gruppentheorie, Vieweg (1931).

[16] With "physical" we mean in this context: "Along the lines familiar to most physicists with some knowledge of the theory of angular momentum." If in some place in the following sections, jumps in the reasoning are made which are a little large for the reader, he is advised to look at the analogous situation in the theory of angular momentum and the point will almost always become clear (at least this was our experience).

[17] D. R. Speiser, in Proceedings of the Istanbul International Summer School 1962 (to be published).

[18] R. H. Dalitz, Lectures, University of Chicago, summer 1962.

[19] We will use the Einstein summation convention.

[20] α^+ denotes the Hermitian conjugate of α, α^T the transpose, and α^* the complex conjugate.

tions of the elements of the tensor A with p upper and q lower indices. The transformation properties of B, C, and D are, however, different from a tensor with p upper and q lower indices. The tensor A is therefore reducible, unless B, C, and D are identically zero.

We find that $B = 0$ when $A_{ij\cdots k}^{i\beta\cdots\gamma} = 0$, thus when the trace of A with respect to the indices α and i is zero; $C = 0$ when A is symmetric in the indices α and β; and $D = 0$ when A is symmetric in the indices i and j.

It is now clear how to construct bases for irreducible representations of $SU(3)$. We take such linear combinations $P(p,q)$ of the monomials $M(p,q)$ that these polynomials $P(p,q)$ are

(1) totally symmetric in all p upper indices,
(2) totally symmetric in all q lower indices,
(3) traceless.[21]

These polynomials $P_{ij\cdots k}^{\alpha\beta\cdots\gamma}$ form a basis for the IR $D(p,q)$ of $SU(3)$. The dimension N of $D(p,q)$, i.e., the number of basis vectors is

$$N = (1 + p)(1 + q)[1 + \tfrac{1}{2}(p + q)] . \quad (2.3)$$

Proof: A tensor, with only upper indices symmetric in these p indices, has $\frac{1}{2}(p + 1)(p + 2)$ linearly independent components. This can be seen in the following way. Due to the symmetry requirement, the order of the indices is irrelevant. We could, therefore, arrange the indices in such a way that we have first all the ones, then all the twos, and finally all the threes. Let us assume that we have α indices equal to one, then α could run from zero to p. For the twos and threes are so left $(p - \alpha)$ indices. We could make up, therefore, $(p - \alpha + 1)$ different combinations, with α ones and the rest of the indices twos and/or threes. In total there are thus

$$\sum_{\alpha=0}^{p} (p - \alpha + 1) = \tfrac{1}{2}(p + 1)(p + 2)$$

different components. A tensor with only lower indices and symmetric in these q indices has $\frac{1}{2}(q + 1)(q + 2)$ linearly independent components. A mixed tensor totally symmetric in its p upper and q lower indices has therefore $N_1 = \frac{1}{4}(p + 1)(p + 2)(q + 1)(q + 2)$ linearly independent components. The requirement that the trace should be zero gives further restrictions. The trace is a tensor with $(p - 1)$ upper and $(q - 1)$ lower indices. The trace of a mixed tensor, totally symmetric in its p upper and totally symmetric in its q lower indices, has N_2 linearly independent components, where $N_2 = \frac{1}{4}p(p + 1)q(q + 1)$. All these components should be identically zero. A traceless tensor symmetric in its p upper and symmetric in its q lower indices has therefore $N = N_1 - N_2$ linearly independent components, *q.e.d.*

A way to denote an IR is to write $\{N\}$, e.g., $D(1,1) = \{8\}$, $D(2,2) = \{27\}$, $D(3,0) = \{10\}$, etc. When more than one IR has the same dimension we could distinguish them by stars, primes, etc. For example, $D(p,q)$ and $D(q,p)$ $(p > q)$ have the same dimension. We denote then $D(p,q) = \{N\}$ and $D(q,p) = \{N^*\}$.

3. GENERATORS OF THE GROUP

In Sec. 2 we have shown that a suitable basis for the IR $D(p,q)$ of $SU(3)$ is formed by the N polynomials $P(p,q)$. There polynomials span a linear vector space V_N. A transformation α of $SU(3)$ in the space C_3 corresponds to a transformation U in the space V_N. These transformations U form the IR $D(p,q)$ of $SU(3)$, they are unitary[22] $(U^+ = U^{-1})$ and unimodular[23] (det $U = 1$).

Any unitary transformation U can be written as

$$U = e^{iH} , \quad (3.1)$$

where H is Hermitian; $H = H^+$. Also the inverse is true; for any Hermitian H the U defined by (3.1) is unitary. The requirement of unimodularity implies

$$\mathrm{Tr}\, H = 0 . \quad (3.2)$$

Also here the inverse is true; (3.2) ensures that U defined by (3.1) is unimodular.

We can also write the transformations α in the space C_3 in the form (3.1) with the condition (3.2). In a three-dimensional space, there exist nine linearly independent Hermitian operators but only eight traceless ones. To these eight operators in C_3 (the generators of the group) correspond eight Hermitian traceless operators F_i in V_N. We write, therefore,[3]

$$H = \sum_{i=1}^{8} \alpha_i F_i , \quad (3.3)$$

where $F_i = F_i^+$, Tr $F_i = 0$, and α_i is real. To obtain the commutation relations for F_i, it is more con-

[21] Due to the symmetry requirements 1 and 2, every polynomial has only one trace. By trace we mean here only the contraction of one upper and one lower index and not the contraction of two upper or two lower indices.

[22] We can always choose the basis such that the matrices are unitary. See for example L. S. Pontrjagin, *Topologische Gruppen* (B. G. Teubner Verlagsgesellschaft, Leipzig, 1957), Vol. 1, Sec. 32.

[23] The representations have to be unimodular, because otherwise the unimodular matrices would form an invariant subgroup. The group $SU(3)$ does not have an invariant subgroup, the representations are therefore unimodular.

venient to express (3.3) slightly differently. We introduce a set of nine traceless operators[24] A^i_k, which are defined such that their representation in C_3 is given by

$$(A^i_k)_{\mu\nu} = \delta_{i\nu}\delta_{k\mu} - \tfrac{1}{3}\,\delta_{ik}\delta_{\mu\nu} \qquad (3.4)$$

($i,k,\mu,\nu = 1$, 2, or 3). These operators satisfy

$$A^i_k = (A^k_i)^+ . \qquad (3.5)$$

Moreover, they are not totally independent but

$$A^1_1 + A^2_2 + A^3_3 = 0 . \qquad (3.6)$$

We can now write (3.3) in terms of these A^i_k; we get

$$H = \sum \beta^k_i A^i_k , \qquad (3.7)$$

where the hermiticity of H requires

$$\beta^i_k = (\beta^k_i)^* . \qquad (3.8)$$

It is easy to see that these matrices A^i_k satisfy the commutation relations

$$[A^i_k, A^j_l] = \delta^i_l A^j_k - \delta^j_k A^i_l . \qquad (3.9)$$

To these 9 operators in C_3 correspond 9 operators in every space V_N satisfying the relations (3.5) to (3.9). Then they are, of course, not any more 3 by 3 but N by N matrices.

We will introduce here still another notation for the generators of the group, which we will use throughout the rest of this paper. With the help of this new notation the connection between the F_i (e.g., Gell-Mann's notation[3]) and the A^i_k (e.g., Okubo's notation[10]) is readily made.

We denote

$$\begin{aligned} &F_1 = I_1 , \quad F_4 = K_1 , \quad F_6 = L_1 , \quad F_8 = M , \\ &F_2 = I_2 , \quad F_5 = K_2 , \quad F_7 = L_2 , \\ &F_3 = I_3 . \end{aligned} \qquad (3.10)$$

We can form then the operators

$$\begin{aligned} I_\pm &= I_1 \pm iI_2 , \\ K_\pm &= K_1 \pm iK_2 , \\ L_\pm &= L_1 \pm iL_2 . \end{aligned} \qquad (3.11)$$

The operators A^i_k are then expressed in terms of these operators

$$\begin{aligned} &A^1_1 = I_3 + \tfrac{1}{3}\sqrt{3}M , \quad A^2_1 = I_+ , \quad A^1_2 = I_- , \\ &A^2_2 = -I_3 + \tfrac{1}{3}\sqrt{3}M , \quad A^3_1 = K_+ , \quad A^1_3 = K_- , \\ &A^3_3 = -\tfrac{2}{3}\sqrt{3}M , \quad A^3_2 = L_+ , \quad A^2_3 = L_- . \end{aligned} \qquad (3.12)$$

[24] These operators A^i_k differ from the ones given by Okubo (Ref. 10) by an over-all minus sign.

For completeness we will also give the relation between our set of generators and the set used by Behrends *et al.*[1]

$$\begin{aligned} &H_1 = (1/\sqrt{3})I_3 \quad E_1 = (1/\sqrt{3})I_+ \quad E_{-1} = (1/\sqrt{3})I_- \\ &H_2 = (1/\sqrt{3})M \quad E_2 = (1/\sqrt{3})K_+ \quad E_{-2} = (1/\sqrt{3})K_- \\ &\qquad\qquad E_3 = (1/\sqrt{3})L_+ \quad E_{-3} = (1/\sqrt{3})L_- \end{aligned} \qquad (3.13)$$

4. COMMUTATION RELATIONS

We note from the commutation relations (3.9) that I_3 and M are two commuting operators. Because the rank of the group is two, there exist no other linear operators commuting with these two. We will combine them into a vector $\mathbf{E} = (I_3, M)$. The commutation relations can then be written as

$$\begin{aligned} [\mathbf{E}, I_\pm] &= \pm\mathbf{i}I_\pm , \\ [\mathbf{E}, K_\pm] &= \pm\mathbf{k}K_\pm , \\ [\mathbf{E}, L_\pm] &= \pm\mathbf{l}L_\pm , \end{aligned} \qquad (4.1a)$$

and

$$\begin{aligned} [I_+, I_-] &= 2\mathbf{i}\cdot\mathbf{E} , \\ [K_+, K_-] &= 2\mathbf{k}\cdot\mathbf{E} , \\ [L_+, L_-] &= 2\mathbf{l}\cdot\mathbf{E} , \end{aligned} \qquad (4.1b)$$

where the unit vectors $\mathbf{i}$, $\mathbf{k}$, and $\mathbf{l}$ are defined by

$$\mathbf{i} = (1,0) , \quad \mathbf{k} = (\tfrac{1}{2}, \tfrac{1}{2}\sqrt{3}) , \quad \mathbf{l} = (-\tfrac{1}{2}, \tfrac{1}{2}\sqrt{3}) .$$

The other commutation relations, less symmetric in form, are

$$\begin{aligned} &[I_-, K_+] = L_+ , \quad [K_-, I_+] = L_- , \\ &[I_+, L_+] = K_+ , \quad [L_-, I_-] = K_- , \\ &[K_+, L_-] = I_+ , \quad [L_+, K_-] = I_- , \end{aligned} \qquad (4.1c)$$

and the rest is zero.

We will introduce the operators[24a]

$$P_i = e^{i\pi I_2} , \quad P_k = e^{i\pi K_2} , \quad P_l = e^{i\pi L_2} . \qquad (4.2)$$

Then

$$\begin{aligned} &P_i^{-1} I_\pm P_i = -I_\mp , \quad P_i^{-1} K_\pm P_i = L_\pm , \\ &P_k^{-1} I_\pm P_k = L_\mp , \quad P_k^{-1} K_\pm P_k = -K_\mp , \\ &P_l^{-1} I_\pm P_l = K_\pm , \quad P_l^{-1} K_\pm P_l = -I_\pm , \\ &\qquad P_i^{-1} L_\pm P_i = -K_\pm , \\ &\qquad P_k^{-1} L_\pm P_k = -I_\mp , \\ &\qquad P_l^{-1} L_\pm P_l = -L_\mp . \end{aligned} \qquad (4.3)$$

[24a] Quite extensive use of these operators is made by C. A. Levinson, H. J. Lipkin, and S. Meshkov, Phys. Letters **1**, 44, 125, and 307 (1962); Nuovo Cimento **23**, 236 (1962); Phys. Rev. Letters **10**, 361 (1962). See also A. J. Macfarlane, E. C. G. Sudarshan, and C. Dullemond, Nuovo Cimento (to be published).

These relations can easily be proved in the following way:

$$P_k^{-1}I_\pm P_k = I_\pm + i\pi[I_\pm,K_2] + ((i\pi)^2/2!)[[I_\pm,K_2],K_2] + \cdots$$
$$= I_\pm \cos \tfrac{1}{2}\pi + L_\mp \sin \tfrac{1}{2}\pi = L_\mp ,$$

and analogously for the other relations. From the relations (4.3) follows directly that

$$P_i^{-1}P_kP_i = P_l \ , \quad P_i^{-1}P_lP_i = P_k^{-1} ,$$
$$P_k^{-1}P_iP_k = P_l^{-1} , \quad P_k^{-1}P_lP_k = P_i \ ,$$
$$P_l^{-1}P_iP_l = P_k \ , \quad P_l^{-1}P_kP_l = P_i^{-1} . \qquad (4.4)$$

Moreover, we have the relations

$$P_i^{-1}\mathbf{E}P_i = \mathbf{E} - 2\mathbf{i}(\mathbf{i}\cdot\mathbf{E}) ,$$
$$P_k^{-1}\mathbf{E}P_k = \mathbf{E} - 2\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) ,$$
$$P_l^{-1}\mathbf{E}P_l = \mathbf{E} - 2\mathbf{l}(\mathbf{l}\cdot\mathbf{E}) . \qquad (4.5)$$

The relations (4.5) can easily be proved as follows:

$$P_k^{-1}\mathbf{E}P_k = \mathbf{E} + i\pi[\mathbf{E},K_2] + ((i\pi)^2/2!)[[\mathbf{E},K_2],K_2] + \cdots ,$$
$$= \mathbf{E} + \mathbf{k}(\mathbf{k}\cdot\mathbf{E})(\cos\pi - 1) + \mathbf{k}K_1 \sin\pi ,$$
$$= \mathbf{E} - 2\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) .$$

5. COMPLETE SET OF COMMUTING OPERATORS

To denote the different eigenstates, it is convenient to label them with the eigenvalues of a complete set of commuting operators which are linearly independent. The set consisting of I_3 and M can be extended with $I^2 = I_1^2 + I_2^2 + I_3^2$ and with two more operators F^2 and G^3, called Casimir operators.[25,26] These are[10]

$$F^2 = \tfrac{1}{2} \sum_{\mu\nu} A_\mu^\nu A_\nu^\mu = \sum_{i=1}^{8} F_i^2 ,$$
$$G^3 = \sum_{\mu\nu\lambda} A_\mu^\nu A_\nu^\lambda A_\lambda^\mu .$$

The operators F^2 and G^3 have the property that they commute with every F_i. According to Schur's lemma[27,28] these operators are constants for an irreducible representation. The IR's can, therefore, also conveniently be labeled by the eigenvalues f^2 and g^3 of these operators F^2 and G^3. Of course, the set (p,q) is equivalent with the set (f^2, g^3). The states within an IR can be labeled by the eigenvalues $T(T+1)$, T_z, and m of I^2, I_3, and M. Also for the eigenvalues of $\mathbf{E} = (I_3, M)$ we will use the vector notation $\mathbf{e} = (T_z, m)$. We will denote by $|\mathbf{e},\gamma)$ an eigenstate of the operator $\mathbf{E}$ belonging to the eigenvalue $\mathbf{e}$. The label γ describes the unspecified other quantum numbers. For an IR, we can also define the highest eigenvalue $\mathbf{e}_H$ and the highest eigenstate $|p, q, T, e_H, \gamma)$. This highest eigenvalue is that eigenvalue $\mathbf{e}$ within the IR which has the largest T_z; the highest eigenstate is the eigenstate corresponding to the highest eigenvalue.

6. TWO THEOREMS

We are now in the position to state some useful theorems.

Theorem 1: Let $|\mathbf{e}, \gamma)$ be an eigenstate of $\mathbf{E}$. If $K_+|\mathbf{e}, \gamma)$ is different from zero, then $K_+|\mathbf{e}, \gamma)$ is also an eigenstate of $\mathbf{E}$ with the eigenvalue $\mathbf{e} + \mathbf{k}$.

Proof: From $[\mathbf{E}, K_+] = \mathbf{k}K_+$ follows

$$\mathbf{E}K_+|\mathbf{e},\gamma) = K_+(\mathbf{E} + \mathbf{k})|\mathbf{e},\gamma) = (\mathbf{e} + \mathbf{k})K_+|\mathbf{e},\gamma) ,$$
Q.E.D.

Analogously we have, if $K_-|\mathbf{e}, \gamma) \neq 0$, then $K_-|\mathbf{e}, \gamma)$ has the eigenvalue $\mathbf{e} - \mathbf{k}$; if $I_\pm|\mathbf{e}, \gamma) \neq 0$ then $I_\pm|\mathbf{e}, \gamma)$ has the eigenvalue $\mathbf{e} \pm \mathbf{i}$, and if $L_\pm|\mathbf{e}, \gamma) \neq 0$ then $L_\pm|\mathbf{e}, \gamma)$ has the eigenvalue $\mathbf{e} \pm \mathbf{l}$.

This theorem has a simple geometrical interpretation in a two-dimensional eigenvalue diagram (Fig. 1). In this eigenvalue diagram every eigenvalue is

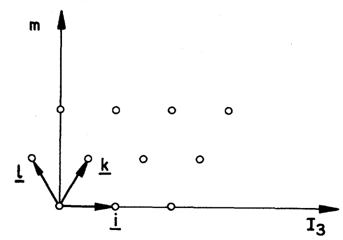

FIG. 1. Part of a two-dimensional eigenvalue diagram showing the regular pattern (Theorem 1).

represented by a point $\mathbf{e} = (T_z, m)$. These points form a regular pattern; the distances between neighboring points being $\mathbf{i}$, $\mathbf{k}$, or $\mathbf{l}$.

Theorem 2: Let $|\mathbf{e}, \gamma)$ be an eigenstate of $\mathbf{E}$, then $P_i|\mathbf{e}, \gamma)$ is also an eigenstate of $\mathbf{E}$ with the eigenvalue $\mathbf{e} - 2\mathbf{i}(\mathbf{i}\cdot\mathbf{e})$. The degeneracy of the state $P_i|\mathbf{e}, \gamma)$ is the same as the degeneracy of the state $|\mathbf{e}, \gamma)$.

Proof: From $P_i^{-1}EP_i = \mathbf{E} - 2\mathbf{i}(\mathbf{i}\cdot\mathbf{E})$ follows

$$\mathbf{E}P_i|\mathbf{e},\gamma) = P_i\{\mathbf{E} - 2\mathbf{i}(\mathbf{i}\cdot\mathbf{E})\}|\mathbf{e},\gamma)$$
$$= \{\mathbf{e} - 2\mathbf{i}(\mathbf{i}\cdot\mathbf{e})\}P_i|\mathbf{e},\gamma)$$

[25] H. B. G. Casimir, Proc. Roy. Acad. Amsterdam **34**, 844 (1931).
[26] M. Hamermesh, *Group Theory and its Application to Physical Problems* (Addison-Wesley Publishing Company, Inc., New York, 1962), Secs. 8–13.
[27] I. Schur, Sitzber. Preuss. Akad. Wiss. Physik. math. Kl. **24**, 406 (1905).
[28] Ref. 26, Secs. 8–14, lemma II.

Moreover, P_i is a unitary operator; it conserves, therefore, the multiplicity of the state.

Analogously we find that $P_k|\mathbf{e}, \gamma)$ and $P_l|\mathbf{e}, \gamma)$ have the eigenvalues $\mathbf{e} - 2\mathbf{k}(\mathbf{k}\cdot\mathbf{e})$ and $\mathbf{e} - 2\mathbf{l}(\mathbf{l}\cdot\mathbf{e})$. The degeneracy of these states is again the same as the degeneracy of $|\mathbf{e}, \gamma)$.

Also this theorem has a simple geometrical interpretation in the two-dimensional eigenvalue diagram. If $\mathbf{e}_1$ is an eigenvalue, then also the values $\mathbf{e}_2$, $\mathbf{e}_3$, and $\mathbf{e}_4$, obtained from $\mathbf{e}_1$ by reflection with respect to lines through the origin perpendicular to the $\mathbf{i}$, $\mathbf{k}$, and $\mathbf{l}$ directions, are eigenvalues [see Fig. 2(a)]. Once more applying P_i gives two more eigenvalues $\mathbf{e}_5$ and $\mathbf{e}_6$. So, in general, the existence of one eigenvalue implies the existence of six eigenvalues[29,30] all with the same degeneracy.

There are two exceptions. The first exception is when the eigenvalue $\mathbf{e}_1$ lies on one of the reflection lines [see Fig. 2(b)]. In this case the existence of one eigenvalue implies only the existence of three eigenvalues. The second exception is when the eigenvalue $\mathbf{e}_1$ lies in the center; $\mathbf{e}_1 = (0,0)$. In this case no other eigenvalues are implied.

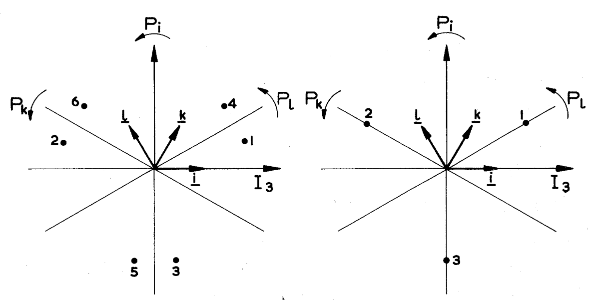

FIG. 2. Two-dimensional eigenvalue diagrams showing the results of the operators P_i, P_k, and P_l on the state $\phi(1)$ (Theorem 2). (a) When $\mathbf{e}(1)$ is arbitrary,
$P_i\phi(1) = \phi(2)$, $P_i\phi(3) = \phi(5)$,
$P_k\phi(1) = \phi(3)$, $P_i\phi(4) = \phi(6)$.
$P_l\phi(1) = \phi(4)$,
(b) When $\mathbf{e}(1)$ lies on the reflection line P_l,
$P_i\phi(1) = \phi(2)$,
$P_k\phi(1) = \phi(3)$,
$P_l\phi(1) = \phi(1)$.

We note that the operator P_i is the charge symmetry operator. The symmetries implied by the operators P_k and P_l are, therefore, generalizations of the principle of charge symmetry.

[29] In Racah's lecture notes (Ref. 30) is shown that not only $\mathbf{e}$ and $\mathbf{e} - \mathbf{i}(\mathbf{i}\cdot\mathbf{e})$ are eigenvalues, but the whole chain (isomultiplet) connecting these two eigenvalues.

[30] G. Racah, *Group Theory and Spectroscopy* (Institute for Advanced Study, Princeton, New Jersey, 1951).

7. IRREDUCIBLE REPRESENTATIONS I

In this section we consider the simplest irreducible representations which are of interest to us.

D(0,0) = {1}. This IR consists of only one state and because of theorem 2, the eigenvalue $\mathbf{e}$ belonging to this single state is $\mathbf{e} = (0,0)$. In the octet model, one identifies the hypercharge operator $Y = S + B$ with the operator

$$Y = (2/\sqrt{3})M\,. \tag{7.1}$$

This state is, therefore, an isosinglet state with $Y = 0$. The operators F_i we can represent by 1×1 matrices which are all identically zero (traceless).

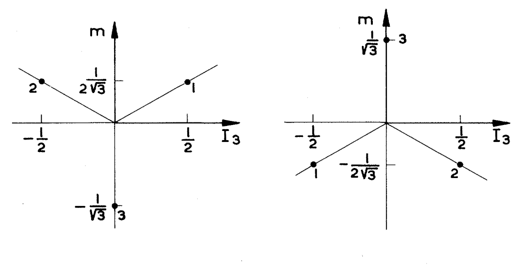

FIG. 3. Eigenvalue diagrams of the IR's with $N = 3$. (a) The irreducible representation $D(1,0) = \{3\}$. (b) The irreducible representation $D(0,1) = \{3^*\}$.

D(1,0) = {3}. There are three possibilities for this case according to theorem 2. The first possibility is three degenerate eigenvalues at the origin $I = Y = 0$. For this case, all the operators F_i are identically zero. Because they are 3×3 matrices, this case is reducible. The two other possibilities have eigenvalue diagrams as depicted in Fig. 3. Here we have to make a choice. We make the conventional choice [Fig. 3(a)]. We leave the eigenvalue diagram [Fig. 3(b)] for the contragredient representation $D(0,1) = \{3^*\}$. We are, therefore, able to write down

$$I_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } M = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

For the other matrices we have first to define the relative phases between these three states. We will do this the following way:

(1) Within an isomultiplet we use the Condon and Shortley phase convention.[31] This establishes that

$$I_+\phi_1 = 0\,, \quad I_+\phi_2 = \phi_1\,, \quad I_+\phi_3 = 0\,.$$

[31] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, England, 1935). See also Ref. 37.

(2) The relative phases between the different isomultiplets can also be defined rather easily. We require also for the operators $K_\pm$ the phase choice of Condon and Shortley. Therefore,

$$K_+\phi_1 = 0\,,\quad K_+\phi_2 = 0\,,\quad K_+\phi_3 = \phi_1\,.$$

The matrix elements of the operators $I_\pm$ and $K_\pm$ are now totally defined and, therefore, also of $L_+ = [I_-, K_+]$. We can identify $\phi_1 = x^1$, $\phi_2 = x^2$, and $\phi_3 = x^3$. These states ϕ_1, ϕ_2, and ϕ_3 in $SU(3)$ are equivalent to the states α (spin up) and β (spin down) in $SU(2)$.

D(0,1) = {3*}. The eigenvalue diagram is like Fig. 3(b). We can, therefore, write

$$I_3 = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \text{and}\ M = \frac{1}{2\sqrt{3}}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

With our phase conventions we have now

$$I_+\phi_i = \delta_{i1}\phi_2\,,\quad K_+\phi_i = \delta_{i1}\phi_3\,.$$

We can identify[32] $\phi_1(3^*) = -x_1$, $\phi_2(3^*) = x_2$, and $\phi_3(3^*) = x_3$.

All the IR $D(p,q)$ can be formed by the direct product of p times $D(1,0)$ and q times $D(0,1)$. This direct product contains of course more than the IR $D(p,q)$. However, it is clear that the highest eigenstate $\phi(e_H)$ of $D(p,q)$ can be formed in only one way

$$\phi(e_H) = \underbrace{x^1 y^1 \cdots z^1}_{p\ \text{factors}}\ \underbrace{u_2 v_2 \cdots w_2}_{q\ \text{factors}}\,. \tag{7.2}$$

The highest eigenvalue is, therefore,

$$\mathbf{e}_H = \left(\frac{p+q}{2},\ \frac{p-q}{2\sqrt{3}}\right). \tag{7.3}$$

Moreover, it is clear that this highest eigenvalue is nondegenerate. This highest eigenstate has hypercharge

$$Y = (p-q)/3\,. \tag{7.4}$$

Physically the hypercharge is an integer. If we want to restrict ourselves to the octet model, then not all the IR's of $SU(3)$ are interesting. The only IR's realizable in nature[33] are IR's for which $p - q = 3n$, where $n = 0, \pm1, \pm2, \pm3, \cdots$ etc.

These are, therefore, the representations

$$\{1\} = D(0,0),\quad \{10\} = D(3,0),\quad \{28\} = D(6,0),\ \text{etc.},$$
$$\{10^*\} = D(0,3),\ \{28^*\} = D(0,6),\ \text{etc.},$$
$$\{8\} = D(1,1),\quad \{35\} = D(4,1),\quad \text{etc.},$$
$$,\ \{35^*\} = D(1,4),\quad \text{etc.},$$
$$\{27\} = D(2,2),\quad \{81\} = D(5,2),\quad \text{etc.},$$
$$,\ \{81^*\} = D(2,5),\quad \text{etc.},$$
$$\{64\} = D(3,3),\quad \text{etc.}$$

To stay closer to the physics, we will work in the following solely with the hypercharge operator Y and its eigenvalues. The vertical scale in the eigenvalue diagrams we will, however, compress in such a way that the unit of length along this vertical axis still corresponds, as if we have plotted the eigenvalues of M. In this way we preserve the high symmetry in these eigenvalue diagrams.

To be able to refer simply to the different eigenstates of an IR we will enumerate them from 1 to N. We will choose the following ordering for the states:

(a) within an isomultiplet, the states are ordered so that I_z decreases;

(b) the isomultiplets belonging to the same Y are ordered so that I decreases;

(c) the groups for different Y are ordered such that Y decreases.

For the contragredient representation $\{N^*\}$, we adopt an opposite convention for (a) and (c); we order the states within a multiplet such that I_z increases and the groups of different Y we order such that Y increases. In this way the eigenvalue diagram for $\{N^*\}$ is the inverse with respect to the origin ($I_z = 0$, $Y = 0$) of the eigenvalue diagram for $\{N\}$.

To be able to define later uniquely the Clebsch–Gordan coefficients of $SU(3)$, it is necessary first of all to define precisely the relative phases within the IR's. We adopt, therefore, the following convention:

(1) The relative phases *within* a definite isomultiplet are determined by the Condon and Shortley phase convention.[31] Then

$$I_+\phi(I,I_z,Y) = [(I-I_z)(I+I_z+1)]^{\frac{1}{2}}\phi(I,I_z+1,Y),$$
$$I_-\phi(I,I_z,Y) = [(I+I_z)(I-I_z+1)]^{\frac{1}{2}}\phi(I,I_z-1,Y), \tag{7.5}$$

and

$$P_i\phi(I,I_z,Y) = (-)^{I+I_z}\phi(I,-I_z,Y)\,. \tag{7.6}$$

(2) The relative phases *between* the different isomultiplets we define then with the help of the operators $K_\pm$. Biedenharn[33a] has pointed out that

$$K_+\phi(I,I_z,Y) = b_+\phi(I+\tfrac{1}{2},I_z+\tfrac{1}{2},Y+1) + b_-\phi(I-\tfrac{1}{2},I_z+\tfrac{1}{2},Y+1), \tag{7.7}$$

[32] See Sec. 8 for the specific phases.

[33] This comes from our special choice (7.1) for the hypercharge operator.

[33a] L. C. Biedenharn, Phys. Letters **3**, 69 and 254 (1962); J. Math. Phys. **4**, 436 (1963). See also G. E. Baird and L. C. Biedenharn, J. Math. Phys. **4**, 1449 (1963).

where

$$b_+ = \left\{\frac{(I + I_z + 1)[\frac{1}{3}(p - q) + I + \frac{1}{2}Y + 1][\frac{1}{3}(p + 2q) + I + \frac{1}{2}Y + 2][\frac{1}{3}(2p + q) - I - \frac{1}{2}Y]}{2(I + 1)(2I + 1)}\right\}^{\frac{1}{2}} \quad (7.8a)$$

and

$$b_- = \left\{\frac{(I - I_z)[\frac{1}{3}(q - p) + I - \frac{1}{2}Y][\frac{1}{3}(p + 2q) - I + \frac{1}{2}Y + 1][\frac{1}{3}(2p + q) + I - \frac{1}{2}Y + 1]}{2I(2I + 1)}\right\}^{\frac{1}{2}} . \quad (7.8b)$$

We require thus always $b_\pm$ to be real and positive.

This uniquely defines the relative phases between all the states of an IR.

8. CONTRAGREDIENT REPRESENTATIONS

The representation $D(p,q)$ is called contragredient to the representation $D(q,p)$. These representations are intimately connected.

If $U = e^{i\alpha_i F_i}$ is a representation of an element of $SU(3)$, then so is $U^* = (U^{-1})^T$. Now

$$(U^{-1})^T = e^{-i\alpha_i F_i^T} = e^{i\alpha_i F_i'} .$$

Therefore, we could choose [actually we do not, see Eq. (8.1)] the generators F_i' of the contragredient representation to be

$$F_i' = -F_i^T .$$

Then we have

$$I_3' = -I_3 , \quad I_\pm' = -I_\mp ,$$
$$Y' = -Y , \quad K_\pm' = -K_\mp ,$$
$$L_\pm' = -L_\mp ,$$

and the relation between the eigenstates is

$$\phi(\{N^*\},I,I_z,Y) = \eta\phi^*(\{N\},I, -I_z, -Y)$$

where η is an over-all phase factor.

This choice is certainly *inconvenient,* because it implies that not all of the elements of the matrices $I_\pm$ and $K_\pm$ are positive, which is required due to our phase conventions (1) and (2). We have to take the following choices for the matrices F_i'' of the contragredient representation $\{N^*\}$ if we have the matrices F_i of the representation $\{N\}$.

$$I_3'' = -I_3 \quad \text{and} \quad Y'' = -Y . \quad (8.1a)$$

This is required because of the eigenvalue diagram and the specific ordering of the different states. Our phase conventions require the choice

$$I_\pm'' = I_\mp$$
$$K_\pm'' = K_\mp . \quad (8.1b)$$

Then the commutation relations (4.1c) require

$$L_\pm'' = -L_\mp . \quad (8.1c)$$

The relation between the eigenstates of the representations $\{N\}$ and $\{N^*\}$ is then

$$\phi(\{N^*\},I,I_z,Y) = \eta(-)^{I_z+\frac{1}{2}Y}\phi^*(\{N\},I, -I_z, -Y) . \quad (8.2)$$

Here η is an over-all phase which could conveniently be defined by the condition[34]

$$\phi(\{N^*\},I,0,0) = \phi^*(\{N\},I,0,0) \quad (8.3)$$

then $\eta = 1$.

9. IRREDUCIBLE REPRESENTATIONS II

In this section we will discuss the IR's which we will use later on.

D(1,1) = {8}. One can immediately construct the eigenvalue diagram [Fig. 4(a)]. The highest eigenvalue $\mathbf{e}_3$ is $I_z = 1$, $Y = 0$, hence $I = 1$. Because of theorem 2 this eigenvalue ensures the existence of six eigenvalues $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_5$, $\mathbf{e}_7$, and $\mathbf{e}_8$ (all nondegenerate). The existence of $\mathbf{e}_4$ ($I = 1$, $I_z = 0$, $Y = 0$) is also implied by the existence of $\mathbf{e}_3$, because $\mathbf{e}_3$, $\mathbf{e}_4$, and $\mathbf{e}_5$ form the $I = 1$ isomultiplet. We count so in total 7 eigenstates. The missing eighth state $\mathbf{e}_6$ can be nothing else than a $I = 0$, $Y = 0$ state. The matrices $I_\pm$ are given by Eq. (7.5), the matrices K_+ by Eqs. (7.7) and (7.8).

D(3,0) = {10}. The eigenvalue diagram is given in Fig. 4(b). The highest eigenvalue $\mathbf{e}_1$ is $I = \frac{3}{2}$, $I_z = \frac{3}{2}$, $Y = 1$. Theorem 2 implies the existence of $\mathbf{e}_4$ and $\mathbf{e}_{10}$. The eigenvalues $\mathbf{e}_2$ and $\mathbf{e}_3$ have to exist to make up the $I = \frac{3}{2}$, $Y = 1$ isomultiplet. However, the existence of $\mathbf{e}_2$ and $\mathbf{e}_3$ implies the existence of $\mathbf{e}_5$, $\mathbf{e}_7$, $\mathbf{e}_8$, and $\mathbf{e}_9$ due to theorem 2. The last eigenvalue $\mathbf{e}_6$ is necessary to complete the $I = 1$, $Y = 0$ multiplet.

D(0,3) = {10*}. The eigenvalue diagram is given in Fig. 4(c). The matrix K_+ is easily determined from the corresponding matrix of the representation $\{10\}$. We have

$$K_+(10^*) = [K_+(10)]^\dagger .$$

[34] In the octet model every IR possesses a state with $I_z = Y = 0$. See also Sec. 14.

The representations

$$D(2,2) = \{27\}\ , \quad D(6,0) = \{28\}\ ,$$
$$D(4,1) = \{35\}\ , \quad D(3,3) = \{64\}\ , \quad \text{etc.}\ ,$$

and the corresponding conjugate representations $\{28^*\}$, $\{35^*\}$, etc., can be obtained along the same lines. The eigenvalue diagrams are given in Figs. 4(d)–4(g).

10. CLEBSCH-GORDAN COEFFICIENTS

When one forms the product representation of two IR's $D(p_1,q_1)$ and $D(p_2,q_2)$, then this product

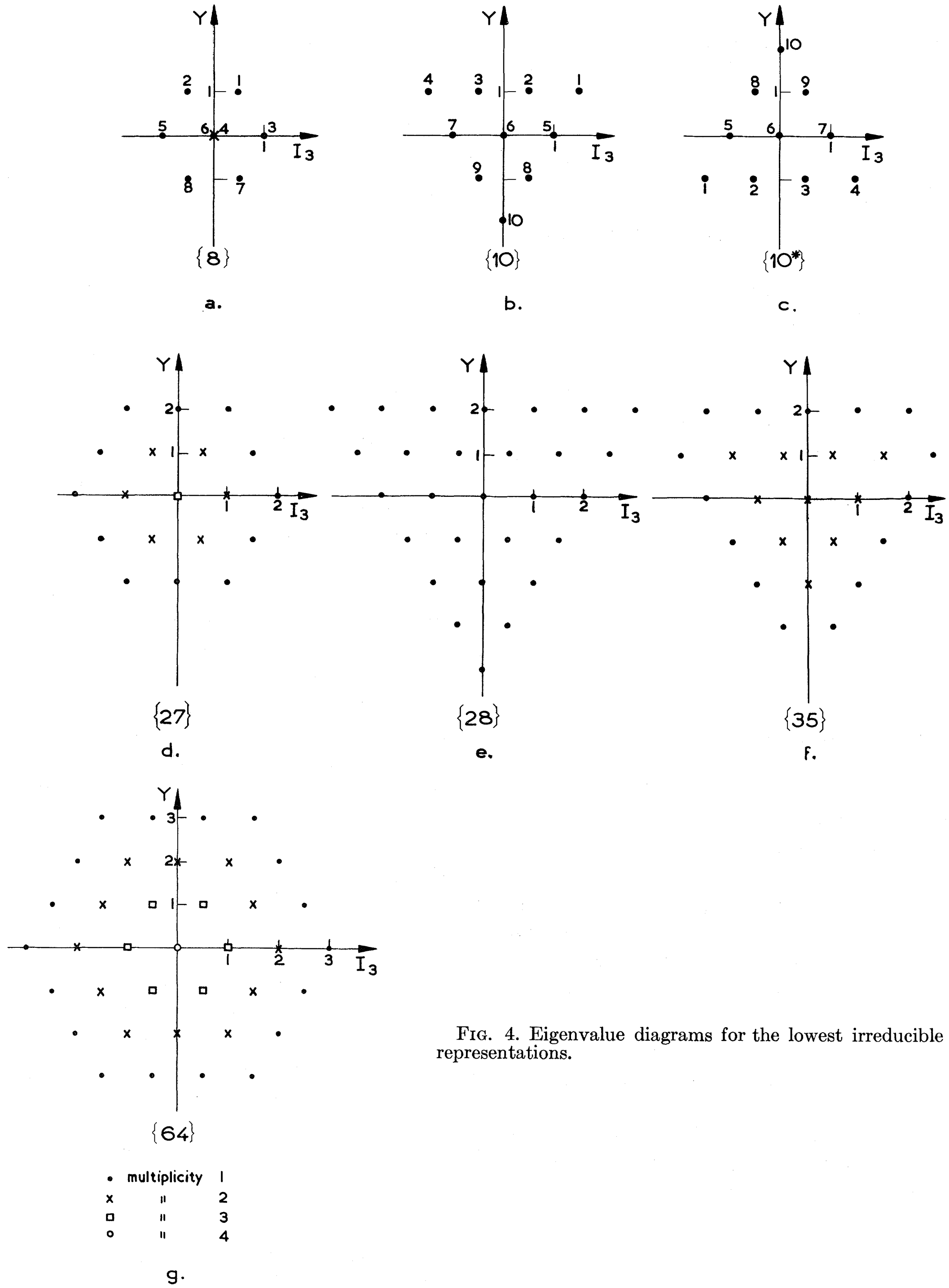

FIG. 4. Eigenvalue diagrams for the lowest irreducible representations.

representation is, in general, reducible. Symbolically one writes

$$D(p_1,q_1) \otimes D(p_2,q_2) = \sum_{P,Q} \oplus\, \sigma(P,Q) D(P,Q)\,, \quad (10.1)$$

where $\sigma(P,Q)$ is an integer. The series on the right-hand side of Eq. (10.1) is called the Clebsch–Gordan series (CG series). This symbolical equation states that the representation $D(P,Q)$ is contained $\sigma(P,Q)$ times in the direct product of the IR's $D(p_1,q_1)$ and $D(p_2,q_2)$.

We have seen in Sec. 5 that a complete set of operators necessary to specify uniquely the states of an IR is G^3, F^2, I^2, I_3, and Y. The states of the product representation $D(p_1,q_1) \otimes D(p_2,q_2)$ are, therefore, completely specified by the eigenvalues of the 10 linearly independent, commuting operators

$$G^3(1),\ G^3(2),\ F^2(1),\ F^2(2),\ I^2(1),\ I^2(2),\ I_3(1),\ I_3(2),\ Y(1), \text{ and } Y(2). \quad \text{(I)}$$

If we define the operators

$$F_i = F_i(1) + F_i(2) \quad (i = 1,\cdots,8) \quad (10.2)$$

then a (noncomplete) set of commuting operators is

$$G^3,\ G^3(1),\ G^3(2),\ F^2,\ F^2(1),\ F^2(2),\ I^2,\ I_3, \text{ and } Y.$$

However, this makes only 9 operators. We need another operator to make this set complete. This operator Γ is unfortunately not contained in the group. We have to go outside the group to find this operator[35] which is necessary to distinguish the different $D(P,Q)$ for the same P and Q, when $\sigma(P,Q) > 1$. A complete set of commuting operators is therefore

$$G^3,\ G^3(1),\ G^3(2),\ F^2,\ F^2(1),\ F^2(2),\ I^2,\ I_3,\ Y, \text{ and } \Gamma. \quad \text{(II)}$$

In the following we will use a shorthand notation when there is no chance for confusion. We denote then the eigenvalues g^3 and f^2 of G^3 and F^2 collectively by μ, the eigenvalues I, I_z, and Y by ν, and the eigenvalue of Γ by γ. We denote the eigenstates of the representation $D(p_1,q_1)$ by $\phi^{(\mu_1)}_{\nu_1}$, the eigenstates of $D(p_2,q_2)$ by $\phi^{(\mu_2)}_{\nu_2}$. We write the eigenstates of the set (II) in the product representation as

$$\psi\begin{pmatrix}\mu_1\,\mu_2\,\mu_\gamma \\ \nu\end{pmatrix}.$$

The dimension of the representation $D(p_i,q_i)$ is N_i.

In the N_1N_2 dimensional product space, we can take as basis the vectors $\phi^{(\mu_1)}_{\nu_1}\,\phi^{(\mu_2)}_{\nu_2}$ of the product representation or the direct sum of the vectors

$$\psi\begin{pmatrix}\mu_1\,\mu_2\,\mu_\gamma \\ \nu\end{pmatrix}$$

of the different IR's contained in the product of $D(p_1,q_1) \otimes D(p_2,q_2)$. These two different sets of orthonormal basis vectors are connected by a unitary transformation[36]

$$\psi\begin{pmatrix}\mu_1\,\mu_2\,\mu_\gamma \\ \nu\end{pmatrix} = \sum_{\nu_1,\nu_2} \begin{pmatrix}\mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu\end{pmatrix} \phi^{(\mu_1)}_{\nu_1}\phi^{(\mu_2)}_{\nu_2}\,. \quad (10.3)$$

The coefficients $\left(\begin{smallmatrix}\mu_1 & \mu_2 & \mu\alpha \\ \nu_1 & \nu_2 & \nu\end{smallmatrix}\right)$ are the Clebsch–Gordan coefficients (CG coefficients) of $SU(3)$.

We could have looked at the above problem in another way which is very useful. The product states $\phi^{(\mu_1)}_{\nu_1}\,\phi^{(\mu_2)}_{\nu_2}$ are eigenstates of the operators of the set (I). Therefore, they are also eigenstates of $I_z = I_{1z} + I_{2z}$ and $Y = Y_1 + Y_2$, but not of G^3, F^3, Γ, and I^2. With the help of the Clebsch–Gordan coefficients $C^{I_1\,I_2\,I}_{I_{1z}I_{2z}I_z}$ of $SU(2)$, we can construct the eigenfunctions χ of the operators $G^3(1)$, $G^3(2)$, $F^2(1)$, $F^2(2)$, I, I_z, and Y, but not yet of G^3, F^2, and Γ. Then

$$\chi\begin{pmatrix}\mu_1 & \mu_2 & \\ I_1Y_1 & I_2Y_2 & II_zY\end{pmatrix} = \sum_{I_{1z}I_{2z}} C^{I_1\,I_2\,I}_{I_{1z}I_{2z}I_z}\phi^{(\mu_1)}_{\nu_1}\phi^{(\mu_2)}_{\nu_2}\,. \quad (10.4)$$

Now we can combine the different χ to obtain eigenstates of the set (II)

$$\psi\begin{pmatrix}\mu_1\,\mu_2\,\mu_\gamma \\ \nu\end{pmatrix} = \sum_{\substack{I_1,Y_1\\ I_2,Y_2}} \left(\begin{matrix}\mu_1 & \mu_2 \\ I_1Y_1 & I_2Y_2\end{matrix}\middle|\begin{matrix}\mu_\gamma \\ IY\end{matrix}\right) \times \chi\begin{pmatrix}\mu_1 & \mu_2 & \\ I_1Y_1 & I_2Y_2 & II_zY\end{pmatrix}. \quad (10.5)$$

The coefficients

$$\left(\begin{matrix}\mu_1 & \mu_2 \\ I_1Y_1 & I_2Y_2\end{matrix}\middle|\begin{matrix}\mu_\gamma \\ IY\end{matrix}\right)$$

are called isoscalar factors.[13] Note the dependence of these isoscalar factors on the total isospin I.

Comparison of (10.4) and (10.5) with (10.3) shows

$$\begin{pmatrix}\mu_1\,\mu_2\,\mu_\gamma \\ \nu_1\,\nu_2\,\nu\end{pmatrix} = C^{I_1\,I_2\,I}_{I_{1z}I_{2z}I_z}\left(\begin{matrix}\mu_1 & \mu_2 \\ I_1Y_1 & I_2Y_2\end{matrix}\middle|\begin{matrix}\mu_\gamma \\ IY\end{matrix}\right). \quad (10.6)$$

Because the CG coefficients of $SU(2)$ are well known, it is sufficient to give the isoscalar factors to specify the CG coefficients of $SU(3)$ uniquely. Care has to be taken, however, that the CG coefficients of $SU(2)$ have the correct phase factors.[31]

[35] This is about all that can be said about this operator Γ. In practice, one uses symmetry properties of the wave function.

[36] It turns out that one can choose the different arbitrary phases in such a way that all the CG coefficients are real. The transformation matrix is then a real orthogonal matrix.

To define uniquely the isoscalar factors, one has to define the relative phase of the basis vectors of the IR $D(P,Q)$ in the CG series (10.1) with respect to the basis vectors of the product representation $D(p_1,q_1) \otimes D(p_2,q_2)$. We will take these phase factors always real; this will result then in real isoscalar factors. To decide on the sign of the phase, we consider the highest eigenstate $\phi_{\nu_H}^{(\mu)}$ of the IR $D(P,Q)$, then

$$\phi_{\nu_H}^{(\mu)} = \sum_{\nu_1,\nu_2} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu_H \end{pmatrix} \phi_{\nu_1}^{(\mu_1)} \phi_{\nu_2}^{(\mu_2)} .$$

Among the different CG coefficients, we choose the one with the largest possible I_1 to be positive. If this is not sufficient to decide, we take from the coefficients with the largest possible I_1 the one with the largest possible I_2 positive. This convention was sufficient to determine the phases in the cases met here. Perhaps this is not, in general, sufficient; however, this convention can easily be generalized. We have chosen this convention as the most direct generalization of the phase convention for the CG coefficients of $SU(2)$. There one requires

$$C_{j_1\ j-j_1\ j}^{j_1\ j_2\ j} > 0 .$$

11. ORTHOGONALITY RELATIONS

The CG coefficients of $SU(3)$ form a real orthogonal matrix. Therefore,

$$\phi_{\nu_1}^{(\mu_1)} \phi_{\nu_2}^{(\mu_2)} = \sum_{\mu\nu\gamma} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} \psi \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ & & \nu \end{pmatrix} , \quad (11.1)$$

and

$$\sum_{\nu_1\nu_2} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma'{}' \\ \nu_1 & \nu_2 & \nu' \end{pmatrix} = \delta_{\mu\mu'}\delta_{\gamma\gamma'}\delta_{\nu\nu'} , \quad (11.2a)$$

$$\sum_{\mu\gamma\nu} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1' & \nu_2' & \nu \end{pmatrix} = \delta_{\nu_1\nu_1'}\delta_{\nu_2\nu_2'} . \quad (11.2b)$$

The orthogonality relations of the CG coefficients of $SU(2)$ are well known.[37] Therefore, the orthogonality relations for the isoscalar factors are

$$\sum_{\substack{I_1Y_1\\I_2Y_2}} \left(\begin{array}{cc|c} \mu_1 & \mu_2 & \mu_\gamma \\ I_1Y_1 & I_2Y_2 & IY \end{array} \right) \left(\begin{array}{cc|c} \mu_1 & \mu_2 & \mu_\gamma'{}' \\ I_1Y_1 & I_2Y_2 & IY' \end{array} \right) = \delta_{\mu\mu'}\delta_{\gamma\gamma'}\delta_{YY'} , \quad (11.3a)$$

and

$$\sum_{\mu\gamma Y} \left(\begin{array}{cc|c} \mu_1 & \mu_2 & \mu_\gamma \\ I_1Y_1 & I_2Y_2 & IY \end{array} \right) \left(\begin{array}{cc|c} \mu_1 & \mu_2 & \mu_\gamma \\ I_1'Y_1' & I_2'Y_2' & IY \end{array} \right) = \delta_{I_1I_1'}\delta_{I_2I_2'}\delta_{Y_1Y_1'}\delta_{Y_2Y_2'} . \quad (11.3b)$$

[37] M. E. Rose, *Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

12. CLEBSCH–GORDAN SERIES

The direct product $D \otimes D$ of two IR's of $SU(3)$ can be decomposed in several IR's of $SU(3)$. This is formally described by the Clebsch–Gordan series (10.1). To find the different $D(P,Q)$ with their multiplicities $\sigma(P,Q)$ we will follow the method of Speiser.[17,38]

We start by making a (p,q) coordinate system (Fig. 5). In this coordinate system we can represent every

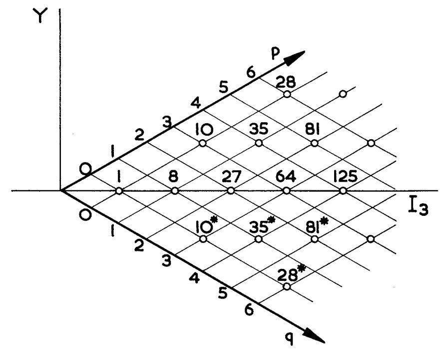

FIG. 5. Coordinate system in which we can represent every irreducible representation $\{N\} = D(p,q)$ by a point (p,q).

IR $D(p,q)$ by the point (p,q). We make this coordinate system oblique; the p and q axes make angles of 30° (see Fig. 5) with a horizontal line, which we will call the I_3 axis. The axis perpendicular to the I_3 axis is called the Y axis. The unit of length along the p and q axis we take to be $l/\sqrt{3}$.

Next we can reflect this figure about the p axis and about the q axis. Reflecting now about the Y axis, we obtain the "lattice" as shown in Fig. 6. We give a positive weight to the nonshaded sextants and to the shaded ones a negative weight.

In order to obtain the Clebsch–Gordan series of the direct product $D(p_1,q_1) \otimes D(p_2,q_2)$ we need the eigenvalue diagram of $D(p_1,q_1)$ or $D(p_2,q_2)$ with the multiplicities of the eigenvalues. We will assume that we have the eigenvalue diagram of $D(p_1,q_1)$ drawn on scale; the unit of length for I_3 is l, the unit of length for Y is $\frac{2}{3}\sqrt{3}l$. We place this eigenvalue diagram of $D(p_1,q_1)$ on top of Fig. 6 in such a way that the eigenvalue $(I_3,Y) = (0,0)$ coincides with the lattice point $\{N_2\} = D(p_2,q_2)$ in the first positive weight sextant and that the I_3 axis of the eigenvalue diagram coincides with the I_3 axis of the lattice. The eigenvalues of $D(p_1,q_1)$ coincide now all with points of the

[38] This method can be used for all the IR's of $SU(3)$ and not only for the subset of IR's used by us. Because we are only interested in the octet model, we give this restrictive version.

lattice representing IR's, except those eigenvalues which fall on the p, q, or Y axes. We can now state Speiser's theorem:

Every IR covered by an eigenvalue of $D(p_1,q_1)$, in the above described way, is contained in the direct product $D(p_1,q_1) \otimes D(p_2,q_2)$ as many times as the multiplicity of the eigenvalue which covers it and with a sign equal to the weight of the sextant. The contributions of the negative weight sextants have therefore to be subtracted from (instead of added to) the Clebsch–Gordan series. Eigenvalues covering the p, q, and Y axes can be neglected.

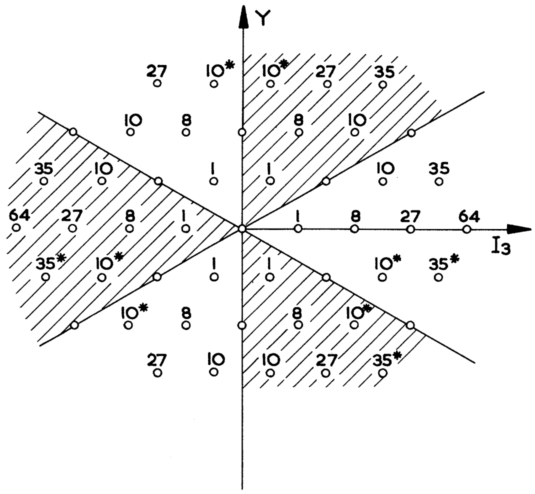

FIG. 6. The lattice. The nonshaded areas have positive weight, the shaded areas negative weight.

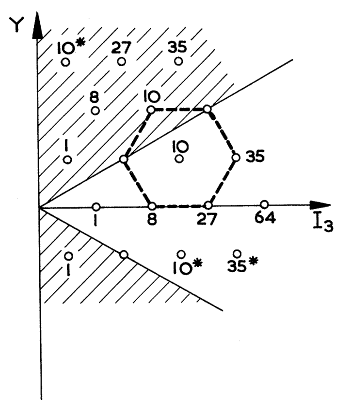

FIG. 7. Determination of the CG series for the direct product $\{8\} \otimes \{10\}$.

To demonstrate the procedure we will obtain the CG series of $\{8\} \otimes \{10\}$. We will do this in two ways.

First, we will place the center of the eigenvalue diagram [Fig. 4(a)] of $\{8\}$ on top of the point $\{10\}$ in a positive-weight section of the lattice (see Fig. 7). This point (IR) is covered by an eigenvalue of multiplicity two. This gives, therefore, a contribution to the CG series of $\oplus\ 2 \times \{10\}$. The points (IR's) $\{8\}$, $\{27\}$, and $\{35\}$ in the positive-weight sextant, and the $\{10\}$ in the negative-weight sextant are covered by eigenvalues of multiplicity one. They give, therefore, a contribution

$$\{8\} \oplus \{27\} \oplus \{35\} \ominus \{10\}$$

to the CG series. We can neglect the two eigenvalues falling on the p axis. The complete CG series becomes now

$$\begin{aligned} \{8\} \otimes \{10\} &= \{8\} \oplus \{27\} \oplus \{35\} \ominus \{10\} \oplus 2 \\ &\quad \times \{10\} \\ &= \{8\} \oplus \{10\} \oplus \{27\} \oplus \{35\}\,. \end{aligned}$$

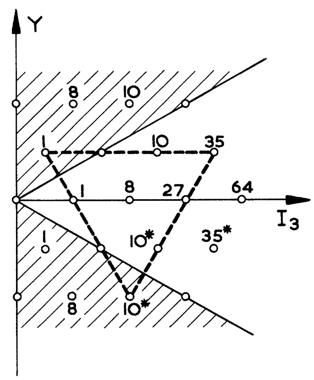

FIG. 8. Determination of the CG series for the direct product $\{10\} \otimes \{8\}$.

To obtain the CG series the other way, we place the (0,0) eigenvalue of the eigenvalue diagram [Fig. 4(b)] of $\{10\}$ on the point $\{8\}$ in a positive-weight sextant of the lattice (see Fig. 8). All the eigenvalues of $\{10\}$ have multiplicity one. The positive-weight sextant gives the contribution to the CG series

$$\{1\} \oplus \{8\} \oplus \{10\} \oplus \{10^*\} \oplus \{27\} \oplus \{35\}\,.$$

The negative-weight sextants give the contribution

$$\ominus \{1\} \ominus \{10^*\}\,.$$

We may neglect again the two eigenvalues of $\{10\}$ falling on the p and q axes. The complete CG series is, therefore,

$$\{10\} \otimes \{8\} = \{8\} \oplus \{10\} \oplus \{27\} \oplus \{35\}$$

as we have seen before.

13. IRREDUCIBLE REPRESENTATIONS AND IRREDUCIBLE TENSOR OPERATORS

Under a transformation α of $SU(3)$ the vector x in the space C_3 transforms according to

$$x' = \alpha x\,. \tag{13.1}$$

The basis vectors $\psi_\nu^{(\mu)}$ of the IR $\{\mu\} = D(p,q) = \{N\}$ transform then as

$$\psi_\nu'^{(\mu)} = U\psi_\nu^{(\mu)} = \sum_{\nu'=1}^{N} \psi_{\nu'}^{(\mu)} D_{\nu'\nu}^{(\mu)*}(\alpha)\,. \tag{13.2}$$

Here $D^{(\mu)}$ is a unimodular unitary matrix.

Therefore,

$$\sum_\nu D_{\nu\nu'}^{(\mu)}(\alpha) D_{\nu\nu''}^{(\mu)*}(\alpha) = \delta_{\nu'\nu''} \tag{13.3}$$

and

$$\psi_\nu^{(\mu)} = \sum_{\nu'} D_{\nu\nu'}^{(\mu)}(\alpha) \psi_{\nu'}'^{(\mu)}\,. \tag{13.4}$$

As already pointed out in Sec. 3, the matrices $D^{(\mu)}$ form the IR $\{\mu\} = D(p,q)$ of the group $SU(3)$. In that same section, we have seen that every transformation of $SU(3)$ is characterized by eight real parameters α_i. The matrices $D^{(\mu)}(\alpha)$ are, therefore, functions of the α_i. It is possible to define a density function $\rho(\alpha_i)$ such that[39]

$$\int d\alpha_1 \cdots d\alpha_8 \rho(\alpha_1,\cdots,\alpha_8) = 1 \tag{13.5}$$

if the integral is performed over all the elements of the group $SU(3)$. We write[40]

$$d\Omega = d\alpha_1 \cdots d\alpha_8 \rho(\alpha_1,\cdots,\alpha_8)\,.$$

Then one can also show that[39,41]

$$\int d\Omega D_{ik}^{(\mu)}(\alpha) D_{jl}^{(\nu)*}(\alpha) = \frac{1}{N_\mu} \delta_{\mu\nu}\delta_{ij}\delta_{kl}\,. \tag{13.6}$$

From the transformation properties (13.2) and (13.4) of the basis vectors of the IR's and the definition (10.3) of the CG coefficients we find the relation

$$D_{\nu_1\lambda_1}^{(\mu_1)}(\alpha) D_{\nu_2\lambda_2}^{(\mu_2)}(\alpha) = \sum_{\mu\nu\lambda\gamma} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \lambda_1 & \lambda_2 & \lambda \end{pmatrix} D_{\nu\lambda}^{(\mu)}(\alpha) \tag{13.7}$$

and the inverse relation

$$\sum_{\substack{\nu_1\nu_2 \\ \lambda_1\lambda_2}} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_2 & \mu'_{\gamma'} \\ \nu_1 & \nu_2 & \lambda \end{pmatrix} D_{\nu_1\lambda_1}^{(\mu_1)}(\alpha) D_{\nu_2\lambda_2}^{(\mu_2)}(\alpha) = \delta_{\mu\mu'}\delta_{\gamma\gamma'} D_{\nu\lambda}^{(\mu)}(\alpha)\,. \tag{13.8}$$

In (13.2) we defined the transformation of the basis vectors $\psi_\nu^{(\mu)}$ of the IR $D(p,q)$. We can now also define irreducible tensor operators of rank μ. This is a set of N_μ operators $T_\nu^{(\mu)}$ which transform under a transformation α of $SU(3)$ as

$$T_\nu'^{(\mu)} = UT_\nu^{(\mu)}U^{-1} = \sum_{\nu'} T_{\nu'}^{(\mu)} D_{\nu'\nu}^{(\mu)*}(\alpha) \tag{13.9}$$

in complete analogy with Eq. (13.2).

14. SYMMETRY PROPERTIES OF THE CG COEFFICIENTS

In this section we will study some of the symmetry properties of the CG coefficients of $SU(3)$ and of the isoscalar factors. Unfortunately, our lack of knowledge of the operator Γ will reflect itself here in the impossibility of defining rigorously some over-all phase factors. Fortunately, these phase factors are rather unimportant and can be fixed afterwards.

I

If the IR $\{\mu\}_\gamma$ appears in the CG series of $\{\mu_1\} \otimes \{\mu_2\}$ then it will also appear in the CG series $\{\mu_2\} \otimes \{\mu_1\}$, because these series are identical.

Therefore,

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} = \xi_1 \begin{pmatrix} \mu_2 & \mu_1 & \mu_\gamma \\ \nu_2 & \nu_1 & \nu \end{pmatrix} \tag{14.1}$$

where the $\xi_1 = \pm 1$ according to our phase convention as given in Sec. 10. These $\xi_1 = \xi_1(\mu_1,\mu_2,\mu_\gamma)$ are independent of the "magnetic" quantum numbers ν_1, ν_2, and ν. Therefore, we can take the highest eigenvalue ν_H of $\{\mu\}_\gamma$ and determine there the value of ξ_1.

Because of the property of the CG coefficients of $SU(2)$

$$C_{I_{1z}I_{2z}I_z}^{I_1 I_2 I} = (-)^{I_1+I_2-I} C_{I_{2z}I_{1z}I_z}^{I_2\, I_1\, I}\,, \tag{14.2}$$

we obtain for the isoscalar factors the relation

$$\left(\begin{matrix} \mu_1 & \mu_2 \\ I_1Y_1 & I_2Y_2 \end{matrix}\middle|\begin{matrix} \mu_\gamma \\ IY \end{matrix}\right) = \xi_1(-)^{I_1+I_2-I} \left(\begin{matrix} \mu_2 & \mu_1 \\ I_2Y_2 & I_1Y_1 \end{matrix}\middle|\begin{matrix} \mu_\gamma \\ IY \end{matrix}\right)\,. \tag{14.3}$$

From this relation (14.3) we obtain directly that

$$\left(\begin{matrix} \mu_1 & \mu_1 \\ I_1Y_1 & I_1Y_1 \end{matrix}\middle|\begin{matrix} \mu_\gamma \\ I_2Y_1 \end{matrix}\right) = 0 \quad \begin{matrix} \text{for } \xi_1 = 1 & \text{if } 2I_1 - I = \text{odd} \\ \text{for } \xi_1 = -1 & \text{if } 2I_1 - I = \text{even}\,. \end{matrix}$$

---

[39] Ref. 26, Chap. 8.

[40] For $SU(2)$ we have

$$\int d\Omega = \frac{1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta d\beta \int_0^{2\pi} d\gamma\,,$$

where α, β, and γ are the three Euler angles.

[41] For $SU(2)$ this is the well-known relation

$$\int d\Omega D_{M_1M_1'}^{(J_1)*} D_{M_2M_2'}^{(J_2)*} = \frac{1}{2J_1+1} \delta_{J_1J_2}\delta_{M_1M_2}\delta_{M_1'M_2'}\,.$$

II

Consider the integral

$$I = \int d\Omega D^{(\mu_3)^*}_{\nu_3\lambda_3}(\alpha) D^{(\mu_2)}_{\nu_2\lambda_2}(\alpha) D^{(\mu_1)}_{\nu_1\lambda_1}(\alpha) \,. \qquad (14.4)$$

Using (13.7) and (13.6) we obtain

$$I = \frac{1}{N_3} \sum_{\gamma} \begin{pmatrix} \mu_1 & \mu_2 & \mu_{3\gamma} \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_2 & \mu_{3\gamma} \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} . \qquad (14.5)$$

From (8.2) we can write

$$\phi_{\nu}^{(\mu^*)} = \eta(-)^{\bar{\nu}} \phi_{-\nu}^{(\mu)^*} , \qquad (14.6)$$

where

$$\bar{\nu} = I_z + \tfrac{1}{2}Y \quad \text{and} \quad -\nu = (I, -I_z, -Y) \,.$$

The phase factor η is only dependent on μ but not on ν. Equation (13.4) together with (14.6) gives

$$D^{(\mu^*)^*}_{\nu'\nu}(\alpha) = (-)^{\bar{\nu}+\bar{\nu}'} D^{(\mu)}_{-\nu'-\nu}(\alpha) \,. \qquad (14.7)$$

Using (14.7) we can rewrite (14.4) as

$$I = (-)^{\bar{\nu}_2+\bar{\lambda}_2+\bar{\nu}_3+\bar{\lambda}_3} \int d\Omega D^{(\mu_2^*)^*}_{-\nu_2-\lambda_2}(\alpha) D^{(\mu_3^*)}_{-\nu_3-\lambda_3}(\alpha) D^{(\mu_1)}_{\nu_1\lambda_1}(\alpha)$$

$$= \frac{1}{N_2} \sum_{\gamma'} (-)^{\bar{\nu}_2+\bar{\nu}_3} \begin{pmatrix} \mu_1 & \mu_3^* & \mu_{2\gamma'}^* \\ \nu_1 & -\nu_3 & -\nu_2 \end{pmatrix}$$

$$\times \begin{pmatrix} \mu_1 & \mu_3^* & \mu_{2\gamma'}^* \\ \lambda_1 & -\lambda_3 & -\lambda_2 \end{pmatrix} (-)^{\bar{\lambda}_2+\bar{\lambda}_3} .$$

Comparing this last expression for I with (14.5) and noting that this equation should hold independently of the values of the "magnetic" quantum numbers ν_1, ν_2, ν_3, λ_1, λ_2, and λ_3 gives in case of only one quantum number γ

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_{3\gamma} \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix} = \xi_2(-)^{I_{1z}+Y_1/2} \left(\frac{N_3}{N_2}\right)^{\frac{1}{2}} \begin{pmatrix} \mu_1 & \mu_3^* & \mu_{2\gamma'}^* \\ \nu_1 & -\nu_3 & -\nu_2 \end{pmatrix} . \qquad (14.8)$$

It is expected that this formula holds, or can be made to hold by a suitable choice of Γ, in case there are different eigenvalues γ belonging to the same μ_3. For example, in the case $\{8\} \otimes \{8\} = \{8\}_1 \otimes \{8\}_2$ and $\{8\} \otimes \{27\} = \{27\}_1 \otimes \{27\}_2$, (14.8) does hold. The arbitrary phase $\xi_2 = \xi_2(\mu_1,\mu_2,\mu_3) = \pm 1$ is again independent of the magnetic quantum numbers. This phase can be determined in every specific case by considering the highest eigenvalue.

For the CG coefficients of $SU(2)$ we have the relation

$$C^{I_1\, I_2\, I_3}_{I_{1z} I_{2z} I_{3z}} = (-)^{I_1-I_{1z}} \left(\frac{2I_3+1}{2I_2+1}\right)^{\frac{1}{2}} C^{I_1\quad I_3\quad I_2}_{I_{1z}-I_{3z}-I_{2z}} .$$

Therefore, the relation for the isoscalar factors is

$$\left(\begin{array}{cc|c} \mu_1 & \mu_2 & \mu_{3\gamma} \\ I_1Y_1 & I_2Y_2 & I_3Y_3 \end{array}\right) = \xi_2(-)^{I_1+Y_1/2} \left(\frac{(2I_2+1)N_3}{(2I_3+1)N_2}\right)^{\frac{1}{2}}$$

$$\times \left(\begin{array}{cc|c} \mu_1 & \mu_3^* & \mu_{2\gamma'}^* \\ I_1Y_1 & I_3 - Y_3 & I_2 - Y_2 \end{array}\right) . \qquad (14.9)$$

From (14.8) and $\left(\begin{smallmatrix} \mu & 1 & \mu \\ \nu & 0 & \nu \end{smallmatrix}\right) = 1$

follows

$$\begin{pmatrix} \mu & \mu^* & 1 \\ \nu & -\nu & 0 \end{pmatrix} = (-)^{I_z+I_H+\frac{1}{2}(Y+Y_H)} N^{-\frac{1}{2}} \qquad (14.10)$$

if the highest eigenvalue of $\{\mu\}$ has $I = I_H$ and $Y = Y_H$; and $\{\mu\} = \{N\}$. We find also

$$\left(\begin{array}{cc|c} \mu & \mu^* & 1 \\ IY & I - Y & 00 \end{array}\right) = (-)^{I+I_H+\frac{1}{2}(Y+Y_H)} \left(\frac{2I+1}{N}\right)^{\frac{1}{2}} . \qquad (14.11)$$

From (14.9) we obtain moreover that

$$\left(\begin{array}{cc|c} \mu_1 & \mu_2 & \mu_{2\gamma}^* \\ I_1 2Y_2 & I_2 - Y_2 & I_2Y_2 \end{array}\right) = 0 \quad \begin{array}{l} \text{for} \quad \xi_2 = 1 \\ \text{if} \quad I_1 + Y_2 = \text{odd} \,, \\ \text{for} \quad \xi_2 = -1 \\ \text{if} \quad I_1 + Y_2 = \text{even} \,. \end{array}$$

III

From the definition (10.3) of the CG coefficients, from their reality property, and the relation (14.6) we deduce

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_{3\gamma} \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix} = \xi_3 \begin{pmatrix} \mu_1^* & \mu_2^* & \mu_{3\gamma}^* \\ -\nu_1 & -\nu_2 & -\nu_3 \end{pmatrix} , \qquad (14.12)$$

where ξ_3 is independent of the magnetic quantum numbers ν_1, ν_2, and ν_3, and $\xi_3 = \pm 1$. It is important to note here that the phase convention (8.3) cannot always be applied consistently at the same time to $\{\mu_1\}$, $\{\mu_2\}$, and $\{\mu_3\}$. Therefore, in certain cases $\xi_3 = -1$. Because of the relation

$$C^{I_1\, I_2\, I_3}_{I_{1z} I_{2z} I_{3z}} = (-)^{I_1+I_2-I_3} C^{\;I_1\quad I_2\quad I_3}_{-I_{1z}-I_{2z}-I_{3z}} ,$$

we get for the isoscalar factors

$$\left(\begin{array}{cc|c} \mu_1 & \mu_2 & \mu_{3\gamma} \\ I_1Y_1 & I_2Y_2 & I_3Y_3 \end{array}\right) = \xi_3(-)^{I_1+I_2-I_3}$$

$$\times \left(\begin{array}{cc|c} \mu_1^* & \mu_2^* & \mu_{3\gamma}^* \\ I_1 - Y_1 & I_2 - Y_2 & I_3 - Y_3 \end{array}\right) . \qquad (14.13)$$

Again the ξ_3 is easily determined in any particular case by considering the highest eigenstates.

15. WIGNER–ECKART THEOREM

The Wigner–Eckart[15] theorem can also be stated for $SU(3)$. This theorem concerns the matrix element $(\phi^{(\mu_3)}_{\nu_3}, T^{(\mu_2)}_{\nu_2} \phi^{(\mu_1)}_{\nu_1})$ of an irreducible tensor operator

$T^{(\mu_2)}_{\nu_2}$ between two basis states, $\phi^{(\mu_1)}_{\nu_1}$ and $\phi^{(\mu_3)}_{\nu_3}$ of IR's. It reads

$$(\phi^{(\mu_3)}_{\nu_3},T^{(\mu_2)}_{\nu_2}\phi^{(\mu_1)}_{\nu_1}) = \sum_{\gamma} \begin{pmatrix} \mu_1 & \mu_2 & \mu_{3\gamma} \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix} (\mu_3||T^{(\mu_2)}||\mu_1)_\gamma . \quad (15.1)$$

We have to sum here over γ. The right-hand side contains, therefore, as many terms as the IR $\{\mu_3\}$ is contained in the product $\{\mu_1\} \otimes \{\mu_2\}$. The Eq. (15.1) is a theorem in as far as it predicts the dependence of the matrix element on ν_1, ν_2, and ν_3. At the same time (15.1) is a definition of the reduced matrix elements $(\mu_3||T^{(\mu_2)}||\mu_1)_\gamma$.

We can prove this theorem the following way. From the Eqs. (13.4) and (13.9) follows

$$(\phi^{(\mu_3)}_{\nu_3},T^{(\mu_2)}_{\nu_2}\phi^{(\mu_1)}_{\nu_1}) = \sum_{\lambda_1\lambda_2\lambda_3} D^{(\mu_3)*}_{\nu_3\lambda_3}D^{(\mu_2)}_{\nu_2\lambda_2}D^{(\mu_1)}_{\nu_1\lambda_1}(\phi'^{(\mu_3)}_{\lambda_3},T'^{(\mu_2)}_{\lambda_2}\phi'^{(\mu_1)}_{\lambda_1}).$$

Making use of the combination property (13.7) of the D functions one obtains

$$(\phi^{(\mu_3)}_{\nu_3},T^{(\mu_2)}_{\nu_2}\phi^{(\mu_1)}_{\nu_1}) = \sum_{\substack{\lambda_1\lambda_2\lambda_3\\ \mu\nu\lambda\gamma}} D^{(\mu_3)*}_{\nu_3\lambda_3} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \lambda_1 & \lambda_2 & \lambda \end{pmatrix} \times D^{(\mu)}_{\nu\lambda}(\phi'^{(\mu_3)}_{\lambda_3},T'^{(\mu_2)}_{\lambda_2}\phi'^{(\mu_1)}_{\lambda_1}) .$$

The matrix elements are independent of our choice of coordinate system; they are, therefore, independent of the eight real transformation parameters $\alpha_1,\cdots,\alpha_8$. Thus we can perform the integration over $d\Omega$. After making use of (13.6) we obtain

$$(\phi^{(\mu_3)}_{\nu_3},T^{(\mu_2)}_{\nu_2}\phi^{(\mu_1)}_{\nu_1}) = \sum_{\gamma} \begin{pmatrix} \mu_1 & \mu_2 & \mu_{3\gamma} \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix} \frac{1}{N_3} \sum_{\lambda_1\lambda_2\lambda_3} \begin{pmatrix} \mu_1 & \mu_2 & \mu_{3\gamma} \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} \times (\phi^{(\mu_3)}_{\lambda_3},T^{(\mu_2)}_{\lambda_2}\phi^{(\mu_1)}_{\lambda_1}) .$$

This gives us the form (15.1) of the Wigner–Eckart theorem if one defines

$$(\mu_3||T^{(\mu_2)}||\mu_1)_\gamma = \frac{1}{N_3} \sum_{\lambda_1\lambda_2\lambda_3} \begin{pmatrix} \mu_1 & \mu_2 & \mu_{3\gamma} \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} (\phi^{(\mu_3)}_{\lambda_3},T^{(\mu_2)}_{\lambda_2}\phi^{(\mu_1)}_{\lambda_1}) .$$

The reduced matrix element is obviously independent of the quantum numbers ν_1, ν_2, and ν_3.

16. THE MASS FORMULA

We will consider the mass formula specifically here for the octet (the IR $\{8\}$). The discussion for every other IR can be done along the same lines. In the octet model one assumes that the strongest interactions are invariant under transformations belonging to the group $SU(3)$. In the absence of any other interactions the particles belonging to the same IR of $SU(3)$ should have, therefore, the same mass, the unitary multiplet mass. The symmetry of these strongest interactions (unitary symmetry) is broken by some unknown weaker mechanism but in such a way that the isospin I and the hypercharge Y are still conserved. Through the action of this unitary symmetry breaking mechanism, the mass degeneracy of the particles belonging to the same unitary multiplet will be removed. If one assumes the simplest form for the symmetry breaking interaction, then one can derive the Okubo mass formula[10,42] for fermions and for bosons[43]

$$M = M_0 + M_1Y + M_2\{I(I+1) - \tfrac{1}{4}Y^2\} \quad (16.1)$$

$$m^2 = m_0^2 + m_2^2\{I(I+1) - \tfrac{1}{4}Y^2\} . \quad (16.2)$$

These formulas give the Gell-Mann–Okubo mass relations

$$M_N + M_\Xi = \tfrac{3}{2}M_\Lambda + \tfrac{1}{2}M_\Sigma , \quad (16.3a)$$

$$m_K^2 = \tfrac{3}{4}\, m_\eta^2 + \tfrac{1}{4}\, m_\pi^2 , \quad (16.3b)$$

$$m_{K^*}^2 = \tfrac{3}{4}\, m_\omega^2 + \tfrac{1}{4}\, m_\rho^2 . \quad (16.3c)$$

The relations (16.3a) and (16.3b) are very well satisfied; the relation (16.3c), however, is not so well satisfied.

If one introduces then the electromagnetic interaction, this has to be done in such a way that T_3 and Y are still conserved. The presence of this interaction results in mass differences between the different members of the same isomultiplet. If one introduces the electromagnetic interaction, but neglects the presence of the unknown unitary symmetry breaking mechanism, one can derive a relationship between the mass differences of the members of the isomultiplets which make up the same unitary multiplet. For the baryons one obtains[11]

$$M(\Xi^0) - M(\Xi^-) + M(p) - M(n) = M(\Sigma^+) - M(\Sigma^-) . \quad (16.4)$$

This relation is also very well satisfied.

We have seen that when the weaker symmetry breaking interaction is switched on, the unitary multiplets split up in isomultiplets. As this breakdown of unitary symmetry has to be done in such a way as to preserve the selection rules for isospin and hypercharge, every such mass operator must have the form

$$0_1 = \sum_\mu T^{(\mu)}_{0,0,0} , \quad (16.5)$$

---

[42] The use of the mass for fermions and the (mass)^2 for bosons in formulas like (16.1) and (16.2) seems first to be suggested by R. P. Feynman. It is related to the fact that in the Lagrangian the mass term for bosons is $m^2\phi^+\phi$ and for fermions $M\bar{\psi}\psi$.

[43] We note that for bosons a term linear in Y cannot occur.

where the summation goes over all the physically allowable IR's of $SU(3)$, therefore[43a] $\mu = 1, 8, 27$, etc. The breakdown due to the electromagnetic interaction has to be such that T_3, as well as Y, are conserved. The most general operator[43b] achieving this is, therefore,

$$0_2 = \sum_{\mu, I} T^{(\mu)}_{I,0,0}\,, \tag{16.6}$$

where μ runs over all the possible IR's and I over all the values within an IR consistent with $Y = 0$. One notices that the operator 0_1 is a special case of 0_2. Hence, the general mass operator must have the form

$$\left.\begin{matrix} M \\ m^2 \end{matrix}\right\} = \sum_{\mu, I} T^{(\mu)}_{I,0,0} \tag{16.7}$$

where M is the mass operator for fermions and m^2 for bosons. For the IR $\{8\}$ only the irreducible tensor operators belonging to the IR's $\{1\}$, $\{8\}$, $\{10\}$, $\{10^*\}$, and $\{27\}$ give nonvanishing expectation values. Therefore, if we restrict ourselves only to these operators out of (16.7) which conserve the total I spin, then

$$M(\text{or } m^2) = T^{(1)}_{0,0,0} + T^{(8)}_{0,0,0} + T^{(27)}_{0,0,0}\,. \tag{16.8}$$

Using the Wigner–Eckart theorem (15.1) we obtain for the isomultiplet masses of the baryons

$$\begin{aligned} M_N &= a_1 - (\sqrt{5}/10)a_{8_1} + \tfrac{1}{2}\,a_{8_2} + (1/3\sqrt{5})a_{27}\,,\\ M_\Xi &= a_1 - (\sqrt{5}/10)a_{8_1} - \tfrac{1}{2}\,a_{8_2} + (1/3\sqrt{5})a_{27}\,,\\ M_\Lambda &= a_1 - (\sqrt{5}/5)a_{8_1} \qquad\quad - (1/\sqrt{5})a_{27}\,,\\ M_\Sigma &= a_1 + (\sqrt{5}/5)a_{8_1} \qquad\quad - (1/9\sqrt{5})a_{27}\,, \end{aligned} \tag{16.9}$$

where

$$a_{\mu\gamma} = (8||T^{(\mu)}_{0,0,0}||8)_\gamma\,. \tag{16.10}$$

We have here four masses and four constants. Solving (16.9) for the constants gives

$$\begin{aligned} a_1 &= \tfrac{1}{8}\,[2M_N + 2M_\Xi + M_\Lambda + 3M_\Sigma]\,,\\ a_{8_1} &= (1/\sqrt{5})[3M_\Sigma - M_\Lambda - M_N - M_\Xi]\\ a_{8_2} &= M_N - M_\Xi\,,\\ a_{27} &= -(9/8\sqrt{5})[3M_\Lambda + M_\Sigma - 2M_N - 2M_\Xi]\,. \end{aligned} \tag{16.11}$$

The Gell-Mann–Okubo mass relation (16.3a) is based on the assumption that the mass differences transform as $T^{(8)}_{0,0,0}$, i.e., $a_{27} = 0$. In practice this condition is rather well satisfied because [44–46] $a_1 = 1150.84$ MeV/c^2, $a_{8_1} = 91.34$ MeV/c^2, $a_{8_2} = -379.54$ MeV/c^2, and $a_{27} = 11.9$ MeV/c^2.

The mass formula for the electromagnetic mass differences can be obtained by introducing those operators $T^{(\mu)}_{I,0,0}$, which exhibit explicitly the breakdown of isotopic spin conservation. Thus

$$T = T^{(8)}_{1,0,0} + T^{(10)}_{1,0,0} + T^{(10^*)}_{1,0,0} + T^{(27)}_{1,0,0} + T^{(27)}_{2,0,0}\,.$$

We get

$$\begin{aligned} M_p &= M_N + (\sqrt{15}/90)(9b_{8_1} - 4b_{27})\\ &\qquad + (1/2\sqrt{3})b_{8_2} - (1/\sqrt{15})(b_{10} - b_{10^*})\,,\\ M_n &= M_N - (\sqrt{15}/90)(9b_{8_1} - 4b_{27})\\ &\qquad - (1/2\sqrt{3})b_{8_2} + (1/\sqrt{15})(b_{10} - b_{10^*})\,,\\ M_{\Xi^0} &= M_\Xi - (\sqrt{15}/90)(9b_{8_1} - 4b_{27})\\ &\qquad + (1/2\sqrt{3})b_{8_2} - (1/\sqrt{15})(b_{10} - b_{10^*})\,,\\ M_{\Xi^-} &= M_\Xi + (\sqrt{15}/90)(9b_{8_1} - 4b_{27})\\ &\qquad - (1/2\sqrt{3})b_{8_2} + (1/\sqrt{15})(b_{10} - b_{10^*})\,,\\ M_\Lambda &= M_\Lambda\,,\\ M_{\Sigma^0} &= M_\Sigma + \tfrac{4}{9}\,c_{27}\,,\\ M_{\Sigma^+} &= M_\Sigma - \tfrac{2}{9}\,c_{27} + (1/\sqrt{3})b_{8_2}\\ &\qquad + (1/\sqrt{15})(b_{10} - b_{10^*})\,,\\ M_{\Sigma^-} &= M_\Sigma - \tfrac{2}{9}\,c_{27} - (1/\sqrt{3})b_{8_2}\\ &\qquad - (1/\sqrt{15})(b_{10} - b_{10^*})\,, \end{aligned} \tag{16.12}$$

where

$$b_{\mu_\gamma} = (8||T^{(\mu)}_{1,0,0}||8)_\gamma \tag{16.13}$$

and

$$c_{\mu_\gamma} = (8||T^{(\mu)}_{2,0,0}||8)_\gamma\,.$$

We notice from (16.12) that we have here essentially four arbitrary constants $(9b_{8_1} - 4b_{27})$, b_{8_2}, $(b_{10} - b_{10^*})$,

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[43a] The IR's $\{10\}$ and $\{10^*\}$ do not have states with $I = I_z = Y = 0$.

[43b] Analogous results can be found in M. A. Rashid and I. I. Yamanaka, Phys. Rev. (to be published).

[44] We use here the following masses:

$$\begin{aligned} M_p &= 938.21 \pm 0.01 \text{ MeV}/c^2\,,\\ M_n &= 939.51 \pm 0.01 \text{ MeV}/c^2\,,\\ M_\Lambda &= 1115.36 \pm 0.14 \text{ MeV}/c^2\,,\\ M_{\Sigma^+} &= 1189.40 \pm 0.20 \text{ MeV}/c^2\,,\\ M_{\Sigma^0} &= 1191.5 \pm 0.5 \text{ MeV}/c^2\,,\\ M_{\Sigma^-} &= 1195.96 \pm 0.30 \text{ MeV}/c^2\,,\\ M_{\Xi^0} &= 1315.8 \pm 0.8 \text{ MeV}/c^2\,,\\ M_{\Xi^-} &= 1321.0 \pm 0.5 \text{ MeV}/c^2\,. \end{aligned}$$

The first six masses are from tables from W. H. Barkas and A. H. Rosenfeld, University of California, Berkeley (1960), UCRL-8050. The Ξ^- mass is from Ref. 45 and the Ξ^0 mass from Ref. 46.

[45] L. Bertanza, V. Brisson, P. L. Connolly, E. L. Hart, I. S. Mittra, G. C. Moneti, R. R. Rau, N. P. Samios, I. O. Skillicorn, S. S. Yamamoto, M. Goldberg, L. Gray, J. Leitner, S. Lichtman, and J. Westgard, *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962), p. 437.

[46] D. H. Stork, Bull. Am. Phys. Soc. **8**, 46 (1963).

and c_{27}. However, there are also only four independent mass differences

$$(M_n - M_p)\,, \quad (M_{\Sigma^+} - M_{\Sigma^-})\,, \quad (M_{\Xi^0} - M_{\Xi^-})\,,$$

and

$$[M_{\Sigma^0} - \tfrac{1}{2}(M_{\Sigma^+} + M_{\Sigma^-})]\,.$$

The relation (16.4) between the electromagnetic mass differences can be obtained by assuming $b_{10} - b_{10^*} = 0$. From the experimentally observed masses[44] one obtains $b_{8_1} - \frac{4}{9} b_{27} = (2.52 \pm 0.7)$ MeV$/c^2$, $b_{8_2} = 22.6$ MeV$/c^2$, $b_{10} - b_{10^*} = (-0.12 \pm 0.6)$ MeV$/c^2$.

For the vector mesons, using[47]

$$m_\omega = 782 \text{ MeV}/c^2$$
$$m_\rho = 750 \text{ MeV}/c^2\,,$$

(16.3c) predicts $m_{K^*} = 774$ MeV$/c^2$. Now there exist two K^* resonances, one[6] at 888 MeV$/c^2$ and the other[5] at 730 MeV$/c^2$. None of these values is very close to 774 MeV$/c^2$. However, let us rewrite Eq. (16.9) for the vector mesons. We get

$$\begin{aligned} m_\omega^2 &= m_0^2 - 2m_1^2 - 9m_2^2 \\ m_\rho^2 &= m_0^2 + 2m_1^2 - m_2^2 \\ m_{K^*}^2 &= m_0^2 - m_1^2 + 3m_2^2 \end{aligned} \qquad (16.14)$$

where

$$m_0^2 = a_1\,, \quad m_1^2 = (\sqrt{5}/10)a_{8_1}\,, \quad m_2^2 = -(1/9\sqrt{5})a_{27}\,, \quad \text{and} \quad a_{8_2} = 0\,.$$

The assumption made by Okubo[10] to obtain (16.2) was setting arbitrarily $m_2^2 = 0$. One could just as well set arbitrarily $m_1^2 = 0$. This leads to the mass relation

$$m_{K^*}^2 = \tfrac{3}{2} m_\rho^2 - \tfrac{1}{2} m_\omega^2\,. \qquad (16.15)$$

This formula predicts the K^* at 733 MeV$/c^2$, surprisingly close to the observed 730 MeV$/c^2$ $K - \pi$ resonance. This might imply that the mass differences in the vector meson octet do not transform according to the representation $\{8\}$, but perhaps according to the representation $\{27\} = D(2,2)$.

Another explanation[48] is that the ω does not belong to the vector octet, but is a unitary singlet. The vector octet consists of the ρ, K^* (888 MeV$/c^2$), and another $I = 0$ vector meson ϕ. The mass relation (16.3c) predicts $m_\phi^2 = \frac{4}{3} m_{K^*}^2 - \frac{1}{3} m^2$, or $m_\phi = 927$ MeV$/c^2$. Due to the presence of the ω, with about the same mass and with the same quantum numbers as the ϕ, this level is pushed up and perhaps this ϕ meson is the 1020-MeV resonance[49] in the $K\overline{K}$ system.

17. YUKAWA COUPLINGS

In the unitary symmetry model of strong interactions very definite relations are predicted between the different meson baryon coupling constants. We shall restrict ourselves here to the Yukawa-type coupling between the baryon octet and the pseudoscalar meson octet. Generalizations to other couplings are obvious. In unitary spin space we shall denote the wave function of the baryons by B and of the mesons by M.

We make the following assignments[50]:

$$\begin{aligned} B_1 &= p\,, & M_1 &= K^+\,, \\ B_2 &= n\,, & M_2 &= K^0\,, \\ B_3 &= -\Sigma^+\,, & M_3 &= -\pi^+\,, \\ B_4 &= \Sigma^0\,, & M_4 &= \pi^0\,, \\ B_5 &= \Sigma^-\,, & M_5 &= \pi^-\,, \\ B_6 &= \Lambda\,, & M_6 &= \eta\,, \\ B_7 &= \Xi^0\,, & M_7 &= \overline{K}^0\,, \\ B_8 &= \Xi^-\,, & M_8 &= -K^-\,. \end{aligned} \qquad (17.1)$$

The antiparticles belong to the conjugate representation. As the meson particle representation is equivalent to the antiparticle representation, one must be careful with the phases. We note that our above assignment for the mesons is consistent with the phase conventions (8.2) and (8.3) for the conjugate representation.

With the same conventions the antibaryon wave functions B^+ are

$$\begin{aligned} B_1^+ &= -(\Xi^-)^+\,, & B_3^+ &= -(\Sigma^-)^+\,, \\ B_2^+ &= (\Xi^0)^+\,, & B_4^+ &= (\Sigma^0)^+\,, \\ B_7^+ &= n^+\,, & B_5^+ &= (\Sigma^+)^+\,, \\ B_8^+ &= -p^+\,, & B_6^+ &= \Lambda^+\,. \end{aligned} \qquad (17.2)$$

We assume an interaction Lagrangian of the Yukawa type

$$\mathcal{L}_{\text{int}} = -g(B^+B)M\,.$$

The Lagrangian should be a unitary singlet, i.e., belong to the IR $\{1\}$. The mesons transform as the

[47] B. P. Gregory, *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962), p. 779.

[48] J. J. Sakurai, Phys. Rev. Letters **9**, 472 (1962).

[49] P. Schlein, W. E. Slater, L. T. Smith, D. H. Stork, and H. K. Ticho, Phys. Rev. Letters **10**, 368 (1963); P. L. Connolly, E. L. Hart, K. W. Lai, G. London, G. C. Moneti, R. R. Rau, N. P. Samios, I. O. Skillicorn, S. S. Yamamoto, M. Goldberg, M. Gundzik, J. Leitner, and S. Lichtman, Phys. Rev. Letters **10**, 371 (1963).

[50] We assign $B_3 = -\Sigma^+$ in analogy with $M_3 = -\pi^+$. This gives $\Sigma \cdot \pi = \Sigma^+\pi^- + \Sigma^0\pi^0 + \Sigma^-\pi^+$.

IR {8}. To preserve unitary symmetry the invariant $\mathcal{J} = B^+B$ must also transform as the IR {8}. However, there are two ways to couple $\{8\} \otimes \{8\}$ to {8}. The two possible currents $\mathcal{J}^{(1)}$ and $\mathcal{J}^{(2)}$ are given by

$$\mathcal{J}_\nu^{(\gamma)} = \sum_{\nu_1\nu_2} \begin{pmatrix} 8 & 8 & 8_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} B_{\nu_1}^+ B_{\nu_2} . \qquad (17.3)$$

The interaction Lagrangian is, therefore,

$$\mathcal{L}_{\text{int}} = -(g_1\mathcal{J}^{(1)} + g_2\mathcal{J}^{(2)})M = -\mathcal{J}M , \qquad (17.4)$$

where

$$\mathcal{J}M = \sum_\nu \begin{pmatrix} 8 & 8 & 1 \\ \nu & -\nu & 0 \end{pmatrix} \mathcal{J}_\nu M_{-\nu} . \qquad (17.5)$$

Introducing the constants

$$g_p = [(\sqrt{30}/40)g_1 + (\sqrt{6}/24)g_2] , \qquad (17.6)$$

$$\alpha = (\sqrt{6}/24)(g_2/g_p) , \qquad (17.7)$$

we write the interaction Lagrangian as

$$\begin{aligned}\mathcal{L}_{\text{int}} = {} & g_{NN\pi}(N_1^+\boldsymbol{\tau}N_1)\cdot\boldsymbol{\pi} + g_{\Xi\Xi\pi}(N_2^+\boldsymbol{\tau}N_2)\cdot\boldsymbol{\pi} \\ & + g_{\Lambda\Sigma\pi}(\Lambda^+\boldsymbol{\Sigma} + \boldsymbol{\Sigma}^+\Lambda)\cdot\boldsymbol{\pi} - ig_{\Sigma\Sigma\pi}(\boldsymbol{\Sigma}^+ \times \boldsymbol{\Sigma})\cdot\boldsymbol{\pi} \\ & + g_{NN\eta}(N_1^+N_1)\eta + g_{\Xi\Xi\eta}(N_2^+N_2)\eta \\ & + g_{\Lambda\Lambda\eta}(\Lambda^+\Lambda)\eta + g_{\Sigma\Sigma\eta}(\boldsymbol{\Sigma}^+\cdot\boldsymbol{\Sigma})\eta \\ & + g_{N\Lambda K}\{(N_1^+K)\Lambda + \Lambda^+(K^+N_1)\} \\ & + g_{\Xi\Lambda K}\{(N_2^+K_c)\Lambda + \Lambda^+(K_c^+N_2)\} \\ & + g_{N\Sigma K}\{\boldsymbol{\Sigma}^+\cdot(K^+\boldsymbol{\tau}N_1) + (N_1^+\boldsymbol{\tau}K)\cdot\boldsymbol{\Sigma}\} \\ & + g_{\Xi\Sigma K}\{\boldsymbol{\Sigma}^+(K_c^+\boldsymbol{\tau}N_2) + (N_2^+\boldsymbol{\tau}K_c)\cdot\boldsymbol{\Sigma}\} , \end{aligned} \qquad (17.8)$$

where

$$\begin{aligned} & g_{NN\pi} = g_p , \quad g_{\Xi\Xi\pi} = -g_p(1-2\alpha_p) , \\ & g_{\Lambda\Sigma\pi} = \tfrac{2}{3}\sqrt{3}g_p(1-\alpha_p) , \quad g_{\Sigma\Sigma\pi} = 2g_p\alpha_p , \\ & g_{NN\eta} = \tfrac{1}{3}\sqrt{3}g_p(4\alpha_p - 1) , \\ & g_{\Xi\Xi\eta} = -\tfrac{1}{3}\sqrt{3}g_p(1+2\alpha_p) , \\ & g_{\Sigma\Sigma\eta} = \tfrac{2}{3}\sqrt{3}g_p(1-\alpha_p) , \quad g_{\Lambda\Lambda\eta} = -\tfrac{2}{3}\sqrt{3}g_p(1-\alpha_p) , \\ & g_{N\Lambda K} = -\tfrac{1}{3}\sqrt{3}g_p(1+2\alpha_p) , \\ & g_{\Xi\Lambda K} = \tfrac{1}{3}\sqrt{3}g_p(4\alpha_p - 1) , \quad g_{N\Sigma K} = g_p(1-2\alpha_p) , \\ & g_{\Xi\Sigma K} = -g_p . \end{aligned} \qquad (17.9)$$

We have used here the notation

$$N_1 = \begin{pmatrix} p \\ n \end{pmatrix} , \quad N_2 = \begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix} , \quad K = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix} ,$$

$$K_c = \begin{pmatrix} \overline{K}^0 \\ -K^- \end{pmatrix} , \qquad (17.10)$$

Λ, $\boldsymbol{\Sigma}$, $\boldsymbol{\pi}$, and η for the isospin wave functions of the particles. To illustrate the procedure followed we will consider in more detail the coupling of the η meson with the baryons. We have $\eta = M^6$. Equations (17.4), (17.5), and (14.10) give that

$$\mathcal{L}_{\text{int}} = (1/2\sqrt{2})\mathcal{J}_6 M_6 .$$

Using the isoscalars from Table II, we obtain

$$\begin{aligned}\mathcal{J}_6^{(1)} = {} & (1/2\sqrt{5})\{B_1^+B_8 + B_8^+B_1 - B_2^+B_7 - B_7^+B_2\} \\ & - (1/\sqrt{5})\{B_3^+B_5 - B_4^+B_4 + B_5^+B_3 , \\ & - (1/\sqrt{5})B_6^+B_6 \\ = {} & (-1/2\sqrt{5})\{(\Xi^-)^+\Xi^- + p^+p + (\Xi^0)^+\Xi^0 \\ & + n^+n\} + (1/\sqrt{5})\{(\Sigma^-)^+\Sigma^- + (\Sigma^0)^+\Sigma^0 \\ & + (\Sigma^+)^+\Sigma^+\} - (1/\sqrt{5})\Lambda^+\Lambda \\ = {} & -(1/2\sqrt{5})\{N_1^+N_1 + N_2^+N_2\} \\ & + (1/\sqrt{5})\{\boldsymbol{\Sigma}^+\cdot\boldsymbol{\Sigma} - \Lambda^+\Lambda\} \end{aligned}$$

and

$$\begin{aligned}\mathcal{J}_6^{(2)} &= \tfrac{1}{2}\{B_1^+B_8 - B_8^+B_1 - B_2^+B_7 + B_7^+B_2\} \\ &= \tfrac{1}{2}\{N_1^+N_1 - N_2^+N_2\} . \end{aligned}$$

Therefore, using (17.6), (17.7), and (17.9), one obtains

$$\begin{aligned}\mathcal{J}_6 = g_1\mathcal{J}_6^{(1)} + g_2\mathcal{J}_6^{(2)} = {} & 2\sqrt{2}\Big[\Big(-\frac{g_1}{4\sqrt{10}} + \frac{g_2}{4\sqrt{2}}\Big)N_1^+N_1 \\ & + \Big(-\frac{g_1}{4\sqrt{10}} - \frac{g_2}{4\sqrt{2}}\Big)N_2^+N_2 \\ & + (1/2\sqrt{10})g_1(\boldsymbol{\Sigma}^+\cdot\boldsymbol{\Sigma} - \Lambda^+\Lambda)\Big] \\ = {} & 2\sqrt{2}[\tfrac{1}{3}\sqrt{3}g_p(4\alpha_p - 1)N_1^+N_1 \\ & -\tfrac{1}{3}\sqrt{3}g_p(1+2\alpha_p)N_2^+N_2 \\ & + \tfrac{2}{3}\sqrt{3}g_p(1-\alpha_p)(\boldsymbol{\Sigma}^+\cdot\boldsymbol{\Sigma} - \Lambda^+\Lambda)] \\ = {} & 2\sqrt{2}[g_{NN\eta}N_1^+N_1 + g_{\Xi\Xi\eta}N_2^+N_2 + g_{\Sigma\Sigma\eta}\boldsymbol{\Sigma}^+\cdot\boldsymbol{\Sigma} \\ & + g_{\Lambda\Lambda\eta}\Lambda^+\Lambda] . \end{aligned}$$

This leads directly to the interaction Lagrangian (17.8).

Experimentally one knows $g_{NN\pi}$ very well[51] and one has a pretty fair idea about $g_{\Lambda\Sigma\pi}$ and $g_{\Sigma\Sigma\pi}$. This establishes that α_p is small[52] and perhaps[53] $\alpha_p \sim \frac{1}{4}$.

Also the coupling of the vector mesons to the baryons is described by a formula like (17.8). We have only to make the replacement

$$\boldsymbol{\pi} \to \boldsymbol{\varrho} , \quad \eta \to \omega(\text{or } \phi) , \quad K \to K^* , \quad \text{and } K_c \to K_c^* .$$

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[51] T. Spearman, Nucl. Phys. **16**, 402 (1960); G. Salzman and H. Schnitzer, Phys. Rev. **113**, 1153 (1959).

[52] J. J. de Swart and C. K. Iddings, Phys. Rev. **130**, 319 (1963).

[53] A. W. Martin and K. C. Wali, Phys. Rev. **130**, 2455 (1963).

However, we have now two coupling constants $g_v^{(e)}$ and $g_v^{(m)}$ and two factors $\alpha_v^{(e)}$ and $\alpha_v^{(m)}$ for the electric type and magnetic type of coupling.

In case there should exist a meson ϕ which is a unitary singlet, then the interaction Lagrangian of this meson with the baryon octet is

$$\mathcal{L}_{\text{int}} = g\{N_1^+N_1 + \Lambda^+\Lambda + \mathbf{\Sigma}^+\cdot\mathbf{\Sigma} + N_2^+N_2\}\phi\,. \quad (17.11)$$

A very nice special case of the vector meson octet-baryon octet coupling is obtained by setting $\alpha_v^{(e)} = 1$. We recover then the universal coupling of the $I = 1$ ρ meson to the isospin current and the universal coupling of the $I = 0$ octet vector meson ω (or ϕ) to the hypercharge current as proposed by Sakurai.[54] A unitary singlet vector meson is universally coupled to the baryonic current.

18. TABLES OF ISOSCALAR FACTORS

In this section, we present Tables I–VI of isoscalar factors for the following cases:

$$\{8\} \otimes \{8\} = \{27\} \oplus \{10\} \oplus \{10^*\} \oplus \{8\}_1 \oplus \{8\}_2 \oplus \{1\}$$

$$\{8\} \otimes \{10\} = \{35\} \oplus \{27\} \oplus \{10\} \oplus \{8\}$$

$$\{8\} \otimes \{27\} = \{64\} \oplus \{35\} \oplus \{35^*\} \oplus \{27\}_1 \oplus \{27\}_2 \oplus \{10\} \oplus \{10^*\} \oplus \{8\}$$

$$\{10\} \otimes \{10\} = \{35\} \oplus \{28\} \oplus \{27\} \oplus \{10^*\}$$

$$\{10\} \otimes \{10^*\} = \{64\} \oplus \{27\} \oplus \{8\} \oplus \{1\}\,.$$

With the help of the symmetry properties (14.3), (14.9), and (14.13), these tables can be extended quite a bit. In fact, already some of the entries in the tables are redundant, because they can be obtained with the help of Eqs. (14.3), (14.9), or (14.13) from other parts of the table. However, we feel that omitting these numbers would sometimes be confusing. In Table I we will give for some cases the phases ξ_1, ξ_2, and ξ_3, this will facilitate the extension of the tables of isoscalar factors for these cases.

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[54] J. J. Sakurai, Ann. Phys. (N. Y.) 11, 1 (1960).

TABLE I. Phase factors to be used in Eqs. (14.1), (14.3), (14.8), (14.9), (14.12), and (14.13).

μ_1	μ_2	μ_3	ξ_1	ξ_2	ξ_3
8	8	27	1	−1	1
		10	−1	−1	1
		10*	−1	1	1
		8_1	1	1	1
		8_2	−1	−1	−1
		1	1	−1	1
8	10	35	1	−1	1
		27	−1	1	1
		10	−1	−1	−1
		8	1	1	−1
8	27	64	1	−1	1
		35	−1	−1	1
		35*	−1	1	1
		27_1	1	1	1
		27_2	−1	−1	−1
		10	1	−1	−1
		10*	1	1	−1
		8	1	−1	1
8	10*	35*	1	−1	1
		27	−1	−1	1
		10*	−1	−1	−1
		8	1	−1	−1
10	10	28	1	1	1
		35	−1	−1	1
		27	1	1	1
		10*	−1	−1	1
10	10*	64	1	1	1
		27	−1	1	−1
		8	1	1	1
		1	−1	1	−1
10*	10*	28*	1	−1	1
		35*	−1	−1	1
		27	1	−1	1
		10	−1	−1	1

TABLE II. Isoscalar factors for $\{8\} \otimes \{8\}$. Given are the isoscalar factors

$$\left(\begin{matrix} 8 & 8 \\ I_1Y_1 & I_2Y_2 \end{matrix}\middle|\begin{matrix}\mu_\gamma \\ I\,Y\end{matrix}\right)$$

for the CG series $\{8\} \otimes \{8\} = \{27\} \oplus \{10\} \oplus \{10^*\} \oplus \{8\}_1 \oplus \{8\}_2 \oplus \{1\}$.

$Y = 2 \quad I = 1$

I_1, Y_1; I_2, Y_2	27	μ_γ
$\frac{1}{2}$, 1; $\frac{1}{2}$, 1	1	

$Y = 2 \quad I = 0$

I_1, Y_1; I_2, Y_2	10*	μ_γ
$\frac{1}{2}$, 1; $\frac{1}{2}$, 1	−1	

TABLE II. (*Continued*)

$Y = 1 \quad I = \frac{3}{2}$

$I_1, Y_1; I_2, Y_2$	27	10	μ_γ
$\frac{1}{2}, 1; 1, 0$	$\sqrt{2}/2$	$-\sqrt{2}/2$	
$1, 0; \frac{1}{2}, 1$	$\sqrt{2}/2$	$\sqrt{2}/2$	

$Y = 1 \quad I = \frac{1}{2}$

$I_1, Y_1; I_2, Y_2$	27	8_1	8_2	10^*	μ_γ
$\frac{1}{2}, 1; 1, 0$	$\sqrt{5}/10$	$3\sqrt{5}/10$	$1/2$	$-1/2$	
$1, 0; \frac{1}{2}, 1$	$-\sqrt{5}/10$	$-3\sqrt{5}/10$	$1/2$	$-1/2$	
$\frac{1}{2}, 1; 0, 0$	$3\sqrt{5}/10$	$-\sqrt{5}/10$	$1/2$	$1/2$	
$0, 0; \frac{1}{2}, 1$	$3\sqrt{5}/10$	$-\sqrt{5}/10$	$-1/2$	$-1/2$	

$Y = 0 \quad I = 2$

$I_1, Y_1; I_2, Y_2$	27	μ_γ
$1, 0; 1, 0$	1	

$Y = 0 \quad I = 1$

$I_1, Y_1; I_2, Y_2$	27	8_1	8_2	10	10^*	μ_γ
$\frac{1}{2}, 1; \frac{1}{2}, -1$	$\sqrt{5}/5$	$-\sqrt{30}/10$	$\sqrt{6}/6$	$-\sqrt{6}/6$	$\sqrt{6}/6$	
$\frac{1}{2}, -1; \frac{1}{2}, 1$	$\sqrt{5}/5$	$-\sqrt{30}/10$	$-\sqrt{6}/6$	$\sqrt{6}/6$	$-\sqrt{6}/6$	
$1, 0; 1, 0$	0	0	$\sqrt{6}/3$	$\sqrt{6}/6$	$-\sqrt{6}/6$	
$1, 0; 0, 0$	$\sqrt{30}/10$	$\sqrt{5}/5$	0	$1/2$	$1/2$	
$0, 0; 1, 0$	$\sqrt{30}/10$	$\sqrt{5}/5$	0	$-1/2$	$-1/2$	

$Y = 0 \quad I = 0$

$I_1, Y_1; I_2, Y_2$	27	8_1	1	8_2	μ_γ
$\frac{1}{2}, 1; \frac{1}{2}, -1$	$\sqrt{15}/10$	$\sqrt{10}/10$	$1/2$	$\sqrt{2}/2$	
$\frac{1}{2}, -1; \frac{1}{2}, 1$	$-\sqrt{15}/10$	$-\sqrt{10}/10$	$-1/2$	$\sqrt{2}/2$	
$1, 0; 1, 0$	$-\sqrt{10}/20$	$-\sqrt{15}/5$	$\sqrt{6}/4$	0	
$0, 0; 0, 0$	$3\sqrt{30}/20$	$-\sqrt{5}/5$	$-\sqrt{2}/4$	0	

$Y = -1 \quad I = \frac{3}{2}$

$I_1, Y_1; I_2, Y_2$	27	10^*	μ_γ
$\frac{1}{2}, -1; 1, 0$	$\sqrt{2}/2$	$-\sqrt{2}/2$	
$1, 0; \frac{1}{2}, -1$	$\sqrt{2}/2$	$\sqrt{2}/2$	

$Y = -1 \quad I = \frac{1}{2}$

$I_1, Y_1; I_2, Y_2$	27	8_1	8_2	10	μ_γ
$\frac{1}{2}, -1; 1, 0$	$-\sqrt{5}/10$	$-3\sqrt{5}/10$	$1/2$	$1/2$	
$1, 0; \frac{1}{2}, -1$	$\sqrt{5}/10$	$3\sqrt{5}/10$	$1/2$	$1/2$	
$\frac{1}{2}, -1; 0, 0$	$3\sqrt{5}/10$	$-\sqrt{5}/10$	$-1/2$	$1/2$	
$0, 0; \frac{1}{2}, -1$	$3\sqrt{5}/10$	$-\sqrt{5}/10$	$1/2$	$-1/2$	

$Y = -2 \quad I = 1$

$I_1, Y_1; I_2, Y_2$	27	μ_γ
$\frac{1}{2}, -1; \frac{1}{2}, -1$	1	

$Y = -2 \quad I = 0$

$I_1, Y_1; I_2, Y_2$	10	μ_γ
$\frac{1}{2}, -1; \frac{1}{2}, -1$	1	

TABLE III. Isoscalar factors for $\{8\} \otimes \{10\}$. Given are the isoscalar factors

$$\begin{pmatrix} 8 & 10 & \mu_\gamma \\ I_1 Y_1 & I_2 Y_2 & I\ Y \end{pmatrix}$$

for the CG series $\{8\} \otimes \{10\} = \{35\} \oplus \{27\} \oplus \{10\} \oplus \{8\}$.

$Y = 2 \quad I = 2$

$I_1, Y_1; I_2, Y_2$	35	μ_γ
$\frac{1}{2}, 1; \frac{3}{2}, 1$	1	

$Y = 2 \quad I = 1$

$I_1, Y_1; I_2, Y_2$	27	μ_γ
$\frac{1}{2}, 1; \frac{3}{2}, 1$	-1	

$Y = 1 \quad I = \frac{5}{2}$

$I_1, Y_1; I_2, Y_2$	35	μ_γ
$1, 0; \frac{3}{2}, 1$	1	

$Y = 1 \quad I = \frac{3}{2}$

$I_1, Y_1; I_2, Y_2$	35	27	10	μ_γ
$1, 0; \frac{3}{2}, 1$	$-1/4$	$-\sqrt{5}/4$	$\sqrt{10}/4$	
$0, 0; \frac{3}{2}, 1$	$\sqrt{5}/4$	$-3/4$	$-\sqrt{2}/4$	
$\frac{1}{2}, 1; 1, 0$	$\sqrt{10}/4$	$\sqrt{2}/4$	$1/2$	

$Y = 1 \quad I = \frac{1}{2}$

$I_1, Y_1; I_2, Y_2$	27	8	μ_γ
$1, 0; \frac{3}{2}, 1$	$\sqrt{5}/5$	$-2\sqrt{5}/5$	
$\frac{1}{2}, 1; 1, 0$	$-2\sqrt{5}/5$	$-\sqrt{5}/5$	

$Y = 0 \quad I = 2$

$I_1, Y_1; I_2, Y_2$	35	27	μ_γ
$1, 0; 1, 0$	$\sqrt{3}/2$	$1/2$	
$\frac{1}{2}, -1; \frac{3}{2}, 1$	$1/2$	$-\sqrt{3}/2$	

$Y = 0 \quad I = 1$

$I_1, Y_1; I_2, Y_2$	35	27	10	8	μ_γ
$1, 0; 1, 0$	$-\sqrt{3}/6$	$-3\sqrt{5}/10$	$\sqrt{3}/3$	$\sqrt{30}/15$	
$0, 0; 1, 0$	$\sqrt{2}/2$	$-\sqrt{30}/10$	0	$-\sqrt{5}/5$	
$\frac{1}{2}, 1; \frac{1}{2}, -1$	$\sqrt{3}/3$	$\sqrt{5}/5$	$\sqrt{3}/3$	$\sqrt{30}/15$	
$\frac{1}{2}, -1; \frac{3}{2}, 1$	$-\sqrt{3}/6$	$\sqrt{5}/10$	$\sqrt{3}/3$	$-2\sqrt{30}/15$	

TABLE III. (*Continued*)

$Y=0 \quad I=0$

$I_1, Y_1; I_2, Y_2$	27	8	μ_γ
$1, 0; 1, 0$	$\sqrt{10}/5$	$-\sqrt{15}/5$	
$\frac{1}{2}, 1; \frac{1}{2}, -1$	$-\sqrt{15}/5$	$-\sqrt{10}/5$	

$Y=-1 \quad I=\frac{3}{2}$

$I_1, Y_1; I_2, Y_2$	35	27	μ_γ
$1, 0; \frac{1}{2}, -1$	$\sqrt{2}/2$	$\sqrt{2}/2$	
$\frac{1}{2}, -1; 1, 0$	$\sqrt{2}/2$	$-\sqrt{2}/2$	

$Y=-1 \quad I=\frac{1}{2}$

$I_1, Y_1; I_2, Y_2$	35	27	10	8	μ_γ
$1, 0; \frac{1}{2}, -1$	$-1/4$	$-7\sqrt{5}/20$	$\sqrt{2}/4$	$\sqrt{5}/5$	
$0, 0; \frac{1}{2}, -1$	$3/4$	$-3\sqrt{5}/20$	$\sqrt{2}/4$	$-\sqrt{5}/5$	
$\frac{1}{2}, 1; 0, -2$	$\sqrt{2}/4$	$3\sqrt{10}/20$	$1/2$	$\sqrt{10}/5$	
$\frac{1}{2}, -1; 1, 0$	$-1/2$	$\sqrt{5}/10$	$\sqrt{2}/2$	$-\sqrt{5}/5$	

$Y=-2 \quad I=1$

$I_1, Y_1; I_2, Y_2$	35	27	μ_γ
$1, 0; 0, -2$	$1/2$	$\sqrt{3}/2$	
$\frac{1}{2}, -1; \frac{1}{2}, -1$	$\sqrt{3}/2$	$-1/2$	

$Y=-2 \quad I=0$

$I_1, Y_1; I_2, Y_2$	35	10	μ_γ
$0, 0; 0, -2$	$\sqrt{2}/2$	$\sqrt{2}/2$	
$\frac{1}{2}, -1; \frac{1}{2}, -1$	$-\sqrt{2}/2$	$\sqrt{2}/2$	

$Y=-3 \quad I=\frac{1}{2}$

$I_1, Y_1; I_2, Y_2$	35	μ_γ
$\frac{1}{2}, -1; 0, -2$	1	

TABLE IV. Isoscalar factors for $\{8\} \otimes \{27\}$. Given are the isoscalar factors

$$\left(\begin{array}{cc|c} 8 & 27 & \mu_\gamma \\ I_1 Y_1 & I_2 Y_2 & I\ Y \end{array}\right)$$

for the CG series $\{8\} \otimes \{27\} = \{64\} \oplus \{35\} \oplus \{35^*\} \oplus \{27\}_1 \oplus \{27\}_2 \oplus \{10\} \oplus \{10^*\} \oplus \{8\}$.

$Y=3 \quad I=\frac{3}{2}$

$I_1, Y_1; I_2, Y_2$	64	μ_γ
$\frac{1}{2}, 1; 1, 2$	1	

$Y=3 \quad I=\frac{1}{2}$

$I_1, Y_1; I_2, Y_2$	35*	μ_γ
$\frac{1}{2}, 1; 1, 2$	-1	

$Y=2 \quad I=2$

$I_1, Y_1; I_2, Y_2$	64	35	μ_γ
$\frac{1}{2}, 1; \frac{3}{2}, 1$	$\sqrt{6}/3$	$-\sqrt{3}/3$	
$1, 0; 1, 2$	$\sqrt{3}/3$	$\sqrt{6}/3$	

$Y=2 \quad I=1$

$I_1, Y_1; I_2, Y_2$	64	35*	27_1	27_2	μ_γ
$\frac{1}{2}, 1; \frac{3}{2}, 1$	$\sqrt{14}/21$	$-2/3$	$\sqrt{70}/14$	$\sqrt{6}/6$	
$\frac{1}{2}, 1; \frac{1}{2}, 1$	$2\sqrt{70}/21$	$\sqrt{5}/6$	$-\sqrt{14}/28$	$\sqrt{30}/12$	
$1, 0; 1, 2$	$-\sqrt{21}/21$	$-\sqrt{6}/6$	$-\sqrt{105}/14$	$1/2$	
$0, 0; 1, 2$	$\sqrt{14}/7$	$-1/2$	$-\sqrt{70}/28$	$-\sqrt{6}/4$	

$Y=2 \quad I=0$

$I_1, Y_1; I_2, Y_2$	35*	10*	μ_γ
$\frac{1}{2}, 1; \frac{1}{2}, 1$	$-\sqrt{30}/6$	$-\sqrt{6}/6$	
$1, 0; 1, 2$	$\sqrt{6}/6$	$-\sqrt{30}/6$	

$Y=1 \quad I=\frac{5}{2}$

$I_1, Y_1; I_2, Y_2$	64	35	μ_γ
$\frac{1}{2}, 1; 2, 0$	$\sqrt{3}/3$	$-\sqrt{6}/3$	
$1, 0; \frac{3}{2}, 1$	$\sqrt{6}/3$	$\sqrt{3}/3$	

$Y=1 \quad I=\frac{3}{2}$

$I_1, Y_1; I_2, Y_2$	64	35	35*	27_1	27_2	10	μ_γ
$\frac{1}{2}, 1; 2, 0$	$\sqrt{7}/21$	$-1/12$	$-\sqrt{5}/6$	$5\sqrt{42}/56$	$\sqrt{10}/8$	$-5\sqrt{2}/12$	
$\frac{1}{2}, 1; 1, 0$	$5\sqrt{7}/21$	$-5/12$	$\sqrt{5}/6$	$-3\sqrt{42}/56$	$\sqrt{10}/8$	$-\sqrt{2}/12$	
$1, 0; \frac{3}{2}, 1$	$-\sqrt{21}/63$	$7\sqrt{3}/36$	$-\sqrt{15}/9$	$-5\sqrt{14}/56$	$\sqrt{30}/8$	$5\sqrt{6}/36$	
$1, 0; \frac{1}{2}, 1$	$5\sqrt{42}/63$	$5\sqrt{6}/18$	$\sqrt{30}/18$	$\sqrt{7}/7$	0	$\sqrt{3}/9$	
$0, 0; \frac{3}{2}, 1$	$\sqrt{105}/21$	$-\sqrt{15}/12$	$-\sqrt{3}/3$	$\sqrt{70}/56$	$-\sqrt{6}/8$	$\sqrt{30}/12$	
$\frac{1}{2}, -1; 1, 2$	$\sqrt{35}/21$	$\sqrt{5}/6$	$-1/3$	$-\sqrt{210}/28$	$-\sqrt{2}/4$	$-\sqrt{10}/6$	

$Y=1 \quad I=\frac{1}{2}$

$I_1, Y_1; I_2, Y_2$	64	35*	27_1	27_2	10*	8	μ_γ
$\frac{1}{2}, 1; 1, 0$	$\sqrt{35}/21$	$-\sqrt{10}/6$	$3\sqrt{105}/70$	$1/2$	$-1/3$	$2\sqrt{5}/15$	
$\frac{1}{2}, 1; 0, 0$	$2\sqrt{35}/21$	$\sqrt{10}/6$	$-\sqrt{105}/70$	$1/2$	$1/3$	$\sqrt{5}/15$	
$1, 0; \frac{3}{2}, 1$	$-\sqrt{42}/63$	$\sqrt{3}/9$	$-\sqrt{14}/7$	0	$-\sqrt{30}/9$	$2\sqrt{6}/9$	
$1, 0; \frac{1}{2}, 1$	$-\sqrt{210}/63$	$-5\sqrt{15}/36$	$-19\sqrt{70}/280$	$\sqrt{6}/8$	$7\sqrt{6}/36$	$\sqrt{30}/45$	
$0, 0; \frac{1}{2}, 1$	$\sqrt{210}/21$	$-\sqrt{15}/12$	$-13\sqrt{70}/280$	$-\sqrt{6}/8$	$-\sqrt{6}/12$	$-\sqrt{30}/15$	
$\frac{1}{2}, -1; 1, 2$	$-2\sqrt{7}/21$	$\sqrt{2}/12$	$-\sqrt{21}/28$	$\sqrt{5}/4$	$-\sqrt{5}/6$	$-2/3$	

TABLE IV. (*Continued*)

$Y=0 \quad I=3$

$I_1,\ Y_1;\ I_2,\ Y_2$	64	μ_γ
$1,\ 0;\ 2,\ 0$	1	

$Y=0 \quad I=2$

$I_1,\ Y_1;\ I_2,\ Y_2$	64	35	35*	27_1	27_2	μ_γ
$\frac{1}{2},\ 1;\ \frac{3}{2},\ -1$	$2\sqrt{21}/21$	$-\sqrt{3}/3$	$\sqrt{3}/6$	$-\sqrt{210}/28$	$\sqrt{2}/4$	
$\frac{1}{2},\ -1;\ \frac{3}{2},\ 1$	$2\sqrt{21}/21$	$\sqrt{3}/6$	$-\sqrt{3}/3$	$-\sqrt{210}/28$	$-\sqrt{2}/4$	
$1,\ 0;\ 1,\ 0$	$\sqrt{210}/21$	$\sqrt{30}/12$	$\sqrt{30}/12$	$\sqrt{21}/14$	0	
$1,\ 0;\ 2,\ 0$	0	$\sqrt{2}/4$	$-\sqrt{2}/4$	0	$\sqrt{3}/2$	
$0,\ 0;\ 2,\ 0$	$\sqrt{7}/7$	$-1/2$	$-1/2$	$\sqrt{70}/14$	0	

$Y=0 \quad I=1$

$I_1,\ Y_1;\ I_2,\ Y_2$	64	35	35*	27_1	27_2	10	10*	8	μ_γ
$\frac{1}{2},\ 1;\ \frac{3}{2},\ -1$	$2\sqrt{35}/63$	$-1/9$	$-5/18$	$3\sqrt{14}/28$	$\sqrt{30}/12$	$-2\sqrt{5}/9$	$-\sqrt{5}/9$	$4/9$	
$\frac{1}{2},\ -1;\ \frac{3}{2},\ 1$	$-2\sqrt{35}/63$	$5/18$	$1/9$	$-3\sqrt{14}/28$	$\sqrt{30}/12$	$-\sqrt{5}/9$	$-2\sqrt{5}/9$	$-4/9$	
$\frac{1}{2},\ 1;\ \frac{1}{2},\ -1$	$10\sqrt{7}/63$	$-\sqrt{5}/9$	$2\sqrt{5}/9$	$-3\sqrt{70}/70$	$\sqrt{6}/6$	$-1/9$	$4/9$	$2\sqrt{5}/45$	
$\frac{1}{2},\ -1;\ \frac{1}{2},\ 1$	$10\sqrt{7}/63$	$2\sqrt{5}/9$	$-\sqrt{5}/9$	$-3\sqrt{70}/70$	$-\sqrt{6}/6$	$-4/9$	$1/9$	$2\sqrt{5}/45$	
$0,\ 0;\ 1,\ 0$	$5\sqrt{7}/21$	$-\sqrt{5}/6$	$-\sqrt{5}/6$	$-\sqrt{70}/70$	0	$1/3$	$-1/3$	$-2\sqrt{5}/15$	
$1,\ 0;\ 0,\ 0$	$10\sqrt{7}/63$	$2\sqrt{5}/9$	$2\sqrt{5}/9$	$4\sqrt{70}/70$	0	$2/9$	$-2/9$	$-\sqrt{5}/45$	
$1,\ 0;\ 1,\ 0$	0	$\sqrt{30}/12$	$-\sqrt{30}/12$	0	$1/2$	$\sqrt{6}/6$	$\sqrt{6}/6$	0	
$1,\ 0;\ 2,\ 0$	$-\sqrt{14}/63$	$\sqrt{10}/36$	$\sqrt{10}/36$	$-\sqrt{35}/14$	0	$5\sqrt{2}/18$	$-5\sqrt{2}/18$	$2\sqrt{10}/9$	

$Y=0 \quad I=0$

$I_1,\ Y_1;\ I_2,\ Y_2$	64	27_1	27_2	8	μ_γ
$\frac{1}{2},\ 1;\ \frac{1}{2},\ -1$	$2\sqrt{21}/21$	$\sqrt{210}/70$	$\sqrt{2}/2$	$2\sqrt{15}/15$	
$\frac{1}{2},\ -1;\ \frac{1}{2},\ 1$	$-2\sqrt{21}/21$	$-\sqrt{210}/70$	$\sqrt{2}/2$	$-2\sqrt{15}/15$	
$1,\ 0;\ 1,\ 0$	$-\sqrt{21}/21$	$-4\sqrt{210}/70$	0	$2\sqrt{15}/15$	
$0,\ 0;\ 0,\ 0$	$2\sqrt{7}/7$	$-4\sqrt{70}/70$	0	$-\sqrt{5}/5$	

$Y=-1 \quad I=\frac{5}{2}$

$I_1,\ Y_1;\ I_2,\ Y_2$	64	35*	μ_γ
$\frac{1}{2},\ -1;\ 1,\ 0$	$\sqrt{3}/3$	$-\sqrt{6}/3$	
$1,\ 0;\ \frac{3}{2},\ -1$	$\sqrt{6}/3$	$\sqrt{3}/3$	

$Y=-1 \quad I=\frac{3}{2}$

$I_1,\ Y_1;\ I_2,\ Y_2$	64	35*	35	27_1	27_2	10*	μ_γ
$\frac{1}{2},\ -1;\ 2,\ 0$	$-\sqrt{7}/21$	$1/12$	$\sqrt{5}/6$	$-5\sqrt{42}/56$	$\sqrt{10}/8$	$-5\sqrt{2}/12$	
$\frac{1}{2},\ -1;\ 1,\ 0$	$5\sqrt{7}/21$	$-5/12$	$\sqrt{5}/6$	$-3\sqrt{42}/56$	$-\sqrt{10}/8$	$\sqrt{2}/12$	
$1,\ 0;\ \frac{3}{2},\ -1$	$\sqrt{21}/63$	$-7\sqrt{3}/36$	$\sqrt{15}/9$	$5\sqrt{14}/56$	$\sqrt{30}/8$	$5\sqrt{6}/36$	
$1,\ 0;\ \frac{1}{2},\ -1$	$5\sqrt{42}/63$	$5\sqrt{6}/18$	$\sqrt{30}/18$	$\sqrt{7}/7$	0	$-\sqrt{3}/9$	
$0,\ 0;\ \frac{3}{2},\ -1$	$\sqrt{105}/21$	$-\sqrt{15}/21$	$-\sqrt{3}/3$	$\sqrt{70}/56$	$\sqrt{6}/8$	$-\sqrt{30}/12$	
$\frac{1}{2},\ 1;\ 1,\ -2$	$\sqrt{35}/21$	$\sqrt{5}/6$	$-1/3$	$-\sqrt{210}/28$	$\sqrt{2}/4$	$\sqrt{10}/6$	

$Y=-1 \quad I=\frac{1}{2}$

$I_1,\ Y_1;\ I_2,\ Y_2$	64	35	27_1	27_2	10	8	μ_γ
$\frac{1}{2},\ -1;\ 1,\ 0$	$-\sqrt{35}/21$	$\sqrt{10}/6$	$-3\sqrt{105}/70$	$1/2$	$-1/3$	$-2\sqrt{5}/15$	
$\frac{1}{2},\ -1;\ 0,\ 0$	$2\sqrt{35}/21$	$\sqrt{10}/6$	$-\sqrt{105}/70$	$-1/2$	$-1/3$	$\sqrt{5}/15$	
$1,\ 0;\ \frac{3}{2},\ -1$	$-\sqrt{42}/63$	$\sqrt{3}/9$	$-\sqrt{14}/7$	0	$\sqrt{30}/9$	$2\sqrt{6}/9$	
$1,\ 0;\ \frac{1}{2},\ -1$	$\sqrt{210}/63$	$5\sqrt{15}/36$	$-19\sqrt{70}/280$	$\sqrt{6}/8$	$7\sqrt{6}/36$	$-\sqrt{30}/45$	
$0,\ 0;\ \frac{1}{2},\ -1$	$\sqrt{210}/21$	$-\sqrt{15}/12$	$-13\sqrt{70}/280$	$\sqrt{6}/8$	$\sqrt{6}/12$	$-\sqrt{30}/15$	
$\frac{1}{2},\ 1;\ 1,\ -2$	$2\sqrt{7}/21$	$-\sqrt{2}/12$	$\sqrt{21}/28$	$\sqrt{5}/4$	$-\sqrt{5}/6$	$2/3$	

$Y=-2 \quad I=2$

$I_1,\ Y_1;\ I_2,\ Y_2$	64	35*	μ_γ
$\frac{1}{2},\ -1;\ \frac{3}{2},\ -1$	$\sqrt{6}/3$	$-\sqrt{3}/3$	
$1,\ 0;\ 1,\ -2$	$\sqrt{3}/3$	$\sqrt{6}/3$	

$Y=-2 \quad I=1$

$I_1,\ Y_1;\ I_2,\ Y_2$	64	35	27_1	27_2	μ_γ
$\frac{1}{2},\ -1;\ \frac{3}{2},\ -1$	$-\sqrt{14}/21$	$2/3$	$-\sqrt{70}/14$	$\sqrt{6}/6$	
$\frac{1}{2},\ -1;\ \frac{1}{2},\ -1$	$2\sqrt{70}/21$	$\sqrt{5}/6$	$-\sqrt{14}/28$	$-\sqrt{30}/12$	
$1,\ 0;\ 1,\ -2$	$\sqrt{21}/21$	$\sqrt{6}/6$	$\sqrt{105}/14$	$1/2$	
$0,\ 0;\ 1,\ -2$	$\sqrt{14}/7$	$-1/2$	$-\sqrt{70}/28$	$\sqrt{6}/4$	

$Y=-2 \quad I=0$

$I_1,\ Y_1;\ I_2,\ Y_2$	35	10	μ_γ
$\frac{1}{2},\ -1;\ \frac{1}{2},\ -1$	$\sqrt{30}/6$	$-\sqrt{6}/6$	
$1,\ 0;\ 1,\ -2$	$\sqrt{6}/6$	$\sqrt{30}/6$	

$Y=-3 \quad I=\frac{3}{2}$

$I_1,\ Y_1;\ I_2,\ Y_2$	64	μ_γ
$\frac{1}{2},\ -1;\ 1,\ -2$	1	

$Y=-3 \quad I=\frac{1}{2}$

$I_1,\ Y_1;\ I_2,\ Y_2$	35	μ_γ
$\frac{1}{2},\ -1;\ 1,\ -2$	1	

TABLE V. Isoscalar factors for {10} ⊗ {10}. Given are the isoscalar factors

$$\begin{pmatrix} 10 & 10 & \mu_\gamma \\ I_1Y_1 & I_2Y_2 & I\,Y \end{pmatrix}$$

for the CG series {10} ⊗ {10} = {35} ⊕ {28} ⊕ {27} ⊕ {10*}.

$Y = 2\quad I = 3$

$I_1,\ Y_1;\ I_2,\ Y_2$	28	μ_γ
$\frac{3}{2},\ 1;\ \frac{3}{2},\ 1$	1	

$Y = 2\quad I = 2$

$I_1,\ Y_1;\ I_2,\ Y_2$	35	μ_γ
$\frac{3}{2},\ 1;\ \frac{3}{2},\ 1$	-1	

$Y = 2\quad I = 1$

$I_1,\ Y_1;\ I_2,\ Y_2$	27	μ_γ
$\frac{3}{2},\ 1;\ \frac{3}{2},\ 1$	1	

$Y = 2\quad I = 0$

$I_1,\ Y_1;\ I_2,\ Y_2$	10*	μ_γ
$\frac{3}{2},\ 1;\ \frac{3}{2},\ 1$	-1	

$Y = 1\quad I = \frac{5}{2}$

$I_1,\ Y_1;\ I_2,\ Y_2$	28	35	μ_γ
$\frac{3}{2},\ 1;\ 1,\ 0$	$\sqrt{2}/2$	$\sqrt{2}/2$	
$1,\ 0;\ \frac{3}{2},\ 1$	$\sqrt{2}/2$	$-\sqrt{2}/2$	

$Y = 1\quad I = \frac{3}{2}$

$I_1,\ Y_1;\ I_2,\ Y_2$	35	27	μ_γ
$\frac{3}{2},\ 1;\ 1,\ 0$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	
$1,\ 0;\ \frac{3}{2},\ 1$	$-\sqrt{2}/2$	$\sqrt{2}/2$	

$Y = 1\quad I = \frac{1}{2}$

$I_1,\ Y_1;\ I_2,\ Y_2$	27	10*	μ_γ
$\frac{3}{2},\ 1;\ 1,\ 0$	$\sqrt{2}/2$	$\sqrt{2}/2$	
$1,\ 0;\ \frac{3}{2},\ 1$	$\sqrt{2}/2$	$-\sqrt{2}/2$	

$Y = 0\quad I = 2$

$I_1,\ Y_1;\ I_2,\ Y_2$	28	35	27	μ_γ
$\frac{3}{2},\ 1;\ \frac{1}{2},\ -1$	$\sqrt{5}/5$	$\sqrt{2}/2$	$\sqrt{30}/10$	
$\frac{1}{2},\ -1;\ \frac{3}{2},\ 1$	$\sqrt{5}/5$	$-\sqrt{2}/2$	$\sqrt{30}/10$	
$1,\ 0;\ 1,\ 0$	$\sqrt{15}/5$	0	$-\sqrt{10}/5$	

$Y = 0\quad I = 1$

$I_1,\ Y_1;\ I_2,\ Y_2$	35	27	10*	μ_γ
$\frac{3}{2},\ 1;\ \frac{1}{2},\ -1$	$-\sqrt{6}/6$	$-\sqrt{2}/2$	$-\sqrt{3}/3$	
$\frac{1}{2},\ -1;\ \frac{3}{2},\ 1$	$-\sqrt{6}/6$	$\sqrt{2}/2$	$-\sqrt{3}/3$	
$1,\ 0;\ 1,\ 0$	$-\sqrt{6}/3$	0	$\sqrt{3}/3$	

$Y = 0\quad I = 0$

$I_1,\ Y_1;\ I_2,\ Y_2$	27	μ_γ
$1,\ 0;\ 1,\ 0$	1	

$Y = -1\quad I = \frac{3}{2}$

$I_1,\ Y_1;\ I_2,\ Y_2$	28	35	27	10*	μ_γ
$\frac{3}{2},\ 1;\ 0,\ -2$	$\sqrt{5}/10$	1/2	$3\sqrt{5}/10$	1/2	
$0,\ -2;\ \frac{3}{2},\ 1$	$\sqrt{5}/10$	$-1/2$	$3\sqrt{5}/10$	$-1/2$	
$1,\ 0;\ \frac{1}{2},\ -1$	$3\sqrt{5}/10$	1/2	$-\sqrt{5}/10$	$-1/2$	
$\frac{1}{2},\ -1;\ 1,\ 0$	$3\sqrt{5}/10$	$-1/2$	$-\sqrt{5}/10$	1/2	

$Y = -1\quad I = \frac{1}{2}$

$I_1,\ Y_1;\ I_2,\ Y_2$	35	27	μ_γ
$1,\ 0;\ \frac{1}{2},\ -1$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	
$\frac{1}{2},\ -1;\ 1,\ 0$	$-\sqrt{2}/2$	$\sqrt{2}/2$	

$Y = -2\quad I = 1$

$I_1,\ Y_1;\ I_2,\ Y_2$	28	35	27	μ_γ
$1,\ 0;\ 0,\ -2$	$\sqrt{5}/5$	$\sqrt{2}/2$	$\sqrt{30}/10$	
$0,\ -2;\ 1,\ 0$	$\sqrt{5}/5$	$-\sqrt{2}/2$	$\sqrt{30}/10$	
$\frac{1}{2},\ -1;\ \frac{1}{2},\ -1$	$\sqrt{15}/5$	0	$-\sqrt{10}/5$	

$Y = -2\quad I = 0$

$I_1,\ Y_1;\ I_2,\ Y_2$	35	μ_γ
$\frac{1}{2},\ -1;\ \frac{1}{2},\ -1$	-1	

$Y = -3\quad I = \frac{1}{2}$

$I_1,\ Y_1;\ I_2,\ Y_2$	28	35	μ_γ
$\frac{1}{2},\ -1;\ 0,\ -2$	$\sqrt{2}/2$	$\sqrt{2}/2$	
$0,\ -2;\ \frac{1}{2},\ -1$	$\sqrt{2}/2$	$-\sqrt{2}/2$	

$Y = -4\quad I = 0$

$I_1,\ Y_1;\ I_2,\ Y_2$	28	μ_γ
$0,\ -2;\ 0,\ -2$	1	

TABLE VI. Isoscalar factors for {10} ⊗ {10*}. Given are the isoscalar factors

$$\left(\begin{matrix} 10 & 10^* \\ I_1 Y_1 & I_2 Y_2 \end{matrix}\middle| \begin{matrix} \mu_\gamma \\ I\ Y \end{matrix}\right)$$

for the CG series {10} ⊗ {10}* = {64} ⊕ {27} ⊕ {8} ⊕ {1}.

$Y = 3 \quad I = \frac{3}{2}$

$I_1, Y_1; I_2, Y_2$	64	μ_γ
$\frac{3}{2}$, 1; 0, 2	1	

$Y = 2 \quad I = 2$

$I_1, Y_1; I_2, Y_2$	64	μ_γ
$\frac{3}{2}$, 1; $\frac{1}{2}$, 1	1	

$Y = 2 \quad I = 1$

$I_1, Y_1; I_2, Y_2$	64	27	μ_γ
$\frac{3}{2}$, 1; $\frac{1}{2}$, 1	$\sqrt{21}/7$	$2\sqrt{7}/7$	
1, 0; 0, 2	$2\sqrt{7}/7$	$-\sqrt{21}/7$	

$Y = 1 \quad I = \frac{5}{2}$

$I_1, Y_1; I_2, Y_2$	64	μ_γ
$\frac{5}{2}$, 1; 1, 0	1	

$Y = 1 \quad I = \frac{3}{2}$

$I_1, Y_1; I_2, Y_2$	64	27	μ_γ
$\frac{5}{2}$, 1; 1, 0	$\sqrt{14}/7$	$\sqrt{35}/7$	
1, 0; $\frac{1}{2}$, 1	$\sqrt{35}/7$	$-\sqrt{14}/7$	

$Y = 1 \quad I = \frac{1}{2}$

$I_1, Y_1; I_2, Y_2$	64	27	8	μ_γ
$\frac{3}{2}$, 1; 1, 0	$\sqrt{7}/7$	$4\sqrt{35}/35$	$\sqrt{10}/5$	
1, 0; $\frac{1}{2}$, 1	$2\sqrt{7}/7$	$\sqrt{35}/35$	$-\sqrt{10}/5$	
$\frac{1}{2}$, −1; 0, 2	$\sqrt{14}/7$	$-3\sqrt{70}/35$	$\sqrt{5}/5$	

$Y = 0 \quad I = 3$

$I_1, Y_1; I_2, Y_2$	64	μ_γ
$\frac{3}{2}$, 1; $\frac{3}{2}$, −1	1	

$Y = 0 \quad I = 2$

$I_1, Y_1; I_2, Y_2$	64	27	μ_γ
$\frac{3}{2}$, 1; $\frac{3}{2}$, −1	$\sqrt{7}/7$	$\sqrt{42}/7$	
1, 0; 1, 0	$\sqrt{42}/7$	$-\sqrt{7}/7$	

$Y = 0 \quad I = 1$

$I_1, Y_1; I_2, Y_2$	64	27	8	μ_γ
$\frac{3}{2}$, 1; $\frac{3}{2}$, −1	$\sqrt{21}/21$	$\sqrt{14}/7$	$\sqrt{6}/3$	
1, 0; 1, 0	$\sqrt{210}/21$	$3\sqrt{35}/35$	$-2\sqrt{15}/15$	
$\frac{1}{2}$, −1; $\frac{1}{2}$, +1	$\sqrt{210}/21$	$-4\sqrt{35}/35$	$\sqrt{15}/15$	

$Y = 0 \quad I = 0$

$I_1, Y_1; I_2, Y_2$	64	27	8	1	μ_γ
$\frac{3}{2}$, 1; $\frac{3}{2}$, −1	$\sqrt{35}/35$	$\sqrt{210}/35$	$\sqrt{10}/5$	$\sqrt{10}/5$	
1, 0; 1, 0	$2\sqrt{105}/35$	$\sqrt{70}/14$	0	$-\sqrt{30}/10$	
$\frac{1}{2}$, −1; $\frac{1}{2}$, 1	$3\sqrt{70}/35$	$-\sqrt{105}/35$	$-\sqrt{5}/5$	$\sqrt{5}/5$	
0, −2; 0, 2	$2\sqrt{35}/35$	$-3\sqrt{210}/70$	$\sqrt{10}/5$	$-\sqrt{10}/10$	

$Y = -1 \quad I = \frac{5}{2}$

$I_1, Y_1; I_2, Y_2$	64	μ_γ
1, 0; $\frac{3}{2}$, −1	1	

$Y = -1 \quad I = \frac{3}{2}$

$I_1, Y_1; I_2, Y_2$	64	27	μ_γ
1, 0; $\frac{3}{2}$, −1	$\sqrt{14}/7$	$\sqrt{35}/7$	
$\frac{1}{2}$, −1; 1, 0	$\sqrt{35}/7$	$-\sqrt{14}/7$	

$Y = -1 \quad I = \frac{1}{2}$

$I_1, Y_1; I_2, Y_2$	64	27	8	μ_γ
1, 0; $\frac{3}{2}$, −1	$\sqrt{7}/7$	$4\sqrt{35}/35$	$\sqrt{10}/5$	
$\frac{1}{2}$, −1; 1, 0	$2\sqrt{7}/7$	$\sqrt{35}/35$	$-\sqrt{10}/5$	
0, −2; $\frac{1}{2}$, 1	$\sqrt{14}/7$	$-3\sqrt{70}/35$	$\sqrt{5}/5$	

$Y = -2 \quad I = 2$

$I_1, Y_1; I_2, Y_2$	64	μ_γ
$\frac{1}{2}$, −1; $\frac{3}{2}$, −1	1	

$Y = -2 \quad I = 1$

$I_1, Y_1; I_2, Y_2$	64	27	μ_γ
$\frac{1}{2}$, −1; $\frac{3}{2}$, −1	$\sqrt{21}/7$	$2\sqrt{7}/7$	
0, −2; 1, 0	$2\sqrt{7}/7$	$-\sqrt{21}/7$	

$Y = -3 \quad I = \frac{3}{2}$

$I_1, Y_1; I_2, Y_2$	64	μ_γ
0, −2; $\frac{3}{2}$, −1	1	