The Structure of Light Nuclei*

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CONTENTS

I. INTRODUCTION

THE search for empirical regularities in the systematics of stable nuclei dates back to Aston's (A19)[1] early mass spectograph measurements. A number of stability rules formulated by Harkins (cf. H31 for a summary as well as references to earlier work) were complicated by the nuclear model popular at that time, in which the nucleus was regarded as a collection of A protons whose charges were partially balanced by $A - Z$ electrons.

Shortly following Chadwick's (C32) detection of the neutron and its acceptance as an elementary particle, the neutron–proton nuclear model was advanced independently by Heisenberg (H32) and Iwanenko (I32). Physicists soon began to speculate on the nature of a neutron–proton interaction (H32a, M33, H33) that could produce the binding of the deuteron and more complex nuclei (W33, F35, F35a).

The basic nuclear problem was then defined. The nucleus consists of A interacting particles (Z protons and $N = A - Z$ neutrons) of approximately equal mass, whose description should be given by solving Schrödinger's equation[2]

$$-\frac{\hbar^2}{2M}\sum_{i=1}^{A}\nabla_i^2\psi + \sum_{i<j}V_{ij}\psi = E\psi \qquad \text{(I.1)}$$

where V_{ij} represents the interaction between the ith and jth nucleons. Our problem may now be seen to be twofold: (1) To find the V_{ij} explicitly; (2) from this interaction operator deduce the physical characteristics of the various nuclear systems in order to compare them with experiment. The natural procedure is to start by determining V_{ij} through analysis of the two-body problem and proceed to step 2. Regrettably, the nucleon–nucleon interaction is sufficiently complicated that the two-body problem provides insufficient information as to its detailed character, and one must study complex nuclei in step 1 as well as step 2. In addition, even if V_{ij} were explicitly known, the methods for solving Eq. (I.1) for a complex system are not trivial.

Consequently, the nuclear theorist must play a double game with Eq. (I.1), and two very general procedures have become popular. The first method involves assuming an interaction operator (or selecting one from meson theory), solving Eq. (I.1) in a reasonably straightforward manner in order to derive the physical properties of the nuclear systems, and finally checking the results against experiment. This method is less useful for complex nuclei ($A \gtrsim 4$) where "reasonable" methods of computation become open to question. The second procedure, often used in complex nuclei, is to make certain simplifying assumptions concerning the nature of the wave function ψ. In clearer language, one constructs a model which possesses the salient physical properties of nuclear systems. One then attempts to justify this model in terms of the interaction operator, and thus, in turn, from the model learn something of the important features of nuclear forces.

Before proceeding we must note that writing Eq. (I.1) is in itself assuming a certain model, generally designated as the "two-body interaction model." We suppress, in this model, consideration of many-body forces (P39). Such neglect is justifiable on three counts. No evidence points to a necessity of introducing such forces. Though the two-body interaction model hardly yields 100% agreement with observation at this time, agreement is good enough to warrant continued work with two-body forces alone. Recent calculations on infinite nuclear matter (B57) with two-body interactions give reasonable results in the light of our knowledge of the interior of heavy nuclei. If many-body forces were needed anywhere, one would expect them to be essential in the nuclear

* Supported in part by grants from the National Science Foundation and the Research Council of the University of Nebraska.

[1] See the bibliography that appears at the end of this paper.

[2] Relativistic corrections are ~5% in H^2, and slightly more in heavier nuclei. The theory is not yet sufficiently refined to warrant such corrections, and they are usually not invoked.

many-body problem. Lastly, the introduction of many-body forces would introduce so many new parameters into the problem that one could fit all of the data easily, but such fits would be of dubious significance. Hence, although many-body forces may someday be required by meson theory, it is very doubtful that they will ever be introduced phenomenologically.

Prior to the neutron–proton model it had been suggested that the protons might, to a reasonable approximation, occupy independent orbitals (B30) in much the same manner that the extranuclear electrons occupy individual orbitals in the Hartree model of the atom. In the neutron–proton model irregular periodicities reminiscent of shell structure are not too difficult to recognize (G34). Bartlett (B32, B32a) recorded evidence of the existence of s, p, and d shells in light nuclei, and recognized that the $1p$ shell is being completed in the region between $A = 4$ and $A = 16$.

It was soon recognized that nuclei with certain numbers of protons or neutrons (magic numbers) possessed unusual stability and abundance (E34, G34a). The set of magic numbers generally accepted today are

$$N \text{ or } Z = 2, 6, 8, 14, 20, 28, 50, 82, \text{and } 126. \quad \text{(I.2)}$$

We shall not review the evidence for these numbers in its entirety, but rather refer the reader to one of the excellent books (F55, M55) on this subject for a detailed discussion. The major points are briefly as follows:

1. Stability and abundance. Nuclei with N or Z magic have unusually large binding energies and are correspondingly highly abundant. If N and Z are both magic ($_2He_2$, $_6C_6$, $_8O_8$, $_{14}Si_{14}$, $_{20}Ca_{20}$, $_{20}Ca_{28}$, and $_{82}Pb_{126}$) this stability is all the greater. In particular, the last nucleon to complete a magic number is very tightly bound. Furthermore, nuclei with $Z(N)$ magic possess an unusually large number of stable isotopes (isotones).

2. Neutron capture cross sections. Nuclei with neutron number one short of a magic number exhibit a large neutron capture cross section, while nuclei with a magic number of neutrons have a small neutron capture cross section.

3. *Islands of isomerism.* Long-lived isomers (half-life near one second) appear just prior to the completion of a magic number of protons or neutrons in the periodic table.

4. Electric quadrupole moments. The nuclear quadrupole moments tend to be small near a magic number and large far from the magic numbers. This indicates that a nucleus with N or Z magic is not easily deformed from a spherical shape.

5. Delayed neutron emission. Delayed neutron emitters ($_{36}Kr_{51}$, $_{54}Xe_{83}$, and $_8O_9$) occur when N is one greater than a magic number, emphasizing the instability of the last neutron in such nuclei.

The data clearly point toward an interpretation of the magic numbers in terms of shell closures, analogous to the noble gases of atomic physics.

The primitive nuclear model therefore became an independent particle model, in which each nucleon was to be considered to move independently of every other nucleon (at least to a first approximation) and definite assignments of single-particle orbitals could be made. Such a nuclear model lacks the intuitive justification which impelled the corresponding atomic model to success. Electron motion in the atom is dominated to a fair approximation by the strong central field of the nucleus. Nuclear interactions are generated by the nucleons themselves. The nucleon density is relatively high and the forces basically attractive (though the existence of an infinite repulsion at short distances is strongly suspected). In Sec. V we will discuss the theoretical justification of the independent-particle model in terms of nuclear forces. Until then we simply accept the model as an interesting hypothesis which yields striking confirmation with observations.

Much emphasis in the 1930's was placed on the development of the explicit nature of the nucleon-nucleon interaction. Construction of variational trial wave functions in terms of determinants of single-particle orbitals provided a method of checking nuclear forces against the properties of nuclei (particularly in the range $4 < A \leqslant 16$), and so the independent-particle model found much use. The effect of an inverted doublet splitting on nuclear levels was treated, but not considered a major effect because the Thomas term (the predominant source of such a splitting known at the time) was not large (I36, D36, B38).

Despite the emphasis on shell structure in the 1930's the model did not enjoy overwhelming success at that time. Although the shell model had many qualitative features in agreement with observations, quantitative comparisons were found to lack acceptable accuracy. For example, one could only obtain about 70% of the binding energy of nuclei in the $1p$ shell.

At the same time alternate nuclear models, whose basic philosophy was quite different from the shell model, had more success. The liquid-drop model (W35, B36) proved useful in describing nuclear re-

actions and explaining the trend of the nuclear stability curve. The α-particle model held promise since it appeared to explain numerous features of nuclei (binding energy per α bond was nearly constant for nuclei with an integral number of α particles, α activity of heavier nuclei was a natural consequence, and some success was had in explaining excited states) and if adopted would reduce the complexity of calculations for medium weight nuclei. Both the liquid-drop and the α-particle models imply that strong correlations between particle motions are important in nuclei, contrary to the assumptions of the independent-particle model.

We shall see later that the existence of collective effects in nuclear motions is not a cogent argument against shell structure. Justification of the shell model was long in coming, however, and interest in this subject dwindled until Mayer (M49) and Haxel, Jensen, and Suess (H49) emphasized the importance of jj coupling.

A major requirement of the shell model is that it reproduce the magic numbers. Shell closures at 2, 8, and 20 are predicted by simple central wells which are relatively flat near the origin such as the harmonic oscillator (see Table I-1) or square well (F55). The magic numbers 50 and 82 may be obtained by including a central elevation (E34, F49) in the potential well which increases with increasing A.

The jj coupling model reproduces the magic numbers in a striking manner. If one adds to the harmonic oscillator potential a spin-orbit interaction:

$$V_{\text{s.o.}}(r)\mathbf{s}\cdot\mathbf{l} \tag{I.3}$$

the levels are split as shown in Table I-2. Diagonal

TABLE I-1. Level order in harmonic oscillator shells.

Oscillator shells	l	Number of particles In shell	In the oscillator well
I	0	2	2
II	1	6	8
III	2,0	12	20
IV	3,1	20	40
V	4,2,0	30	70
VI	5,3,1	42	112
VII	6,4,2,0	56	168

matrix elements of the spin-orbit term in single-particle states may be evaluated by the relation

$$(l\,\tfrac{1}{2}\,jm|\mathbf{l}\cdot\mathbf{s}|l\,\tfrac{1}{2}\,jm) = -W(jl\,\tfrac{1}{2}\,1;\,\tfrac{1}{2}\,l) \times (\tfrac{1}{2}\,||\mathbf{s}||\,\tfrac{1}{2})(l||\mathbf{l}||l)\,, \tag{I.4}$$

where

$$(\tfrac{1}{2}\,||\mathbf{s}||\,\tfrac{1}{2}) = \sqrt{2}[C^{1/2\ 1\ 1/2}_{1/2\ 0\ 1/2}]^{-1}\,(\tfrac{1}{2}\,\tfrac{1}{2}\,|s_z|\,\tfrac{1}{2}\,\tfrac{1}{2}) = (\tfrac{3}{2})^{1/2} \tag{I.4a}$$

$$\begin{aligned}(l||\mathbf{l}||l) &= (2l+1)^{1/2}\,[C^{l1l}_{l0l}]^{-1}\,(ll|l_z|ll)\\ &= [l(l+1)(2l+1)]^{1/2}\,.\end{aligned} \tag{I.4b}$$

The Racah function $W(abcd;ef)$ and the vector coupling coefficients C^{abc}_{def} are treated in detail in several modern texts (W59, E57a, F59). Evaluation of the Racah function yields:

TABLE I-2. Level order in oscillator shells with an added $\mathbf{l}\cdot\mathbf{s}$ interaction.

Nuclear shells	l_j	Number of particles In shell	In nucleus
I	$s_{1/2}$	2	2
II	$p_{3/2}$	4	6
III	$p_{1/2}$	2	8
IV	$d_{5/2}$	6	14
V	$s_{1/2},d_{3/2}$	6	20
VI	$f_{7/2}$	8	28
VII	$p_{3/2},p_{1/2},f_{5/2},g_{9/2}$	22	50
VIII	$d_{5/2},s_{1/2},d_{3/2},g_{7/2},h_{11/2}$	32	82
IX	$h_{9/2},f_{7/2},f_{5/2},p_{3/2},p_{1/2},i_{13/2}$	44	126

$$\begin{aligned}(l\,\tfrac{1}{2}\,jm|\mathbf{l}\cdot\mathbf{s}|l\,\tfrac{1}{2}\,jm) &= \tfrac{1}{2}\,[j(j+1) - l(l+1) - \tfrac{3}{4}]\\ &= l/2 \qquad \text{if } j = l + \tfrac{1}{2}\\ &= -(l+1)/2 \text{ if } j = l - \tfrac{1}{2}\,,\end{aligned} \tag{I.4c}$$

Far more obvious methods of deriving Eq. (I.4c) are usually employed. We have inserted the more general procedure because later, when we consider the origin of the $\mathbf{l}\cdot\mathbf{s}$ term from two-body interactions, it is essential for maximum clarity.

We see that the magic numbers immediately appear when $V_{\text{s.o.}}(r)$ is given a sign so that our spin-orbit term is attractive in $j = l + \frac{1}{2}$ states. In addition to yielding the magic numbers, the jj coupling model predicts spins and parities throughout the periodic table with impressive accuracy as well as giving a fair insight into the sign and magnitude of magnetic–dipole moments and the sign of electric–quadrupole moments. For a detailed comparison of shell-model predictions with known nuclear states we once more refer to established references (F55, M55). Tables I-3 and I-4 contain nuclear data for light nuclei. Magnetic moments may be compared to the single-particle values (Schmidt lines):

$$\mu_{j=l+1/2} = (j - \tfrac{1}{2})g_l + \tfrac{1}{2}\,g_s \tag{I.5a}$$

TABLE I-3. Angular momenta, parities, moments, and probable shell-model configurations for the ground states of stable nuclei in the $1p$ shell.

Element	$J\pi$	μ	Q	neutron configuration $1s_{1/2}$	$1p_{3/2}$	$1p_{1/2}$	proton configuration $1s_{1/2}$	$1p_{3/2}$	$1p_{1/2}$
$_3Li_3$	1^+	0.82	0.01	2	1		2	1	
$_3Li_4$	$3/2^-$	3.26		2	2		2	1	
$_4Be_5$	$3/2^-$	−1.18	0.02	2	3		2	2	
$_5B_5$	3^+	1.80		2	3		2	3	
$_5B_6$	$3/2^-$	2.69	0.05	2	4		2	3	
$_6C_6$	0^+	0	0	2	4		2	4	
$_6C_7$	1/2	0.70		2	4	1	2	4	
$_7N_7$	1^+	0.40	0.16	2	4	1	2	4	1
$_7N_8$	$1/2^-$	−0.28		2	4	2	2	4	1
$_8O_8$	0^+	0	0	2	4	2	2	4	2

$$\mu_{j=l-1/2} = [j/(j+1)][(j+\tfrac{3}{2})g_l - \tfrac{1}{2}g_s], \quad \text{(I.5b)}$$

while single-particle estimates for the electric–quadrupole moment are

$$\langle jj|Q_{op}|jj\rangle = -\frac{(2j-1)}{2(j+1)}\langle r^2\rangle . \quad \text{(I.5c)}$$

Some obvious features of Tables I-3 and I-4 are worth repeating here. In the $1p$ shell jj coupling is not yet firmly established and the assigned configurations in Table I-3 cannot be considered to be more than a caricature of the physical states for most cases. Actually the spin-orbit term seems to build up in the $1p$ shell (I53) and exerts a dominant influence at $A \geqslant 15$.

Shell model predictions in the region $16 \leqslant A \leqslant 40$ appear to be reasonably valid. A glaring exception is F^{19} with a perfect $2s_{1/2}$ state in the middle of a $1d_{5/2}$ shell. In $_{11}Na_{12}$ and $_{10}Ne_{11}$ we note a curious tendency for three like nucleons in a j shell to couple to $J = j - 1$. Though irregularities such as these require explanation, they are not really inconsistent with shell structure. On the other hand, numerous irregularities in nuclei are immediately explained by the shell model. Consulting Table I-4 we note that argon does not possess an odd mass isotope (the only such element with $Z < 58$). Argon isotopes with expected stability on the basis of the neutron excess curve are Ar^{37} and Ar^{39}. In both cases the isobars Cl^{37} and K^{39} have greater binding due to the fact that they fill a neutron shell at $N = 20$ (a similar situation arises near $N = 82$ for $_{58}Ce$).

Mayer (M50) has emphasized the importance of a "pairing energy" which increases with increasing j. This effect may be described as a strong tendency for two like nucleons to couple to $J = 0$. Thus one finds, for example, that in the range $65 \leqslant N \leqslant 75$ the neutrons fill $1h_{11/2}$ orbitals in pairs while a single odd neutron goes into a $3s_{1/2}$ orbital. A similar situation arises in the range $111 \leqslant N \leqslant 125$ with competing $1i_{13/2}$ and $3p_{1/2}$ orbitals. The effect of pairing correlations in light nuclei is not as obvious, but, as we shall see in Sec. IV, this effect gives rise to the utility of a seniority classification of shell-model configurations which plays an important role in the theoretical interpretation of nuclear energy levels.

In spite of its initial success it soon became clear that the shell model was inadequate in describing

TABLE I-4. Angular momenta, parities, moments, and probable shell-model configurations for ground states of stable odd A nuclei in the $1d_{5/2}$ and $2s_{1/2}$ $1d_{3/2}$ shells (about a closed O^{16} core).

Element	$J\pi$	μ	Q	neutron configuration $1d_{5/2}$	$2s_{1/2}$	$1d_{3/2}$	proton configuration $1d_{5/2}$	$2s_{1/2}$	$1d_{3/2}$
$_8O_9$	$5/2^+$	−1.89	−0.005	1					
$_9F_{10}$	$1/2^+$	2.63		2				1	
$_{10}Ne_{11}$	$3/2^+$			3			2		
$_{11}Na_{12}$	$3/2^+$	−2.22	0.1	4			3		
$_{12}Mg_{13}$	$5/2^+$	−0.88		5			4		
$_{13}Al_{14}$	$5/2^+$	3.64	0.149	6			5		
$_{14}Si_{15}$	$1/2^+$	−0.55		6	1		6		
$_{15}P_{16}$	$1/2^+$	1.13		6	2		6	1	
$_{16}S_{17}$	$3/2^+$	0.64	−0.055	6	2	1	6	2	
$_{16}S_{19}$	$3/2^+$	1.00	0.038	6	2	3	6	2	
$_{17}Cl_{18}$	$3/2^+$	0.82	−0.078	6	2	2	6	2	1
$_{17}Cl_{20}$	$3/2^+$	0.68	−0.061	6	2	4	6	2	1
$_{19}K_{20}$	$3/2^+$	0.39		6	2	4	6	2	3

certain phenomena. Although the sign of the electric-quadrupole moments are generally given correctly, the shell model often predicts a value which is an order of magnitude smaller than that observed; experimental transition rates for $E2$ radiation are often correspondingly larger than those given by theory; and clearly no independent-particle model is capable of describing the fission process.

Consequently it is not found possible to attribute all nuclear properties to the nucleons in unfilled shells and one must investigate the collective motion of the particles constituting the nuclear core. Foldy and Milford (F50) and Bohr (B52) treated the nuclear core as an incompressible fluid, capable of surface oscillations, coupled to the motion of the nucleons in unfilled shells. The theory has been extended by Bohr and Mottelson (B53) with considerable improvement in the prediction of nuclear properties.

Collective effects manifest themselves principally in heavier nuclei. Indications exist, however, that weak collective effects produce essential corrections to the shell model in nuclei as light as O^{17}; while definite evidence for strongly coupled surface oscillations near aluminum have been found. In addition, the collective model is of interest in the study of light nuclei simply from the standpoint of the investigation of the relationships of all nuclear models to an exact description of nuclear structure.

II. NUCLEAR SIZE PARAMETERS

The earliest estimate of nuclear size was made, quite naturally, by Rutherford in his classic discovery of the atomic nucleus. The only conclusion that could be reached at the time was that the nuclear charge appeared to be contained within a sphere of radius $\sim 10^{-11}$cm.

As theoretical interpretations of nuclear data and the experimental techniques for accumulating it improved, considerable attention focused on determining the nuclear density distribution. Several methods were popular in the period 1930 to 1950. The theory of the penetration of charged particles (G28, C29) through a Coulomb barrier yielded information concerning the size of α radioactive nuclei, and stable nuclei where reaction yields initiated by charged particles were known. The scattering cross section for fast neutrons approaches just twice the geometric cross section:

$$\sigma = 2\pi R^2 \tag{II.1}$$

when the neutron wavelength is small compared to the nuclear radius. In mirror nuclei ($Z' = N$, $N' = Z$) the mass difference should be due to the Coulomb repulsion of the protons. If one assumes the protons to be uniformly distributed within a sphere of radius R, the Coulomb energy of Z protons is

$$E_c(Z) = 3Z(Z-1)e^2/5R\,. \tag{II.2}$$

The Coulomb energy difference between mirror pairs is then

$$\begin{aligned}\Delta(Z',Z) &= E_c(Z') - E_c(Z) \\ &= (3e^2/5R)[Z'(Z'-1) - Z(Z-1)]\,,\end{aligned} \tag{II.3}$$

from which one may estimate R if $\Delta(Z', Z)$ is known.

The above methods yield fairly consistent results[3] (see B52a for a summary) but possess a common defect. The nucleon distribution is not determined in detail. Only one parameter (R) describing the size may be discerned in the data. Consequently, one must begin the analysis by assuming that the form of the distribution is known. In addition, several of the methods (neutron scattering, reaction yields, and α decay) require some knowledge of the range of nuclear forces, and really measure the "nuclear force radius" (D55) rather than the radius of the nucleon distribution.

Many modern methods of determining nuclear size parameters employ particles which only interact with the nucleus electromagnetically. A prominent example is the scattering of electrons by nuclei. In Born approximation the electron differential cross section for scattering per unit solid angle (Ω) is given by

$$\sigma(\theta) = \left[\frac{Ze^2\cos\theta/2}{2E\sin^2\theta/2}\right]^2 |F|^2\,, \tag{II.4}$$

where F is the nuclear form factor for a spherically symmetric charge distribution (if the wavelength of the electrons is very short compared to the radius of the first Bohr orbit, the atomic form factor averages out):

$$F(q) = \frac{4\pi}{Z}\int_0^\infty \rho(r)\,\frac{\sin qr}{qr}\,r^2 dr\,, \tag{II.5}$$

q is the magnitude of the electron momentum change divided by $\hbar$:

$$q = (2E/\hbar c)\sin\theta/2\,, \tag{II.6}$$

E is the energy of the incident electron, and $\rho(r)$ is the nuclear charge density. The criterion for validity of the Born approximation in high energy ($v \cong c$)

[3] The data are fitted with a uniform distribution of nucleons within a sphere of radius $R = r_0 A^{1/3}$ where $r_0 \cong 1.45 \times 10^{-13}$ cm.

electron scattering is that $Z \ll 137$. This condition is not adequately fulfilled for heavy nuclei, where accurate phase-shift calculations must be performed, but in the region of very light nuclei ($Z \leqslant 8$) it will suffice.[4]

Hofstadter and his collaborators have performed electron scattering experiments on numerous nuclei, and extensive review papers on this work along with other determinations of nuclear size appear in the literature (H56, H57, H57a, E61). We shall confine our interest to the work on nuclei with $Z \leqslant 8$ where specific connections with shell-model parameters have been made.

Meyer-Berkout, Ford, and Green (M59) have made a detailed analysis of electron scattering experiments on nuclei in the $1p$ shell, and attempt to fit the charge distributions to a variety of reasonable functions, and Ehrenberg, Hofstadter, Meyer-Berkout, Ravenhall, and Sobottka (E59) have performed an extensive analysis on C^{12} and O^{16}. Several functions yield an acceptable fit to the data (a Fourier series of about 6 terms, an Hermite function of about 4 terms, a Fermi function, or an oscillator function). Of particular interest to us is the fit with harmonic oscillator functions, because of its direct connection with the shell model.

If one considers a harmonic oscillator well, with two protons in the $1s$ state and $Z-2$ protons in the $1p$ state, the charge density is given by

$$\rho_{\text{h.o.}}(\mathbf{r}) = \frac{2e}{\pi^{3/2}a^3}\left\{1 + \tfrac{1}{3}(Z-2)\frac{r^2}{a^2}\right\}\exp\left(-\frac{r^2}{a^2}\right), \tag{II.7}$$

where

$$a = (\hbar/m\omega)^{1/2} \tag{II.7a}$$

and $\hbar\omega$ is the oscillator well parameter. A good fit is obtained with this simple function for nuclei in the range $4 \leqslant A \leqslant 16$, with the exception of Li^6. The rms radius of this charge distribution is given by

$$(\langle r^2\rangle)^{1/2} = a[(5Z-4)/2Z]^{1/2}. \tag{II.7b}$$

Numerical results for nuclei on which experiments have been performed are collected in Table II-1.

Two corrections should be applied to Eq. (II.7) before making a connection with shell-model parameters. The proton is not a point charge, and its extension is not negligible compared to the size of the nucleus. A Gaussian charge distribution for the proton

$$\rho_p(\mathbf{r}) = \pi^{-3/2}a_p^{-3}\exp(-r^2/a_p^2), \tag{II.8}$$

with rms radius

$$r_p = (3/2)^{1/2}a_p \cong 0.8(10^{-13})\text{ cm}, \tag{II.8a}$$

gives a reasonable (though not the best) fit to electron-scattering data and yields a particularly simple result when folded into $\rho_{\text{h.o.}}(r)$ to obtain the distribution of charge:

$$\rho_c(\mathbf{r}) = \int \rho_{\text{h.o.}}(\mathbf{r}')\rho_p(\mathbf{r}-\mathbf{r}')d\mathbf{r}' = \frac{2}{\pi^{3/2}(a^2+a_p^2)^{3/2}} \times\left\{1 + \frac{(Z-2)}{3}\left[\frac{3a_p^2}{2(a^2+a_p^2)} + \frac{a^2r^2}{(a^2+a_p^2)^2}\right]\right\} \times \exp[-r^2/(a^2+a_p^2)]. \tag{II.9}$$

Tassie and Barker (T58) have pointed out that in addition to this correction one must account for the fact that the center of mass of the nucleus does not coincide with the center of the shell-model potential well. If the levels of an oscillator well are filled in order of increasing energy, the motion of the center of mass within the well is described by (E55):

$$\psi_{\text{c.m.}} = (A/\pi^3a^6)^{1/4}\exp(-AR^2/2a^2), \tag{II.10}$$

where $R = A^{-1}\Sigma_i r_i$. To subtract out the center of mass motion one need only multiply the form factor corresponding to $\rho_c(\mathbf{r})$:

$$F_c(q) = \{1 - [(Z-2)/6Z]a^2q^2\}\exp[-(a^2+a_p^2)q^2/4] \tag{II.11}$$

by a factor $\exp(-q^2a^2/4A)$. The charge density then becomes (T58):

$$\rho'(\mathbf{r}) = \frac{2e}{\pi^{3/2}a_m^3} \times\left[1 + \frac{(Z-2)}{2}\left(1 - \frac{a^2}{a_m^2}\right) + \frac{(Z-2)}{3}\frac{a^2r^2}{a_m^4}\right] \times \exp(-r^2/a_m^2), \tag{II.12}$$

Table II-1. Shell-model parameters for light nuclei deduced from electron scattering (all distances are in units of 10^{-13} cm).

Element	$(\langle r^2\rangle)^{1/2}$	$(\langle r^2\rangle - r_p^2)^{1/2}$	a
He^4	1.61	1.44	1.32
Li^6	2.80	2.68	. . .
Be^9	2 20	2.05	1.51
B^{11}	2.25	2.10	1.50
C^{12}	2.40	2.27	1.58
N^{14}	2.45	2.32	1.59
O^{16}	2.65	2.52	1.71

[4] Born approximation is not valid near a diffraction minimum even in light nuclei. Since one knows just where and how it fails, however, one may still employ the simple relation in Eq. (II.4) without obtaining misleading charge distributions.

where

$$a_m^2 = [(A-1)/A]a^2 + a_p^2 . \qquad \text{(II.12a)}$$

The rms radius of the charge distribution is now given by

$$\langle r^2 \rangle = \tfrac{3}{2}[(A-1)/A]a^2 + [(Z-2)/Z]a^2 + r_p^2 . \qquad \text{(II.13)}$$

Values of a corresponding to acceptable rms radii are shown in Table II-1. These may be compared to those quoted in Table I of reference (H57) where finite proton size and center of mass corrections were not made. The difference is only $\sim 4\%$, because the two corrections act in opposite directions. We note a curious tendency for a to increase throughout the $1p$-shell. The large size of O^{16} is particularly surprising since this nucleus represents a doubly magic shell, and one generally tends to associate tighter binding with a smaller size.

In Li^6 the fit with an oscillator distribution function is so poor that assignment of an a value is not justified. The charge distribution in Li^6 may be fitted by assuming that the $1s$ and $1p$ nucleons move in different oscillator wells (B58, J60). The function

$$\rho(\mathbf{r}) = (2e/\pi^{3/2})\{b_s^{-3}\exp(-r^2/b_s^2) + (r^2/3b_p^2) \times \exp(-r^2/b_p^2)\} \qquad \text{(II.14)}$$

yields an acceptable fit with the parameters

$$b_s = 2.65 \times 10^{-13}\,\text{cm}, \quad b_p = 1.07 \times 10^{-13}\,\text{cm} \quad (B58) \qquad \text{(II.14a)}$$

or

$$b_s = 1.72 \times 10^{-13}\,\text{cm}, \; b_p = 2.14 \times 10^{-13}\,\text{cm} \quad (J60) . \qquad \text{(II.14b)}$$

The form factor corresponding to Eq. (II.14) used was corrected for the finite size of the proton and center-of-mass motion (J60):

$$F(q) = 2\exp[-q^2(\tfrac{5}{6}b_a^2 + a_p^2)/4] + (1 - b_p^2q^2/6) \times \exp[-q^2(\tfrac{5}{6}b_p^2 + a_p^2)/4] . \qquad \text{(II.14c)}$$

It was assumed that the center-of-mass correction is approximately similar in form to that in Eq. (II.11). The latter values (II.14b) would appear to be the more likely since it is hard to believe that the $1p$ protons move in a narrower well than the $1s$ protons. Jancovici (J56) has performed a variational calculation on O^{16} to determine to what extent the $1p$ nucleons deform the $1s$ shell. He obtained a large s-shell dilatation (in order to provide better overlap between $1s$ and $1p$ orbitals) in support of the parameters in Eq. (II.14b), although it is somewhat surprising that only two loosely bound $1p$ nucleons produce such a large dilatation of the closed s shell in Li^6.

Beyond O^{16} the nuclear charge distributions have generally been fit to a Fermi function. It is difficult to clearly differentiate between an oscillator distribution and a Fermi function in the range $16 \leqslant A \leqslant 40$ due to the insensitivity of the scattering to the charge near the origin. A central elevation or depression in $\rho(r)$ is consequently not easy to verify.

The electron-scattering experiments appear, therefore, to yield strong justification for the use of harmonic-oscillator wave functions in calculations on light nuclei, as well as an estimate of specific size parameters.

If one now returns to the old picture of nuclear size, in which the nucleons are uniformly distributed within a sphere of radius $R = r_0A^{1/3}$, one finds that a reasonable approximation is given by $r_0 \cong 1.25 \times 10^{-13}$ cm instead of the previously accepted value of $\cong 1.45 \times 10^{-13}$ cm. It is surprising, therefore, that good agreement was obtained for Coulomb energies in the old model, which should yield an estimate $\sim 15\%$ too low. Bethe and Bacher (B36a) long ago pointed out that Eq. (II.2) is too naive in that it neglects the effect of the Pauli exclusion principle. The nuclear wave function must be antisymmetric in the proton coordinates and this tends to keep the protons apart and reduce the Coulomb energy. For the time being, let us neglect the neutrons and consider the wave function to be a Slater determinant for the protons only:

$$\psi = (Z!)^{-1/2}|u_1u_2\cdots u_s| . \qquad \text{(II.15)}$$

The Coulomb energy is then given by

$$E_c(Z) = \int\cdots\int\psi^*\left(\sum_{i<j}\frac{e^2}{r_{ij}}\right)\psi\mathbf{dr}_1\mathbf{dr}_2\cdots\mathbf{dr}_Z = \tfrac{1}{2}Z(Z-1)\int\cdots\int\psi^*\left(\frac{e^2}{r_{12}}\right)\psi\mathbf{dr}_1\mathbf{dr}_2\cdots\mathbf{dr}_Z \qquad \text{(II.16)}$$

$$E_c(Z) = \tfrac{1}{2}\sum_{i,j}\int\cdots\int\left[u_i^*(1)u_j^*(2)\frac{e^2}{r_{12}}u_i(1)u_j(2) - u_i^*(1)u_j^*(2)\frac{e^2}{r_{12}}u_j(1)u_i(2)\right]\mathbf{dr}_1\mathbf{dr}_2 , \qquad \text{(II.16a)}$$

where the sum is permitted to go over i and j independently, since the self-energy term with $i = j$ cancels out. The first term in Eq. (II.16a) gives Eq. (II.2) (one must deduct the self-energy terms with $i = j$ if this term is considered alone). The

second term is the exchange term which estimates the reduction of the Coulomb energy due to the exclusion principle. Bethe and Bacher estimate the Coulomb energy from Eq. (II.16a) using plane waves, obtaining

$$E_c(Z) = (e^2/R)(\tfrac{3}{5} Z^2 - 0.460\, Z^{4/3}) . \quad \text{(II.17)}$$

Further investigations of the exchange term (S57, S58, S60), employing more realistic wave functions, indicate that the $Z^{4/3}$ dependence is approximately valid, but the constant must be reduced for light nuclei (S58) where a uniform proton distribution is not a good approximation.

Although Eq. (II.17) exhibits the general trend of Coulomb energies, shell-model effects appear in the data which require more detailed calculations for specific nuclear systems in order to be understood. An alternation of the second differences in the Coulomb energy with odd–even Z is a symptom of the pairing correlation (C54), while the vanishing of second differences in the Coulomb energy may be attributed to major shell crossings (K58) (see Table II-2).

Feenberg and Goertzel (F46) obtained an expression for the Coulomb energy on the basis of supermultiplet theory which to a first approximation explains the odd–even alternation with Z. The total number of proton bonds is given by

$$N_t = \tfrac{1}{2} Z(Z-1) , \quad \text{(II.18)}$$

TABLE II-2. Coulomb energy differences and second differences for mirror nuclei. Energies are given in MeV.

Nucleus of larger Z	$\Delta(Z)$	$\Delta\Delta(Z)$
$_2He^3$	0.76	0.76
$_3Li^5$	0.80	0.04
$_4Be^7$	1.64	0.85
$_5B^9$	1.85	0.21
$_6C^{11}$	2.76	0.91
$_7N^{13}$	3.00	0.24
$_8O^{15}$	3.49	0.48
$_9F^{17}$	3.55	0.06
$_{11}Ne^{19}$	4.04	0.49
$_{11}Na^{21}$	4.30	0.27
$_{12}Mg^{23}$	4.89	0 57
$_{13}Al^{25}$	5.03	0.16
$_{14}Si^{27}$	5.60	0.57
$_{15}P^{29}$	5.77	+0.16
$_{16}S^{31}$	5.70	−0.07
$_{17}Cl^{33}$	5.94	+0.24
$_{18}Ar^{35}$	6.19	0.25

Of these

$$N_p = \tfrac{1}{2}\{Z + \tfrac{1}{2}[(-1)^Z - 1]\} \quad \text{(II.19)}$$

represent bonds between "paired" protons (that is, proton pairs where the spin of the pair couples to zero). The orbital part of the wave function for the N_p pairs is symmetric, while the remaining $N_t - N_p$ bonds are $\frac{3}{4}$ orbitally antisymmetric and $\frac{1}{4}$ symmetric. The Coulomb energy is then written

$$\begin{aligned} E_c(Z) &= (N_t - N_p)(\tfrac{3}{4} L_a + \tfrac{1}{4} L_s) + N_p L_s \\ &= N_t(\tfrac{3}{4} L_a + \tfrac{1}{4} L_s) + N_p(\tfrac{3}{4} L_s - \tfrac{3}{4} L_a) \\ &= \tfrac{1}{2} Z(Z-1)(\tfrac{3}{4} L_a + \tfrac{1}{4} L_s) \\ &\quad + \tfrac{1}{2}\{Z + \tfrac{1}{2}[(-1)^Z - 1]\}(\tfrac{3}{4} L_s - \tfrac{3}{4} L_a) , \end{aligned} \quad \text{(II.20)}$$

where L_s is the Coulomb energy of a symmetric bond and L_a the Coulomb energy of an antisymmetric bond.

For Coulomb energy differences in mirror pairs, Eq. (II.20) yields

$$\begin{aligned} \Delta(Z) = E_c(Z) - E_c(Z-1) &= (Z-1)(\tfrac{3}{4} L_a + \tfrac{1}{4} L_s) \\ &\quad + \tfrac{3}{8}[1 + (-1)^Z](L_s - L_a) \end{aligned} \quad \text{(II.20a)}$$

and for second differences

$$\begin{aligned} \Delta\Delta(Z) \equiv \Delta(Z) - \Delta(Z-1) &= (\tfrac{3}{4} L_a + \tfrac{1}{4} L_s) \\ &\quad + (-1)^Z(\tfrac{3}{4} L_s - \tfrac{3}{4} L_a) . \end{aligned} \quad \text{(II.20b)}$$

The alternation of second differences in mirror nuclei, displayed in Table II-2 is then qualitatively explained by Eq. (II.20b). Quantitative agreement could not be expected since the parameters L_s and L_a certainly must depend upon the specific nucleon orbitals under consideration. Furthermore, consulting Table II-1, it is seen that shell-model parameters for a definite orbital within a shell may undergo considerable variation from one nucleus to another (it is quite generally assumed that there is no difference in the parameters of mirror nuclei owing to the charge independence of nuclear forces).

Carlson and Talmi (C54) have revisited the effect of the pairing property on nuclear Coulomb energies employing jj-coupled harmonic oscillator wave functions. The possible effect of the neutrons is suppressed and they consider only the proton state of lowest seniority.[5] Reasonable agreement with electron scattering data is found in the region $6 \leqslant Z \leqslant 15$, but the theory was not satisfactory for nuclei with $A \leqslant 11$. Unna (U58) made a similar calculation using proper isobaric spin wave functions which include the effect of the neutrons, but found that this has a minor influence on the results.

Sengupta (S61) has pointed out that the pairing effect in Coulomb energy is significantly different in

[5] The seniority quantum number is discussed in detail in Sec. IV.

the jj and LS coupling schemes. After reworking the problem for selected nuclei in the $1p$ shell (Li^6, Be^9, B^{11}, and N^{14}) with LS-coupled oscillator wave functions he finds reasonable agreement with the results of the electron-scattering experiments for all cases except Be^9, where the electron scattering data is best fit with an rms radius of 2.26×10^{-13} cm, while the Coulomb energy difference for the mirror pair Be^9–B^9 is best fit with an rms radius of 2.48×10^{-13} cm.

The discrepancy at $A = 9$ may be due to the Thomas–Ehrman (E51, T51) effect. If the separation energy of the last nucleon is large the use of wave functions corresponding to an infinite potential well (such as the harmonic oscillator well) appears justified. For small separation energy, however, one must take into account the fact that the potential goes rapidly to zero beyond the nuclear radius and that this may strongly influence the wave functions. Consequently, one finds that Coulomb energies are reduced when calculated with wave functions for a finite well (this effect is particularly pronounced in s states). The separation energy for the last neutron in Be^9 is only 1.6 MeV, and the Thomas–Ehrman effect would appear to be applicable.

Further reduction in the Coulomb energy of nuclei could be caused by the finite size of the proton. Dalitz and Downs (D58) have estimated this effect in He^3, using the proton distribution in Eq. (II.8) and a simple s-state function for He^3:

$$\psi = N \exp\{-(r_{12}^2 + r_{13}^2 + r_{23}^2)/2R\} . \quad \text{(II.21)}$$

They then obtain for the Coulomb energy the expression

$$E_c(2) = \frac{e^2}{R}\left(\frac{2}{\pi}\right)^{1/2}\left[1 + \frac{2}{3}\frac{r_p^2}{R^2}\right]^{-1/2} \quad \text{(II.22)}$$

leading to a reduction in the Coulomb energy $\sim 10\%$. Ohmura and Ohmura (O61) have reported that when a repulsive core is included in the nucleon-nucleon interaction this effect is substantially reduced. Consequently, the reduction of Coulomb energies due to the finite size of the proton is probably not too important beyond the $1s$ shell.

III. THE 1s SHELL

In most attempts to calculate the properties of nuclear states, a completely rigorous solution of Eq. (I.1) is not attempted due to the complexity involved in treating a system composed of several interacting particles. In the first s shell (represented by the dynamically stable nuclei, $_1H^2$, $_1H^3$, $_2He^3$, and $_2He^4$), however, one is dealing with a sufficiently small number of nucleons so that calculations of reliable accuracy are attainable.

To accomplish such solutions one must start with an explicit nucleon–nucleon interaction operator. A completely acceptable interaction operator is not yet available, but much is known concerning its probable character from analysis of the two-body problem. We shall not review this analysis in detail (cf. H57b, M61, and G60 for a thorough discussion), but simply summarize the results in order to obtain a general form for the interaction:

$$\begin{aligned} V_{ij} = {} & P_0(i,j)J_0(r_{ij}) + P_1(i,j)J_1(r_{ij}) + P_2(i,j)J_2(r_{ij}) \\ & + P_3(i,j)J_3(r_{ij}) + [\tfrac{1}{4}(1 - \boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)J_4(r_{ij}) \\ & + \tfrac{1}{4}(3 + \boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)J_5(r_{ij})]S_{ij} + [\tfrac{1}{4}(1 - \boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)J_6(r_{ij}) \\ & + \tfrac{1}{4}(3 + \boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)J_7(r_{ij})]\mathbf{l}_{ij}\cdot(\boldsymbol{\delta}_i + \boldsymbol{\delta}_j) . \end{aligned} \quad \text{(III.1)}$$

The $P_k(i,j)$ are projection operators:

$$P_0(i,j) = \tfrac{1}{16}(1 - \boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j)(3 + \boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)$$
(singlet–even state), (III.1a)

$$P_1(i,j) = \tfrac{1}{16}(3 + \boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j)(1 - \boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)$$
(triplet–even state), (III.1b)

$$P_2(i,j) = \tfrac{1}{16}(1 - \boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j)(1 - \boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)$$
(singlet–odd state), (III.1c)

$$P_3(i,j) = \tfrac{1}{16}(3 + \boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)(3 + \boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j)$$
(triplet–odd state), (III.1d)

$\boldsymbol{\delta}$ is the Pauli spin operator, $\boldsymbol{\tau}$ the isobaric spin operator, and $\mathbf{l}_{ij}$ is the relative orbital angular momentum between particles i and j:

$$\hbar\mathbf{l}_{ij} = (\mathbf{r}_i - \mathbf{r}_j) \times (\mathbf{p}_i - \mathbf{p}_j) . \quad \text{(III.1e)}$$

The radial functions $J_k(r_{ij})$ are generally assumed to be of the form of a Yukawa, Gaussian, exponential, square-well, or some similar short-range ($\sim 10^{-13}$ cm) function. The fit to nucleon–nucleon scattering at intermediate energy is much improved if the potential becomes strongly repulsive at a distance near 0.4×10^{-13} cm, and good fits to the data have been made assuming a Yukawa radial dependence around an infinitely repulsive (hard) core of this radius[6] (G60).

The introduction of a hard core of radius c into the nucleon–nucleon interaction presents a nontrivial complication in solving Eq. (I.1) by imposing the

[6] Recently it has been suggested (S61a, C59) that a velocity dependent potential which becomes strongly repulsive at intermediate energies could fit the data and replace the "hard core" hypothesis. Extensive calculations on nuclei in the $1s$ shell with such an interaction have not yet been reported.

boundary condition

$$\Psi = 0 \quad \text{when} \quad r_{ij} \leqslant c \tag{III.2}$$

for any of the r_{ij}.

The noncentral components of the interaction reduce the variety of absolute quantum numbers, complicating the wave functions considerably. In fact, there are only three good quantum numbers left:

(1) J, the total angular momentum,
(2) T, the isobaric spin,
(3) π, the parity.

If one were dealing only with central forces, the orbital angular momentum L and spin S would also be good quantum numbers; and, in addition, if one deals only with Wigner and Majorana exchange interactions, the wave function would also be characterized by the partition quantum number [λ] (W59) for the positions of the particles.[7]

Although L, S, and [λ] are not good quantum numbers with realistic nuclear interactions, they are sometimes useful in the construction of nuclear wave functions. In the deuteron, for example, one has $J = 1+$ and $T = 0$. The wave function may be expanded into ${}^3S_1$ and ${}^3D_1$ components, consistent with the absolute quantum numbers (for a two-body system [λ] and π are redundant). Eliminating the center-of-mass motion by the transformation

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 , \quad \mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2 , \tag{III.3}$$

the Hamiltonian becomes

$$H = -(\hbar^2/Mr^2)[r(\partial^2/\partial r^2)r - L_{\text{op}}^2] + V(r) . \tag{III.4}$$

In the ground state of the deuteron, only triplet–even forces are effective, so that the interaction operator becomes

$$V(r) = J_1(r) + J_4(r)S_{12} + J_6(r)\mathbf{L}_{\text{op}}\cdot(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) . \tag{III.4a}$$

The wave function may then be expanded in the form

$$(\mathcal{Y}_{LSJM} = \sum C^{LSJ}_{\mu\mu' M} Y_{L\mu}\chi_{S\mu'}):$$
$$\psi = [u(r)/r]\mathcal{Y}_{0111} + [w(r)/r]\mathcal{Y}_{2111} . \tag{III.5}$$

Substitution of Eq. (III.5) into $H\psi = E\psi$ yields

$$-\frac{\hbar^2}{M}\left\{\frac{d^2u}{dr^2}\mathcal{Y}_{0111} + \frac{d^2w}{dr^2}\mathcal{Y}_{2111} - \frac{6w}{r^2}\mathcal{Y}_{2111}\right\}$$
$$+ J_1(r)\{u\mathcal{Y}_{0111} + w\mathcal{Y}_{2111}\} + J_4(r)\{\sqrt{8}u\mathcal{Y}_{2111}$$
$$+ [\sqrt{8}\mathcal{Y}_{0111} - 2\mathcal{Y}_{2111}]w\} + J_6(r)[-6w\mathcal{Y}_{2111}]$$
$$= E[u\mathcal{Y}_{0111} + w\mathcal{Y}_{2111}] , \tag{III.6}$$

where we have used the relationships

$$S_{12}\mathcal{Y}_{0111} = \sqrt{8}\,\mathcal{Y}_{2111} \tag{III.6a}$$

$$S_{12}\mathcal{Y}_{2111} = \sqrt{8}\,\mathcal{Y}_{0111} - 2\mathcal{Y}_{2111} \tag{III.6b}$$

$$\mathbf{L}_{\text{op}}\cdot(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\mathcal{Y}_{LSJM} = (J^2 - L^2 - S^2)\mathcal{Y}_{LSJM} . \tag{III.6c}$$

Utilizing the orthonormality of the $\mathcal{Y}_{LSJM}$, one obtains a pair of coupled differential equations for u and w:

$$-\frac{\hbar^2}{M}\frac{d^2u}{dr^2} + J_1u + \sqrt{8}\,J_4w = Eu \tag{III.7a}$$

$$-\frac{\hbar^2}{M}\left[\frac{d^2w}{dr^2} - \frac{6w}{r^2}\right] + J_1w + J_4(\sqrt{8}\,u - 2w)$$
$$- 6J_6w = Ew , \tag{III.7b}$$

subject to the normalization

$$\int_0^\infty (u^2 + w^2)dr = 1 . \tag{III.7c}$$

The effect of the tensor–even potential (J_4) is to couple the ${}^3S_1$ and ${}^3D_1$ states, which nicely explains the nonvanishing electric-quadrupole moment of the deuteron (R41) and makes the introduction of J_4 into the interaction operator a necessity. Rarita and Schwinger (R41) found that with a ${}^3D_1$ admixture of about 4%, one could obtain the correct magnetic–dipole moment of H^2 as well as its quadrupole moment. More recent calculations on the two-body system (G60) attempt to fit the rms radius (1.96×10^{-13} cm after finite proton size corrections are made) of H^2, now available from electron-scattering experiments. Such fits appear to require a ${}^3D_1$ state admixture of about 7%, leaving a serious discrepancy in the magnetic dipole moment. The velocity dependent spin-orbit interaction, found to be essential in interpreting polarization effects in proton–proton scattering at 310 MeV, produces a correction to the magnetic moment which yields a further discrepancy (F57b, S57a). Most estimates of corrections due to meson effects (H57b) appear to reduce agreement with experiment, and it seems that much work must

[7] The partition quantum number [λ] denotes the symmetry class of the function. If one, for example, considers a function of three variables, three symmetry classes are possible:

[λ] = [3] (one symmetric function):
$U([3]) = u(1)\,u(2)\,u(3)$,
[λ] = [111] (one antisymmetric function):
$U([111]) = \det|u(1)\,v(2)\,w(3)|$,
and
[λ] = [21] (two functions of intermediate symmetry):
$U_a([21]) = u(1)\,[u(2)\,v(3) - u(3)\,v(2)]$
$U_b([21]) = 2v(1)\,u(2)\,u(3) - u(1)\,[u(2)\,v(3) + u(3)\,v(2)]$

still be done before the magnetic moment of H^2 is fully understood.

If the radial functions $J_k(r)$ are nonsingular, Eqs. (III.7) may be solved by standard techniques of numerical integration. For example, one may start by assuming an eigenvalue E [$u(r)$ must behave $\sim r$ near the origin, hence, $u'(o)$ is known], a value for $u(o)$, and then simply use the coupled equations for u and w to calculate these functions out to their asymptotic forms. The eigenvalue E which properly connects the asymptotic solution [where the $J_k(r) = 0$] to the solution at the origin solves the equations. If a hard core is present, this procedure is complicated by the boundary condition in Eq. (III.2). Now one must also guess a value for $u'(c)$ in performing the iterative numerical integrations. Such a procedure is unsatisfactory, due to the fact that a guess for $u'(c)$ that is only slightly wrong produces a very poor asymptotic solution. A revised procedure has been formulated by Laurikainen and Varho (L59). They begin by solving the asymptotic equations with an assumed eigenvalue E [such a solution provides the slope of $u(r)$ and $w(r)$ where $r \gg 0$]. Then u and w are computed from Eqs. (III.7) back to the point where they vanish (say at $r = c'$). When $c' = c$ the assumed eigenvalue is the one which solves Eqs. (III.7).

Thus, computing the properties of the deuteron is reduced to a straightforward procedure which we shall now attempt to extend to the triton and the α particle. The triton quantum numbers ($J\pi = \frac{1}{2}+$, $T = 1/2$) are consistent with LS states $^2S_{1/2}$, $^2P_{1/2}$, $^4P_{1/2}$, and $^4D_{1/2}$; while those of the α particle ($J\pi = 0+$, $T = 0$) are compatible with $^1S_0$, $^3P_0$, and $^5D_0$. The problem is complicated by a substantial variety of linearly independent states for each L and S, over which some confusion exists in the literature (G42, S53). A very neat and systematic way to classify the states of the triton has been devised by Derrick and Blatt (D58), and we shall reproduce their argument here.

The key to a systematic formulation of the problem lies in selecting coordinates which display a simple behavior under permutation of the three particles. Derrick and Blatt select the three interparticle distances (forming the three sides of a triangle):

$$x_1 = r_{23}, \quad x_2 = r_{13}, \quad x_3 = r_{12} \tag{III.8}$$

and the three Euler angles ($\alpha\beta\gamma$) describing the orientation of this triangle in space (the three center-of-mass coordinates are, of course, eliminated).

It remains to select the normal orientation of the triangle (where $\alpha = \beta = \gamma = 0$). Obviously we may orient the triangle so that it lies in the xy plane in normal position, but a definition of the direction of the x and y axes in a manner which is invariant to the permutation of particles requires more thought. Two possibilities were seriously considered:

1. Compute the two principal moments of inertia of the triangle, and let the x axis coincide with the axis of the larger moment of inertia.

2. The x axis may be chosen as the invariant "Euler line" within the triangle (this is the line through the following three points: the center of gravity, the intersection of normals drawn from the vertices to the opposite sides, and the intersection of the normals drawn from the midpoints of the sides).

Derrick and Blatt adopt the first of these possibilities, because it makes their final expressions less complicated. This choice has the disadvantage that the x and y axes are only defined up to a $\pm$ sign, and some of the Euler-angle wave functions are double valued. This turns out to be a problem only in odd-parity states, however, and it is of no concern in considering the triton.

The orbital parts of the triton wave function are described by the Euler angles $\alpha\beta\gamma$ and are just the representation coefficients of the rotation group (W59, E57a, F59):

$$D^L_{\mu M L}(\alpha\beta\gamma) \tag{III.9}$$

where L is the orbital angular momentum, M_L its z component in the space-fixed axes, and μ its z component in the body axes of the triangle. The index μ labels different functions which cannot be obtained from each other by rotation of the space axes; consequently, there are $2L + 1$ distinct orbital states corresponding to each value of L. The parity of these states is $(-1)^\mu$, and thus in the triton we require the states where μ is even. This leads to orbital functions:

$$S \text{ state } ([\lambda] = [3]) = \sqrt{2}/4\pi \tag{III.10a}$$

$$P \text{ state } ([\lambda] = [111]) = (i\sqrt{6}/4\pi) D^1_{0M}(\alpha\beta\gamma) \tag{III.10b}$$

$$D \text{ state } ([\lambda] = [3]) = (\sqrt{10}/4\pi) D^2_{0M}(\alpha\beta\gamma) \tag{III.10c}$$

$$D \text{ state } ([\lambda] = [3]) = -(\sqrt{5}/4\pi) \times [D^2_{2M}(\alpha\beta\gamma) + D^2_{-2M}(\alpha\beta\gamma)] \tag{III.10d}$$

$$D \text{ state } ([\lambda] = [111]) = -i(\sqrt{5}/4\pi) \times [D^2_{2M}(\alpha\beta\gamma) - D^2_{-2M}(\alpha\beta\gamma)], \tag{III.10e}$$

each of which belongs to a definite symmetry class $[\lambda]$.

These orbital states must be vector coupled to spin-isobaric spin functions so that $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is a good quantum number. An analysis of the spin-isobaric spin functions for a three-nucleon system is displayed in Table III-1. We find that the following

TABLE III-1. Classification of spin-isobaric spin functions for three nucleons.

Weight	$(m_s m_\tau)_1$	$(m_s m_\tau)_2$	$(m_s m_\tau)_3$	ST	$[\lambda]$
1	$\frac{1}{2}\ \frac{1}{2}$	$\frac{1}{2}\ \frac{1}{2}$	$\frac{1}{2}\ \frac{1}{2}$	$(\frac{3}{2}\frac{3}{2})^1$	[3]
3	$\frac{1}{2}\ \frac{1}{2}$	$\frac{1}{2}\ \frac{1}{2}$	$\frac{1}{2}\ -\frac{1}{2}$	$(\frac{3}{2}\frac{1}{2})^2$	[21]
3	$\frac{1}{2}\ \frac{1}{2}$	$\frac{1}{2}\ -\frac{1}{2}$	$\frac{1}{2}\ \frac{1}{2}$	$(\frac{1}{2}\frac{3}{2})^2$	[21]
3	$\frac{1}{2}\ \frac{1}{2}$	$\frac{1}{2}\ -\frac{1}{2}$	$-\frac{1}{2}\ \frac{1}{2}$	$(\frac{1}{2}\frac{1}{2})^4$	[111] + [21]
6	$\frac{1}{2}\ \frac{1}{2}$	$-\frac{1}{2}\ \frac{1}{2}$	$\frac{1}{2}\ \frac{1}{2}$		+ [3]

spin states are consistent with $T = \frac{1}{2}$:

$S = \frac{1}{2}$:
one symmetric function,
one antisymmetric function,
one set of two functions with intermediate symmetry. (III.11a)

$S = \frac{3}{2}$:
one set of two functions of intermediate symmetry. (III.11b)

One can therefore construct a variety of independent spectroscopic states, each to be then combined with some function of the internal coordinates x_1, x_2, and x_3 in a fashion that makes the final wave function antisymmetric with respect to the interchange of any pair of nucleons. Since internal functions of any symmetry may be constructed, this

TABLE III-2. Classification of states for the triton.

Spectroscopic classification	Permutation symmetry: Internal	Euler angles	Spin-isobaric spin	$\lvert\mu\rvert$
$^2S_{1/2}$	[3]	[3]	[111]	0
$^2S_{1/2}$	[111]	[3]	[3]	0
$^2S_{1/2}$	[21]	[3]	[21]	0
$^2P_{1/2}$	[3]	[111]	[3]	0
$^2P_{1/2}$	[111]	[111]	[111]	0
$^2P_{1/2}$	[21]	[111]	[21]	0
$^4P_{1/2}$	[21]	[111]	[21]	0
$^4D_{1/2}$	[21]	[3]	[21]	0
$^4D_{1/2}$	[21]	[3]	[21]	2
$^4D_{1/2}$	[21]	[111]	[21]	2

places no restriction on the spectroscopic states consistent with the total angular momentum J. A summary of the states thus obtained is given in Table III-2.

We see that ten distinct functions exist for the triton. Hence, a rigorous treatment will involve solving a set of ten coupled differential equations (each involving the three variables x_1, x_2, and x_3) analogous to Eqs. (III.7) for the deuteron. This is too laborious a task to be practical even for modern electronic computers, and one must resort to a variational calculation for the internal wave functions. The spin-isobaric spin sums and the integrations over angles have been done by Derrick (D60, D60a), and a full calculation using the Brueckner–Gammel (B57) potential has been carried out by Derrick, Mustard, and Blatt (D61). They find slightly too much binding energy, and a Coulomb radius that is much too large; which would appear to invalidate the Brueckner–Gammel potential. Werntz (W61) has performed a similar calculation with the Gartenhaus potential, and found it too weak to bind the triton at all.

Cohen (C60, C61) has attempted an analogous program for the α particle. Coordinates similar to

TABLE III-3. Classification of states for the α particle.

Spectroscopic classification	Permutation symmetry: Internal	Spin-orbit	Isobaric spin
$^1S_0$	[4]	[22]	[22]
$^1S_0$	[22]	[22]	[22]
$^1S_0$	[1111]	[22]	[22]
$^3P_0$	[4]	[22]	[22]
$^3P_0$	[22]	[22]	[22]
$^3P_0$	[22]	[1111]	[22]
$^3P_0$	[31]	[211]	[22]
$^3P_0$	[31]	[31]	[22]
$^3P_0$	[211]	[31]	[22]
$^3P_0$	[211]	[211]	[22]
$^3P_0$	[1111]	[22]	[22]
$^5D_0$	[4]	[22]	[22]
$^5D_0$	[22]	[22]	[22]
$^5D_0$	[22]	[4]	[22]
$^5D_0$	[31]	[31]	[22]
$^5D_0$	[211]	[31]	[22]
$^5D_0$	[1111]	[22]	[22]

those of Derrick and Blatt may be used (C59a), but result in an extremely complicated kinetic energy operator. Consequently, Cohen employs

$$\mathbf{r} = \tfrac{1}{2}(\mathbf{r}_3 + \mathbf{r}_4 - \mathbf{r}_1 - \mathbf{r}_2)\,, \qquad \text{(III.12a)}$$

$$\boldsymbol{\varrho}_1 = (1/\sqrt{2})(\mathbf{r}_2 - \mathbf{r}_1)\,, \qquad \text{(III.12b)}$$

and

$$\boldsymbol{\varrho}_2 = (1/\sqrt{2})(\mathbf{r}_4 - \mathbf{r}_3)\,. \qquad \text{(III.12c)}$$

The results of his classification are given in Table III-3. Since seventeen functions are found one must once again be satisfied with a variational calculation. No such calculation has as yet been reported with a

modern interaction operator, possibly because of the failure of the Brueckner–Gammel and Gartenhaus potential to yield the observed properties of the triton.

If one attempts to extend this type of calculation to heavier nuclei, clearly the number of coupled functions will increase rapidly, making such extrapolations unfeasible. It seems desirable to formulate a treatment of the problem which generates the correct linear combination of functions automatically.

Bolsterli and Feenberg (B56) have devised a perturbation procedure, with this aim, which appears to be fairly accurate. The zero-order Hamiltonian is taken to be a sum of single-particle harmonic oscillator Hamiltonians with a uniform displacement in energy:

$$H_0 = \tfrac{1}{2}\hbar\omega \sum_{i=1}^{A} (p_i^2 + q_i^2) + U$$
$$= \sum_{i=1}^{A} H_{\text{osc}}(i) + U\,. \qquad \text{(III.13)}$$

The perturbation operator is

$$W = \sum_{i<j} V_{ij} - \tfrac{1}{2}\hbar\omega(\textstyle\sum_i q_i^2 - AQ^2) - U\,, \qquad \text{(III.14)}$$

where $\mathbf{Q} = A^{-1}\Sigma\mathbf{q}_i$ and the depth of U is adjusted so that $W_{00} = 0$. The center-of-mass term is subtracted out of W so that we will not mix states having different center-of-mass motion from that of the zero-order wavefunction. The second-order energy shift

$$E - E_0 = \sum_{n\neq 0} \frac{|W_{on}|^2}{E - E_n}\,, \qquad \text{(III.15)}$$

with the aid of the identities

$$\frac{1}{E - E_n} = -\int_0^\infty e^{\lambda(E-E_n)}d\lambda \qquad \text{(III.16)}$$

$$e^{-\lambda E_n}\psi_n = e^{-\lambda H_0}\psi_n \qquad \text{(III.17)}$$

may be expressed in the form:

$$E - E_0 = -\int_0^\infty e^{\lambda E}(Ve^{-\lambda H_0}V)_{00}d\lambda - \frac{|V_{00}|^2}{E - E_0}$$
$$+ (E - E_0 - 2\hbar\omega)^{-1}\,[(M^2)_{00} - (M_{00})^2$$
$$- 2(MV)_{00} + 2M_{00}V_{00}]\,, \qquad \text{(III.18)}$$

where

$$E_0 = V_{00} - M_{00} + (H_{\text{osc}})_{00} \qquad \text{(III.18a)}$$

and

$$M = \tfrac{1}{2}\hbar\omega(\textstyle\sum_i q_i^2 - AQ^2)\,. \qquad \text{(III.18b)}$$

The operator $e^{-\lambda H_0}$ may be evaluated by

$$\exp\,[-\mu(p^2 + q^2)]f(\mathbf{q}) = \left(\frac{k}{2\pi g}\right)^{3/2}\iiint f(\mathbf{v})$$
$$\times \exp\,[-(1/2\,g)(q^2 + v^2 - 2k\mathbf{q}\cdot\mathbf{v})]d\mathbf{v}\,, \qquad \text{(III.19)}$$

where $g = \tanh 2\mu$ and $k = \operatorname{sech} 2\mu$. The approximate eigenvalue E includes the energy associated with center-of-mass motion in the oscillator well, and consequently, one adjusts $\hbar\omega$ to minimize the internal energy of the physical system ($E - \frac{3}{2}\hbar\omega$).

This procedure has the advantage of involving matrix elements of only the zero-order wave function. Guided by the shell model, one may hope to be able to choose a zero-order configuration which yields a reasonable first approximation to the actual physical state, and thus the configuration mixing will be small and a perturbation expansion justified.[8] A simplified form of the interaction operator in Eq. (III.1) has been fitted to the properties of H^2, H^3, He^3, and He^4 (G59) and the calculation of third-order terms for H^2 indicate that, at least in the $1s$ shell, extension of the perturbation series beyond second order is not essential (G56, F57, G57).

A major difficulty in applying perturbation theory to the states of light nuclei appears when a repulsive core is incorporated into the interaction operator. Suppose, for example, that we must deal with a short-range repulsive term of the form

$$V_R \sum_{i<j} J_R(r_{ij}) \qquad \text{(III.20)}$$

and we wish to use second-order perturbation theory. The quadratic terms in V_R appearing in second order are attractive and if V_R is made sufficiently large, these terms will dominate the repulsive terms linear in V_R. Obviously, one must exercise great caution in applying perturbation expansions to strong repulsive interactions, and if a hard core is present, conventional perturbation theory is invalid.

Conventional perturbation expansions may be modified in a number of ways to remedy the situation.[9] Instead of the trial wave function used to derive the Brillouin–Wigner perturbation expansion

$$\Psi = \frac{1}{N}\left[\psi_0 + \sum_{n\neq 0}\frac{\psi_n W_{n0}}{E - E_n} + \cdots\right], \qquad \text{(III.21)}$$

[8] Conceivably, a perturbation expansion could converge slowly even if the zero-order function is a good approximation. Such a situation probably obtains in infinite nuclear matter where a large number of configurations are needed to describe the physical state. In very light nuclei ($A \gtrsim 16$), however, a perturbation calculation may be expected to be a reasonable approach to the problem (F58).

[9] Levinger *et al.* (L60) have treated the hard core within the framework of second-order perturbation theory employing a pseudopotential.

one may begin with the prescription

$$\Psi = \frac{1}{N} \prod_{i<j} s_{ij}(r_{ij}) \left[\psi_0 + \sum_{n \neq 0} \frac{\psi_n W'_{n0}}{E - E_n} + \cdots \right]. \tag{III.22}$$

In Eq. (III.22) W' may be any interaction operator (so long as it will not violate the symmetry and boundary conditions of the problem); for example, it may be taken to be the interaction operator of the problem with troublesome singular parts (such as the repulsive core) eliminated. Then only linear terms in the repulsive core appear in the perturbation expansion, and these may be simplified by introducing the correlation term s_{ij} which is zero when $r_{ij} \leqslant c$ and goes to unity rapidly as $r_{ij} \to \infty$ (C59b).

A revised form of the trial function in Eq. (III.22) has been applied to a variety of problems involving hard cores, including the deuteron problem with the Brueckner–Gammel potential (K62). Average energy denominators were used, and considerable liberty with the W' was taken in order to simplify the matrix elements. The trial function was taken to be of the form

$$\begin{aligned} \Psi = \sum_{n=0}^{N_0} r^n \{ & \lambda_n [1 - \exp k_n(r - c)] \exp a_n r + S_{12} \lambda'_n \\ & \times [1 - \exp k'_n(r - c)] \exp a'_n r \} \quad \text{if} \quad r \geqslant c \\ = 0 \, , & \qquad \text{if} \quad r \leqslant c \end{aligned} \tag{III.23}$$

where λ_n, λ'_n, k_n, k'_n, a_n, and a'_n are variational parameters. It was found necessary to take $N_0 = 1$ in order to obtain two figure accuracy in the energy eigenvalue.

Note added in proof. Recently, Blatt, Derrick, and Lyness (B62) have reported an error in the coding for the triton calculation described in this section (D61b). The corrected energy eigenvalue for H^3 is -5.7 MeV, with Coulomb radius 2.50×10^{-13} cm. Since the calculated eigenvalue no longer lies below the experimental binding energy, the conclusion that this work invalidates the Brueckner–Gammel potential is too strong.

IV. SHELL-MODEL CALCULATIONS

The nuclear shell model has achieved wide popularity among nuclear theorists owing to its easily visualizable character and the success in interpreting experimental results. The assumption that, to a first approximation, each nucleon moves in an average potential independent of the motion of other nucleons is an attractive one, due to one's familiarity with the Hartree–Fock theory of atomic structure. As we have pointed out in Sec. I, the *a priori* reasoning that made the independent-particle model reasonable in describing atomic spectra, is lacking in the nuclear problem. In the early days of the shell model it was proposed that the Pauli exclusion principle plays a vital role in that it prevents a nucleon from being scattered into nearby orbitals which are occupied by other nucleons. Only in recent years, however, has this conjecture been tested by actual calculations on nuclear systems.

In this section we describe calculations that begin by postulating the shell model, and within its framework, try to interpret the observed features of nuclei. No attempt will be made to justify the model, other than by comparing it with experiment.

1. The Single-Particle Model and the Single-Particle Functions

In the most primitive version of the shell model one neglects all residual interaction that may exist between nucleons in their individual orbits. No attempt is made to introduce correlations between particles except for the obviously essential points that the angular momenta must be vector coupled, so that J is a good quantum number, and the total wave function must be antisymmetric with respect to the interchange of any two nucleons. The ground-state coupling rules for angular momenta are very simple:

1. In even-even nuclei $J = 0$,
2. In odd A nuclei the total angular momentum is attributed to the last nucleon added to the shell ($J = j$),
3. The rule becomes somewhat more complicated for odd–odd nuclei where the last proton and the last neutron must be vector coupled. Nordheim (N50) has devised the criteria

a. $J = |j_p - j_n|$ if $j_p + j_n + l_p + l_n$ is even,

b. $J > |j_p - j_n|$ if $j_p + j_n + l_p + l_n$ is odd.

The single-particle orbitals are specified by

$$\psi_{nljm} = R_{nl}(r) \sum C^{l\,1/2\,j}_{\mu\mu'm} Y_{l\mu}(\theta,\varphi) \chi_{1/2\,\mu'}(\sigma) \, , \tag{IV.1}$$

in addition to an isobaric spin variable.

The radial wave function R_{nl} is generally taken to be the solution of the equation

$$\begin{aligned} -\frac{\hbar^2}{2M} \left[\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d}{dr} R_{nl} \right) - \frac{l(l+1)}{r^2} R_{nl} \right] \\ + [V_c(r) + V_{so}(r) \langle \boldsymbol{\sigma} \cdot \mathbf{l} \rangle - E_{nl}] R_{nl} = 0 \, . \end{aligned} \tag{IV.2}$$

It is frequently assumed that V_c is either a square well or infinite harmonic oscillator potential, since solutions for these potentials are readily available in

a simple form. Both potentials have the desirable features of being relatively flat near the center and rising rapidly near the edge of the nucleus. On the other hand, each has unrealistic characteristics. The walls of the oscillator well extend on up to infinity, so that the asymptotic form of the radial function $\sim \exp(-k r^2)$. Actual nucleon orbitals certainly fall off $\sim \exp(-k r)$ as r approaches infinity. The eigenfunctions for the square well behave properly at infinity, but in this case the well's sharp corners are unrealistic and force up the energy eigenvalue of orbitals with high n or l, since for these much of the wave function lies outside the well. By contrast the harmonic-oscillator orbitals of high n and l are too tightly bound. We see, therefore, that the oscillator and square wells each yield a level scheme deviating from that found in nuclei, but the deviations are in opposite directions. Consequently, a level scheme derived by interpolating between the two and then adding a term in $\mathbf{l}\cdot\boldsymbol{\sigma}$ is found to reproduce the levels found in nuclei to a reasonable first approximation (M55, F55).

Several attempts (R56, G55, G56a) have been made to reproduce the observed nuclear levels with a more realistic interaction. A square well with a diffuse boundary, of the form (G55, G56a)

$$\begin{aligned} V_c &= -V_0 \quad \text{if} \quad r \leqslant a \\ &= -V_0 \exp[(a-r)/\delta a] \quad \text{if} \quad r \geqslant a \end{aligned} \tag{IV.3}$$

or (R56)

$$V_c = -V_0[1 + \exp \alpha(r-a)]^{-1} \tag{IV.4}$$

have been found to fulfill the purpose fairly well. If one tries to interpret the spin-orbit interaction as the Thomas relativistic correction

$$[V_c(2M^2c^2)r]^{-1}(dV_c/dr)\mathbf{L}\cdot\mathbf{S}\,, \tag{IV.5}$$

one finds that this term yields a splitting in the right direction, but it must be reinforced by about a factor of 40 to yield the right magnitude.[10]

In deriving the proton levels, the Coulomb repulsion must be inserted into Eq. (IV.2), and consequently the well depth V_0 must be increased to ensure beta stability. This anomaly leads to a proton distribution of slightly [the effect is $\sim 3\%$ in Au^{197} (R56)] smaller dimension than the neutron distribution. Johnson and Teller (J54) predicted such an effect; basing their argument on the notion that due to the Coulomb barrier the classical turning point of the proton, as it hits the nuclear surface, is actually inside that of the neutron. Recent experiments (J57) on the absorption of negative K mesons by nuclei indicate that in the surface region where the nuclear density is less than 10% its normal value, the density of neutrons is still approximately equal to the density of protons, casting grave doubt on the Johnson–Teller effect. A revision of the proton well which exhibits equal proton and neutron densities at the nuclear surface has not yet been reported.

The best indication of what the actual average potential seen by a nucleon in a light nucleus is like, has been made by Brueckner, Lockett, and Rotenberg[11] (B61). They find that the $V_c(r)$ term in Eq. (IV.2) must be replaced by a nonlocal interaction:

$$V_c(r)\psi(r) \rightarrow \int V(r,r')\psi(r')dr'\,, \tag{IV.6}$$

where $V(r,r')$ is of such complexity that the motivation in writing Eq. (IV.2) is nearly lost. An encouraging feature of their calculation is that the single-particle orbitals derived closely resemble harmonic oscillator eigenfunctions.

This result is in excellent agreement with the experimental evidence described in Sec. II. The primary value of Eq. (IV.2) may then be considered to be that it can provide one with a workable set of single-particle orbitals. In light nuclei, which concern us here, the indications that harmonic oscillator eigenfunctions closely approximate nuclear orbitals appear to be nearly overwhelming.

The resemblance is a remarkably fortunate one, in that harmonic-oscillator orbitals possess many analytic properties which greatly simplify shell-model calculations. We shall review a few of these properties here.

The single-particle oscillator Hamiltonian is given by

$$H = -(\hbar^2/2m)\nabla^2 + \tfrac{1}{2}kr^2\,, \tag{IV.7}$$

which may be conveniently rewritten as

$$H = \tfrac{1}{2}\hbar\omega(p^2 + q^2)\,, \tag{IV.7a}$$

where $\omega^2 = k/m$, $\mathbf{q} = (k/\hbar\omega)^{1/2}r = \alpha^{1/2}r$, and $\mathbf{p} = -i\boldsymbol{\nabla}_q$. The eigenfunctions are well known

$$\varphi_{nlm}(r) = R_{nl}(r)Y_{lm}(\theta,\varphi)\,, \tag{IV.8}$$

[10] A velocity dependent force of the type $\mathbf{L}\cdot\mathbf{S}$ introduces a correction to the magnetic dipole moment of odd proton nuclei owing to the fact that $\mathbf{p}$ must be replaced by $\mathbf{p} - (e/c)\mathbf{A}$ when the nucleus is in an electromagnetic field. It is difficult to interpret this correction in a serious manner, due to the doubt concerning the actual origin of the doublet splitting in nuclei (see Sec. VI).

[11] This calculation will be discussed in more detail in Sec. V.

where the Y_{lm} are the usual spherical harmonics, and (T52):

$$R_{nl}(r) = \alpha^{(2l+3)/4}\left[\frac{2^{l-n+2}(2l+2n+1)!!}{\pi^{1/2}\,n!}\right]^{1/2} r^l \times \exp(-\tfrac{1}{2}\alpha r^2)\sum_{k=0}^{n}\frac{(-1)^k 2^k n!(\alpha r^2)^k}{k!(n-k)!(2l+2k+1)!!} \tag{IV.8a}$$

[we use the notation $(2l-1)!! = (2l-1)\cdot(2l-3)\cdots 3\cdot 1$], where n is the number of radial nodes excluding the one at the origin and l is the orbital angular momentum. The energy eigenvalue is

$$E_\Lambda = (\Lambda + \tfrac{3}{2})\hbar\omega\,, \tag{IV.9}$$

where $\Lambda = 2n + l$. Each Λ_i shell may accommodate

$$N(\Lambda_i) = (\Lambda_i + 1)(\Lambda_i + 2) \tag{IV.10}$$

like nucleons (with spin $\frac{1}{2}$), so that if one fills all levels up to some $\Lambda_{\max}$ then

$$\sum_{\Lambda_i=0}^{\Lambda\max} N(\Lambda_i) = \tfrac{1}{3}(\Lambda_{\max}+1)(\Lambda_{\max}+2)(\Lambda_{\max}+3) \tag{IV.10a}$$

like nucleons occupy the well.

When calculating the expectation value of the nuclear interaction energy one often encounters matrix elements of the form

$$F(n_1l_1m_1,n_2l_2m_2,n_1'l_1'm_1',n_2'l_2'm_2') = \int\cdots\int \varphi^*_{n1l1m1}(r_1) \times \varphi^*_{n2l2m2}(r_2)V(|\mathbf{r}_1-\mathbf{r}_2|)\varphi_{n1'l1'm1'}(r_1) \times \varphi_{n2'l2'm2'}(r_2)\mathbf{dr}_1\mathbf{dr}_2\,, \tag{IV.11}$$

just as one does in the theory of atomic spectra (C35). The two most important types of these terms are the direct

$$R(n_1l_1m_1,n_2l_2m_2) = F(n_1l_1m_1,n_2l_2m_2,n_1l_1m_1,n_2l_2m_2)\,, \tag{IV.11a}$$

and exchange

$$G(n_1l_1m_1,n_2l_2m_2) = F(n_1l_1m_1,n_2l_2m_2,n_2l_2m_2,n_1l_1m_1) \tag{IV.11b}$$

integrals. In atomic physics, these integrals are generally evaluated by expanding $V(|\mathbf{r}_1-\mathbf{r}_2|)$ by the Slater method

$$V(r_{12}) = \sum_{k=0}^{\infty} f_k(r_1,r_2)P_k(\cos\theta_{12})\,, \tag{IV.12}$$

where the Legendre polynomial $P_k(\cos\theta_{12})$ may be written as

$$P_k(\cos\theta_{12}) = \frac{4\pi}{2k+1}\sum_q (-1)^q Y_{kq}(1)Y_{k-q}(2)\,. \tag{IV.13}$$

The angular integrations are then easily performed by standard methods (E57a, C35):

$$(l'm'|LM|lm) = \left[\frac{(2l'+1)(2L+1)}{4\pi(2l+1)}\right]^{1/2} C^{l'Ll}_{000}C^{l'Ll}_{m'Mm}\,. \tag{IV.14}$$

In nuclear spectroscopy this method is usually not practical because one works with a variety of potentials for which the coefficients $f_k(r_1,r_2)$ are complicated functions. Furthermore, in nuclear problems noncentral interactions are no longer small and must be considered, making the Slater method lose much of its formal simplicity.

Talmi (T52) has developed an alternate procedure for calculating the F integrals when harmonic oscillator orbitals are used. The coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ are replaced by relative and center-of-mass coordinates:

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2\,,\quad \mathbf{R} = \tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)\,,$$
$$\mathbf{p} = \tfrac{1}{2}(\mathbf{p}_1-\mathbf{p}_2)\,,\quad \mathbf{P} = \mathbf{p}_1+\mathbf{p}_2 \tag{IV.15}$$

for which the oscillator Hamiltonian for two particles

$$H = -(\hbar^2/2m)(\nabla_1^2+\nabla_2^2) + \tfrac{1}{2}k(r_1^2+r_2^2) \tag{IV.16a}$$

readily separates into relative and center-of-mass terms:

$$H = [-(\hbar^2/4m)\nabla_R^2 + kR^2] + [-(\hbar^2/m)\nabla_r^2 + \tfrac{1}{4}kr^2]\,. \tag{IV.16b}$$

Consequently, the eigenfunctions may always be expanded into the form

$$\varphi_{n1l1m1}(\mathbf{r}_1)\varphi_{n2l2m2}(\mathbf{r}_2) = \sum a(n_1l_1m_1,n_2l_2m_2,nlm,n'l'm')\varphi_{nlm}(\mathbf{r})\varphi_{n'l'm'}(\mathbf{R})\,, \tag{IV.17}$$

with the restrictions

$$m_1+m_2 = m+m'$$
$$2n_1+l_1+2n_2+l_2 = 2n+l+2n'+l'\,. \tag{IV.17a}$$

Then, for example, $R(n_1l_1m_1,n_2l_2m_2)$ may be written as

$$R(n_1l_1m_1,n_2l_2m_2) = \sum T(n_1l_1m_1,n_2l_2m_2;nlm) \times \int\varphi^*_{nlm}(\mathbf{r})V(r)\varphi_{nlm}(\mathbf{r})\mathbf{dr}\,, \tag{IV.18}$$

since the integrations over the center-of-mass functions are simply performed. Several authors (L60a, K53, T56, F58, B60) have dealt with the problem of calculating the T (Talmi) coefficients in the literature. As a result of the Talmi expansion one eventu-

ally has to deal only with radial integrals remarkably simple in form

$$J_{2\lambda} = \int_0^\infty r^{2\lambda+2} \exp(-r^2) V(r) dr . \quad \text{(IV.18a)}$$

Clearly from Eqs. (IV.16) such a reduction is possible only with oscillator functions.

A further advantage in using oscillator functions is that only in this case (excluding the trivial example of free particles) may the center-of-mass motion be subtracted out in closed form (L58). As we remarked in Sec. II, the center-of-mass of the nucleus does not coincide with the center of the shell-model potential well. Generally one works problems in which the center of the well is defined to be at rest. Consequently the kinetic energy associated with the motion of the center of mass is included in the calculation and must be subtracted out in order that the energy eigenvalue correspond to the internal energy of the system.

Bethe and Rose (B37) have proven a useful theorem[12] concerning this subtraction for the oscillator well. Consider a well where all of the levels are occupied up to some $\Lambda = k$, the remaining particles all going into $\Lambda = k + 1$ orbitals. The properly antisymmetrized wave function may be written as a sum of terms of the form

$$P_{k+1}(\mathbf{r}_i) \exp\left(-\tfrac{1}{2}\alpha \sum_{i=1}^{A} r_i^2\right) \quad \text{(IV.19)}$$

where $P_{k+1}(\mathbf{r}_i)$ is an $A \times A$ determinant the elements of which are polynomials in the $\mathbf{r}_i$ (in addition to spin and isobaric spin variables) of degree no higher than $k + 1$. The exponential factor may be rewritten, replacing $\mathbf{r}_i$ by $\mathbf{r}_i - \mathbf{R}$ ($\mathbf{R} = A^{-1}\Sigma\mathbf{r}_i$), as

$$\exp(-\tfrac{1}{2}\alpha \textstyle\sum r_i^2) = \exp[-\tfrac{1}{2}\alpha \sum (r_i - R)^2] \times \exp(-\tfrac{1}{2}\alpha A R^2) . \quad \text{(IV.20)}$$

It is also easily seen that

$$P_{k+1}(\mathbf{r}_i - \mathbf{R}) \equiv P_{k+1}(\mathbf{r}_i) . \quad \text{(IV.21)}$$

To prove Eq. (IV.21) merely consider an expansion of $P_{k+1}(\mathbf{r}_i - \mathbf{R})$ into powers of R:

$$P_{k+1}(\mathbf{r}_i - \mathbf{R}) = P_{k+1}(\mathbf{r}_i) + \mathbf{R}\cdot P_k(\mathbf{r}_i) + R^2 P_{k-1}(\mathbf{r}_i) + \cdots , \quad \text{(IV.22)}$$

but

$$P_l(\mathbf{r}_i) \equiv 0 \quad \text{if} \quad l < k + 1 , \quad \text{(IV.22a)}$$

since $P_{k+1}(\mathbf{r}_i)$ had columns corresponding to every linearly independent polynomial of degree less than the maximum that appeared, $(k + 1)$. Consequently the center-of-mass oscillation may be described as being in the $1s$ state:

$$\psi_{\text{c.m.}} = N \exp(-\tfrac{1}{2}\alpha A R^2) \quad \text{(IV.23)}$$

with energy eigenvalue:

$$E_{\text{c.m.}} = \tfrac{3}{2}\hbar\omega . \quad \text{(IV.24)}$$

Obviously, the theorem fails unless all individual orbitals with energy less than the maximum are occupied. Thus, center-of-mass effects cause little difficulty in constructing ground-state wave functions, but can be very troublesome in considering the spectrum of excited states. As an example of this, Elliott and Skyrme (E55) consider the doubly excited states of He^4:

$$\psi_1 = [(1s)^3(2s)]\ {}^1S_0 , \quad \text{(IV.25a)}$$

$$\psi_2 = [(1s)^2(1p)^2]\ {}^1S_0 . \quad \text{(IV.25b)}$$

There appear to be two such states, but both contain an excitation of the center-of-mass. Diagonalizing with respect to R^2, one obtains

$$\psi_1' = (3/4)^{1/2}\psi_1 + (1/4)^{1/2}\psi_2 \quad \text{(IV.26a)}$$

$$\psi_2' = (1/4)^{1/2}\psi_1 - (3/4)^{1/2}\psi_2 . \quad \text{(IV.26b)}$$

In ψ_1' the center-of-mass motion is described by Eq. (IV.23), while ψ_2' represents the original α particle with the center of mass in the $2s$ state. Thus ψ_1' is the only true excited state of the internal system which should be considered. Wave functions such as ψ_2' are known as spurious states, and must generally be subtracted out before even qualitative agreement with the experimental spectrum of energy levels can be attained (V60, B61).

2. Classification of Shell-Model Basis Functions and the Construction of the Energy Matrices

Although the single-particle model has been qualitatively fairly successful in interpreting nuclear moments, beta decay, isomeric transitions, level schemes, and general treads in the systematics of stable nuclei,[13] clearly its assumptions are too naive to yield a final quantitative picture of nuclear properties. Refinement of the shell model must take into explicit account the nucleon–nucleon interaction; at least among nucleons in the unfilled shells, and possibly in the closed shells as well. In addition, the

[12] This theorem apparently was forgotten and later rediscovered and extended by Elliott and Skyrme (E55), whose simplified proof is given here.

[13] The extent of this success has been reviewed in considerable detail elsewhere (F55, M55, E57) and will not be repeated in this paper.

description of a nuclear state in terms of only one configuration of single-particle orbitals could not be strictly realistic, and the effect of configuration interaction should be taken into account.

To carry out these refinements, one must frequently compute matrix elements of the form

$$(\psi|\sum_{i<j} V_{ij}|\psi')\,, \tag{IV.27}$$

where ψ and ψ' are combinations of $A \times A$ determinants of single-particle orbitals u:

$$(A!)^{-1/2}\,|u_1 u_2 \cdots u_A|\,. \tag{IV.28}$$

Such matrix elements may be evaluated by the Slater method, popular in the early days of atomic spectra (C35), as we employed it to obtain an expression for the nuclear Coulomb energy in Eq. (II.16a). In modern calculations the method of fractional parentage (G34b, R43) generally proves to be more practical. Consider an antisymmetric state ψ^k comprising k nucleons in some subshell whose orbitals we shall denote by φ_{jm}. It must always be possible to expand ψ^k in terms of states ψ^{k-1}

$$\psi^k_\alpha = \sum_{\alpha'} (\psi^k_\alpha\{|\psi^{k-1}_{\alpha'})\{\psi^{k-1}_{\alpha'},\varphi_j\}_\alpha\,, \tag{IV.29}$$

where $\{\psi^{k-1}_{\alpha'},\varphi_j\}_\alpha$ represents the antisymmetrized result of vector coupling $\psi^{k-1}_{\alpha'}$ and φ_j to the quantum numbers α designating the state ψ^k_α, and $(\psi^k_\alpha\{|\psi^{k-1}_{\alpha'})$ are the coefficients of fractional parentage. Now our matrix element

$$(\psi^k_\alpha|\sum_{i<j} V_{ij}|\psi^k_\beta) = \tfrac{1}{2}\,k(k-1)(\psi^k_\alpha|V_{12}|\psi^k_\beta) \tag{IV.30}$$

may be expressed as

$$\begin{aligned}(\psi^k_\alpha|\sum_{i<j} V_{ij}|\psi^k_\beta) &= \tfrac{1}{2}\,k(k-1)\\ &\times \sum_{\alpha',\beta'}(\psi^k_\alpha\{|\psi^{k-1}_{\alpha'})(\psi^k_\beta\{|\psi^{k-1}_{\beta'})(\psi^{k-1}_{\alpha'}|V_{12}|\psi^{k-1}_{\beta'})\\ &= \frac{k}{k-2}\sum_{\alpha',\beta'}(\psi^k_\alpha\{|\psi^{k-1}_{\alpha'})(\psi^k_\beta\{|\psi^{k-1}_{\beta'})(\psi^{k-1}_{\alpha'}|\sum_{i<j} V_{ij}|\psi^{k-1}_{\beta'})\,,\end{aligned} \tag{IV.31}$$

so that the matrix elements in the $(\varphi_{jm})^k$ configurations are easily computed if the matrix elements of the $(\varphi_{jm})^{k-1}$ configuration and the appropriate coefficients of fractional parentage are known. Thus, the stage is set to perform a chain calculation for all configurations of the shell starting with the relatively simple evaluation of the two-particle matrix elements

$$(\psi^2_\alpha|V_{12}|\psi^2_\beta)\,. \tag{IV.32}$$

Such a chain calculation may be tedious, however, and can be averted by the further expansion

$$\psi^k_\alpha = \sum_{\alpha',\alpha''}(\psi^k_\alpha\{|\psi^{k-2}_{\alpha'}\psi^2_{\alpha''})\{\psi^{k-2}_{\alpha'},\psi^2_{\alpha''}\}_\alpha\,, \tag{IV.33}$$

which yields an expression for $(\psi^k_\alpha|\sum V_{ij}|\psi^k_\beta)$ directly in terms of two-particle matrix elements

$$\begin{aligned}(\psi^k_\alpha|\sum_{i<j} V_{ij}|\psi^k_\beta) &= \tfrac{1}{2}\,k(k-1)\sum_{\alpha',\alpha'',\beta',\beta''}(\psi^k_\alpha\{|\psi^{k-2}_{\alpha'}\psi^2_{\alpha''})\\ &\times(\psi^k_\beta\{|\psi^{k-2}_{\beta'}\psi^2_{\beta''})(\psi^2_{\alpha''}|V_{12}|\psi^2_{\beta''})\,.\end{aligned} \tag{IV.34}$$

Extensive tables of fractional-parentage coefficients have been compiled (J51, E52, E53, F52b, J54a) in the literature, particularly in the range useful for calculations on light nuclei.

Matrix elements for the kinetic energy operator are considerably simpler to compute, being just the sum of single-particle operators with an appropriate center-of-mass correction. Thus the energy matrix for a system of several nucleons may be deduced without the need of working with, or for that matter even writing down, the wave functions. It is, however, necessary to classify the spectroscopic states allowed in a given configuration.

In the $1s$ shell the simplifying assumptions of the independent-particle model are not essential; and thus, as we saw in Sec. III, a general classification of spectroscopic states without recourse to making l a good quantum number for each nucleon is feasible. One clearly sees, after noting the variety of states obtained for the α particle, that such rigor is impractical for $A > 4$. Consequently, in the region between helium and oxygen ($4 < A \leqslant 16$) it is common practice to assume a ground state configuration which is principally

$$(1s)^4(1p)^{A-4}\,, \tag{IV.35}$$

with perhaps some configuration interaction treated as a higher-order correction. The $1p$ shell represents the region of the periodic table where the doublet splitting builds up toward its dominant character in complex nuclei (I53), and so either LS or jj basis functions may be employed in the calculations for this region.

The earliest detailed analysis of states in the $1p$ shell was made by Wigner, Feenberg, Phillips, and Bardeen (F37, W37a, F37a, B38a) in terms of LS-coupled basis functions

$$\psi_{[\lambda]}(l^n;LSJMTT_3)\,. \tag{IV.36}$$

Since Wigner and Majorana forces dominate the nucleon–nucleon interaction, it is useful to classify the basis functions in terms of the partition quantum

number [λ], which spread out the irreducible representations of the symmetric group (W59, R48, H62). Small components of Heisenberg and Bartlett forces in the interaction will tend to mix functions with different [λ], while tensor and spin-orbit forces will reduce the significance of L and S in the nuclear wave function as well.

The properly antisymmetric LS-coupled states for n nucleons in a p shell are displayed in Table IV-1, as

TABLE IV-1. Classification of states $(p)^n$ in LS coupling.

n	[λ]	S	T	L	n	[λ]	S	T	L
0	[0]	0	0	0	5	[311]	$\frac{1}{2}$	$\frac{1}{2}$	0,2
1	[1]	$\frac{1}{2}$	$\frac{1}{2}$	1			$\frac{1}{2}$	$\frac{3}{2}$	
2	[2]	0	1	0,2			$\frac{3}{2}$	$\frac{1}{2}$	
		1	0				$\frac{3}{2}$	$\frac{3}{2}$	
	[11]	0	0	1		[221]	$\frac{1}{2}$	$\frac{1}{2}$	1
		1	1				$\frac{1}{2}$	$\frac{3}{2}$	
3	[3]	$\frac{1}{2}$	$\frac{1}{2}$	1,3			$\frac{3}{2}$	$\frac{1}{2}$	
	[21]	$\frac{1}{2}$	$\frac{1}{2}$	1,2			$\frac{3}{2}$	$\frac{3}{2}$	
		$\frac{1}{2}$	$\frac{3}{2}$				$\frac{5}{2}$	$\frac{1}{2}$	
		$\frac{3}{2}$	$\frac{1}{2}$				$\frac{5}{2}$	$\frac{3}{2}$	
	[111]	$\frac{1}{2}$	$\frac{1}{2}$	0	6	[42]	0	1	$1,2^2,3,4$
		$\frac{3}{2}$	$\frac{3}{2}$				1	0	
4	[4]	0	0	0,2,4		[411]	0	0	1,3
	[31]	0	1	1,2,3			1	1	
		1	0			[33]	0	0	1,3
		1	1				1	1	
	[22]	0	0	0,2		[321]	0	1	1,2
		0	2				1	0	
		2	0				1^2	1^2	
		1	1				0	2	
	[211]	0	1	1			2	0	
		1	0				1	2	
		1	1				2	1	
		1	2			[222]	0	1	0
		2	1				1	0	
5	[41]	$\frac{1}{2}$	$\frac{1}{2}$	1,2,3,4			1	2	
	[32]	$\frac{1}{2}$	$\frac{1}{2}$				2	1	
		$\frac{1}{2}$	$\frac{3}{2}$				1	3	
		$\frac{3}{2}$	$\frac{1}{2}$				3	1	

they were originally obtained by Hund (H37). Work with LS-coupled basis functions has been extended by Jahn *et al.* (J50, J51, J54a) with some application into the $2s$ $1d$ shell, and a classification of states for the f shell has been made by Flowers (F52).

The LS classification for nucleons is a straightforward extension of the coupling scheme for atoms. A major difference in the ordering of levels comes about due to the fact that the nucleon–nucleon interaction is dominantly attractive and therefore favors low-lying states of maximum orbital symmetry, while in considering the interactions of electrons in atoms the inverse situation obtains.

Beyond oxygen the observed doublet splitting is consistently large enough so that the effective single-body spin-orbit interaction

$$\sum_i \mathbf{l}_i \cdot \mathbf{s}_i \qquad \text{(IV.37)}$$

couples the spin and orbital angular momentum of each particle, splitting the single-particle energy levels with different j. In this case it is clearly better to employ jj-coupled basis functions in the calculations.

As the spin-orbit interaction begins to play a dominant role, one also detects a tendency for two like nucleons in the same jl shell to couple their angular momentum to $J = 0$. This fact is largely responsible for the coupling rules

$$J = 0 \text{ for even–even nuclei}$$

and

$$J = j \text{ for odd } A \text{ nuclei},$$

expressed in Sec. IV.1. The significance of this fact was first noted by Mayer (M50). If one approximates the nuclear interaction by a δ function[14]

$$V_{ij} = -V_0 \delta(\mathbf{r}_i - \mathbf{r}_j), \qquad \text{(IV.38)}$$

the interaction energy of k identical nucleons in the nlj shell is then proportional to the radial integral:

$$I_{nl} = \frac{V_0}{8\pi} \int_0^\infty R_{nl}^4(r) r^2 dr. \qquad \text{(IV.39)}$$

Mayer (M50, M55) has shown that the ground-state interaction energy $E_k(nlj)$ is given by

$$\begin{aligned} E_k(nlj) &= -\tfrac{1}{2} k(2j+1) I_{nl} \qquad k \text{ even} \\ &= -\tfrac{1}{2}(k-1)(2j+1) I_{nl} \qquad k \text{ odd}. \end{aligned} \qquad \text{(IV.40)}$$

Equation (IV.40) was verified by Mayer for $j \leqslant 7/2$, and continues to hold for larger j as well. It appears as if the nucleons are grouped into pairs with $J = 0$, each pair contributing

$$E_2(nlj) = -(2j+1) I_{nl} \qquad \text{(IV.40a)}$$

to the total energy, while an odd nucleon contributes nothing at all.

Racah (R43) has studied interactions with this property in LS coupling. One may define an operator $\overline{Q}$

$$\overline{Q} = \sum_{i<j} \overline{q}_{ij}, \qquad \text{(IV.41)}$$

with the $\overline{q}_{ij}$ defined so that

$$(l^2 LM | \overline{q}_{ij} | l^2 LM) = (2l+1)\delta_{L0}. \qquad \text{(IV.41a)}$$

Now consider a state of the configuration l^n where the expectation value of $\overline{Q}$ does not vanish. Such a state

[14] For a δ-function interaction, Majorana and Wigner forces become equivalent, since one obtains a nonvanishing interaction energy between a pair of particles only if their wave function is symmetric with respect to interchange of the labels of $\mathbf{r}_i$ and $\mathbf{r}_j$.

corresponds[15] to a state of the configuration l^{n-2} with the same S and L. If this state also has $\langle\overline{Q}\rangle \neq 0$ one may go back to l^{n-4} and so on until finally one reaches a state l^s with $\langle\overline{Q}\rangle = 0$, in which there are no paired particles. This number s, common to all states of the sequence described above, is defined as the seniority of the state. A similar operator for jj coupling is easily defined:

$$Q = \sum_{i<j} q_{ij} , \tag{IV.42}$$

where

$$(j^2JM|q_{ij}|j^2JM) = (2j+1)\delta_{J0} , \tag{IV.42a}$$

and one may proceed through an argument analogous to that for $\overline{Q}$, to define the seniority s of a state in the configuration j^n.

Owing to the evidence for a pairing interaction in nuclei, it is profitable to classify the states of a configuration in terms of the seniority quantum number s, as well as J and T. The discussion will be facilitated by the introduction of Racah's (R42) unit tensors $u_{kq}(i)$ defined by their operation on the spin-orbit part of a single-particle wave function:

$$u_{kq}(i)\psi_{jm}(i) = (2j+1)^{-1/2} C^{jkj}_{mq(m+q)}\psi_{j(m+q)}(i) . \tag{IV.43}$$

The u_{kq} are normalized so that Racah's double-barred matrix element is $\delta_{jj'}$:

$$(j'||u_k||j) \equiv (2j+1)[C^{j'kj}_{j'(j-j')j}]^{-1}(j'j'|u_{k(j'-j)}|jj) = \delta jj'. \tag{IV.44}$$

These operators may be written explicitly as

$$u_{00} = (2j+1)^{-1/2} , \tag{IV.45a}$$

$$u_{1q} = [j(j+1)(2j+1)]^{-1/2}J_q , \tag{IV.45b}$$

$$u_{2q} \backsim \sum C^{112}_{\mu\mu' q}J_\mu J_{\mu'} , \tag{IV.45c}$$

and so on. Though explicit formulas for the u_{kq} may be instructive they are not in practice useful, and it is best to rely on the defining Eq. (IV.43).

The $u_{kq}(i)$ form a complete set of operators of the unitary group $U(2j+1)$, while the $\psi_{jm}(i)$ form the basis functions. Similarly, one may define operators for n particles:

$$U_{kq} = \sum_{i=1}^{n} u_{kq}(i) , \tag{IV.46}$$

which generate the basis functions

$$\prod_{i=1}^{n} \psi_{jm_i}(i) . \tag{IV.47}$$

[15] That is, one can express this state in terms of a state of the configuration l^{n-2}, with the same L and S coupled to the state of l^2 with $L = 0$ by means of fractional parentage coefficients.

The three-dimensional rotations (to be referred to as R_3) form a particular subgroup of the unitary transformations, and classification of the functions in (IV.47) in terms of the irreducible representations of R_3 is equivalent to making the total angular momentum a good quantum number (W59, H62). Similarly, classifying the functions in Eq. (IV.47) according to their symmetry [λ], means breaking the general unitary group $U(2j+1)$ down into its irreducible subgroups under permutation of the particles.

In jj coupling we are further interested in the symplectic subgroup of transformations, which leave invariant the antisymmetric bilinear form (H62):

$$\Psi_{J=0}(1,2) = (2j+1)^{-1/2}\sum_{m=-j}^{j}(-1)^{j-m}\psi_{jm}(1)\psi_{j-m}(2)$$
$$= \sum_m C^{jj0}_{m-m0}\psi_{jm}(1)\psi_{j-m}(2) . \tag{IV.48}$$

The importance of classifying the states of a configuration j^n in terms of the symplectic subgroup is easily seen from Mayer's result on pairing energies. If the state is denoted by the seniority quantum number s and further by the reduced isobaric spin t, which is the isobaric spin of the state of smallest number of particle having the same transformation properties as ψ (the j^s configuration) (F52a), then considerable information concerning the ordering of levels within a configuration becomes evident before performing a calculation. It is further common practice to designate the state by a partition number $(\sigma) = (\sigma_1\sigma_2\cdots\sigma_{j+1/2})$, which yields the symplectic symmetry of the state and is analogous to [λ] $= [\lambda_1\lambda_2\cdots\lambda_{2j+1}]$ with pairs of particles that are vector coupled to zero angular momentum subtracted out. Such a classification of states has been

TABLE IV-2. Classification of states for n nucleons in $j = \frac{3}{2}$ orbitals.

n	[λ]	T	$(\sigma_1\sigma_2)$	(s,t)	allowed J values
0	[0]	0	(00)	(0,0)	0
1	[1]	$\frac{1}{2}$	(10)	$(1,\frac{1}{2})$	$\frac{3}{2}$
2	[2]	0	(20)	(2,0)	1,3
	[11]	1	(00)	(0,0)	0
			(11)	(2,1)	2
3	[21]	$\frac{1}{2}$	(10)	$(1,\frac{1}{2})$	$\frac{3}{2}$
			(21)	$(3,\frac{1}{2})$	$\frac{1}{2},\frac{5}{2},\frac{7}{2}$
	[111]	$\frac{3}{2}$	(10)	$(1,\frac{1}{2})$	$\frac{3}{2}$
4	[22]	0	(00)	(0,0)	0
			(11)	(2,1)	2
			(22)	(4,0)	2,4
	[211]	1	(20)	(2,0)	1,3
			(11)	(2,1)	2
	[1111]	2	(00)	(0,0)	0

made for the configurations j^n with $j = 3/2$, $5/2$, and $7/2$ by Flowers (F52a) and we display his results in Tables IV-2 to IV-4. Fractional-parentage coefficients

TABLE IV-3. Classification of states for n nucleons in $j = 5/2$ orbitals.

n	$[\lambda]$	T	$(\sigma_1\sigma_2\sigma_3)$	(s,t)	allowed J values
0	[0]	0	(000)	(0,0)	0
1	[1]	$\frac{1}{2}$	(100)	$(1,\frac{1}{2})$	$\frac{5}{2}$
2	[2]	0	(200)	(2,0)	1, 3, 5
	[11]	1	(000)	(0,0)	0
			(110)	(2,1)	2,4
3	[21]	$\frac{1}{2}$	(100)	$(1,\frac{1}{2})$	$\frac{5}{2}$
			(210)	$(3,\frac{1}{2})$	$\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $(\frac{7}{2})^2$, $\frac{9}{2}$, $\frac{11}{2}$, $\frac{13}{2}$
	[111]	$\frac{3}{2}$	(100)	$(1,\frac{1}{2})$	$\frac{5}{2}$
			(111)	$(3,\frac{3}{2})$	$\frac{3}{2}$, $\frac{9}{2}$
4	[22]	0	(000)	(0,0)	0
			(110)	(2,1)	2,4
			(220)	(4,0)	0, 2^2, 3, 4^2, 5, 6^2, 8
	[211]	1	(200)	(2,0)	1, 3, 5
			(110)	(2,1)	2, 4
			(211)	(4,1)	1, 2, 3^2, 4, 5, 6, 7
	[1111]	2	(000)	(0,0)	0
			(110)	(2,1)	2, 4
5	[221]	$\frac{1}{2}$	(100)	$(1,\frac{1}{2})$	$\frac{5}{2}$
			(210)	$(3,\frac{1}{2})$	$\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $(\frac{7}{2})^2$, $\frac{9}{2}$, $\frac{11}{2}$, $\frac{13}{2}$.
			(111)	$(3,\frac{3}{2})$	$\frac{3}{2}$, $\frac{9}{2}$
			(221)	$(5,\frac{1}{2})$	$\frac{1}{2}$, $\frac{3}{2}$, $(\frac{5}{2})^2$, $(\frac{7}{2})^2$, $(\frac{9}{2})^2$, $(\frac{11}{2})^2$, $\frac{13}{2}$, $\frac{15}{2}$, $\frac{17}{2}$
	[2111]	$\frac{3}{2}$	(100)	$(1,\frac{1}{2})$	$\frac{5}{2}$
			(210)	$(3,\frac{1}{2})$	$\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $(\frac{7}{2})^2$, $\frac{9}{2}$, $\frac{11}{2}$, $\frac{13}{2}$.
			(111)	$(3,\frac{3}{2})$	$\frac{3}{2}$, $\frac{9}{2}$
	[11111]	$\frac{5}{2}$	(100)	$(1,\frac{1}{2})$	$\frac{5}{2}$
6	[222]	0	(200)	(2,0)	1, 3, 5
			(211)	(4,1)	1, 2, 3^2, 4, 5, 6, 7
			(222)	(6,0)	1, 3^2, 4, 5, 6, 7, 9
	[2211]	1	(000)	(0,0)	0
			$(110)^2$	$(2,1)^2$	2^2, 4^2
			(220)	(4,0)	0, 2^2, 3, 4^2, 5, 6^2, 8
			(211)	(4,1)	1, 2, 3^2, 4, 5, 6, 7
	[21111]	2	(200)	(2,0)	1, 3, 5
			(110)	(2,1)	2, 4
	[111111]	3	(000)	(0,0)	0

TABLE IV-4. Classification of states for n like nucleons ($T = n$) for $j = \frac{7}{2}$ orbitals.

n	$[\lambda]$	$(\sigma_1\sigma_2\sigma_3\sigma_4)$	(s,t)	allowed J values
0	[0]	(0000)	(0,0)	0
1	[1]	(1000)	$(1,\frac{1}{2})$	$\frac{7}{2}$
2	[11]	(0000)	(0,0)	0
		(1100)	(2,1)	2, 4, 6
3	[111]	(1000)	$(1,\frac{1}{2})$	$\frac{7}{2}$
		(1110)	$(3,\frac{3}{2})$	$\frac{3}{2}$, $\frac{5}{2}$, $\frac{9}{2}$, $\frac{11}{2}$, $\frac{15}{2}$
4	[1111]	(0000)	(0,0)	0
		(1100)	(2,1)	2, 4, 6
		(1111)	(4,2)	2, 4, 5, 8

for these configurations have been tabulated by Edmonds and Flowers (E52).

In terms of Racah's unit tensors, one may write the seniority operator as

$$q_{12} = + \sum_{k=0}^{2j} (-1)^k (2k+1) u_k(1) \cdot u_k(2) \,. \qquad \text{(IV.49)}$$

The Casimir operator[16] (R51) for the unitary group is given by (E52)

$$\mathcal{G}(U) = \sum_{k=0}^{2j} (2k+1) U_k \cdot U_k \qquad \text{(IV.50)}$$

which has eigenvalues in the states j^n $[\lambda]$:

$$\begin{aligned} g[\lambda] &= \langle j^n[\lambda] | \mathcal{G}(U) | j^n[\lambda] \rangle \\ &= ng[1] + 2 \sum_{i<j} \sum_{k=0}^{2j} (2k+1) \langle u_k(i) \cdot u_k(j) \rangle \\ &= -2T(T+1) + \tfrac{1}{2} n(4j + 8 - n) \,, \end{aligned} \qquad \text{(IV.51)}$$

and similarly, the Casimir operator for the symplectic subgroup is (E52)

$$\mathcal{G}(Sp) = 2 \sum_{\text{odd } k} (2k+1) U_k \cdot U_k \,, \qquad \text{(IV.52)}$$

and has eigenvalues

$$\begin{aligned} g(\sigma) &= \langle j^n[\lambda](\sigma) | \mathcal{G}(Sp) | j^n[\lambda](\sigma) \rangle \\ &= ng(1) + 4 \sum_{i<j} \sum_{\text{odd } k} (2k+1) \langle u_k(i) \cdot u_k(j) \rangle \\ &= -2t(t+1) + \tfrac{1}{2} s(4j + 8 - s) \,. \end{aligned} \qquad \text{(IV.53)}$$

Consequently, the expectation value of Q becomes

$$\langle Q \rangle = \tfrac{1}{2} \{ g[\lambda] - ng[1] - g(\sigma) + ng(1) \} \,. \qquad \text{(IV.54)}$$

It is now possible to examine the types of interaction operators which possess the pairing property found by Mayer. Racah and Talmi (R52) first investigated this question, searching for an interaction whose matrix elements have the form

$$(ns\Gamma | V | ns'\Gamma') = \delta_{ss'} \{ (ss\Gamma | V | ss\Gamma') + \tfrac{1}{2} (n - s) E_0 \} \,. \qquad \text{(IV.55)}$$

Such an interaction is diagonal in the seniority quantum number, with each zero-coupled pair of nucleons contributing E_0 to the interaction energy. The interaction operator may be expanded in terms of the unit tensors:

$$V_{12} = \sum_k \mathcal{V}_k u_k(1) \cdot u_k(2) \,, \qquad \text{(IV.56)}$$

where the $\mathcal{V}_k$ are integrals on the radial wave func-

[16] Casimir's operator for a semisimple group of elements X_μ is (R51, H62)

$$C = g^{\rho\sigma} X_\rho X_\sigma \,,$$

where $g^{\rho\sigma}$ is the metric matrix and C may be shown to commute with every element of the group. The Casimir operator for R_3, for example, is simply the square of the total angular momentum.

tions of the nucleons. The interaction operator now takes the form

$$V = \sum_{i<j} V_{ij} = \tfrac{1}{2} \sum_k \mathcal{V}_k [U_k \cdot U_k - \sum_i u_k(i) \cdot u_k(i)]$$
$$= \tfrac{1}{2} \sum_k \mathcal{V}_k U_k \cdot U_k - \tfrac{1}{2} n \sum_k \mathcal{V}_k u_k(1) \cdot u_k(1) . \quad \text{(IV.57)}$$

This interaction clearly has the pairing property expressed in Eq. (IV.55) if the first term is diagonal in the seniority and independent of the total number of particles. Racah (R43) has shown that these conditions hold if we are summing only over tensors of odd rank in Eq. (IV.57).

Now consider a Wigner interaction, which may be expanded as

$$V(r_{12}) = \textstyle\sum_k V_k(r_1, r_2) C_k(1) \cdot C_k(2) , \quad \text{(IV.58)}$$

where

$$C_{kq} = [4\pi/(2k+1)]^{1/2} Y_{kq} . \quad \text{(IV.58a)}$$

Only terms with even k will contribute to the matrix elements

$$\int \cdots \int R^*_{n_1 l_1}(r_1) R^*_{n_2 l_2}(r_2) V(r_{12}) R_{n_1 l_1}(r_1) R_{n_2 l_2}(r_2) d\mathbf{r}_1 d\mathbf{r}_2 \quad \text{(IV.59)}$$

due to the selection rule on parity. Consequently if we consider the interaction

$$\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 V(r_{12}) \quad \text{(IV.60)}$$

(which for like particles represents a mixture of one part Wigner interaction plus two parts Majorana), only terms of odd k in Eq. (IV.57) yield a nonvanishing contribution. As we have pointed out previously, in the limit of δ-function forces there is no distinction between Wigner and Majorana interactions for like nucleons, and thus, in this limit the seniority is always a good quantum number.

Though this situation is not precisely realized for nuclear interactions, it is true that one has far weaker interaction in orbitally antisymmetric states than in the symmetric states. For this reason the seniority is almost a good quantum number, and clearly an examination of nuclear systematics reveals the existence of the pairing property.

French (F60) has examined the criteria for the existence of states of good seniority in terms of the symplectic symmetry of the nuclear wave function, finding that in addition to being composed of a sum of scalar products of odd-rank tensors the interaction may also have an additive multiple of T^2, and of course, a simple additive constant.

3. Intermediate Coupling

A rudimentary refinement of the single-particle model involves a careful investigation of the mode of coupling between the angular momenta for the individual orbitals. In the $1p$ shell for example one may regard the states of normal parity[17] as belonging to the configuration

$$(1s)^4 (1p)^{A-4} , \quad \text{(IV.61)}$$

and for the time being ignore the effect of configuration mixing. It is then possible, after consulting Tables IV-1 and IV-2, to note which states of this configuration are consistent with the particular values of total angular momentum and isobaric spin associated with a given nuclear level, and then proceed to try and calculate the physical properties of this level.

To illustrate the procedure consider the ground state of Li^6 which has $J = 1+$ and $T = 0$. If LS coupling is used, one may assume the wave function to be the linear combination

$$\psi(Li^6) = a\psi(^3S_1) + b\psi(^1P_1) + c\psi(^3D_1) , \quad \text{(IV.62)}$$

while if jj basis functions are employed one has

$$\psi(Li^6) = a'\psi(p^2_{1/2}) + b'\psi(p^2_{3/2}) + c'\psi(p_{1/2}p_{3/2}) , \quad \text{(IV.63)}$$

where all states belong to the $(1s)^4\ (1p)^2$ configuration, and the closed $1s$ shell contributes nothing to the total angular momentum and the isobaric spin. If one adopts an explicit interaction operator [of the form in Eq. (III.1)] it is then possible to set up and diagonalize the energy matrix; the solution for the lowest eigenvalue yielding an estimate for the amplitudes $a\ b\ c$ (or $a'\ b'\ c'$).

Evidently, one makes no ultimate physical error in this case by specializing to either LS or jj basis function, since one may effect a transformation between the two schemes employing the well-known 9-j coefficients (E57a, F59):

$$\psi_{LSJM} = \sum_{j_1, j_2} [(2L+1)(2S+1)(2j_1+1)(2j_2+1)]^{1/2} \times \begin{Bmatrix} l_1 & l_2 & L \\ s_1 & s_2 & S \\ j_1 & j_2 & J \end{Bmatrix} \psi_{j_1 j_2 JM} , \quad \text{(IV.64)}$$

and in both cases we employ the complete set of states belonging to the configuration $(1s)^4\ (1p)^2$. The energy separations of the various multiplets of the

[17] Such states are defined as those having a parity equal to the parity of the orbital in the shell raised to the power of the number of nucleons in the shell. In the $1p$ shell these states have parity $(-1)^{A-4}$.

$1p$ shell have been studied in LS coupling by Feenberg and Phillips (F37a) and in jj coupling by Kurath (K52). If one considers only central interactions, the energy differences may be expressed in terms of the direct integral

$$L = \int\cdots\int \psi^*_{11m}(\mathbf{r}_1)\psi^*_{11m'}(\mathbf{r}_2)V_{12}(r_{12})\psi_{11m}(\mathbf{r}_1)\psi_{11m'} \times (\mathbf{r}_2)d\mathbf{r}_1 d\mathbf{r}_2 \quad \text{(IV.65a)}$$

and the exchange integral

$$K = \int\cdots\int \psi^*_{11m}(\mathbf{r}_1)\psi^*_{11m'}(\mathbf{r}_2)V_{12}(r_{12})\psi_{11m}(\mathbf{r}_2)\psi_{11m'} \times (\mathbf{r}_1)\mathbf{dr}_1\mathbf{dr}_2 \quad \text{(IV.65b)}$$

for the $1p$ shell.

Although the choice between LS and jj basis functions may be a matter of pure expediency in calculations simple enough so that the complete set of states belonging to a configuration may be used, cases arise where a knowledge of which coupling scheme better approximates the physical system is needed. To illustrate this, consider the $J = 3+$, $T = 0$ states of the configuration

$$(1s)^4(1p)^6 \quad \text{(IV.66)}$$

which approximate the ground state of B^{10}. Consulting Table IV-1, we find a total of nine states with the desired properties:

$$\begin{aligned}
[\lambda] &= [42]: \ {}^3D_3 \text{ (twice)}, {}^3F_3, {}^3G_3 \\
[\lambda] &= [411]: \ {}^1F_3 \\
[\lambda] &= [33]: \ {}^1F_3 \\
[\lambda] &= [321]: \ {}^3D_3, {}^5P_3, {}^5D_3 ,
\end{aligned} \quad \text{(IV.67)}$$

and the problem becomes quite difficult even if we only choose the functions of maximum orbital symmetry ($[\lambda] = [42]$). On the other hand, if the $\mathbf{l}\cdot\boldsymbol{\sigma}$ interaction is strong enough so that the $p_{3/2}$ orbital lies appreciably lower than the $p_{1/2}$ orbital and the level is well approximated by jj coupling, then the configuration will be

$$(1s_{1/2})^4(1p_{3/2})^6 \quad \text{(IV.68)}$$

(which means that we have two holes in the $p_{3/2}$ shell). Referring to Table IV-2, we find that such a state is derived from the single function

$$[\lambda] = [2] , \quad (\sigma) = (20) , \quad (p_{3/2})^{-2} , \quad \text{(IV.69)}$$

and so the validity of jj coupling would result in a great simplification of the problem (Z53).

Clearly, jj coupling must be more realistic than LS coupling beyond Ca^{40}, since it is required to reproduce the proper shell closures in heavier nuclei. The necessity of introducing the $\mathbf{l}\cdot\boldsymbol{\sigma}$ interaction in order to obtain the magic number N or $Z = 28$, originally led to an investigation of the possibility that the doublet splitting builds up in the $1f$ shell (F52), but the jj-coupling approach is also needed to yield the shell closure at Z or $N = 14$ and works reasonably well throughout the $1d$ $2s$ shell.

Inglis (I53) first observed that a transition from LS to jj coupling takes place through the $1p$ shell. A ground-state angular momentum of $3+$ is favored for Li^6 in jj coupling, and the observed value of $1+$ indicates that the high orbital symmetry of the ${}^3S_1$ state dominates the wave function of this level. The magnitude of the doublet splitting for the dynamically stable nuclei in the $1p$ shell is shown in Table IV-5. The small separation of the doublet states at $A = 7$ substantiates LS coupling near the beginning of the shell. The splitting is seen to rapidly increase through the shell, and the inversion in sign of the doublet in going from $A = 11$ to $A = 13$ verifies the closure of the $p_{3/2}$ subshell at C^{12}.[18] Clearly, however, the spectrum of excited levels does not conform consistently to either coupling scheme even near the end of the shell. Furthermore, Lane (L53, L54, L55, L55a) has demonstrated that the reaction data and decay widths of nuclei in the $1p$ shell are consistent only with intermediate coupling.

In order to investigate the range of validity of either extreme one must perform calculations in intermediate coupling employing a complete set of jj or LS basis functions. Such calculations are impeded by our incomplete understanding of the origin of the doublet separations in terms of two-body interactions. This enigma led Inglis (I53) to propose a basic model for performing the intermediate coupling calculations in which the Hamiltonian is taken to be a sum of two-body central interactions

TABLE IV-5. The separation between $J = \frac{3}{2}-$ and $J = \frac{1}{2}-$ states for dynamically stable nuclei in the $1p$ shell.

Nucleus	$E(\frac{1}{2}-) - E(\frac{3}{2}-)$ MeV	Nucleus	$E(\frac{1}{2}-) - E(\frac{3}{2}-)$ MeV
Li^7	0.478	C^{11}	1.99
Be^7	0.431	C^{13}	−3.68
Be^9	3.04	N^{13}	−3.51
B^9	2.82	N^{15}	−6.33
B^{11}	2.14	O^{15}	−6.16

[18] Paradoxically the fairly equal level spacing in C^{12} looks like LS coupling, the jj model predicting a large gap above the ground state followed by relatively closely spaced levels derived from the configuration $(p_{3/2})^{-1}\, p_{1/2}$.

and a single particle spin-orbit term:

$$H = \sum_i T_i + \sum_{i<j} [W + MP_M(i,j) + BP_B(i,j) + HP_H(i,j)]V(r_{ij}) + a \sum_i \mathbf{s}_i \cdot \mathbf{l}_i \qquad \text{(IV.70)}$$

where $P_M(i,j)$ is the Majorana operator which exchanges space coordinates, the Bartlett operator $P_B(i,j)$ exchanges spin coordinates

$$P_B(i,j) = \tfrac{1}{2}(1 + \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j), \qquad \text{(IV.71)}$$

and the Heisenberg operator is given by

$$P_H = P_M P_B . \qquad \text{(IV.72)}$$

The relationship between this way of writing the central interaction and the one employed in Eq. (III.1) using projection operators

$$\sum_{i<j} [V_0 P_0(i,j) + V_1 P_1(i,j) + V_2 P_2(i,j) + V_3 P_3(i,j)]V(r_{ij}) \qquad \text{(IV.73)}$$

is easily found:

$$P_0 = \tfrac{1}{4}(1 + P_M)(1 - P_B) = \tfrac{1}{4}(1 + P_M) - \tfrac{1}{4}(P_B + P_H), \qquad \text{(IV.74a)}$$

$$P_1 = \tfrac{1}{4}(1 + P_M)(1 + P_B) = \tfrac{1}{4}(1 + P_M) + \tfrac{1}{4}(P_B + P_H), \qquad \text{(IV.74b)}$$

$$P_2 = \tfrac{1}{4}(1 - P_M)(1 - P_B) = \tfrac{1}{4}(1 - P_M) - \tfrac{1}{4}(P_B - P_H), \qquad \text{(IV.74c)}$$

$$P_3 = \tfrac{1}{4}(1 - P_M)(1 + P_B) = \tfrac{1}{4}(1 - P_M) + \tfrac{1}{4}(P_B - P_H). \qquad \text{(IV.74d)}$$

The parameters are usually normalized so that

$$W + M + B + H = 1 . \qquad \text{(IV.75)}$$

Once a force mixture is decided upon, the spectrum of states belonging to a configuration may be deduced from three parameters; the direct integral L, the exchange integral K, and the strength of the spin-orbit interaction. In the long-range approximation, where $V(r_{12})$ may be taken as a constant over the volume of the nucleus, $K = 0$ by the orthogonality of the single-particle orbitals. On the other hand, in the zero-range approximation, where $V(r_{12}) = V_0\delta(\mathbf{r}_1 - \mathbf{r}_2)$, one obtains $L = 3K$. The actual ratio must lie between the two extremes. Hummel and Inglis (H51) estimate that $L = 6K$ with oscillator functions, while Kurath (K56) obtained $L = 6.8K$. At any rate, the resulting spectrum is not critically sensitive to this ratio over the range of its uncertainty. The ratio a/K may then be used as a parameter which is adjusted to obtain the best agreement with the observed level order.

The contribution of the central interaction is sensitive to the orbital symmetry of the wave function, and consequently, if $K > a$, LS coupling will be favored. On the other hand, if $a > K$, the spin-orbit interaction will be effective in splitting single-particle $j = l \pm \frac{1}{2}$ levels, and jj coupling is then preferred. We must note that the above inequalities are symbolic and must not be taken literally. The real point is to find by a detailed calculation which term is most *effective* in determining the properties of the spectrum.

Inglis performed complete intermediate coupling calculations for two particles ($A = 6$) and two holes ($A = 14$) in the $1p$ shell. In the more complicated cases, further removed from a shell closure, he made an estimate of what the spectrum should look like by computing the two extremes (LS and jj) and interpolating between them. The exchange mixture used was

$$W = H = 0 , \quad M = 0.8 , \quad B = 0.2 , \qquad \text{(IV.76)}$$

which closely resembles the Rosenfeld (R48a) mixture

$$W = -0.13 , \; M = 0.93 , \; H = -0.26 , \; B = 0.46 , \qquad \text{(IV.77)}$$

originally designed to produce saturation of nuclear binding energies as well as the singlet–triplet splitting observed in the deuteron. In Li^6 the ground state is over 90% $^3S_1$ for reasonable values of $a/K(\sim 2)$, indicating that LS coupling is the more reasonable extreme toward the beginning of the shell. In N^{14} the admixtures of $^3S_1$, $^1P_1$, and $^3D_1$ vary more radically with the strength of the spin-orbit term, showing a preference for $^3S_1$ for small a and crossing over to about 80% $^3D_1$ for $a/K \sim 4$.

Kurath (K56) has extended the intermediate coupling calculations to nuclei throughout the $1p$-shell. A typical set of parameters fitting the data is given in Table IV-6. We note the lack of fluctuation of K with mass number compared with the rapid increase in a through the shell. The parameters in Be^9 are not certain because of the lack of experimental data at the time the calculation was done. The parameters listed are essentially in agreement with those of French, Halbert, and Pandya (F55a) on Be^9, however, and probably represent the best fit within the context of the model.[19]

[19] Analysis of the states of Be^9 in terms of the α-particle model (K60) will be discussed later.

Reasonable agreement with the observed spectra is obtained up to excitation energies of about 6 MeV. For highly excited states, the mixing of higher

TABLE IV-6. Summary of intermediate coupling parameters for the $1p$ shell.

A	L/K	K (MeV)	a (MeV)
6	5.8	−1.2	−1.6
7	5.8	−1.2	−1.6
8	5.8	−1.2	−2.4
9	5.8	−1.2	−1.8
10	6.8	−0.9	−4.2
11	6.8	−0.9	−5.4
12	6.8	−0.9	−4.5
13	6.8	−0.9	−4.8
14	5.8	−0.8	−4.0

configurations probably becomes significant. Owing to the fact that parity is a good quantum number, the lowest configurations that may interact with $(1s)^4\,(1p)^{A-4}$ involve excitations of $2\hbar\omega$ in the oscillator wave functions:

$$\begin{aligned}&(1s)^4\,(1p)^{A-5}\,(2p)\,,\\&(1s)^4\,(1p)^{A-5}\,(1f)\,,\\&(1s)^3\,(1p)^{A-4}\,(2s)\,,\\&(1s)^3\,(1p)^{A-4}\,(1d)\\&(1s)^2\,(1p)^{A-2}\,.\end{aligned}\qquad\text{(IV.78)}$$

(We note that all of these configurations involve excitation of the center of mass and therefore contain spurious states which must be subtracted out of the final calculations.)

The static electromagnetic moments in the $1p$ shell are consistent with the parameters deduced from spectrum of excited states (K56, K59). An examination of the radiative transition widths (K57), however, yield $E2$ transitions which are generally too slow (although $M1$ transitions are in agreement), indicating a need for incorporating some collective distortion into the wave functions.

Meshkov and Ufford (M56) and Soper (S57b) have attempted to deduce the force mixture from the spectrum of Li^6, and then apply the obtained interaction to the states of Li^7. To illustrate the general procedure, we review the details of the calculation by Soper.

First, the energy of a proton and a neutron for Li^6 in the presence of the $(1s)^4$ core, but in the absence of any interaction with each other, is deduced by comparing the separation energy for

$$Li^6 \rightarrow He^4 + p + n\,,\qquad\text{(IV.79a)}$$

with

$$Li^5 \rightarrow He^4 + p\,,\qquad\text{(IV.79b)}$$

$$He^5 \rightarrow He^4 + n\,.\qquad\text{(IV.79c)}$$

This new "energy zero" is found to lie 8.011 MeV above the ground state of Li^6. The energy levels used by Soper are shown in Table IV-7.

TABLE IV-7. Energy levels of Li^6 used by Soper (*S57b*).

T	J	Energy (MeV) measured from the ground state	Energy (MeV) measured from "zero"
0	1	0	−8.011
0	3	2.189	−5.822
1	0	3.57	−4.44
0	2	4.52	−3.49
1	2	5.31	−2.7
0	1	5.4	−2.6

Matrix elements for the central interaction in the states of the configuration $(1p)^2$ have been computed by Feenberg and Philips (F37a).

$T = 0$:

$$\begin{aligned}&\langle {}^3S_1|V_{12}|{}^3S_1\rangle = L + 2K\\&\langle {}^3D_J|V_{12}|{}^3D_J\rangle = L - K\\&\langle {}^1P_1|V_{12}|{}^1P_1\rangle = (L - 3K)(W - M - B + H)\,.\end{aligned}\qquad\text{(IV.80a)}$$

$T = 1$:

$$\begin{aligned}&\langle {}^1S_0|V_{12}|{}^1S_0\rangle = (L + 2K)(W + M - B - H)\\&\langle {}^1D_2|V_{12}|{}^1D_2\rangle = (L - K)(W + M - B - H)\\&\langle {}^3P_J|V_{12}|{}^3P_J\rangle = (L - 3K)(W - M - B + H)\,,\end{aligned}\qquad\text{(IV.80b)}$$

while matrix elements of $a\,\mathbf{l}\cdot\mathbf{s}$ are given in Table IV-8. The energy matrices for the states in Table IV-7 may then be set up, and putting in the experimental values for the energy eigenvalues, solved for the parameters L, K, a, W, M, B, and H. Unfortunately, the $T = 0$, $J = 1$ state at 5.4 MeV is too broad to allow a precise determination of the singlet–odd interaction, which is arbitrarily equated to zero. The parameters which fit the spectrum are then found to be

$$W = 0.40\,,\quad M = 0.33\,,\quad B = 0.17\,,\quad H = 0.10\,,\qquad\text{(IV.81a)}$$

$$L = -5.45\text{ MeV}\,,\quad K = -1.18\text{ MeV}\,,$$
$$a = -1.554\text{ MeV}\,,\qquad\text{(IV.81b)}$$

which, upon comparison with Table IV-6, are seen

TABLE IV-8. Matrix elements of a $\Sigma \mathbf{l}_i \cdot \mathbf{s}_i$ in the states of the $(1p)^2$ configuration, obtained from Elliott (E53a) by setting his $6\,D' = a$.

	$^1S_0$	$^3P_0$	$^3S_1$	$^3D_1$	$^1P_1$	$^3P_1$	$^3D_2$	$^3P_2$	$^1D_2$	$^3D_3$
$^1S_0$	0	$-\sqrt{2}\,a$	0	0	0	0	0	0	0	0
$^3P_0$	$-\sqrt{2}\,a$	$-a$	0	0	0	0	0	0	0	0
$^3S_1$	0	0	0	0	$-\sqrt{\frac{3}{2}}\,a$	0	0	0	0	0
$^3D_1$	0	0	0	0	$-\frac{3}{2}a$	$-\sqrt{\frac{5}{6}}\,a$	0	0	0	0
$^1P_1$	0	0	$-\sqrt{\frac{3}{2}}\,a$	$-\frac{3}{2}a$	0	0	0	0	0	0
$^3P_1$	0	0	0	$-\sqrt{\frac{5}{6}}\,a$	0	$-\frac{1}{2}a$	0	0	0	0
$^3D_2$	0	0	0	0	0	0	$-\frac{1}{2}a$	0	0	0
$^3P_2$	0	0	0	0	0	0	0	$\frac{1}{2}a$	$a/\sqrt{2}$	0
$^1D_2$	0	0	0	0	0	0	0	$a/\sqrt{2}$	0	0
$^3D_3$	0	0	0	0	0	0	0	0	0	a

to be in excellent agreement with those found by Kurath (K56). Meshkov and Ufford (M56) deduce a similar force mixture

$$W = M = 0.40\,, \quad B = 0.20\,, \quad H = 0\,, \quad \text{(IV.81c)}$$

entirely within agreement with that of Soper when the uncertainty in the singlet–odd interaction is considered. Both mixtures give satisfactory results when applied to Li^7.

The mixtures given in Eqs. (IV.81) are not at all similar to the Rosenfeld or Inglis mixtures, but the calculations do not appear to be sensitive to the difference. In some measure this may be due to the near equivalence of Wigner and Majorana terms for short-range interactions, which was remarked upon in the preceding section.

More noteworthy is the deviation of the above interactions from the simple Serber mixture

$$W = M = 0.5\,, \quad B = H = 0\,. \quad \text{(IV.82)}$$

The relatively large components of Bartlett and Heisenberg forces in the Soper and Meshkov–Ufford mixtures are difficult to reconcile with the results of low-energy n–p scattering. This discrepancy may possibly be attributed to the neglect of the tensor interaction, which in second order provides a sizeable contribution to the triplet–even interaction but has little effect on low energy n–p scattering processes (G59). The ratio of triplet–even to singlet–even interaction is given by

$$(W + M + B - H)/(W + M - B + H)\,, \quad \text{(IV.83)}$$

so that one may compensate for the neglect of the second-order effects of the tensor force by reinforcing the mixture of Bartlett interaction. Lyons (L57) has shown that the primary effect of tensor forces in Li^6 is to produce a splitting similar to that of a single-body $\mathbf{l}\cdot\mathbf{s}$ term, so that the only additional noticeable effect in the intermediate coupling calculations will be the reinforcement of the triplet–even interaction.

The success of the intermediate coupling calculations in interpreting low-lying states of normal parity in the $1p$ shell has led Elliott and Flowers (E55a) and Redlich (R55) to perform similar investigations for two and three nucleons in the $2s$ $1d$ shell. Although the jj model generally seems to work quite well beyond O^{16}, the ground state of F^{19} (as we saw in Sec. I) is a glaring exception to the usual rules, the last nucleon clearly appearing to be in the $2s_{1/2}$ orbital while a $1d_{5/2}$ state is definitely predicted. This suggests that the tendency toward maximum orbital symmetry is still playing a dominant role in determining the coupling scheme for this nuclide, and an intermediate coupling calculation is required to clear the matter up.

The problem is complicated by the fact that one must deal with two orbitals, the relevant parameters being deduced from the O^{17} spectrum. The 3/2+ excited state of O^{17} lies 5.08 MeV above the 5/2+ ground state. If these states are interpreted as pure $d_{3/2}$ and $d_{5/2}$ orbitals in addition to a closed $(1s)^4$ $(1p)^{12}$ core, the magnitude of the spin-orbit coupling parameter is immediately determined to be

$$a = -(2/5)5.08 \cong -2\ \text{MeV}\,. \quad \text{(IV.84)}$$

The 1/2+ first excited level of O^{17} lies at 0.88 MeV, yielding a separation of the $1d$ and $2s$ orbitals in the absence of a spin-orbit interaction of 1.125 MeV. One is now faced with the determination of eight radial integrals rather than the two (L and K) needed in the p shell. Determination of the integrals directly from the data is therefore impractical, and one must proceed to assume logical parameters for the shape and range of the nuclear interaction as well as the single-particle orbitals. Elliott and Flowers choose a Rosenfeld mixture with Yukawa shape

$$V(r_{ij}) = V_c(r_0/r_{ij}) \exp(-r_{ij}/r_0) \quad \text{(IV.85)}$$

having range $r_0 = 1.37 \times 10^{-13}$ cm. Harmonic oscillator orbitals are used with a variety of length parame-

TABLE IV-9. Percentage analysis of wave functions for $A = 18$.

T	J	Orbital Symmetry [2]	Orbital Symmetry [11]	jj-Configuration $(d_{5/2})^2$	jj-Configuration $d_{5/2}\,d_{3/2}$	jj-Configuration $(d_{3/2})^2$	jj-Configuration $(d_{3/2}\,s_{1/2})$	jj-Configuration $(s_{1/2})^2$
0	1	97	3	34	32	4	0	30
1	0	86	14	79	...	6	...	15

ters, a typical value being 1.64×10^{-13} cm. Comparing this value with those displayed for the $1p$ shell in Table II-1, we find it to be quite reasonable, being between the length parameters deduced from the electron scattering experiments on N^{14} and O^{16}.

The results obtained by Elliott and Flowers for the compositions of the wave functions for $A = 18$ and $A = 19$ are shown in Tables IV-9 and IV-10. A definite tendency toward maximum orbital symmetry rather than spin-orbit splitting is indicated, undoubtedly due in part to the close competition between $2s_{1/2}$ and $1d_{5/2}$ orbitals. It is found, however, that a *restriction* to maximum orbital symmetry would be much too severe, since the spin-orbit forces are not properly accounted for unless they are allowed to mix states of different symmetry. Reasonable agreement with the experimental spectra is obtained if V_c is chosen to be 40 MeV.[20]

The observation of low-lying excited states with negative parity in O^{17}, F^{18}, and F^{19} is disturbing, since

TABLE IV-10. Percentage analysis of wave functions for $A = 19$.

T / J	$\frac{1}{2}$ / $\frac{1}{2}$	$\frac{1}{2}$ / $\frac{3}{2}$	$\frac{1}{2}$ / $\frac{5}{2}$	$\frac{3}{2}$ / $\frac{1}{2}$	$\frac{3}{2}$ / $\frac{3}{2}$	$\frac{3}{2}$ / $\frac{5}{2}$
Orbital symmetry						
[3]	92	84	86	0	0	0
[21]	8	16	14	90	90	90
[111]	0	0	0	10	10	10
jj configuration						
$(\frac{5}{2})^3$	7	16	39	..	62	82
$(\frac{5}{2})^2(\frac{3}{2})$	5	47	26	..	3	7
$(\frac{5}{2})^2(\frac{1}{2})$	38	7	10	92	33	..
$(\frac{5}{2})(\frac{3}{2})(\frac{1}{2})$	21	5	3	8	2	..
$(\frac{5}{2})(\frac{1}{2})^2$	..	19	20	..	..	11
$(\frac{3}{2})(\frac{1}{2})^2$	..	6	2	..	..	0
$(\frac{1}{2})^3$	29	..	..	..	..	..

these states surely involve excitation of the core. No excitation of the $(1s)^4\ (1p)^{12}$ core was considered; however, it is found necessary to invoke some weak collective effects of the core in order to explain the $E2$ transitions for these nuclei. This corresponds to Kurath's (K57) findings in the $1p$ shell and this point will be elaborated upon in Sec. VII.

[20] Bilaniuk and Hough (B57a) find remarkable agreement with the calculations, determining some of the ratios of the mixing coefficients experimentally from the $O^{17}(d,p)O^{18}$ reaction.

States of unnatural parity have been investigated in intermediate coupling by Elliott and Flowers (E57b) for O^{16} and N^{16}, and by Hallert and French (H57c) for N^{15}. The lowest expected configurations are

$$(1s)^4(1p)^{4-5}(2s) \qquad \text{(IV.86a)}$$

$$(1s)^4(1p)^{4-5}(1d)\,, \qquad \text{(IV.86b)}$$

with possibly some distortion of the $1s$ shell for N^{15}

$$(1s)^3(1p)^{12}\,. \qquad \text{(IV.86c)}$$

We note that since these are not "ground-state" configurations, spurious states involving the excitation of the center of mass must be deducted from the spectrum.

Owing to the fact that one must work with $1p$, $2s$, and $1d$ orbitals in these calculations, the technical difficulties far exceed those encountered in investigating states of normal parity. A nontrivial complication arises from the fact that the doublet splitting at the end of the $1p$ shell is considerably greater than at the beginning of the $2s$ $1d$ shell. The $p_{3/2}$–$p_{1/2}$ separation estimated from N^{15} is 6.33 MeV, yielding a spin-orbit strength of

$$a_p \cong -4.2 \text{ MeV}\,, \qquad \text{(IV.87a)}$$

while in O^{17} the $d_{5/2}$–$d_{3/2}$ splitting [previously considered in Eq. (IV.84)] yields

$$a_d \cong -2.0 \text{ MeV}\,. \qquad \text{(IV.87b)}$$

Reasonable agreement with the low-lying states for $A = 16$ and $A = 15$ is found; but several highly excited states are not predicted at all, indicating the possible importance of configurations of triple $(3\hbar\omega)$ excitation. Decay strengths are found to be sensitive to the ratio a_p/a_d, causing some discrepancy with experiment (K59a).

In the N^{15} calculation Halbert and French paid particular attention to the suggestion of Lane (unpublished) that the $2s$ or $1d$ nucleon is very weakly coupled to the nuclear core. Such weak coupling results in states of unnatural parity in a nucleus with mass number $A + 1$ that are expressible simply as one state of the nucleus A vector coupled to a single

$2s$ or $1d$ orbital φ:

$$\psi_{A+1} = \{\psi_A, \varphi\} . \qquad \text{(IV.88)}$$

Contrary to this model, they find that the $(1s)^4$ $(1p)^{10}$ core in N^{15} is not strongly associated with a particular state of N^{14}.

Examples of this weak coupling have been found, however, in other nuclei. In a $B^{10}(d,p)B^{11}$ experiment by Bilaniuk and Hensel (B58a) a pair of single-particle s levels were observed at 9.19 and 9.28 MeV. Bilaniuk and French (B60a) have interpreted these as 7/2+ and 5/2+ levels involving a $2s_{1/2}$ orbital weakly coupled to the $J = 3+$ ground state of N^{14} to give an "s-particle doublet." Kurath and Lawson (K61a) have found evidence for such weakly coupled states in the spectrum of C^{13}.

We may conclude from our survey of the intermediate coupling calculations that the Hamiltonian displayed in Eq. (IV.70) yields remarkably good agreement with the spectra of nuclei in the range $6 \leqslant A \leqslant 19$. This is surprising since all of the complex noncentral terms of the "exact" interaction given in Eq. (III.1) are represented only by the simple single-body $\mathbf{l}\cdot\mathbf{s}$ term. As we saw in Sec. IV.1, the Thomas term arising from the effective single-particle potential is too small by about a factor of 40 to yield the observed doublet splittings, and consequently, the $\mathbf{l}\cdot\mathbf{s}$ interaction considered here cannot be more than a caricature of the actual force which produces these splittings. Balashov (B59) has considered the effect of using a two-body spin-orbit force

$$(\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j)\cdot\mathbf{l}_{ij} \qquad \text{(IV.89)}$$

among the $1p$ nucleons in the Li isotopes [in addition to an effective single-body interaction that each $1p$ nucleon has with the $(1s)^4$ core]. Agreement is only slightly improved and this refinement does not appear to be an essential one. Feingold (F56) and Lyons (L57) have performed calculations on the Li isotopes with the tensor interaction, finding that the resultant spectra and mixing of states closely resemble those yielded by the simple $\mathbf{l}\cdot\mathbf{s}$ term. It appears, therefore, that the salient characteristics of the noncentral components of the actual nucleon–nucleon interaction may be characterized by the single-body $\mathbf{l}\cdot\mathbf{s}$ term used in the intermediate coupling calculations.

A detailed discussion of the origin of the doublet splitting in nuclei is reserved for Sec. VI.

4. Order of States Within a Configuration

Once a particular coupling scheme is decided upon, one may examine the order of levels for a particular nucleus in terms of the configurations dictated by the shell model. The success of this analysis will, in turn, reflect on the validity of the chosen mode of coupling as well as the general assumptions of the shell model. In this section we shall concentrate on jj coupling, since the results of the intermediate coupling calculations indicate that the effective $\mathbf{l}\cdot\mathbf{s}$ term is always strong enough to make a restriction to *pure LS* coupling unrealistic.

First, let us consider the ground-state coupling rules given in Sec. IV.1. For even–even nuclei the general rule $J = 0$ is an immediate consequence of the pairing effect. Each nucleon tends to be paired with a like nucleon in the same "j" shell to give $J_{12} = 0$, and so, quite naturally a $J = 0$ ground state is energetically favorable.

The ground-state rule holding for most odd A nuclei is equally well explained by this effect. All nucleons couple in pairs to give zero angular momentum except for the last nucleon added to the shell. The total angular momentum is then attributed to this last nucleon yielding $J = j$. A substantial number of exceptions to this rule do, however, exist. Typical examples in light nuclei were pointed out in Sec. I, where it was noted that $_{10}Ne_{11}$ and $_{11}Na_{12}$ both have $J = 3/2$ even, though there is no doubt that the $1d_{5/2}$ subshell is being filled. Throughout the periodic table, cases like this are present, where the odd group beyond a shell closure consists of three nucleons and the total angular momentum is $j - 1$ rather than j. Such exceptions may be explained (B53) in terms of the coupling of the odd group of particles to a strongly deformed nuclear core (this point will be elaborated on in Sec. VII).

Guided by the preceding coupling rules, one may view the ground-state angular momentum of an odd–odd nucleus to be simply the vector sum of the angular momenta of the last proton (j_p) and the last neutron (j_n) to be added to each shell, and consequently, lying between the limits

$$|j_p - j_n| \leqslant J \leqslant j_p + j_n . \qquad \text{(IV.90)}$$

Defining the Nordheim number

$$N \equiv j_p + j_n + l_p + l_n , \qquad \text{(IV.91)}$$

one may formulate the coupling rules (N50)

$$J = |j_p - j_n| \quad \text{if} \quad N \quad \text{is even} \qquad \text{(IV.92a)}$$

$$J > |j_p - j_n| \quad \text{if} \quad N \quad \text{is odd} . \qquad \text{(IV.92b)}$$

The shell model offers no interpretation of these rules unless one introduces the residual interaction between the proton and neutron into the problem. For this reason, odd–odd nuclei provide a unique oppor-

tunity to study the residual n–p interaction in the nucleus. The low-lying excited states of these nuclei may be regarded as arising from the recoupling of j_p and j_n, and configuration interaction probably plays a smaller role in determining the level order here than in any other nuclear species.

De-Shalit (D53) and Schwartz (S54) originally attempted to interpret Nordheim's rules in terms of a residual n–p interaction of the form

$$V_{12} = V_0(r_{12}) + V_1(r_{12})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 . \qquad \text{(IV.93)}$$

If one assumes that the low-lying spectra are simply due to the coupling of the last neutron and the last proton added to the shell, the matrix elements of interest are given by

$$\begin{aligned}&\langle j_p j_n JM|V_0 + \boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_n V_1|j_p j_n JM\rangle\\ &\quad = \langle j_p j_n JM|V_0 + V_1|j_p j_n JM\rangle\\ &\qquad - 2(2j_p+1)(2j_n+1)\begin{Bmatrix} l_p & l_n & J\\ j_n & j_p & \frac{1}{2}\end{Bmatrix}^2\\ &\qquad \times \langle l_p l_n JM|V_1|l_p l_n JM\rangle \qquad \text{(IV.94)}\end{aligned}$$

after the $\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_n$ term is disposed of by transforming to LS coupling [through Eq. (IV.64), and using the well-known relationship

$$\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\chi_{SM}(1,2) = 2[S(S+1) - \tfrac{3}{2}]\chi_{SM}(1,2) . \qquad \text{(IV.95)}$$

The 6-j symbol is simply related to the Racah function (E57a)

$$\begin{Bmatrix} a & b & c\\ d & e & f\end{Bmatrix} = (-1)^{a+b+d+e}W(abed;cf) . \qquad \text{(IV.96)}$$

The first term in Eq. (IV.94) yields the characteristic spectrum of a Wigner force, and always provides a $J = |j_p - j_n|$ ground state, as long as each of the Slater integrals of the force are negative. The first excited state almost always has $J = j_p + j_n$. The second term in Eq. (IV.94) is found to be much smaller than the first ($V_1 \sim 0.1V_0$) when doublet splittings in nuclei with j_p or $j_n = 1/2$ are examined; the variation of this term with J is much more pronounced, however, and thus the level order is largely determined by the square of the 6-j symbol (D60b):

$$\begin{aligned}\begin{Bmatrix} l_p & l_n & J\\ j_n & j_p & \frac{1}{2}\end{Bmatrix}^2 &= (1/\epsilon)[J(J+1) - (j_n - j_p)\\ &\quad \times (j_n - j_p \pm 1)] \text{ if } j_p = l_p \pm \tfrac{1}{2}\,, j_n = l_n \mp \tfrac{1}{2}\,,\\ &= (1/\epsilon)[(j_p + j_n)(j_p + j_n + 1) - J(J+1)]\\ &\qquad \text{if } j_p = l_p + \tfrac{1}{2}\,, j_n = l_n + \tfrac{1}{2}\,,\\ &= (1/\epsilon)[(l_p + l_n)(l_p + l_n + 1) - J(J+1)]\\ &\qquad \text{if } j_p = l_p - \tfrac{1}{2}\,, j_n = l_n - \tfrac{1}{2}\,, \qquad \text{(IV.97)}\end{aligned}$$

where

$$\epsilon = (j_p + l_p + \tfrac{1}{2})(j_p + l_p + \tfrac{3}{2})(j_n + l_n + \tfrac{1}{2}) \times (j_n + l_n + \tfrac{3}{2}) . \qquad \text{(IV.97a)}$$

Using a δ-function interaction

$$V_{12} = V_c[(1-\alpha) + \alpha\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2]\delta(\mathbf{r}_1 - \mathbf{r}_2)\,, \quad 0 \leqslant \alpha \leqslant 1\,, \qquad \text{(IV.98)}$$

it was found (D53, S54) that for even N ($j_p = l_p \pm \frac{1}{2}$, $j_n = l_n \mp \frac{1}{2}$), $J = |j_p - j_n|$ remains the ground state as α increases. Furthermore, it is an isolated ground state, the excited states being clustered at a relatively high excitation. For odd N ($j_p = l_p \pm \frac{1}{2}, j_n = l_n \pm \frac{1}{2}$), $J = |j_p - j_n|$ is still the ground state for $\alpha = 0$, but it rises rapidly with increasing α, and even for small values of α, $J = j_p + j_n$ is generally found to be the ground state. It is not an isolated ground state, however, as the excited levels are relatively nearby, explaining the "weakness" of Nordheim's rule for odd N.

A satisfactory interpretation of Nordheim's rules is thereby obtained in terms of a strongly attractive Wigner interaction plus a relatively weak spin-dependent interaction favoring triplet states.[21] These rules are not sufficiently refined to be generally applicable, however, since inherent within them is the assumption that J simply results from the coupling of a single j_p and j_n. We have already seen that exceptions to the rule $J = j$ for odd A nuclei are present, and consequently, it must not always be accurate to assume that in each odd group of an odd–odd nucleus all nucleons but one couple to zero. In other words the Nordheim rules are applicable only when each odd group has seniority 1.

Brennan and Bernstein (B60b) have formulated a revised set of coupling rules in terms of the odd group model. Here one assumes that the p protons in the j_p shell couple to angular momentum J_p, and likewise, the n neutrons in the j_n shell couple to J_n. Both J_p and J_n are deduced experimentally from the neighboring odd A nuclei. It is then found that if

$$(2j_p + 1 - 2p)(2j_n + 1 - 2n) > 0\,, \qquad \text{(IV.99)}$$

then one may substitute for Nordheim's rules the criteria:

$$J = |J_p - J_n| \quad \text{if } N \text{ is even}\,, \qquad \text{(IV.100a)}$$

$$J = |J_p \pm J_n| \quad \text{if } N \text{ is odd}\,. \qquad \text{(IV.100b)}$$

[21] Such an interaction characterizes the effect of the tensor force as well as the central forces. For the δ-function interaction in Eq. (IV.98), α is related to the force mixture parameters by $\alpha = \frac{1}{2}(H + B) = \frac{1}{2}(1 - W - M)$.

If one has a particle-hole configuration:

$$(2j_p + 1 - 2p)(2j_n + 1 - 2n) < 0\,, \quad \text{(IV.101)}$$

there does not appear to be a clear rule concerning the ground-state spin.[22] De-Shalit and Walecka (D60b) have demonstrated that the coupling rules in Eqs. (IV.100) follow simply from the examination of a few Racah coefficients, as long as the dominant contribution of the interaction comes from low multipoles of a Slater expansion.

The spectrum of excited states in odd–odd nuclei has been treated in considerable detail by de-Shalit and Walecka (D61). They find it useful to write the interaction in the form

$$V_{12} = V_0(r_1,r_2,\cos\theta_{12}) + \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 V_1(r_1,r_2,\cos\theta_{12})\,, \quad \text{(IV.102)}$$

and then derive an explicit expression for the distribution of the relative angle between two particles. In addition to the modified Nordheim rules given in Eqs. (IV.100), it is also possible to show that the ground-state angular momentum is even or odd according to whether the parity of the nucleus is odd or even.

In deriving the level order of excited states for a given nucleus, including as many nucleons as possible in the analysis, is found to be profitable. Certainly all nucleons in unfilled shells must be included, and all too often excitation of the core must be considered as well. Such considerations frequently yield additional selection rules (for seniority change, etc.), for electromagnetic transitions (F52a, T60), and significant corrections to the static magnetic and quadrupole moments (M51, F52e).

The single-particle model cannot be expected to predict reasonable spectra because it treats all states within a given configuration as being degenerate. The first logical step in removing this degeneracy is to introduce a two-body interaction between nucleons in unfilled shells. Edmonds and Flowers (E52) have demonstrated, by group theoretical methods, that the diagonal matrix elements of a pure Wigner interaction

$$V_W = \sum_{i<j} V_{ij}(r_{ij}) \quad \text{(IV.103)}$$

in a state of the configuration j^k characterized by the quantum numbers J,T,s, and t may be expanded into a series of the form

$$E(j^k;JTst) = \tfrac{1}{2}k(k-1)E_0 + \tfrac{1}{2}[\tfrac{3}{2}k - 2T(T+1) + k(j+1) - \tfrac{1}{2}g(\sigma)]E_1 + \cdots\,, \quad \text{(IV.104)}$$

where $g(\sigma)$ is the eigenvalue of the Casimir operator introduced in Sec. IV.2:

$$g(\sigma) = (2j+2)s - \tfrac{1}{2}s(s-1) + \tfrac{3}{2}s - 2t(t+1) \quad \text{(IV.105)}$$

and E_0, $E_1,\cdots E_{j-1/2}$ are certain linear combinations of Slater integrals. For short-range forces, E_0 and E_1 are much larger than the remaining terms of the series.[23] If one considers the more general interaction

$$V = \sum_{i<j}[W + MP_M(i,j) + HP_H(i,j) + BP_B(i,j)]V(r_{ij})\,, \quad \text{(IV.106)}$$

an expression in closed form has not been obtained. For the special case of δ-function interactions Elliott and Lane (E57) deduce

$$E(j^k;JTst) = (W + M - H - B) \times [(j+1)k - \tfrac{1}{2}s(2j-s)]E_0 + \text{small terms}\,, \quad \text{(IV.107)}$$

if all nucleons are of the same type ($T = k/2$, $t = s/2$). The difficulty in deriving a general expression lies in the fact that it is awkward to work with Majorana and Bartlett forces in jj coupling where the space and spin portions of the wave function are intractably coupled together. For zero-range forces, however, Majorana forces become equivalent to Wigner forces and Bartlett forces are equivalent to Heisenberg forces, nullifying the difficulty.

We see from Eq. (IV.104) that the seniority and isobaric spin quantum numbers play a dominant role in determining the order of levels within the configuration j^k, the total angular momentum J not even appearing in lowest order. This result cannot be taken too literally by itself, however, due to the fact that it has been derived for the very special conditions of ordinary forces of very short range. In order to test the implications of this, Edmonds and Flowers (E52a) have performed numerical calculations for the energy matrices in configurations j^k with $j = 3/2$, $5/2$, and $7/2$, using both a pure Majorana force and a Rosenfeld exchange mixture. For configurations consisting of only one type of particle, they find an isolated ground state of minimum seniority followed by clusterings of states with the same s and t. For configurations of both protons and neutrons, levels

[22] For a particle-hole interaction, the Wigner force becomes repulsive and the order of levels is inverted. This yields a clustering of levels near the ground state and makes the formulation of a definite rule quite difficult.

[23] In the limit of zero range, $E_0 = 1$, $E_1 = 2$, and $E_2 = 1/15$ (E52) in arbitrary units.

of high angular momentum and large seniority are found to occur quite close to the ground state. For very long ranges, states belonging to the same value of J become degenerate and the seniority quantum number loses its significance in determining level orders. The usefulness of the seniority does clearly persist, however, out to ranges of physical interest.

Calculations attempting to interpret nuclear spectra directly from a two-body interaction operator are strongly impeded by our ignorance of the details of this operator as well as the uncertainty in the parameters of the wave function. Furthermore, there is no guarantee that the interaction between two free nucleons is identical with that contributing the residual interaction in shell-model calculations. Indeed, it is likely that some part of the "true" two-body force is "used up" in determining the effective single-body potential that a particle sees in the shell model, while the remaining effects of this force show up as residual interaction. In order to bypass this difficulty, one may work through a shell with fractional-parentage coefficients as discussed in Sec. IV.2. The two-body matrix elements

$$(j^2 J_{12}|V_{12}|j^2 J_{12}) \qquad \text{(IV.108)}$$

may be determined from the spectrum of the two-body system, and the remaining matrix elements of configurations j^k with $k > 2$ then may be computed from Eq. (IV.34) [alternatively the matrix elements in Eq. (IV.108) could be determined by a least-squares fit to yield optimum agreement with all nuclei in the shell].

An analogous procedure has been formulated by Racah (R52a). Consider any two-body operator t_{ij}. In a configuration of k equivalent nucleons in a state j, one may write

$$(j^k J|\sum_{i<j} t_{ij}|j^k J) = \tfrac{1}{2}k(k-1)(j^k J|t_{12}|j^k J)\,. \qquad \text{(IV.109)}$$

This expression can be further simplified by

$$(j^k J|t_{12}|j^k J) = \sum_{J12} a(j^k;J_{12}J)(j^2 J_{12}|t_{12}|j^2 J_{12})\,, \qquad \text{(IV.110)}$$

where the sum extends over all values of J_{12} allowed in the two-body system and the coefficients a are related to the coefficients of fractional parentage. Now suppose there are N allowed values of J_{12}; we define N independent operators t^1_{ij}, $t^2_{ij}, \cdots t^N_{ij}$ such that we can express the matrix elements in Eq. (IV.108) as

$$(j^2 J_{12}|V_{12}|j^2 J_{12}) = \sum_{n=1}^{N} \alpha_n (j^2 J_{12}|t^n_{12}|j^2 J_{12})\,. \qquad \text{(IV.111)}$$

Since we have N such equations, we can solve for the N unknowns α_k. Combining Eqs. (IV.109), (IV.110), and (IV.111) we find

$$\begin{aligned}(j^k J|\sum_{i<j} V_{ij}|j^k J) &= \tfrac{1}{2}k(k-1)\\ &\times \sum_{J_{12}} a(j^k;JJ_{12})(j^2 J_{12}|V_{12}|j^2 J_{12})\\ &= \sum_{n=1}^{N} \alpha_n (j^k J|\sum_{i<j} t^n_{ij}|j^k J)\,. \qquad \text{(IV.112)}\end{aligned}$$

This relationship provides a greatly simplified method for working through the states of the configurations j^k. The α_n may be determined once again from the two-body system (or to give the best fit for the subshell). Then, if one has chosen a set of operators which have simple matrix elements, the needed energy levels can at once be calculated without the necessity of finding coefficients of fractional parentage. An obvious choice for t^1_{ij} is just the operator which counts pairs of particles and has expectation value $\frac{1}{2}k\ (k-1)$. Another choice is $t^2_{ij} = \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j$, yielding

$$\sum_{i<j} \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j \rightarrow T(T+1) - \tfrac{3}{4}k\,, \qquad \text{(IV.113)}$$

while a third selection could be either the Casimir operator, $g(\sigma)$, or the seniority operator Q.

Unfortunately, there are not always enough operators available with simple expectation values to completely characterize the spectrum. It is, however, always possible to find an expression for the barycentric energy

$$E(j^k,st) = \textstyle\sum_J (2J+1)E(j^k;JTst)/\sum_J (2J+1) \qquad \text{(IV.114)}$$

of a group of levels all having the same (s,t). We first note that in the configuration j^2 only three sets of (s,t) values are present:

$$(s,t) = (0,0),\ (2,0),\ \text{and}\ (2,1)\,. \qquad \text{(IV.115)}$$

It must then always be possible to write $E(j^2;\ st)$ in the form

$$\begin{aligned}E(j^2;st) &= a + b[T(T+1) - \tfrac{3}{2}]\\ &\quad + c[g(\sigma) - 4(j+1)]\,, \qquad \text{(IV.116)}\end{aligned}$$

where a, b, and c are parameters to be determined from the known spectrum. It then follows that

$$\begin{aligned}E(j^k;st) &= \tfrac{1}{2}k(k-1)a + b[T(T+1) - \tfrac{3}{4}k]\\ &\quad + c[g(\sigma) - 2k(j+1)]\,. \qquad \text{(IV.117)}\end{aligned}$$

This equation is not entirely satisfactory by itself since it only indicates the average position of states comprising a given symplectic multiplet (character-

ized by s and t). It can, however, be most useful in an investigation of ground-state binding energies. Even–even nuclei all have ground-state angular momentum $J = 0$, so that $(s,t) = (0,0)$. Furthermore, this is the only (0,0) state of the configuration. Likewise, odd A nuclei usually have the only (1,1) state of a configuration as the ground state. For odd–odd nuclei the ground state is not, generally, an isolated state in (s,t) and an average over a symplectic multiplet must be taken.

Talmi and Thieberger (T56a, T56b) have applied this method to the calculation of binding energies of light nuclei. They fit a function of the form

$$E(j^k;Tst) = kA + \tfrac{1}{2}k(k-1)a + [T(T-1) - \tfrac{3}{4}k]b + [g(\sigma) - 2k(j+1)]c\,, \qquad \text{(IV.118)}$$

where A represents the single-particle energy of a "j" nucleon in the independent-particle model. The results of applying Eq. (IV.118) to light nuclei are displayed in Tables IV-11 to IV-16. The Coulomb contribution to the binding energies for these nuclei was obtained by analysis of mirror systems and then substracted out. The agreement with experiment is found to be excellent in regions where jj coupling is expected to hold, the rms deviation being less than 1%. In nuclei where LS coupling is known to be clearly a better description, agreement is very poor. Thus, for $A = 6$ the experimental values cannot be obtained to within a factor of 2 and these nuclei were, in fact, excluded from the least-squares fit. Again, at the beginning of the $1d_{5/2}$ shell, where, as we saw in Sec. IV.3, intermediate coupling must be employed, comparison with experiment is not as good as it is later in the shell.

TABLE IV-11. Binding energies in the $1p_{3/2}$ shell relative to He^4. Parameters: $A = -0.738$, $a = 0.594$, $b = -5.137$, $2(j+1)c = -3.721$ (all in MeV).

Nucleus	J	Binding energy (MeV) Exp	Theoret (T56b)
$_2He_3$	$\frac{3}{2}$	−0.95	−0.74
$_2He_4$	0	0.93	3.99
$_3Li_3$	0	0.13	2.90
$_3Li_3$	1,3	2.17	4.24
$_3Li_4$	$\frac{3}{2}$	10.95	13.62
$_3Li_5$	2	12.98	13.56
$_3Li_6$	$\frac{3}{2}$	16.53	16.05
$_4Be_3$	$\frac{3}{2}$	9.30	11.92
$_4Be_4$	0	28.19	28.12
$_4Be_5$	$\frac{3}{2}$	29.85	29.76
$_4Be_6$	0	36.66	36.86
$_5B_5$	0	34.70	34.80
$_5B_5$	1,3	36.22	36.14
$_5B_6$	$\frac{3}{2}$	47.90	47.90
$_6C_4$	0	32.02	32.13
$_6C_5$	$\frac{3}{2}$	45.14	45.23
$_6C_6$	0	63.85	63.80

TABLE IV-12. Binding energies in the $1p_{1/2}$ shell relative to C^{12}. Parameters: $A = 5.337$, $a = 2.912$, $b = -1.085$ (all in MeV).

Nucleus	J	Binding energy (MeV) Exp	Theoret (T56b)
$_6C_7$	$\frac{1}{2}$	4.95	5.34
$_6C_8$	0	13.12	13.04
$_7N_6$	$\frac{1}{2}$	1.95	2.31
$_7N_7$	0	10.18	10.02
$_7N_7$	1	12.49	12.19
$_7N_8$	$\frac{1}{2}$	23.32	23.35
$_8O_6$	0	6.54	6.48
$_8O_7$	$\frac{1}{2}$	19.79	19.81
$_8O_8$	0	35.43	35.51

TABLE IV-13. Binding energies in the $1d_{5/2}$ shell relative to O^{16}. Parameters: $A = 4.565$, $a = 0.302$, $b = 3.205$, $2(j+1)c = -2.909$ (all in MeV).

Nucleus	J	Binding energy (MeV) Exp	Theoret (T56b)
$_8O_9$	$\frac{5}{2}$	4.14	4.56
$_8O_{10}$	0	12.21	13.65
$_8O_{11}$	$\frac{5}{2}$	16.16	15.61
$_9F_8$	$\frac{5}{2}$	0.59	0.95
$_9F_{10}$	$\frac{5}{2}$	19.97	21.62
$_{10}Ne_9$	$\frac{5}{2}$	15.85	17.49
$_{10}Ne_{10}$	0	33.05	33.59
$_{10}Ne_{11}$	$\frac{5}{2}$	39.46	39.36
$_{10}Ne_{12}$	0	50.17	49.65
$_{10}Ne_{13}$	$\frac{5}{2}$	55.36	52.83
$_{11}Na_{12}$	$\frac{5}{2}$	58.53	58.03
$_{12}Mg_{12}$	0	70.66	70.41
$_{12}Mg_{13}$	$\frac{5}{2}$	77.99	77.40
$_{12}Mg_{14}$	0	89.10	88.90
$_{13}Al_{12}$	$\frac{5}{2}$	72.96	72.18
$_{13}Al_{13}$	0	84.09	83.69
$_{13}Al_{13}$	1,3,5	84.03	83.44
$_{13}Al_{14}$	$\frac{5}{2}$	97.36	97.68
$_{14}Si_{13}$	$\frac{5}{2}$	91.75	91.96
$_{14}Si_{14}$	0	108.95	110.47

TABLE IV-14. Binding energies in the $2s_{1/2}$ shell relative to Si^{28}. Parameters: $A = 8.606$, $a = 1.978$, $b = -0.316$ (all in MeV).

Nucleus	J	Binding energy (MeV) Exp	Theoret (T56b)
$_{14}Si_{15}$	$\frac{1}{2}$	8.47	8.61
$_{14}Si_{16}$	0	19.09	19.03
$_{15}P_{14}$	$\frac{1}{2}$	2.79	2.81
$_{15}P_{15}$	0	13.28	13.24
$_{15}P_{15}$	1	13.97	13.87
$_{15}P_{16}$	$\frac{1}{2}$	26.38	26.43
$_{16}S_{15}$	$\frac{1}{2}$	20.16	20.21
$_{16}S_{16}$	0	35.24	35.22

The remarkable agreement obtained in most cases is strong confirmation that jj coupling applies and that the quantum numbers T, s, and t do characterize the ground states. Furthermore, it would appear that the central field approximation holds up very well

TABLE IV-15. Binding energies in the $1d_{3/2}$ shell relative to S^{32}. Parameters: $A = 8.658$, $a = 0.118$, $b = -1.815$, $2(j+1)c = -1.759$ (all in MeV).

Nucleus	J	Binding energy (MeV) Exp	Theoret (T56b)
$_{16}S_{17}$	$\frac{3}{2}$	8.65	8.66
$_{16}S_{18}$	0	20.05	20.04
$_{16}S_{19}$	$\frac{3}{2}$	27.07	27.12
$_{16}S_{20}$	0	36.97	36.93
$_{17}Cl_{16}$	$\frac{3}{2}$	2.42	2.38
$_{16}Cl_{17}$	0	13.75	13.77
$_{17}Cl_{18}$	$\frac{3}{2}$	26.46	26.29
$_{17}Cl_{19}$	2	35.03	35.10
$_{17}Cl_{20}$	$\frac{3}{2}$	45.39	45.23
$_{18}Ar_{17}$	$\frac{3}{2}$	19.69	19.58
$_{18}Ar_{18}$	0	34.96	34.83
$_{18}Ar_{19}$	$\frac{3}{2}$	43.79	43.96
$_{18}Ar_{20}$	0	55.54	55.82
$_{19}K_{18}$	$\frac{3}{2}$	36.88	37.00
$_{19}K_{19}$	0	48.88	48.86
$_{19}K_{20}$	$\frac{3}{2}$	61.96	61.85
$_{20}Ca_{19}$	$\frac{3}{2}$	54.32	54.44
$_{20}Ca_{20}$	0	70.29	70.17

and, in fact, the field parameters remain reasonably constant throughout a given subshell. In this respect, nuclear spectroscopy is basically simpler than the corresponding atomic problem where the Slater integrals change when electrons are added to the system. Arima (A58) has repeated this analysis in the $2s_{1/2}$ and $1d_{3/2}$ shells using fractional-parentage

TABLE IV-16. Binding energies of the calcium isotopes relative to Ca^{40}. Parameters: $A = 8.681$, $a = 0.155$, $b = -1.185$, $2(j+1)c = -2.053$ (all in MeV).

Nucleus	J	Binding energy (MeV) Exp	Theoret (T56b)
$_{20}Ca_{21}$	$\frac{7}{2}$	8.37	8.68
$_{20}Ca_{22}$	0	19.85	21.03
$_{20}Ca_{23}$	$\frac{7}{2}$	27.78	28.84
$_{20}Ca_{24}$	0	38.96	40.31
$_{20}Ca_{25}$	$\frac{7}{2}$	46.38	47.24
$_{20}Ca_{26}$	0	57.17	57.84
$_{20}Ca_{27}$	$\frac{7}{2}$	63.87	63.90
$_{20}Ca_{28}$	0	73.77	73.63

coefficients, obtaining equally good agreement with experiment.

When one attempts to do similar calculation for nuclei in which the last neutron occupies a shell different from that of the last proton, a number of technical difficulties present themselves. The isobaric spin T can no longer be introduced as a good quantum number, and so the sum

$$\sum_{i<j} \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j \tag{IV.119}$$

will no longer have meaningful expectation values in the states under consideration. More parameters are needed to describe the situation, and to make matters worse (especially in light nuclei), experimental information is scarce. Goldstein and Talmi (G59a) treat systems with m protons in the j shell and n neutrons in the (higher) j' shell with an expression of the form

(binding energy of nucleus treated)
$-$ (binding energy of preceding closed shells) (IV.120)

$$= E(j^m) + V(j^m,j'^n) + n\alpha_{j'} + V(j'^n) ,$$

where $E(j^m)$ is the single-particle energy of the m protons and is taken from the work of Talmi and Thieberger (T56b); $V(j^m,j'^n)$ is the interaction energy between the protons and the neutrons; $\alpha_{j'}$ is the kinetic energy of a neutron; and $V(j'^n)$ is the interaction energy between the neutrons. The $\alpha_{j'}$ cannot be taken from reference T56b, since these were determined with a closed j shell for the protons, and must therefore be treated as new parameters. The $V(j'^n)$ were taken from reference T56b. The interaction energy $V(j^m,j'^n)$ may be expressed in terms of two-body matrix elements

$$V(j',j,J) = V_J \tag{IV.121}$$

using fractional-parentage coefficients. Application is made to nuclei with protons in the $2s_{1/2}$ shell and neutrons in the $1d_{3/2}$ shell, and those with protons in the $1d_{3/2}$ shell and neutrons in the $1f_{7/2}$ shell. Comparison with experiment is on par with that obtained by Talmi and Thieberger (T56b).

A simple and elegant example of this sort of analysis has been made by Goldstein and Talmi (G56b) and independently by Pandya (P56). Consider the low-lying states of $_{19}K_{21}$ and $_{17}Cl_{21}$. In both cases one has a single neutron in the $f_{7/2}$ shell, while in $_{17}Cl_{21}$ one has a single proton in the $d_{3/2}$ shell, and in $_{19}K_{21}$ a single hole in the $d_{3/2}$ shell. A theorem by Racah (R43, B59a) states that matrix elements of a tensor of odd rank are unchanged if one replaces all particles by holes, while for an even-rank tensor this operation will involve a sign change. Using an explicit analytic form for the coefficients of fractional parentage, a sum over the parent states of a hole configuration may be carried out, leaving a sum over particle states:

$$E_J[j^{-1}j'] = -\sum_{J_0} (2J_0+1) W(jj'j'j;JJ_0) E_{J_0}[jj'] , \tag{IV.122a}$$

and likewise, one has a reciprocal relation

$$E_J[jj'] = -\sum_{J_0}(2J_0+1)W(jj'j'j;JJ_0)E_{J_0}[j^{-1}j'] . \quad \text{(IV.122b)}$$

In a $(d_{3/2})^{\pm 1}f_{7/2}$ configuration one expects states with $J = 5$, 4, 3, and 2 all having negative parity. Utilizing Eqs. (IV.122) the energy levels of $_{17}Cl_{21}$ and $_{19}K_{21}$ may be related to each other in a simple linear manner to an accuracy of about 3%.

The fact that the isobaric spin cannot be regarded as a good quantum number in nuclei where different proton and neutron shells are being filled (which we previously referred to as a difficulty), can, in special cases, prove useful. French (F61) illustrates this by comparing the properties of $_{17}Cl_{17}$ with $_{17}Cl_{21}$. We have already pointed out that in $_{17}Cl_{21}$ the low-lying levels can be attributed to a configuration in which one has one proton in the $1d_{3/2}$ shell and one neutron in the $1f_{7/2}$ shell (the isobaric spin is not a good quantum number). In $_{17}Cl_{17}$ one expects a group of excited levels with one nucleon in each of the orbits $1d_{3/2}$ and $1f_{7/2}$; but now, due to the removal of the closed $1d_{3/2}$ neutron shell, the isobaric spin is again a good quantum number and each level in $_{17}Cl_{17}$ breaks into two levels

$$\psi_J(T=0) = (1/\sqrt{2})[\varphi_J(1,2) + \varphi_J(2,1)] , \quad \text{(IV.123a)}$$

$$\psi_J(T=1) = (1/\sqrt{2})[\varphi_J(1,2) - \varphi_J(2,1)] , \quad \text{(IV.123b)}$$

corresponding to a state in $_{17}Cl_{21}$ described simply by $\varphi(1,2)$. Consequently, if interaction with the closed $1d_{3/2}$ neutron shell is suppressed, a group of the energy levels in $_{17}Cl_{21}$ may be expressed as a simple linear combination of the levels in $_{17}Cl_{17}$:

$$E_J(_{17}Cl_{21}) = \tfrac{1}{2}[E_{J,T=0}(_{17}Cl_{17}) + E_{J,T=1}(_{17}Cl_{17})] . \quad \text{(IV.124)}$$

Unfortunately, the needed levels in $_{17}Cl_{17}$ have not yet been observed, so that Eq. (IV.124) is still experimentally unverified.

It is more difficult to examine levels arising from excited configurations due to the spurious states involving center-of-mass motion and the increasing number of parameters. Unna and Talmi (U58a, T60) have performed a detailed analysis of levels in the $1p_{1/2}$ region which appear to come from the promotion of one or more p nucleons to the $2s$ $1d$ shell. It is frequently found that excited states of unnatural parity in the $1p$ shell may be attributed to the promotion of a $1p$ nucleon to the $2s_{1/2}$ orbital; in fact, it appears that the $2s_{1/2}$ state lies significantly lower than the $1d_{5/2}$ state in this region. Therefore, in a first analysis, only excitations to the $2s_{1/2}$ state were examined. We shall discuss only three of the examples treated; the 0^+ state of O^{16} at 6.06 MeV, and the low $\frac{1}{2}-$ levels of O^{17} and F^{19}.

The first excited state of O^{16} has long been a problem to nuclear theorists. When two $1p_{1/2}$ nucleons are promoted to $2s_{1/2}$ orbitals, Unna and Talmi find two 0^+ excited states lying 13.81 and 16.91 MeV above the ground state. In order to obtain a 0^+ state near the observed 6.06 MeV excitation, they found it necessary to excite four $1p_{1/2}$ nucleons to the $2s_{1/2}$ orbit (obtaining an excitation energy of 6.90 MeV). The reason for this follows from the fact that, using their parameters, the attraction between a pair of $p_{1/2}$ nucleons and a pair of nucleons in the next shell is much smaller than the attraction of two pairs in the same orbit. The $\frac{1}{2}-$ level in O^{17} is similarly interpreted as a $(1p_{1/2})(2s_{1/2})^4$ configuration; while in F^{19} the $\frac{1}{2}-$ state may be explained by raising a single $1p_{1/2}$ nucleon to the $2s_{1/2}$ orbit so that the configuration consists of a $1p_{1/2}$ hole.

The calculations reviewed in this section present a picture of nuclear structure which is startling in its simplicity. In many regions it appears that the nuclear wave function is well represented by jj-coupled single-particle orbitals for which J, T, s, and t are good quantum numbers. In addition, no variation of the radial part of these functions within a subshell is required. Thieberger (T59) has investigated the nature of the two-body interaction that will produce the matrix elements derived in this work. When a hard repulsive core is included in the interaction, it is found that the best force mixture is not far from the Serber type. Furthermore, three-body correlations may be included that yield improved agreement with experiment.

Perhaps the most surprising feature of these calculations is the fact that one need not introduce any configuration interaction to obtain good agreement for binding energies. Upon retrospect, however, one sees that this only means that the two-body matrix elements are always effected by configuration interaction in the same way, that is, independent of the number of nucleons in the shell. The most rigorous test of the derived wave functions is not through their energy eigenvalue, but in examining the expectation values for other nuclear properties such as the static electromagnetic moments and transition rates. Here the individual particle model, in which all nucleons are taken into account, generally provides a better description of the observed system than the single particle model. Nevertheless, significant dis-

crepancies still persist, and one must proceed to take configuration interaction into account.

5. Configuration Interaction

So far we have studied shell-model calculations involving pure configurations:

$$\prod_i (j_i)^{ki} . \qquad \text{(IV.125)}$$

The notion that the true wave function can be reasonably well represented by a single configuration is of course the foundation of the independent particle model. Indeed, we have seen that this assumption must contain a considerable amount of truth since it permits one to interpret a great many features of nuclear spectra.

An exact description of a nuclear level may be obtained in terms of a linear superposition of all possible configurations:

$$\psi = \varphi_0[\prod_i (j_i)^{k_i}] + \sum_{n_i \neq k_i} a_n \varphi_n[\prod_i (j_i)^{n_i}] , \qquad \text{(IV.126)}$$

where $\sum_i n_i = \sum_i k_i = A$, and n symbolically stands for the set of integers n_i. The fact that a consistent picture of nuclear level orders may be obtained with only the leading term φ_0 of this expansion, could indicate that the $|a_n|^2$ are, in fact, all much less than one. Such an inference may, however, not be entirely correct. The only conclusion that can definitely be drawn is that configuration interaction does not play a vital role in determining level orders. This may be due to the fact that each level is effected in approximately the same way by configuration mixing, and then this mixing need not be small.

A more crucial test of the importance of mixed configurations lies in an examination of nuclear properties other than the energy. In calculations of binding energies, the variational stability of the wave function is on the side of the theorist, and it is well known that an acceptable energy eigenvalue may be obtained from a fairly poor wave function.

The static electromagnetic moments provide a useful criterion for the validity of the wave function. If we consider an odd A nucleus with N_0 (odd) nucleons in one group and N_e (even) nucleons in the other group (all filling the j shell), the magnetic moment is given by (T60, B56a)

$$\mu = \beta\mu_e + (1 - \beta)\mu_0 \qquad \text{(IV.127)}$$

if $s = 1$ and $T = |N_e - N_0|/2$. μ_e is the Schmidt moment of a nucleon in the even group, μ_0 is the Schmidt moment of a nucleon in the odd group, and

$$\beta = N_e/(2j + 2)(2T + 2) \qquad \text{if } N_0 > N_e \qquad \text{(IV.127a)}$$

$$\beta = [(2j + 1) - N_e]/(2j + 2)(2T + 2) \text{ if } N_e > N_0 . \qquad \text{(IV.127b)}$$

Magnetic moments for light nuclei are displayed in Table IV-17. Schmidt moments (μ_s) and experimental values are compared with the moment obtained on the basis of the individual particle model where all nucleons in the j shell are taken into account

TABLE IV-17. Magnetic moments in the individual particle model (F52e).

Element	Z	N	Configuration	T	μ_S	μ_I	μ_{exp}
Li	3	4	$(p_{3/2})^3$	$\frac{1}{2}$	3.79	3.04	3.26
Be	4	5	$(p_{3/2})^{-3}$	$\frac{1}{2}$	−1.91	−1.16	−1.18
B	5	6	$(p_{3/2})^{-3}(p_{1/2})^2$	$\frac{1}{2}$	3.79	3.04	2.69
C	6	7	$p_{1/2}$	$\frac{1}{2}$	0.64	0.64	0.70
N	7	8	$(p_{1/2})^{-1}$	$\frac{1}{2}$	−0.26	−0.26	−0.28
O	8	9	$d_{5/2}$	$\frac{1}{2}$	−1.91	−1.91	−1.89
F	9	10	$(s_{1/2})^3$	$\frac{1}{2}$	2.79	2.79	2.63
Mg	12	13	$(d_{5/2})^{-3}$	$\frac{1}{2}$	−1.91	−0.64	−0.96
Cl	17	18	$(d_{3/2})^3$	$\frac{1}{2}$	0.13	0.26	0.82
Cl	17	20	$(d_{3/2})^{-3}$	$\frac{3}{2}$	0.13	0.13	0.68

(F52e). We see that, in general, taking all particles into account improves the theoretical estimate, and in no case yields a poorer value than the single-particle moment. Discrepancies still persist, however, so that one must seek to improve the wave functions further.[24] The need for introducing mixed configurations is also clearly seen in the analysis of stripping reactions (M60), internal conversion coefficients (K59b), and electromagnetic transitions (G59a).

Transition probabilities (both for β and γ decay) furnished early indications of the importance of configuration mixing (D53a). In many nuclei one observes transitions which are forbidden if the shell model assignments are taken literally, while in other cases, experimental half-lives are orders of magnitude greater than shell-model predictions. A famous illustration of this is the beta decay of C^{14}:

$$_6C_8 \rightarrow {}_7N_7 + \beta^- . \qquad \text{(IV.128)}$$

The half-life for this transition is extraordinarily long with $\log ft = 9.03$. The decay is expected to be superallowed ($J\pi = 0+$ in C^{14}, and $J\pi = 1+$ in N^{14}) with $\log ft \sim 5$. The only interpretation of this reduction in the transition probability is that somehow cancellation occurs in the nuclear matrix element. If one adheres to the shell-model configuration at $A = 14$

$$(1s)^4(1p)^{10} \qquad \text{(IV.129)}$$

[24] Several possibilities exist for correcting the magnetic moments other than improving the nuclear wave functions (B56a). None of these show promise of resolving the problem, however, and we shall not consider them here.

the wave functions for C^{14} and N^{14} may be written as the linear combinations

$$\psi_{T=0}({}_7N_7) = a\varphi({}^3S_1) + b\varphi({}^1P_1) + c\varphi({}^3D_1) \quad \text{(IV.130a)}$$

$$\psi_{T=1}({}_6C_8) = a'\varphi({}^1S_0) + b'\varphi({}^3P_0) \ . \quad \text{(IV.130b)}$$

Now the Gamow–Teller matrix element can be written symbolically in the form (F55):

$$(N^{14}|\mathfrak{o}|C^{14}) = \bar{a}a'({}^3S_1|\mathfrak{o}|{}^1S_0) + \bar{b}b'({}^1P_1|\mathfrak{o}|{}^3P_0) \ , \quad \text{(IV.131)}$$

which could be made to vanish for a proper choice of the amplitude factors. An intermediate coupling calculation cannot produce the required cancellation, but it is possible to obtain the required relationship between the amplitudes if one considers the effect of the tensor operator

$$S_{12} = \mathfrak{o}_1 \cdot \mathbf{r}_{12}\mathfrak{o}_2 \cdot \mathbf{r}_{12} - \tfrac{1}{3}\,\mathfrak{o}_1 \cdot \mathfrak{o}_2 \quad \text{(IV.132)}$$

in mixing the states of the shell-model configuration (E56). Visscher and Ferrell (V55) have demonstrated that the strength of the tensor interaction may be determined to produce the desired cancellation. The tensor-force matrix element that is operative in producing the reduction in the beta-decay transition rate may be expressed as the difference between two radial integrals. If the tensor interaction is assumed to have a Yukawa radial dependence, the difference between these two integrals is rather small and one must assume a depth for the interaction so large that one no longer obtains agreement in the ordering of the states of the spectrum. On the other hand, if one uses a field-theoretic dependence for the shape of the tensor interaction

$$e^{-r/a}[a/r + 3(a/r)^2 + 3(a/r)^3] \ , \quad \text{(IV.133)}$$

the required cancellation may be obtained without other complications. Since the radial shape of the tensor interaction is not certain, interpretation of the large *ft* value in C^{14} in terms of the cancellation required in Eq. (IV.131) is in considerable doubt.

Baranger and Meshkov (B58b) have proposed a solution to the problem that appears to have more experimental justification. An analysis of C^{14} (d,t) C^{13} reactions indicate considerable configuration mixing in the C^{14} ground state. The wave functions for C^{14} and N^{14} are then assumed to be of the form

$$\psi = \alpha\varphi[(1s)^4(1p)^{10}] + \beta\varphi[(1s)^4(1p)^8(2s)^2] + \gamma\varphi[(1s)^4(1p)^8(1d)^2] \ . \quad \text{(IV.134)}$$

Amplitude coefficients consistent with the long β-decay half-life are in agreement with the reaction data, although large uncertainty exists in their exact value. Probably the proper resolution of this problem requires both configuration mixing and the tensor interaction.

It is reasonable to expect that configuration mixing will become more prominent in highly excited levels. The ground state should be well represented by a single configuration, if such an assumption is to hold at all. Thus it seems that a systematic investigation of configuration interaction should begin with calculations on ground-state wave functions with the purpose of improving the comparison with static electromagnetic moments. Another advantage in this procedure is that these moments involve the expectation value of a sum of single-particle operators

$$M_{\lambda\mu} = \sum_{i=1}^{A} \mathfrak{M}_{\lambda\mu}(i) \ . \quad \text{(IV.135)}$$

If the mixing coefficients a_n in Eq. (IV.126) are small enough so that, to a first approximation, one need consider only corrections linear in a_n, the expectation value of $M_{\lambda\mu}$ becomes

$$\langle M_{\lambda\mu}\rangle = \langle 0|M_{\lambda\mu}|0\rangle + 2\sum_{n\neq 0} a_n\langle 0|M_{\lambda\mu}|n\rangle \ , \quad \text{(IV.136)}$$

where we have used the fact that $M_{\lambda\mu}$ is Hermitian. The simplification comes from the fact that since $M_{\lambda\mu}$ is the sum of single-particle operators, the states $|n>$ which make a nonvanishing contribution may involve configurations different from $|0>$ by, at most, one nucleon orbital.

We consider then, first of all, configurations in which one nucleon from the shell-model ground-state configuration is promoted from an orbital (jl) to a higher orbital $(j'l')$. It is always prudent to consider a closed core of nucleons whose configuration remains fixed because of its great stability. Certainly this cannot be too far from the case if the core is doubly magic. In most cases it is practical to only consider configuration mixing for the nucleons in the last unfilled subshells, but unfortunately this may not always be the case. To illustrate this, let us examine the situation for ${}_{14}Si_{15}$. The shell-model configuration for the ground state is

$$(1s_{1/2})^4(1p_{3/2})^8(1p_{1/2})^4(1d_{5/2})^{12}(2s_{1/2}) \ . \quad \text{(IV.137)}$$

It is tempting to consider the $1d_{5/2}$ shell and all shells below it as an undistorted core, but this is clearly unrealistic. The $2s_{1/2}$ nucleon will only mix with higher s orbitals, which amounts to no more than improving the radial wave function. That some distortion of the core is required, clearly follows from the fact that the magnetic moment of ${}_{14}Si_{15}$ is far

from the Schmidt line (see Table I-4). One must therefore mix in components of

$$(1s_{1/2})^4(1p_{3/2})^8(1p_{1/2})^4(1d_{5/2})^{11}(2s_{1/2})(1d_{3/2}), \quad \text{(IV.138a)}$$

$$(1s_{1/2})^4(1p_{3/2})^8(1p_{1/2})^4(1d_{5/2})^{10}(2s_{1/2})^3\,, \quad \text{(IV.138b)}$$

and perhaps, even higher configurations.

One might suspect that the configuration in (IV.138a) is particularly important because of the strong radial overlap between $1d_{5/2}$ and $1d_{3/2}$ orbitals. Because of such overlap considerations, it is never really safe to neglect distortion of a core unless it is a closed l shell as well as being a closed j shell, in spite of the large separation between $j = l + \frac{1}{2}$ and $j' = l - \frac{1}{2}$ orbitals due to spin-orbit interaction. Only three such doubly magic cores exist beyond the α particle ($_8O_8$, $_{20}Ca_{20}$, and $_{92}Pb_{126}$); and even for the α-particle configuration mixing is suspected to be $\sim 10\%$.

The mixing of $j = l + \frac{1}{2}$ and $j' = l - \frac{1}{2}$ orbitals will be the most significant in connection with the magnetic-dipole-moment corrections; as well as being anticipated as the most prominent due to overlap of the radial functions. This is easily seen from an examination of the magnetic-moment operator

$$\boldsymbol{\mu}_{op} = \sum_i (g_s^i \mathbf{s}_i + g_l^i \mathbf{l}_i)\,, \quad \text{(IV.139)}$$

which is obviously diagonal in the single-particle l values. Contributions to

$$\langle \boldsymbol{\mu}_{op} \rangle = \langle 0|\boldsymbol{\mu}_{op}|0\rangle + 2\sum_{n\neq 0} a_n \langle 0|\boldsymbol{\mu}_{op}|n\rangle \quad \text{(IV.140)}$$

will come from states $|n>$, where only one nucleon orbital differs from the configuration for the state $|0>$, and this orbital must have the same l value as the original one.

Blin-Stoyle and Perks (B53a, B54) have investigated corrections of this type and found that they generally lead to displacements from the Schmidt moments toward the experimental values. In particular, it was shown that the magnetic moments of $p_{1/2}$ nuclei should not deviate significantly from the Schmidt values, whereas a considerable correction is expected for $s_{1/2}$ nuclei. This corresponds rather well to the observed situation and lends considerable weight to the theory.

More extensive calculations throughout the periodic table have been made by Noya, Arima, and Horie (N59a, H55, A54). Using a two-body interaction of the form

$$V_{12} = (V_0 + V_1 \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\delta(\mathbf{r}_{12})\,, \quad \text{(IV.141)}$$

configuration mixing was computed by second-order perturbation theory. A summary of the results for light nuclei is given in Table IV-18. The consistent improvement over the single-particle moments and the results for the individual-particle model (Table IV-17), verifies the necessity of introducing configuration interaction into the wave functions.

TABLE IV-18. Magnetic moments by configuration mixing.

Nucleus	$J\pi$	Proton configuration	Neutron configuration	Schmidt value	$\mu_{theor.}$ (N59a)	μ_{exp}
$_9F_{10}$	$\frac{1}{2}+$	$2s_{1/2}$	$(1d_{5/2})^2$	2.79	2.73	2.63
$_{15}P_{16}$	$\frac{1}{2}+$	$2s_{1/2}$	...	2.79	1.69	1.13
$_{14}Si_{15}$	$\frac{1}{2}+$	...	$2s_{1/2}$	−1.91	−0.93	−0.56
$_{17}Cl_{18}$	$\frac{3}{2}+$	$1d_{3/2}$	$(1d_{3/2})^2$	0.12	0.63	0.82
$_{17}Cl_{20}$	$\frac{3}{2}+$	$1d_{3/2}$	...	0.12	0.59	0.68
$_{19}K_{20}$	$\frac{3}{2}+$	$(1d_{3/2})^3$		0.12	0.12	0.39
$_{19}K_{22}$	$\frac{3}{2}+$	$(1d_{3/2})^3$	$(1f_{7/2})^2$	0.12	0.14	0.21
$_{16}S_{17}$	$\frac{3}{2}+$		$1d_{3/2}$	1.15	0.65	0.63
$_{16}S_{19}$	$\frac{3}{2}+$		$(1d_{3/2})^3$	1.15	1.03	1.00
$_{13}Al_{14}$	$\frac{5}{2}+$	$(1d_{5/2})^5$	$(1d_{5/2})^6$	4.79	3.39	3.64
$_8O_9$	$\frac{5}{2}+$		$1d_{5/2}$	−1.91	−1.91	−1.89
$_{12}Mg_{13}$	$\frac{5}{2}+$	$(1d_{5/2})^4$	$(1d_{5/2})^5$	−1.91	−0.76	−0.96

On the other hand, configuration interaction does not yield as consistent a picture of the electric quadrupole moments (B59b). This is, perhaps, to be expected, since the quadrupole-moment operator connects single-particle states differing by as many as two units of orbital angular momentum. The collective model, which takes into account large distortions of the core, is needed here.

The calcium isotopes provide excellent objects for more extensive calculations. The $_{20}Ca_{20}$ core fulfills the requirement of being a closed l shell as well as a closed j shell, and jj coupling appears to be valid in this general region (at least for ground states). Levinson and Ford (L55b, F55b, L55c), and Mitler (M61a) have formulated a procedure for deriving the properties of $_{20}Ca_{23}$ from the single-particle spectrum of $_{20}Ca_{21}$ and two-body matrix elements determined from $_{20}Ca_{22}$.

Consider first the following treatment for two interacting particles in a central field (L55b). We write the Schrödinger equation in the form

$$(E - H_0)|\psi\rangle = V|\psi\rangle\,, \quad \text{(IV.142)}$$

where H_0 is the Hamiltonian for the central field due to the core and V represents the two-body interaction. Abbreviating the LS and jj representations by

$$|l_1 l_2 j_1 j_2 JM\rangle = |j_1 j_2 J\rangle \quad \text{(IV.143a)}$$

$$|l_1 l_2 LSJM\rangle = |LSJ\rangle\,, \quad \text{(IV.143b)}$$

and introducing the zero-order solutions

$$H_0|j_1 j_2 J\rangle = \epsilon_J(j_1 j_2)|j_1 j_2 J\rangle\,, \quad \text{(IV.143c)}$$

Eq. (IV.142) can be written in the jj representation as

$$\langle j_1 j_2 J|\psi\rangle = \frac{\langle j_1 j_2 J|V|\psi\rangle}{E - \epsilon_J(j_1 j_2)} . \qquad \text{(IV.144)}$$

With the aid of the unitary transformation coefficients between the jj and LS schemes, Eq. (IV.144) may be written as

$$\sum_{l'_1 l'_2} \left\{ \langle l_1 l_2 LSJ|V|l'_1 l'_2 LSJ\rangle \sum_{j'_1 j'_2} \frac{|\langle l_1 l_2 LSJ|l_1 l_2 j'_1 j'_2 J\rangle|^2}{E - \epsilon_J(j'_1 j'_2)} - \delta(l'_1,l_1)\delta(l'_2,l_2) \right\} \langle l'_1 l'_2 LSJ|\psi\rangle = 0 . \qquad \text{(IV.145)}$$

The determinant of the coefficients inside the bracket is then equated to zero and solved for the $\epsilon_J(j_1 j_2)$. We note that the matrix elements of the two-body interaction appear in the LS representation where they are relatively easy to calculate. Fractional parentage coefficients are employed for extrapolations to systems of more than two nucleons. A Serber exchange mixture was assumed, and since the systems examined contained only neutrons beyond the closed shells, one deals with a pure singlet interaction.

It was further assumed that the radial parts of the $1f$ and $1g$ orbitals overlap 100%, whereas the $1f$ and $2p$ orbitals do not overlap at all (excluding $2p$ orbitals from the analysis). Only four level positions in $_{20}Ca_{22}$ were used for the analysis (0+, 2+, 4+, and 6+), and so in order to find the nine Slater integrals $F^k(k = 0, 1, \cdots 8)$, Levinson and Ford assumed that these integrals varied smoothly with k and expressed F^1, F^3, F^5, F^7, and F^8 in terms of F^0, F^2, F^4, and F^6. The level positions and the ground state magnetic moment of $_{20}Ca_{23}$ were then deduced to within 1% of the experimental values. Amplitude coefficients for the mixed configurations in $_{20}Ca_{22}$ and $_{20}Ca_{23}$ are displayed in Tables IV-19 and IV-20.

TABLE IV-19. Configuration mixing in Ca^{42} based on semi-emperical analysis.

J	$f_{7/2}^2$	$f_{7/2}f_{5/2}$	$f_{5/2}^2$	$g_{9/2}^2$	$g_{7/2}^2$
0	0.87		0.34	−0.31	−0.15
2	0.948	−0.212	0.238	0	0
4	0.915	−0.369	0.170	0	0
6	0.794	−0.610	...	0	0

Mitler (M61a) has reworked this problem, computing the Slater integrals with oscillator functions. He found the results of Levinson and Ford to be in error principally through their neglect of the influence of the $2p$ orbitals. Where previously it was concluded that collective effects were so weak that they could be neglected, Mitler deduces a necessity for invoking a particle-to-surface coupling in order to account for the observed electromagnetic transitions. In addition, the pure singlet–even force is found to be inadequate in accounting for the levels of $_{20}Ca_{23}$, and a weak repulsive triplet–odd interaction as well as a strong tensor component must be included.

TABLE IV-20. Magnetic moment contributions from admixed states of Ca^{43} ($\mu_{exp} = -1.315$).

Admixed state	Amplitude	$\Delta\mu$
$(\frac{7}{2})^2\ [2]\ \frac{5}{2}$	−0.025	0.033
$(\frac{7}{2})^2\ [4]\ \frac{5}{2}$	−0.081	0.211
$(\frac{7}{2})^2\ [6]\ \frac{5}{2}$	−0.108	0.271
$(\frac{5}{2})^2\ [0]\ \frac{7}{2}$	−0.209	−0.117
$(\frac{7}{2})^3$	0.961	−1.764
Center-of-mass motion		0.061
	Total	−1.305

Mitler obtains a magnetic moment for Ca^{43} of −1.825 nuclear magnetons. Arima and Horie's perturbation procedure would give −1.829 nuclear magnetons, which emphasizes the power of their simpler method. It is interesting that Ford and Levinson obtained such close agreement with an inappropriate interaction operator and an incorrect wave function. This work shows the importance of including as many configurations as possible in shell-model calculations, rather than merely trying to guess at which configurations are most important.

Note added in proof. Dawson, Talmi, and Walecka (D62a) have investigated the two-neutron spectrum in O^{18}, generating higher configurations through the Bethe–Goldstone equation and employing the Brueckner–Gammel two-body interaction. The positions for the first five levels is closely reproduced, and the mixing coefficients compare favorably with those obtained by Elliott and Flowers (see Sec. IV.3). This calculation strengthens the argument that the residual interaction to be used in shell-model calculations closely resembles the interaction between two free nucleons.

V. ABSOLUTE CALCULATIONS

We now turn our attention to the general solution of Schrödinger's equation

$$\frac{1}{2M}\left[\sum_{i=1}^{A} p_i^2 - \frac{1}{A}\left(\sum_{i=1}^{A} p_i\right)^2\right]\Psi + \sum_{i<j} V_{ij}\Psi = E\Psi \qquad \text{(V.1)}$$

for a nucleus composed of an arbitrary number of nucleons. These solutions play a vital role in checking our knowledge of the interaction operator V_{ij}, since

the two-body problem does not provide enough information to determine the force accurately. In addition, such calculations should resolve any doubts concerning the validity of nuclear models. The calculations reviewed in the preceding section clearly indicate that the shell model, with all of its refinements, leads to a reasonable interpretation of many nuclear properties. The basic problem of just how literally one may take the shell-model assumptions is still, however, unanswered. To make the question clearer, we define the overlap integral between a model wave function ψ_M and the exact solution of Eq. (V.1):

$$I_M = \int\cdots\int \psi_M \Psi \leqslant 1 , \tag{V.2}$$

the value of which provides a precise criterion for the validity of the model.

Early attempts to perform absolute calculations on the nuclear binding energies involved a Hartree–Fock type of solution, in which one assumes a variational trial function in the form of a determinant of single-particle orbitals

$$\psi = (A!)^{-1/2}|u_1 u_2 \cdots u_A| . \tag{V.3}$$

The guiding philosophy behind this selection is based on the early success of the independent-particle model. These calculations were far from successful, however, yielding far too small binding near the observed nuclear densities (B36a, F37) if they attained binding at all.

This failure to produce reasonable binding energies was largely responsible for the unpopularity of the independent-particle model during the early 1940's. Explanations of the failure were not lacking (F37). The correlation energy due to "clustering" of groups of nucleons had been neglected. The binding energy of a nucleon in the α particle is about 7 MeV. If the binding energy of a nucleon is found to be significantly lower than this, then α-particle clusters must become important in producing the proper energy. A possible remedy, without discarding the independent-particle model, is to take correlations into account by perturbation theory. To this purpose, Eq. (V.1) may be rewritten as

$$\left\{\frac{1}{2M}\left[\sum_{i=1}^{A} p_i^2 - \frac{1}{A}\left(\sum_{i=1}^{A} p_i\right)^2\right] + \sum_{i=1}^{A} V(r_i)\right\}\Psi + \left\{\sum_{i<j} V_{ij} - \sum_i V(r_i)\right\}\Psi = E\Psi , \tag{V.4}$$

where $V(r_i)$ is a ficticious central potential introduced to provide a convenient set of basis functions. In many cases the harmonic oscillator will be employed:

$$V(r) = \tfrac{1}{2} kr^2 \tag{V.5}$$

due to the fact that its eigenfunctions have simple properties and produce a nucleon distribution close to that which is actually observed. The term

$$W = \sum_{i<j} V_{ij} - \sum_i V(r_i) \tag{V.4a}$$

is then treated as a perturbation. First-order perturbation theory is formally identical with the Hartree–Fock approximation, while the higher orders provide correlation energy.[25]

Special care must be taken in these calculations regarding the center-of-mass motion. The kinetic energy of the center of mass (B37)

$$\frac{1}{2MA}\left(\sum_{i=1}^{A} p_i\right)^2 \tag{V.6}$$

has been properly subtracted out of the Hamiltonian in Eq. (V.4), but this is not in itself adequate. When the perturbing terms in the wave function

$$W_{0n}\varphi_n/(E - E_n) \tag{V.7}$$

are considered, one must be careful *not* to include all states φ_n prescribed by the independent-particle model. Some of these states, as was pointed out in Sec. IV.1 (E55), involve the same internal motion of the nucleons that is described by the zero-order wave function, but with the center-of-mass motion excited to a higher state. As a result, the zero-order function would be remixed into the wave function in each order, and consequently, the perturbation expansion would appear not to converge.

Early perturbation treatments of He^4 and Li^6 by Inglis (I37), and O^{16} by Kroeger (K38), demonstrated the importance of second-order energy shifts, but still could only yield about one-half the binding energy of O^{16}.

Second-order perturbation theory treats only two-body correlations. If the clustering of four nucleons is essential, one must go at least to fourth order to include them.

The correlation energy, though significant, does not appear to be the answer to the question. An important reason for the failure of the early Hartree–Fock calculations was the fact that saturation was then believed to be produced by using an exchange

[25] The contribution to the binding from higher orders may, strictly speaking, only be interpreted as correlation energy if a self-consistent field calculation is performed in zero order. If one simply introduces an arbitrary $V(r_i)$, as we do here, part of the higher order effect simply arises from the improvement of the radial wave function.

mixture that caused the direct integrals

$$D = \sum_{l,n} \int \cdots \int u_l^*(1) u_n^*(2) V_{12} u_l(1) u_n(2) d\mathbf{r}_1 d\mathbf{r}_2 \quad \text{(V.8)}$$

to give a nearly vanishing contribution. One finds that $D \sim A^2$ while the kinetic energy varies as $A^{5/3}$, so that the coefficient of D must be nearly zero in the expression for the binding energy of a complex nucleus. On the other hand, the exchange term

$$E = \sum_{l,n} \int \cdots \int u_l^*(1) u_n^*(2) V_{12} u_n(1) u_l(2) d\mathbf{r}_1 d\mathbf{r}_2 \quad \text{(V.9)}$$

varies approximately as $A^{4/3}$, and produced the primary contribution to the binding in early calculations.

Modern interaction operators become repulsive at small nucleon–nucleon separations ($\sim 0.4 \times 10^{-13}$ cm), and need no longer depend upon their exchange nature to produce saturation. A central interaction near the Serber type seems to give the best fit to scattering data, so that the direct integrals now make a strong contribution to the binding energy. Noncentral forces have also been added to the nuclear interaction since the early Hartree–Fock calculations, which may contribute to the binding in higher order.

When perturbation expansions are made, two problems of convergence present themselves. The perturbation series must be terminated at some order. Second-order corrections, which involve two-particle correlations, have clearly been shown to be significant (I37, K38), but the importance of four-particle clusters is in some doubt. Once a definite order of the perturbation is shown to be relevant, there is still some question about how many excited configurations must be included in that order.

The latter convergence problem is resolved by the Bolsterli–Feenberg (B56) perturbation procedure, which was reviewed in Sec. III. The energy shifts for a given order are then generated in closed form by an integral operator. Application of this method has been made to O^{16} (G59). The interaction operator used in the calculation consisted of a Serber central force with a repulsive core and a tensor–even component:

$$V_{12} = J_R \exp(-8r_{12}^2/r_0^2) + J_c(P_0 + P_1) \exp(-r_{12}^2/r_0^2) + \tfrac{1}{4}(1 - \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) S_{12} (r_{12}/r_0)^2 J_S \exp - (r_{12}/r_0)^2 \quad \text{(V.10)}$$

where:

$$J_c = -58.65 \text{ MeV},$$
$$J_s = -107.29 \text{ MeV},$$
$$J_R = +189.75 \text{ MeV},$$

and

$$r_0 = 1.54 \times 10^{-13} \text{ cm}. \quad \text{(V.10a)}$$

The parameters were adjusted to yield a reasonable fit (within 2% of the observed values) to the binding energies of H^2, H^3, and He^4; the rms radii of H^2 and He^4; the Coulomb energy[25a] in He^3; and the electric-quadrupole moment of H^2. A net binding energy of -129.2 MeV was obtained for O^{16}, compared to the experimental value of -127.16 MeV. Agreement is not quite as good as it appears, since Coulomb forces, which contribute about $+14$ MeV to the binding, were neglected.

The rms radius of O^{16} was calculated to be 2.33×10^{-13} cm, compared to an experimental value of 2.57×10^{-13} cm. The mixing of configurations other than the pure shell-model configuration

$$(1s)^4(1p)^{12} \quad \text{(V.11)}$$

was found to be about 18%, so that the overlap integral in Eq. (V.2) is

$$I_M(O^{16}) \approx 0.82. \quad \text{(V.12)}$$

Most of the admixed configurations involve $^5D_0$ states connected to the $^1S_0$ shell-model state by the tensor force.

Although the repulsive core employed was weak by modern standards, the calculation demonstrates the usefulness of the independent-particle model. The zero-order contribution to the energy comes to 87.8 MeV, whereas the second-order tensor terms yield 36.4 MeV, and the remaining 5.0 MeV comes from the central interaction in second order. Thus, the simple shell-model configuration comprises 82% of the wave function and yields nearly 70% of the total binding energy.

After the second-order energy shift is evaluated in a closed integral expression [see Eq. (III.18)], it may be reexpanded into the usual series:

$$\sum_n{}' \frac{|W_{0n}|^2}{E - E_n}. \quad \text{(V.13)}$$

Convergence of the series so obtained is poor; so poor that it is not profitable to try to evaluate the integral expression by such an expansion. Once again this points up the need for including as many excited configurations as possible in shell-model calculations.

The problem of convergence for the perturbation expansion is more difficult. Second-order perturba-

[25a] *Note added in proof.* H. Collard and R. Hofstadter [Bull. Am. Phys. Soc. 1, 489 (1962)] have recently measured the rms radius of He^3, obtaining 1.68 ± 0.17 F. This is in excellent agreement with the value 1.60 F. obtained from the potential in Eq. (V.10).

tion theory accounts only for two-particle correlations. Higher correlations are known to be negligible in infinite nuclear matter at normal density. This result is easily clarified. At its normal density, nuclear matter is saturated and cannot condense. The appearance of complex substructures, such as α particles near the interior of a heavy nucleus, is, therefore, so unlikely that such clusters may be neglected. Near the surface of the nucleus the density is below normal, however, and complex structures may, temporarily, appear. In light nuclei the surface problem is particularly important. The nucleon density in O^{16}, for example, is given approximately in the form

$$\rho(r) = (A + Br^2)e^{-\alpha r^2}, \qquad \text{(V.14)}$$

so that most of the density is below the normal value and correlations involving several particles could become significant.

Da Providencia (D61a) has compared the second-order energy shift in O^{16} with the similar correction in nuclear matter, finding them nearly equal. Correlations are found to increase near the surface, but this appears to have little effect on the convergence of the perturbation series. This is partly attributed to the large spacing between single-particle levels in light nuclei, which reduce the effectiveness of exciting several nucleons out of the shell-model configuration. Another possible argument for neglecting clusters of several particles is that since such clustering is only significant when the density is low, their net effect might not contribute appreciably to the binding energy.

If the repulsive core is required to be infinite, the problem of performing absolute calculations is drastically complicated. In that case, one must impose a new boundary condition:

$$\Psi(\mathbf{r}_1,\mathbf{r}_2,\cdots\mathbf{r}_A) = 0 \quad \text{if} \quad r_{ij} \leqslant c, \qquad \text{(V.15)}$$

where c is the core radius. Since no two nucleons are allowed to approach each other nearer than the limit c, particle correlations are of critical importance. Indeed, it is surprising that the independent-particle model works at all. The resolution of this problem possibly lies in the fact that even if the core is infinite its radius is quite small compared to the mean distance between nucleons, and consequently we may still be dealing with a small effect. This is conjecture, however, and must be proven in order to put the independent-particle model back on a firm foundation.

A possible means of investigating the correlations imposed by a hard core is to choose a variational trial function that is the product of an independent-particle function ψ_0, and a Jastrow (J55) correlation function

$$\Psi = F\psi_0 = \left\{\prod_{i<j} f(r_{ij})\right\}\psi_0, \qquad \text{(V.16)}$$

where

$$f(r) = 0 \quad \text{if} \quad r \leqslant c \qquad \text{(V.16a)}$$

$$f(r) \to 1 \quad \text{as} \quad r \to \infty. \qquad \text{(V.16b)}$$

This trial function satisfies the boundary condition in Eq. (V.15). Hopefully, $f(r_{ij})$ approaches one rapidly as r_{ij} becomes large so that the motion is well approximated by ψ_0. Substituting Eq. (V.16) into (V.1) and integrating we obtain

$$\begin{aligned} E = \{ &-(\hbar^2/2M)\langle\psi_0|F^2 \textstyle\sum_i \nabla_i^2 + F \sum_i (\nabla_i^2 F) \\ &+ F \textstyle\sum_i (\nabla_i F)\cdot\nabla_i - (1/A)F^2 \sum_{i,j} \nabla_i\cdot\nabla_j|\psi_0\rangle \\ &+ \langle\psi_0|F^2 \textstyle\sum_{i<j} V_{ij}|\psi_0\rangle\}/\langle\psi_0|F^2|\psi_0\rangle. \end{aligned} \qquad \text{(V.17)}$$

The only fortunate aspect of this equation appears to be that as long as harmonic oscillator orbitals are used in constructing ψ_0, the center-of-mass motion still subtracts out in the same simple manner as before. The undesirable feature of Eq. (V.17) is the appearance of many-particle terms. For example, consider

$$\langle\psi_0|F^2 \sum_{i<j} V_{ij}|\psi_0\rangle = \tfrac{1}{2}A(A-1)\langle\psi_0|F^2V_{12}|\psi_0\rangle. \qquad \text{(V.18)}$$

In the absence of the correlation function F, diagonal matrix elements of V_{12} become [if ψ_0 can be written in the form of (V.3)]:

$$\begin{aligned} \tfrac{1}{2}A(A-1)\langle\psi_0|V_{12}|\psi_0\rangle = \tfrac{1}{2}\sum_{i<j}\int\cdots\int u_i^*(1)u_j^*(2)V_{12} \\ \times [u_i(1)u_j(2) - u_i(2)u_j(1)]d\mathbf{r}_1 d\mathbf{r}_2, \end{aligned} \qquad \text{(V.18a)}$$

but Eq. (V.18) does not simplify at all.

Dabrowski (D58c) has employed a trial function of this sort in a calculation on O^{16}. The first approximation attempted was to write

$$\langle\psi|H|\psi\rangle = T_0 + \mathcal{U} + C, \qquad \text{(V.19)}$$

where

$$T_0 = \left\langle\psi_0\right| -\frac{\hbar^2}{2M}\sum_i \nabla_i^2\left|\psi_0\right\rangle \qquad \text{(V.19a)}$$

$$\mathcal{U} = \sum_{i<j}\left\langle\psi_0\right|\left\{|f(r_{ij})|^2V_{ij} + \frac{\hbar^2}{M}\left|\frac{df(r_{ij})}{dr_{ij}}\right|\right\}\left|\psi_0\right\rangle. \qquad \text{(V.19b)}$$

T_0 and $\mathcal{U}$ now contain only two-particle correlations while the higher terms are contained in C. These higher terms were then neglected. The $f(r)$ were taken to be:

$$f(r) = 1 - \exp[-\beta(r^2/c^2 - 1)] \quad \text{if} \quad r \geqslant c \tag{V.20}$$

so that expansion into the form (V.18) is easily carried out. Tauber and Wu (T60a) have done a similar calculation for O^{16} and O^{15} using a potential resembling that of Gammel and Thaler (G60). The results appear to be quite reasonable; but Dabrowski has estimated the effect of higher cluster terms in C, using an approximation based on the short range of the correlations, and finds them to be of importance in any attempt at quantitative accuracy.

A powerful procedure for handling systems composed of a large number of strongly interacting particles has been developed by Brueckner and his collaborators (a list of references for the early papers in this series appears in B58c). A detailed exposition of this work will not be given here, but we shall now present a simplified treatment of the Brueckner method in the context of conventional perturbation theory to show where the basic approximations are (R57).

The Hamiltonian is given by

$$H = \sum_i T_i + \sum_{i<j} V_{ij} . \tag{V.21}$$

We introduce a single-particle potential U_i and write the Hamiltonian as

$$H = H_0 + \textstyle\sum_\alpha V'_\alpha , \tag{V.22}$$

where

$$H_0 = \textstyle\sum_i (T_i + U_i) , \tag{V.22a}$$

$$\textstyle\sum_\alpha V'_\alpha = \sum_\alpha (V_\alpha - U_\alpha) , \tag{V.22b}$$

and $\alpha = ij$ is a pair index. U_α is defined so that

$$\textstyle\sum_\alpha U_\alpha = \sum_i U_i , \tag{V.22c}$$

and its exact form is not relevant (for example, one could choose $U_\alpha = [(U_i + U_j)/N - 1]$). Defining the eigenfunctions and eigenvalues of H_0 by

$$H_0\varphi_n = \epsilon_n\varphi_n , \tag{V.23}$$

one may find the solution of Eq. (V.21) from

$$\Psi = \textstyle\sum_n a_n\varphi_n , \tag{V.24a}$$

$$(E - H_0)\Psi = \textstyle\sum_\alpha V'_\alpha\Psi , \tag{V.24b}$$

$$\textstyle\sum_n a_n(E - \epsilon_n)\varphi_n = \sum_\alpha V'_\alpha\Psi . \tag{V.24c}$$

Formal manipulation of Eqs. (V.24) yields ($a_0 \equiv 1$)

$$\Psi = \varphi_0 + \sum_n{}' \frac{\varphi_n\langle\varphi_n| \sum V'_\alpha|\Psi\rangle}{E - \epsilon_n} , \tag{V.25a}$$

$$E = \epsilon_0 + \langle\varphi_0| \sum V'_\alpha|\Psi\rangle , \tag{V.25b}$$

which we emphasize represents an exact formulation of the problem.

The expression for Ψ in Eq. (V.25a) may be iteratively substituted into Eq. (V.25b) to obtain the Brillouin–Wigner perturbation expansion

$$E = \epsilon_0 + \langle\varphi_0| \sum V'_\alpha|\varphi_0\rangle + \sum_n{}' \frac{|\langle\varphi_0| \sum V'_\alpha|\varphi_n\rangle|^2}{E - \epsilon_n} + \cdots , \tag{V.26}$$

which upon introducing the notation

$$\frac{P}{E - H_0} = \sum_{n\neq 0} \frac{|\varphi_n\rangle\langle\varphi_n|}{E - \epsilon_n} \tag{V.27a}$$

$$V' = \textstyle\sum_\alpha V'_\alpha , \tag{V.27b}$$

becomes

$$E = \epsilon_0 + \langle\varphi_0|V' + V'\frac{P}{E - H_0}V' + V'\frac{P}{E - H_0}V'\frac{P}{E - H_0}V' + \cdots|\varphi_0\rangle . \tag{V.28}$$

One may now introduce the operator

$$\begin{aligned} t &= V' + V'\frac{P}{E - H_0}t \\ &= V' + V'\frac{P}{E - H_0}V' + V'\frac{P}{E - H_0}V'\frac{P}{E - H_0}V' + \cdots , \end{aligned} \tag{V.29}$$

defined so that

$$\langle\varphi_m|V'|\Psi\rangle = \langle\varphi_m|t|\varphi_0\rangle . \tag{V.30}$$

Equations (V.25) can then be written as

$$\Psi = \varphi_0 + \sum_n{}' \frac{\varphi_n\langle\varphi_n|t|\varphi_0\rangle}{E - \epsilon_n} , \tag{V.31a}$$

$$E = \epsilon_0 + \langle\varphi_0|t|\varphi_0\rangle . \tag{V.31b}$$

The main approximation made by Brueckner in his treatment of the many-body problem is to write t as a sum of two-body terms:

$$t = \textstyle\sum_\alpha t_\alpha , \tag{V.32}$$

where

$$t_\alpha = V'_\alpha + V'_\alpha\left[\frac{P}{E - H_0}\right]t_\alpha . \tag{V.32a}$$

Correlations between pairs of particles are then treated exactly, while higher correlations are completely neglected. This approximation turns out to work very well in infinite nuclear matter at normal density, basically because of the saturation argument made previously in this section (B58c, B56b, G57b).

The problem formulated thus far does not quite treat two-body correlations correctly, since if one considers the energy to be given by

$$\begin{aligned} E &= \epsilon_0 + \langle\varphi_0| \sum t_\alpha |\varphi_0\rangle \\ &= \epsilon_0 + \langle\varphi_0| \sum_\alpha V'_\alpha \\ &\quad + \sum_\alpha \left(V'_\alpha \left[\frac{P}{(E - H_0)}\right] V'_\alpha\right) + \cdots |\varphi_0\rangle , \end{aligned} \tag{V.33}$$

states which violates the Pauli exclusion principle are spuriously mixed into the wave function. This is easily seen by examining the exact second-order correction

$$\sum_{n\neq 0} \frac{\langle\varphi_0| \sum_\alpha V'_\alpha |\varphi_n\rangle\langle\varphi_n| \sum_\beta V'_\beta |\varphi_0\rangle}{E - \epsilon_n} . \tag{V.34}$$

As long as φ_0 is properly antisymmetrized, the states φ_n are guaranteed to be antisymmetric also since $\sum_\alpha V'_\alpha$ is symmetric in all coordinates. The term

$$\sum_{n\neq 0} \sum_\alpha \frac{|\langle\varphi_0|V'_\alpha|\varphi_n\rangle|^2}{E - \epsilon_n} , \tag{V.35}$$

which actually occurs in Eq. (V.33), does not possess this property, however, because one has effectively chosen only a part of the interaction $\sum_\alpha V'_\alpha$, and so the φ_n in Eq. (V.35) are not guaranteed to be antisymmetric. Corrections for this have been obtained by Bethe and Goldstone (B56b, G57b).

The properties of nuclear matter have been calculated using this method by Brueckner and Gammel (B58c). A potential of the Gammel–Thaler form, but modified to fit the deuteron, is used.[26] This potential is given in Table V-1. The derived properties of nuclear matter are in excellent agreement with our knowledge of the interior of heavy nuclei, yielding a binding energy per nucleon of 14.6 MeV at an equilibrium spacing of 2×10^{-13} cm.

A significant feature of the nuclear matter calculations is the fact that the Pauli principle causes the two-body wave function to approach its uncorrelated form [or to "heal" (G58)] for separations of the order of a Fermi wavelength, even though the hard core requires it to vanish at the core radius. Consider a particular nucleon moving through nuclear matter. As it is scattered by the other nucleons, the nuclear potential attempts to impose on it a new wave number (momentum)

$$k_D \approx (MV_0/\hbar^2)^{1/2} , \tag{V.36}$$

where V_0 is the depth of the two-body potential well.

TABLE V-1. Parameters of the Brueckner–Gammel potential. All radial functions have a Yukawa shape $V_0(a/r)\exp(-r/a)$ outside of a hard core with radius 0.4×10^{-13} cm.

State	V_0(MeV)	$a^{-1}(10^{13}\ \text{cm}^{-1})$
Triplet central even	−877.4	2.0909
Tensor even	−159.4	1.0454
Singlet central even	−434	1.45
LS even	−5000	3.70
Triplet central odd	−14	1.0
Tensor odd	+22	0.8
Singlet central odd	+130	1.00
LS odd	−7315	3.70

This new momentum, however, turns out to be much smaller than the Fermi momentum

$$k_D \ll k_F , \tag{V.37}$$

so that the states into which the nucleon tends to be scattered are already filled. The result of this is that for average nucleon separations, the motion is reasonably described by an independent-particle model.

The extension of this method to problems involving finite nuclei is not trivial. Brueckner, Gammel, and Weitzner (B58d) have devised a method in which the t matrix is assumed to have the same value at each point as nuclear matter at the same density, but the effect of the density gradient is neglected. This assumption of "local uniformity" is based on the observation that the range of nuclear forces is small compared to the falloff distance of the nuclear density. For example, the triplet–even and singlet–even central interaction have ranges of 0.5×10^{-13} and 0.7×10^{-13} cm, respectively, in the Brueckner–Gammel potential, while the nuclear density generally falls from 90% to 10% of its central value in a distance of 2.5×10^{-13} cm.[27] This suggests that the properties of the nuclear medium over any small region of the nucleus should not be very different from the properties of nuclear matter at the same density.

Brueckner, Lockett, and Rotenberg (B61) have calculated the properties of O^{16}, Ca^{40}, and Z^{90}, using the assumption of local uniformity and including some rearrangement energy (B59a). The binding

[26] Several similar potentials have since been tabulated by Gammel and Thaler (G60).

[27] The tensor-even force has a range of 1.25×10^{-13} cm., so that the method is in some doubt in the handling of the configuration mixing effects of this term near the surface.

energy per nucleon predicted by straightforward application of the theory was far too small in magnitude. Under the assumption that this was due to an overestimate of the effect of the repulsive core, the strength of the core terms were arbitrarily reduced in order to reach tighter binding.

The results obtained by reducing the core terms to 0.825 their normal value in O^{16} and 0.90 their normal value in Ca^{40} are shown in Tables V-2 and V-3. The

TABLE V-2. Calculated properties of O^{16} and Ca^{40}.

Element	Binging energy/A Theoret	Exp	rms radii (10^{-13}) cm proton	neutron	total	exp
O^{16}	−4.41	−7.98	2.41	2.38	2.40	2.57
Ca^{40}	−6.12	−8.55	2.91	2.84	2.88	3.49

binding energies are still 3.5 MeV/nucleon short of the experimental value, while the predicted radii are about 8% too small. The order of single-particle levels is found to be in agreement with the shell model, however, and separation energies have been adjusted to be fairly close to the experimental values in three out of four cases. We also note that the proton distribution extends just a bit further than the neutron distribution, in agreement with the conclusions reached in Sec. IV.1. Although the single-particle potential U_i is nonlocal and not very pretty by shell-

TABLE V-3. Calculated energies for O^{16} and Ca^{40} (in MeV). The core strength has been reduced to 0.825 of its normal value in O^{16} and to 0.90 of its normal value in Ca^{40} (B61).

Element	State	Eigenvalue Neutron	Proton	Experimental separation energy Neutron	Proton
O^{16}	$1s_{1/2}$	−44.3	−39.6		
	$1p_{3/2}$	−19.0	−14.6		
	$1p_{1/2}$	−14.9	−10.7	−15.60	−12.11
Ca^{40}	$1s_{1/2}$	−70.1	−60.0		
	$1p_{3/2}$	−44.7	−35.1		
	$1p_{1/2}$	−38.6	−29.2		
	$1d_{5/2}$	−20.6	−11.6		
	$2s_{1/2}$	−16.0	−7.3		
	$1d_{3/2}$	−13.4	−4.9	−15.98	−8.34

model standards, the single-particle orbitals resemble harmonic oscillator eigenfunctions closely enough to once again justify their use in other calculations.

The assumption of local uniformity is suspected to be a primary source of quantitative error in these calculations (R60). It is curious that all calculations on nuclei beyond the 1s shell appear to yield too small a size. The size is, of course, anticipated to be the poorest result of most calculations. Nevertheless, while binding energies can usually be obtained by manipulating parameters in the interaction, no agreement with the observed density of O^{16} has ever been obtained to this author's knowledge. Eden, Emery, and Sampanthar (E58, E59), for example, have performed calculations on O^{16} with a variety of interactions fitted to low-energy two-nucleon data. Binding energies obtained varied from 118 to 256 MeV, but the rms radius could only be raised to 2.0 $\times 10^{-13}$ cm.

It is possible that higher clustering effects have a more significant influence on the nuclear density than has previously been anticipated. At normal density, clusters involving more than two particles are not important. Now consider the energy of a finite number of nucleons as the density is varied. In the neighborhood of the minimum the variation of the energy with changes in density will be very slight. A decrease in the over-all density will result in an enhancement of the contribution from higher clusters, and certainly the density of an actual nucleus does drop slightly to take better advantage of clustering terms. Thus, the effect of clustering, generally neglected in Hartree–Fock calculations, may tend to increase the nuclear size.

If the short-range repulsion in the nucleon–nucleon interaction is not required to be infinite, but merely very strong, procedures for calculating the properties of nuclear systems may be greatly simplified. Clark and Feenberg (C59b) have developed a method for disposing of undesirable core terms. Suppose that the interaction operator has been decomposed into

$$V = V_A + V_R\,, \tag{V.38}$$

where V_R contains all strong short-range repulsions. The wave function is then written as

$$\Psi = e^S\Phi\,, \tag{V.39}$$

where S is a real function of the space coordinates

$$S = \sum_{i<j} S_{ij}(r_{ij})\,. \tag{V.39a}$$

Matrix elements of the Hamiltonian

$$\langle\Psi|H|\Psi\rangle = \int\Psi^*\left[-\frac{\hbar^2}{2M}\sum_i\nabla_i^2 + V_A + V_R\right]\Psi \tag{V.40}$$

can then be written without the V_R term

$$\langle\Psi|H|\Psi\rangle = \int\left[e^{2S}\frac{\hbar^2}{2M}\sum_k\nabla_k\Phi^*\cdot\nabla_k\Phi + e^S\Phi^*V_Ae^S\Phi\right], \tag{V.41}$$

if S is chosen to satisfy

$$(\hbar^2/2M)\sum_k[(\nabla_k S)^2+\nabla_k^2 S]=V_R\,. \quad \text{(V.42)}$$

The method does not apply to hard cores, since Eq. (V.42) will not, generally, be solvable if V_R is infinite over a finite region of space. It is possible to make the repulsion as strong as desired, however, as long as it is only infinite when two particles are in contact.

VI. ORIGIN OF THE DOUBLET SPLITTING

The importance of the spin-orbit interaction in the shell-model Hamiltonian was brought into sharp focus by the success of jj coupling and the intermediate coupling calculations reviewed in Sec. IV.3. The problem concerning the exact origin of the doublet splitting appears to have been first considered by Dancoff and Inglis (D36, I36), long before the significance of this effect in determining nuclear systematics was recognized. The spins and parities of the first two levels in Li^7 were at that time determined to be 3/2− and 1/2−, and enough was known about the ordering of single-particle orbitals to properly identify these as $^2P_{3/2}$ and $^2P_{1/2}$ states of a $(1p)^3$ configuration. A doublet separation such as this must originate from a vector interaction, the simplest example of which is the single-body spin-orbit term

$$\sum_i \mathbf{l}_i\cdot\mathbf{s}_i\,. \quad \text{(VI.1)}$$

The only vector interaction suspected to be influential in nuclear physics was, at that time, the Thomas relativistic correction. Many authors have since reinvestigated this problem and all have come to the same conclusion. The Thomas term yields a doublet separation of the right sign, but is about two orders of magnitude too small.[28]

The problem was revived in 1940 when Dancoff (D40a) attempted to interpret the scattering of nucleons by α particles. The scattering analysis requires both a $p_{3/2}$ and a $p_{1/2}$ phase shift to fit the data, so that once again a doublet splitting is observed. The $p_{1/2}$ phase shift does not pass through 90°, so that it is difficult to assign a definite energy difference to this doublet. It is usually estimated to be about 2.6 MeV (A55). In addition to the Thomas term, Dancoff was aware that the internucleon interaction might involve the tensor operator

$$S_{ij}=\boldsymbol{\delta}_i\cdot\mathbf{n}_{ij}\boldsymbol{\delta}_j\cdot\mathbf{n}_{ij}-\tfrac{1}{3}\,\boldsymbol{\delta}_i\cdot\boldsymbol{\delta}_j\,, \quad \text{(VI.2)}$$

since a term of this form was known to arise from pseudoscalar meson theory with pseudovector coupling.

It is easy to demonstrate that the second-order tensor terms produce doublet splitting. From two Pauli spin operators for particles i and j construct, a tensor of rank two

$$\mathcal{S}_{2m}(ij)=\sum_{\mu+\mu'=m} C^{112}_{\mu\mu' m}\sigma^i_\mu\sigma^j_{\mu'}\,, \quad \text{(VI.3)}$$

where the spherical tensor components are defined in terms of cartesian components by

$$\sigma_0=\sigma_z\,,$$
$$\sigma_{+1}=(1/\sqrt{2})(\sigma_x\pm i\sigma_y)\,. \quad \text{(VI.3a)}$$

Likewise we construct a tensor of rank two from the unit vector $\mathbf{n}_{ij}\equiv\mathbf{n}$

$$R_{2m}(ij)=\sum_{\mu+\mu'=m} C^{112}_{\mu\mu' m}n_\mu n_{\mu'}\,. \quad \text{(VI.4)}$$

The tensor operator may then be expressed as the dot product

$$S_{ij}=\mathcal{S}_2(ij)\cdot R_2(ij)=\sum_m(-1)^m\mathcal{S}_{2m}(ij)R_{2-m}(ij)$$
$$=\sqrt{5}\sum_m C^{220}_{m-m0}\mathcal{S}_{2m}(ij)R_{2-m}(ij)\,. \quad \text{(VI.5)}$$

Second-order terms will involve the product $S_{ij}\,S_{kl}$ which can be expanded making use of the 9-j coefficient:

$$S_{ij}S_{kl}=5\sum_{K=0}^{4}(2K+1)\begin{Bmatrix}2&2&K\\2&2&K\\0&0&0\end{Bmatrix}\sum_{M=-K}^{K}C^{KK0}_{M-M0}$$
$$\times\,\mathcal{S}_{KM}(ijkl)R_{K-M}(ijkl)\,, \quad \text{(VI.6)}$$

where

$$\mathcal{S}_{KM}(ijkl)=\sum_{\mu+\mu'=M}C^{22K}_{\mu\mu' M}\mathcal{S}_{2\mu}(ij)\mathcal{S}_{2\mu'}(kl)\,, \quad \text{(VI.6a)}$$

$$R_{KM}(ijkl)=\sum_{\mu+\mu'=M}C^{22K}_{\mu\mu' M}R_{2\mu}(ij)R_{2\mu'}(kl)\,, \quad \text{(VI.6b)}$$

and

$$\begin{Bmatrix}2&2&K\\2&2&K\\0&0&0\end{Bmatrix}=\frac{1}{5(2K+1)^{1/2}}\,. \quad \text{(VI.6c)}$$

Equation (VI.6) now becomes

$$S_{ij}S_{kl}=\sum_{K=0}^{4}S_K(ijkl)\cdot R_K(ijkl)\,. \quad \text{(VI.7)}$$

If LS basis functions are used, one will need to

[28] The Thomas term would not be small near the hard core of the Gammel–Thaler potential since $\partial V/\partial r\to\infty$. In this region the Thomas correction is inapplicable, however, since it is derived under the criteria $|V|\ll Mc^2$. Actually, the two-body spin-orbit term of meson theory can be thought of as arising from this effect, because both it and the hard core are attributed to the three-pion resonance (B60c).

evaluate matrix elements of the form

$$\langle SLJM|\mathcal{S}_K\cdot R_K|SLJM\rangle = (-1)^K W(JLSK;SL) \times (S||\mathcal{S}_K||S)(L||R_K||L)\,. \quad \text{(VI.8)}$$

This matrix element obviously vanishes unless $S \geqslant K/2$ and $L \geqslant K/2$, so that for a closed shell, only the term with $K = 0$ contributes. The principal effect of a term $S_K\cdot R_K$ with $K \neq 0$ is to produce a spin-orbit splitting. To illustrate this, consider the term with $K = 1$, which produces a doublet splitting. When $S = 1/2$, one can have $J = L \pm 1/2$, and the splitting between these two states is determined by the ratio of two Racah coefficients

$$\frac{W([J = L - \frac{1}{2}]L\ \frac{1}{2}\ K;\ \frac{1}{2}\ L)}{W([J = L + \frac{1}{2}]L\ \frac{1}{2}\ K;\ \frac{1}{2}\ L)} = -\frac{L+1}{L}\,, \quad \text{(VI.8a)}$$

the reduced matrix elements in Eq. (VI.8) retaining the same value for both states. In a like manner, the term with $K = 2$ splits triplets, $K = 3$ quartets, and so on. The term with $K = 0$ is explicitly given by

$$R_{00}(ijkl)\mathcal{S}_{00}(ijkl) = \tfrac{1}{5}\{(\mathbf{n}_{ij}\cdot\mathbf{n}_{kl})^2 - \tfrac{1}{3}\} \times \{\tfrac{1}{2}\,\boldsymbol{\sigma}^i(\boldsymbol{\sigma}^j\cdot\boldsymbol{\sigma}^l)\cdot\boldsymbol{\sigma}^k + \tfrac{1}{2}\,\boldsymbol{\sigma}^i(\boldsymbol{\sigma}^j\cdot\boldsymbol{\sigma}^k)\cdot\boldsymbol{\sigma}^l - \tfrac{1}{3}\,\boldsymbol{\sigma}^i\cdot\boldsymbol{\sigma}^j\boldsymbol{\sigma}^k\cdot\boldsymbol{\sigma}^l\} \quad \text{(VI.9)}$$

and has the properties of a triplet–central interaction. It is this term which produces most of the triplet–singlet splitting in the deuteron. This becomes quite clear after examining the case where $(ij) = (kl)$:

$$R_{00}(ijij)\mathcal{S}_{00}(ijij) \backsim \tfrac{1}{4}\,(3 + \boldsymbol{\sigma}^i\cdot\boldsymbol{\sigma}^j)\,. \quad \text{(VI.9a)}$$

The vector term

$$R_{1M}(ijkl) = (2/5)^{1/2}\mathbf{n}_{ij}\cdot\mathbf{n}_{kl}(\mathbf{n}_{ij}\times\mathbf{n}_{kl})_M \quad \text{(VI.10a)}$$

$$\mathcal{S}_{1M}(ijkl) = (1/40)^{1/2}\{\boldsymbol{\sigma}^j\cdot(\boldsymbol{\sigma}^i\times\boldsymbol{\sigma}^k)_M\boldsymbol{\sigma}^l + \boldsymbol{\sigma}^i\cdot(\boldsymbol{\sigma}^j\times\boldsymbol{\sigma}^k)_M\boldsymbol{\sigma}^l + \boldsymbol{\sigma}^j\cdot(\boldsymbol{\sigma}^i\times\boldsymbol{\sigma}^l)_M\boldsymbol{\sigma}^k + \boldsymbol{\sigma}^i\cdot(\boldsymbol{\sigma}^j\times\boldsymbol{\sigma}^l)_M\boldsymbol{\sigma}^k\} \quad \text{(VI.10b)}$$

is the one of interest in this section, since it has the form required to produce the observed nuclear doublet separations (that is, it is the dot product of two vectors, one of which is constructed from Pauli spin matrices, and the other of orbital coordinates). Such a hypothesis is not a particularly attractive one since it is not clear by inspection that the second-order tensor terms produce a splitting which is even of the proper sign.

The two-body spin-orbit term required in the interpretation of proton–proton scattering (see Table V-1)

$$(\boldsymbol{\sigma}^1 + \boldsymbol{\sigma}^2)\cdot\mathbf{l}_{12} \quad \text{(VI.11)}$$

can easily be seen to produce a splitting whose sign agrees with that which is observed. A debate seems to exist in some quarters as to whether the two-body spin-orbit term is really required since the vector term in Eqs. (VI.10) resembles it so closely, or alternately whether the term in Eqs. (VI.10) is significant. These arguments miss the main point. The actual problem involves the competition between noncentral interactions which operate in orbitally symmetric states with those which operate in orbitally antisymmetric states. There is no doubt that a strong tensor–even term is needed in the interaction. The argument is based on the observed quadrupole moment of the deuteron (R41), and seems incontestable. The evidence for a *spin-orbit–odd* interaction is also very strong (G60). Large polarization effects are observed in proton–proton scattering, which must be attributed to noncentral interactions of the form

$$\mathcal{S}_K(\sigma_1,\sigma_2)\cdot R_K(r_1,r_2)\,, \quad \text{(VI.12)}$$

where $\mathcal{S}_K$ is a spherical tensor of rank K constructed out of Pauli matrices for particles one and two, and likewise, R_K is an orbital spherical tensor. Due to the commutation relations

$$\boldsymbol{\sigma}\times\boldsymbol{\sigma} = 2i\boldsymbol{\sigma}\,, \quad \text{(VI.13)}$$

K can be no larger than two, so that the polarization effects must be interpreted in terms of an interaction either of the form in (VI.11) or (VI.5). Now such noncentral interactions act only in the triplet states for two nucleons, since by the Wigner–Eckart theorem,

$$(S = 0||\mathcal{S}_K||S = 0) = 0 \quad \text{if} \quad K \neq 0\,. \quad \text{(VI.14)}$$

For two protons, the triplet states are all orbitally antisymmetric:

$$^3P\,,\ ^3F\,,\ ^3H\,,\ \cdots\,, \quad \text{(VI.15)}$$

and one must therefore interpret the proton–proton polarization effects in terms of either a tensor–odd or spin-orbit–odd interaction. A tensor–odd interaction turns out to have the wrong energy dependence and cannot fit the scattering data at 310 MeV. In any case, a strong tensor–odd term is not desirable because it would lead to far too large a doublet splitting in $1p$-shell nuclei.

One is then faced with an interaction operator which must have strong tensor–even and spin-orbit–odd components.[29] The question of physical interest with regard to the shell model is then, just how much

[29] A large spin-orbit–even term is included in many of the Gammel–Thaler potentials, but it is not absolutely required to fit the data.

do each of these terms contribute to the observed effects, and do they yield a picture of nuclei which is completely consistent with that found in nature?

Dancoff's (D40a) original calculation on He^5 with tensor forces was disappointing as it yielded a normal P doublet ($P_{1/2}$ ground state, $P_{3/2}$ excited state), rather than the observed inverted P doublet, when the Rarita-Schwinger (R41) square-well potential was used. In addition to being of the wrong sign, the derived splitting was negligible compared with the experimental result.

Feingold (F56) later obtained better agreement with an improved method of calculation, and a more modern interaction operator. The Hamiltonian is written

$$\begin{aligned} H &= H_0 + V + t \\ &= H_0 + W , \end{aligned} \qquad \text{(VI.16)}$$

where V is the two-body central force and t the tensor interaction. One then selects trial function

$$\psi = (1 + \lambda t')\psi_0 , \qquad \text{(VI.17)}$$

where t' is a tensor operator modified from t to yield an improved eigenvalue (this is essentially the well-known Leonard–Jones method). The eigenvalue is then given by:

$$E = \frac{(H)_{00} + 2\lambda(t'H)_{00} + \lambda^2(t'Ht')_{00}}{1 + 2\lambda(t')_{00} + \lambda^2(t't')_{00}} , \qquad \text{(VI.18a)}$$

so that minimization with respect to λ yields

$$E[(t')_{00} + \lambda(t't')_{00}] = (t'H)_{00} + \lambda(t'Ht')_{00} , \qquad \text{(VI.18b)}$$

and Eqs. (VI.18) and (VI.19) may then be solved simultaneously for λ and E. The method is equivalent to second-order perturbation theory with an average energy denominator

$$E - E_0 = \sum_{n\neq 0} \frac{|W_{0n}|^2}{E - E_n} \cong \frac{(W^2)_{00} - (W_{00})^2}{E - E_{av}} . \qquad \text{(VI.18c)}$$

Using a Gaussian-shaped potential, fitted to low-energy two-body data, Feingold obtained an inverted doublet splitting of 380 KeV in Li^7 (experimental value: 480 KeV), and a $^3S_1$–$^1S_0$ separation of 1.4 MeV in Li^6 (experimental value: 3.5 MeV).

The doublet splitting in He^5 was also computed, yielding a result of the right sign and about one-half as large as the observed value. The essential difference between Feingold's calculation and the earlier work of Dancoff appears to be that while Dancoff used a square-well and exponential wave function, Feingold employs a Gaussian well with oscillator functions. Feingold then obtains large contributions in the right direction from the radial integrals at fairly large particle separations.

The He^5 problem is a particularly difficult one since there is no bound state. Several authors (G61, T60b, F56) proceed to assume a bound-state wave function, either fitting the size parameter to the $Li^5 - He^5$ Coulomb energy difference, or using the α-particle parameters. The results so obtained are of the correct sign and magnitude, but it is possible that a theory which assumes a bound state where none exists, may overestimate the effect of the tensor force. Two attempts have been made at phase-shift calculations for n–α scattering. The first (S57c) assumed scattering of neutrons from an α particle which was given as a pure $^1S_0$ state and found that only about 30% of the ^2P splitting could be attributed to the tensor interaction.

Nagata *et al.* (N59b) also computed the phase shifts for n–α scattering using the pion-theoretical potential

$$\begin{aligned} V^{(1\pi)} = {} & (g^2/4\pi)\mu c^2 \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \\ & \times \{\tfrac{1}{3}\,\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + S_{12}[1 + 3/\kappa r + 3/(\kappa r)^2]\} \\ & \times (\kappa r)^{-1} e^{-\kappa r} , \end{aligned} \qquad \text{(VI.19)}$$

where μ is the pion rest mass

$$\kappa^{-1} = \hbar/\mu c , \qquad \text{(VI.19a)}$$

and

$$g^2/4\pi \cong 0.08 . \qquad \text{(VI.19b)}$$

The equations for an isolated α particle are then set up and solved with a trial function which is a mixture of $^1S_0$ and $^5D_0$ states, fitted to give the observed size of the α particle in the $^1S_0$ state. This results in a binding energy of 33 MeV (experimental value 28 MeV) and a 9% admixture of the $^5D_0$ state. The scattering of a neutron from this system (which is assumed to be undistorted by the neutron) is then computed. The computed P-doublet splitting is then found to be about 60% of the observed value. The significant difference between this calculation and the previous one is attributed primarily to including $^5D_0$ states in the α-particle wave function.

The phase-shift calculations are greatly simplified if one does not consider any distortion of the α particle. Takagi *et al.* (T59a) and Nagata *et al.* (N59b) estimate, however, that such distortions may actually contribute strongly to the doublet splitting, and should be considered. The importance of this effect has clearly been demonstrated by Terasawa and Arima (T60b, A60, A60a) and also by Jancovici (J59). The clearest example is found in

Terasawa's (T60b) calculation on He^5. Consider a shell-model representation of the α-particle wave function in jj coupling. The main part of the wave function will be described by the closed-shell configuration

$$(1s_{1/2})^4 . \qquad \text{(VI.20a)}$$

Nuclear forces will then mix in other configurations, two of the lowest being

$$(1s_{1/2})^2(1p_{3/2})^2 \qquad \text{(VI.20b)}$$

$$(1s_{1/2})^2(1p_{1/2})^2 . \qquad \text{(VI.20c)}$$

Now the lowest configurations for the P-doublet levels of He^5 are:

$$(1s_{1/2})^4(1p_{3/2}) \quad \text{for} \quad J = 3/2 \qquad \text{(VI.21a)}$$

$$(1s_{1/2})^4(1p_{1/2}) \quad \text{for} \quad J = 1/2 . \qquad \text{(VI.21b)}$$

The existence of a fifth nucleon in a $1p$ orbital will tend to suppress configuration mixing of the core states because of the Pauli principle, and thereby actually reduce the magnitude of the binding energy of the core. This effect will be greater for the $^2P_{1/2}$ state since the $1p_{1/2}$ shell accommodates fewer nucleons then the $1p_{3/2}$ shell. For instance, if the fifth nucleon is a $1p_{1/2}$ neutron then promotion of two neutrons from the core to $1p_{1/2}$ orbitals is forbidden, but if the fifth nucleon is a $1p_{3/2}$ neutron promotion of two neutrons to the $1p_{3/2}$ shell is still possible. The net result of this is a tendency for the $^2P_{1/2}$ level to lie higher than the $^2P_{3/2}$ level.

We note that this effect will always tend to produce a doublet splitting whose sign is consistent with the ordering of states in the shell model. Unfortunately, however, this does not provide an ironclad proof that the splitting due to the tensor force always has the proper sign. The mixing of other configurations also contribute to the doublet separations, and the only way to find their sign and magnitude seems to be to actually perform a detailed calculation.

A significant feature of the doublet splitting is the fact that it varies in magnitude throughout a given shell. This fact was pointed out in Sec. IV.3 with regard to the $1p$ shell, where the variation is so extreme that the mode of coupling changes in going from Li^6 to N^{15}. Elliott and Lane (E54) have attempted to interpret this variation in terms of a two-body spin-orbit interaction which takes on the same values in states of even and odd orbital symmetry.[30] Oscillator wave functions are employed (with length parameter b) along with a Yukawa shaped potential (range parameter a). The doublet separations for a particle and a hole in the $1p$, $1d$, $2p$, and $1f$ shells were then deduced in terms of the ratio (a/b).

It is very difficult to interpret the results of this calculation due to the uncertainty in the size parameters b. There is a marked tendency for a hole to exhibit a much larger doublet splitting than a particle due to its interaction with the nucleons of an additional shell, and an equally marked tendency for it to increase with increasing quantum numbers (nl) of the orbital.

Feingold (F59a) has emphasized the importance of explaining the small doublet splitting observed in Li^7 (0.478 MeV). A two-body vector force, with the exchange character used by Elliot and Lane to obtain a reasonable ratio for the N^{15}/He^5 doublet separation, yields a ratio for the N^{15}/Li^7 splitting near 4 compared to the experimental value of 13.3. Feingold considered the effect of a three-body vector force of the form

$$(\mathbf{r}_{12}\cdot\mathbf{r}_{13})(\mathbf{r}_{12}\times\mathbf{r}_{13})\cdot\{\boldsymbol{\sigma}^2\times\boldsymbol{\sigma}^3+\tfrac{1}{5}\,i[4\boldsymbol{\sigma}^1(\boldsymbol{\sigma}^2\cdot\boldsymbol{\sigma}^3) - \boldsymbol{\sigma}^2(\boldsymbol{\sigma}^3\cdot\boldsymbol{\sigma}^1) - \boldsymbol{\sigma}^3(\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2)]\} , \qquad \text{(VI.22)}$$

which will arise from the second-order tensor term displayed in Eqs. (VI.10). It was found that agreement with the ratio for ^2P separation in N^{15}/Li^7 could only be obtained if the tensor force has exchange properties not too close to the Serber type, so that a bit more tensor–odd interaction would have to be included in the Gammel–Thaler potential.

A possible interpretation of the small separation for the Li^7 doublet may lie in a consideration of mixing jj configurations. The $3/2-$ level is composed of

$$(p_{3/2})^3 , \quad (p_{3/2})^2p_{1/2} , \quad p_{3/2}(p_{1/2})^2 , \qquad \text{(VI.23a)}$$

while the $1/2-$ level is made up principally of

$$(p_{3/2})^2p_{1/2} , \quad (p_{3/2})^3 , \quad (p_{1/2})^2p_{3/2} , \quad (p_{1/2})^3 . \qquad \text{(VI.23b)}$$

The appearance of the configuration $(p_{3/2})^3$ in the $1/2-$ level may pull the energy down significantly relative to the $3/2-$ ground state. It is not possible to estimate the magnitude of such an effect without going through a calculation of the matrix elements involved.

The determination of size parameters plays a vital role in the spin-orbit problem since doublet separations seem to be especially sensitive to them in the region of physical interest. Often these parameters are fixed by either fitting the rms radius obtained from scattering experiments or the Coulomb energy difference for a pair of mirror nuclei. This procedure

[30] A symmetric spin-orbit interaction (that is one multiplied by $\tau_1\cdot\tau_2$) yields results which are not at all like the observed values.

may lead to serious errors on two counts. It is generally just the zero-order shell-model wave function that is used in the fitting. Arima and Terasawa (A60) have pointed out that the higher order configurations tend to increase the size of the nucleus, and therefore, perhaps the parameters should be chosen so that the zero-order wave function yields a smaller radius and a larger Coulomb energy than that observed in nature. This makes sense, but the magnitude of such an effect in the absence of an absolute calculation is obviously in considerable doubt. Secondly, the errors incurred by calculating matrix elements of an interaction with wave functions which were not derived from that interaction are very hard to estimate. In a binding-energy calculation, variational stability of the wave function relieves the difficulty somewhat, but in estimating the separation between two levels one is not so fortunate.

Estimates of the separation between $1p_{3/2}$ and $1p_{1/2}$ orbitals in O^{16} were computed in connection with two of the absolute calculations treated in Sec. V (B61, G59). The calculations of Breckner, Lockett, and Rotenberg (B61) resulted in far too much doublet splitting. So much in fact that the $1f_{7/2}$ neutron level was pulled down lower than the proton $1d_{3/2}$ level in Ca^{40}, and the spin-orbit–even interaction had to be equated to zero to retain β stability. This done, the $p_{3/2}$–$p_{1/2}$ separation in O^{16} is about 4 MeV (see Table V-3), which one may compare with an energy difference of 6.3 MeV between the 1/2− and 3/2− levels of N^{15}.

Most of this 4 MeV is attributed to the spin-orbit rather than the tensor interaction. Since the spin-orbit force used had an excessively short range ($\sim 0.27 \times 10^{-13}$ cm), the assumption of local uniformity should hold, and the effect of the spin-orbit interaction at the density obtained should be reasonably accurate. The tensor–even force, on the other hand, has a very long range, and it is doubtful that tensor forces were properly taken into account. In addition, if the spin-orbit–even terms are dropped, some modification of the tensor–even terms should be required to retain agreement with n–p scattering data.

In calculating the properties of O^{16} by the Bolsterli–Feenberg perturbation procedure (G62), the energy difference between $1p_{3/2}$ and $1p_{1/2}$ orbitals was found to be 5.6 MeV, which is about 11% short of the 6.3 MeV doublet P separation in N^{15}. It is interesting that the two-body vector term arising from the second-order tensor contribution is of the form

$$i(\mathbf{r}_{12}\cdot\mathbf{r}'_{12})(\mathbf{r}_{12}\times\mathbf{r}'_{12})\cdot(\boldsymbol{\sigma}^1+\boldsymbol{\sigma}^2)\,, \qquad \text{(VI.24)}$$

which closely resembles the usual two-body spin-orbit term of Eq. (VI.11) (Feingold's formalism yields precisely this term in third order). The contribution of this term is quite small, however, and the main effect comes from the three-body term displayed in Eq. (VI.22). This is in agreement with the hypothesis that the effect arises from the Pauli principle since, as we saw in Sec. 5, the three-body terms in the second-order energy shift

$$\langle\varphi_0|V_{12}|\varphi_n\rangle\langle\varphi_n|V_{13}|\varphi_0\rangle \qquad \text{(VI.25)}$$

help to correct violations of the Pauli principle that occur when the two-body terms

$$\langle\varphi_0|V_{12}|\varphi_n\rangle\langle\varphi_n|V_{12}|\varphi_0\rangle \qquad \text{(VI.26)}$$

are considered alone.

After surveying the large number of calculations of doublet separations in various nuclei, it appears that the effects of either the spin-orbit or tensor force are individually large enough to account for the observed splittings. No calculation which adequately treats both forces together has been reported. It is possible that cross terms between the tensor and the spin-orbit terms appear which provide a partial cancellation of the effects that each term yields individually (F61a).

If such cancellations do not occur, then some of the calculations described above must badly overestimate doublet separations. The primary defect with the estimates of the tensor contribution is that the calculations were carried out within the context of second-order perturbation theory. The essential property of the tensor interaction in producing the doublet splitting appears to be its long range (A60, T60b). One is then dealing with a very smoothly varying interaction, and it is hard to believe that higher order corrections would drastically change the second-order result.

The spin-orbit force is of very short range. The second-order term has a sign opposite to that of the first and appears to be larger, invalidating a perturbation expansion (F62). In this case, however, only the two-body terms are significant and the Brueckner method should treat them quite adequately unless cross terms with the central and tensor forces become important. This makes the fact that the spin-orbit-even interaction had to be set equal to zero in the O^{16} and Ca^{40} calculations quite significant, and strongly suggests that the noncentral components of the Brueckner–Gammel potential require detailed revisions.

VII. COLLECTIVE NUCLEAR MOTION

The shell-model calculations reviewed in Sec. IV reveal a remarkably consistent picture of nuclear level orders and some understanding of more subtle nuclear properties, such as static magnetic-dipole moments. A persistent failure of the model in determining electric-quadrupole moments and $E2$ transition rates is, however, revealed with equal clarity. The electric-quadrupole moment of a single proton in a state specified by quantum numbers nlj is

$$\langle nljj|Q_{\text{op}}|nljj\rangle = -[(2j-1)/2(j+1)]\langle nl|r^2|nl\rangle\,, \tag{VII.1}$$

and observed nuclear moments sometimes exceed this by more than an order of magnitude (F55, B53), while $E2$ transitions frequently have lifetimes one hundred times shorter than that given by a single-particle estimate (G51).

The occurrence of nuclear fission provides strong evidence for collective behavior in nuclei, since clearly no independent-particle model could be expected to yield a dynamical description of this process. In addition the success of the liquid-drop model (B36) in the interpretation of nuclear reactions emphasizes the limitations of the shell model.

Rainwater (R51a) treated the nucleus as a "core," plus a single odd nucleon whose angular momentum tends to deform the core into a spheroidal shape. Consider a sphere of charge Z and radius R, which is deformed into a spheroid with symmetry axis a and azimuthal axis b, such that the volume remains unchanged:

$$ab^2 = R^3\,. \tag{VII.2}$$

The quadrupole moment of the spheroid is then

$$Q = \tfrac{4}{5}\,\epsilon Z R^2\,, \tag{VII.3}$$

where

$$\epsilon = (a-b)/R \tag{VII.3a}$$

and terms quadratic in ϵ are neglected. The introduction of such collective distortions of the nuclear core appear to lead to corrections for the nuclear-quadrupole moments which are of the right magnitude (F51).

To clarify the mechanism of the distortion one may assume a potential which is a function of the particle density (D53b), so that surfaces of constant density are also surfaces of constant potential. Consider the anisotropic harmonic oscillator (F55):

$$V(r) = \tfrac{1}{2}\,M\omega^2[\lambda(x^2+y^2) + z^2/\lambda^2]\,, \tag{VII.4}$$

deformed by the parameter λ into a spheroidal shape so that surfaces of constant density are given by

$$(r')^2 = \lambda(x^2+y^2) + z^2/\lambda^2 = \text{const}\,. \tag{VII.5}$$

The energy eigenvalue is immediately seen to be

$$E/\hbar\omega = \lambda^{1/2}[\textstyle\sum (n_x + n_y + 1) - 1] + (1/\lambda)[\sum (n_z + \tfrac{1}{2}) - \tfrac{1}{2}]\,, \tag{VII.6}$$

where the sum goes over the quantum numbers for each particle in the potential well, and the energy of the center-of-mass motion has been deducted.

The eigenvalue is then minimized with respect to λ yielding

$$E/\hbar\omega = \tfrac{3}{2}\,(2W_z)^{1/3}(W_x + W_y)^{2/3} \simeq W - y^2/4W \tag{VII.7}$$

at

$$\lambda = [2W_z/(W_x + W_z)]^{2/3} \cong [1 - y/2W]^2\,, \tag{VII.7a}$$

where

$$W_i = \textstyle\sum (n_i + \tfrac{1}{2}) - \tfrac{1}{2}\,, \tag{VII.7b}$$

$$W = \textstyle\sum_i W_i\,, \tag{VII.7c}$$

and

$$Y = W - 3W_z\,. \tag{VII.7d}$$

Levels tend to fill in order of decreasing W_z within an oscillator shell; thus at the beginning of the shell one has $\lambda > 1$, favoring a prolate spheroidal shape. As the shell is filled, the shape alternates from being prolate to oblate so that either is about equally likely. In regions displaying large deformations one finds that nearly all nuclei have a prolate shape, so that this result is not realistic. Moszkowski (M55a) has repeated the analysis using a distorted square well instead of the harmonic oscillator. Prolate shapes are then found to be more favorable, owing to the fact that a square well tends to lower the energy of states with high orbital angular momentum.

Bohr and Mottelson (B53) have formulated a theory of the collective motion of the nucleons comprising the core coupled to nucleons in unfilled shells, which is founded on the liquid-drop model (B36, H53). The nuclear core is assumed to have a sharply defined surface whose shape is described by

$$R(\theta,\varphi) = R_0[1 + \textstyle\sum_{\lambda,\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\theta,\varphi)]\,. \tag{VII.8}$$

If one treats the core as an irrotational, incompressible fluid, then oscillations of the surface are described by the Hamiltonian

$$H_{\text{so}} = \textstyle\sum_{\lambda,\mu} \tfrac{1}{2}\,[B_\lambda|\dot{\alpha}_{\lambda\mu}|^2 + C_\lambda|\alpha_{\lambda\mu}|^2]\,, \tag{VII.9}$$

where

$$C_\lambda = (\lambda - 1)(\lambda + 2)R_0^2 S - \frac{3}{2\pi}\left(\frac{\lambda - 1}{2\lambda + 1}\right)\frac{Z^2e^2}{R_0^2}\,, \tag{VII.9a}$$

$$B_\lambda = (3/4\pi\lambda)AMR_0^2\,, \tag{VII.9b}$$

and S is the nuclear surface tension. The condition

$$\alpha_{\lambda\mu} = (-1)^\mu \alpha^*_{\lambda-\mu} \tag{VII.10}$$

ensures that R is real. It is obvious that H_{so} describes a set of harmonic oscillators with frequencies

$$\omega_\lambda = (C_\lambda/B_\lambda)^{1/2}\,, \tag{VII.11}$$

when the real and imaginary parts of the $\alpha_{\lambda\mu}$ are treated as independent variables (one must be careful to note that $\alpha_{\lambda\mu}$ is not Hermitian). A standard form is obtained by making use of the transformation (F55)

$$\alpha_{\lambda\mu} = (\hbar/2B_\lambda\omega_\lambda)^{1/2}[b_{\lambda\mu} + (-1)^\mu b^*_{\lambda-\mu}] \tag{VII.12a}$$

$$\dot{\alpha}_{\lambda\mu} = -i\omega_\lambda(\hbar/2B_\lambda\omega_\lambda)^{1/2}[b_{\lambda\mu} - (-1)^\mu b^*_{\lambda-\mu}], \tag{VII.12b}$$

so that the Hamiltonian becomes

$$H_{so} = \tfrac{1}{2}\sum_{\lambda,\mu}\hbar\omega_\lambda(b^*_{\lambda\mu}b_{\lambda\mu} + b_{\lambda\mu}b^*_{\lambda\mu})\,, \tag{VII.13}$$

and the $b^*_{\lambda\mu}$ and $b_{\lambda\mu}$ play the familiar role of creation and destruction operators

$$b^*\delta_{n'}(n) = (n' + 1)^{1/2}\delta_{n'+1}(n) \tag{VII.14a}$$

$$b\delta_{n'}(n) = (n')^{1/2}\delta_{n'-1}(n)\,. \tag{VII.14b}$$

It is clear that this description is reasonable only so long as λ is much smaller than A. From Eqs. (VII.9) we see that the cases where $\lambda = 1$ (corresponding to a displacement of the system as a whole) and $\lambda = 0$ (describing a dilatation or "breathing mode") are not treated by this formulation of the problem.[31] Most of the applications have been made to the modes with $\lambda = 2$, which represent quadrupole distortions, and some calculations indicate that modes with $\lambda = 3$ are significant (D62).

The particles in unfilled shells are often treated by the shell model

$$H_p = \sum_i (T_i + V(r_i))\,, \tag{VII.15}$$

and in more refined treatments some interaction may be introduced between them. The interaction between the core and the "extra" nucleons is taken to be

$$H_{\text{int}} = \sum_i [V(r_i') - V(r_i)] \cong \sum_i r_i \frac{\partial V}{\partial r_i}\sum_{\lambda,\mu}\alpha_{\lambda\mu}Y_{\lambda\mu}(\theta_i,\varphi_i)\,, \tag{VII.16}$$

which becomes

$$H_{\text{int}} = \sum_i \delta(r_i - R_0)DR\sum_{\lambda,\mu}\alpha_{\lambda\mu}Y_{\lambda\mu}(\theta_i,\varphi_i) \tag{VII.16a}$$

if $V(r)$ is a rectangular well of depth D and radius R_0. Evaluation of H_{int} will involve the radial integral

$$\langle nl|\delta(r - R_0)DR|nl\rangle = R_0^3 DR_{nl}^2(R_0)\,, \tag{VII.16b}$$

which turns out to be close to 40 MeV, independent of n and l (F51).

Many applications with various computational techniques were attempted in the early literature (D53b, M54, W53, F53). If H_{int} is small then one may treat it as a perturbation (weak coupling). Considering only quadrupole deformations ($\lambda = 2$) the eigenfunctions of H_{so} can be written immediately:

$$\phi(n_2 n_1 n_0 n_{-1} n_{-2}) = \delta_{n2}(n_2')\cdots\delta_{n-2}(n'_{-2})\,. \tag{VII.17}$$

Each quantum of surface oscillation carries 2 units of angular momentum (F55), and eigenstates of K^2 (the angular momentum operator for the core) are obtained in terms of the ϕ

$$V(\nu KK_z) = \sum_{n_\mu}\delta_\nu(\textstyle\sum n_\mu)\delta_{Kz}(\sum \mu n_\mu) \times (n_2\cdots n_{-2}|\nu KK_z)\phi(n_2\cdots n_{-2})\,. \tag{VII.18}$$

Solutions of H_p are simply Slater determinants for the particles in unfilled shells

$$|u_{j_1 m_1}u_{j_2 m_2}\cdots u_{j_n m_n}|\,. \tag{VII.19}$$

The unperturbed coupled system is then represented by

$$\Phi(\nu KJ;II_z) = \sum (KJK_zM|II_z)U_{JM}V(\nu KK_z)\,. \tag{VII.20}$$

One may then proceed to solve

$$(H_{so} + H_p + H_{\text{int}})\psi(II_z) = E\psi(II_z) \tag{VII.21}$$

by perturbation theory (weak coupling), assuming the zero-order wave function to be

$$\Phi(\nu_{\min}KJ;II_z)\,, \tag{VII.22}$$

where $\nu_{\min}$ is the least number of surfons consistent with the other quantum numbers or apply an inter-

[31] The $\lambda = 0$ "breathing mode" has been treated in some detail (F56a) in attempts to interpret the first excited state of O^{16} at 6.06 MeV ($J\pi = 0+$). It now appears that the breathing mode lies much higher above the ground state (G57c, G57d), and one must seek an interpretation of this level in terms of pair excitations (U58a, V62) as indicated in Sec. IV.4.

mediate coupling procedure in which the wave function is taken to be

$$\psi(II_z) = \sum_{\nu JK} p_{\nu JK}\Phi(\nu JK;II_z) , \quad \text{(VII.23)}$$

where the sum is taken over a sufficient number of states to produce convergence. The $p_{\nu JK}$ are varied to minimize the total energy. Matrix elements needed in applications where the unfilled shells contain only one nucleon have been tabulated by Feenberg (F55).

If the interaction Hamiltonian is very strong, the core will be deformed into a permanently nonspherical equilibrium shape. The quantum number ν then loses its significance to the extent that it no longer makes sense to use it in the classification of basis functions, and the intermediate coupling approach is inappropriate. Oscillations about a spheroidal equilibrium shape are best treated by transforming the coordinates to the body axes of the system

$$Y_{2\nu}(\theta',\varphi') = \sum_\mu D^2_{\mu\nu} Y_{2\mu}(\theta,\varphi) , \quad \text{(VII.24a)}$$

$$\alpha^1_{2\nu} = \sum_\mu D^2_{\mu\nu}\alpha_{2\mu} . \quad \text{(VII.24b)}$$

If the rotated axes coincide with the major axes of the core, $\alpha'_{22} = \alpha'_{2-2}$ and $\alpha'_{11} = \alpha'_{1-1} = 0$, so that the deviations from the equilibrium shape are described by only two parameters, and we may make the substitution

$$\alpha'_{20} = \beta\cos\gamma \quad \text{(VII.24c)}$$

$$\alpha'_{22} = (1/\sqrt{2})\beta\sin\gamma . \quad \text{(VII.24d)}$$

The motion of the core is now described by β, γ, and the three Euler angles which specify the orientation of the nuclear symmetry axes relative to the laboratory system. The interaction Hamiltonian becomes

$$H_{\text{int}} = -DR_0 \sum_i \delta(r_i - R) \times [\beta\cos\gamma Y_{20} + \sqrt{2}\beta\sin\gamma Y_{22}] . \quad \text{(VII.25)}$$

Since $\sum_\mu |\alpha_{2\mu}|^2 = \beta^2$, the potential energy of the surface is simply

$$V_{\text{so}} = \tfrac{1}{2} C_2\beta^2 , \quad \text{(VII.26)}$$

while T_{so} breaks up into a rotational part describing the motion of the symmetry axes

$$T_{\text{rot}} = \sum_\nu \frac{\hbar^2 R_\nu^2}{2\mathcal{I}_\nu} \quad \text{(VII.27a)}$$

and a part describing the vibrations in shape and size of the core

$$T_{\text{vib}} = \tfrac{1}{2} B_2(\dot\beta^2 + \beta^2\dot\gamma^2) . \quad \text{(VII.27b)}$$

The vibrational states are generally very high in the nuclear sprectrum, so that they are hard to observe. The moments of inertia are given by the hydrodynamical model as

$$\mathcal{I}_\nu = 4B_2\beta^2\sin^2(\gamma - \tfrac{2}{3}\pi\nu) , \quad \text{(VII.27c)}$$

and the R_ν are the angular momentum operators for the nuclear core, obeying the usual commutation relations

$$[R_a,R_b] = -iR_{a\times b} . \quad \text{(VII.27d)}$$

In seeking appropriate eigenfunctions which have $I(I+1)$ and I_z as good quantum numbers it is useful to rewrite Eq. (VII.27a) substituting $\mathbf{R} = \mathbf{I} - \mathbf{J}$

$$T_{\text{rot}} = (\hbar^2/2\mathcal{I})(\mathbf{I}^2 + \mathbf{J}^2 - 2\mathbf{I}\cdot\mathbf{J}) + \left(\frac{\hbar^2}{2\mathcal{I}_{z'}} - \frac{\hbar^2}{2\mathcal{I}}\right)(I_{z'} - J_{z'})^2 , \quad \text{(VII.28)}$$

where we set

$$\mathcal{I}_{x'} = \mathcal{I}_{y'} = \mathcal{I} . \quad \text{(VII.28a)}$$

The motion of the nucleons in unfilled shells is assumed to be rapid, compared with the rotation of the core, so that the particle motion adjusts to this rotation "adiabatically." The wave function may then be written as a simple product

$$D(\theta_\nu)\chi(\mathbf{r}_i) , \quad \text{(VII.29)}$$

where D represents the motion of the core, and χ the intrinsic motion of the nucleons in unfilled shells. Since $\mathbf{I}$ induces a rotation of the coupled system, it clearly must leave the intrinsic state of the extra nucleons invariant. Thus $\mathbf{I}$ operates only on D while J operates only on χ.

It follows at once that $D(\theta_\nu)$ must be the proper function for the symmetric top, with eigenvalues

$$\mathbf{I}^2 D^I_{MK} = I(I+1)D^I_{MK} ,$$
$$I_z D^I_{MK} = MD^I_{MK} ,$$
$$I_{z'} D^I_{MK} = KD^I_{MK} , \quad \text{(VII.29a)}$$

where K is the projection of $\mathbf{I}$ on the symmetry axis, and M is the projection of $\mathbf{I}$ on the space-fixed axis. Because the nucleons move in a deformed well, J is not generally a good quantum number. Therefore, the intrinsic wave function χ will be designated by Ω, the component of the angular momentum of the nucleons in unfilled shells on the axis of symmetry. The axial symmetry of the nuclear core is assured by the constraint

$$R_{z'}\psi = 0 \quad \text{(VII.29b)}$$

which implies that

$$(I_{z'} D^I_{MK})\chi_\Omega = D^I_{MK}(J_{z'}\chi_\Omega) \quad \text{(VII.29c)}$$

so we must have

$$K = \Omega\,. \qquad \text{(VII.29d)}$$

Properly normalized eigenfunctions for the coupled system must then be taken as[32]

$$\psi(IKM) = [(2I+1)/16\pi^2]^{1/2} \times \{D^I_{MK}\chi_K + (-1)^{I-J}D^I_{M-K}\chi_{-K}\} \qquad \text{(VII.29e)}$$

to preserve invariance under reflections of the coordinates.

A set of states having the same intrinsic wave function, but differing in I are denoted as a rotational band. Since $I \geqslant K$ the allowed states of a K band are

$$I = K, K+1, \cdots \infty\,, \qquad \text{(VII.29f)}$$

and will obviously be ordered according to increasing values of I. When $K = 0$ the states

$$I = 0,2,4,\cdots \qquad \text{(VII.29g)}$$

have even parity, while the states

$$I = 1,3,5,\cdots \qquad \text{(VII.29h)}$$

have odd parity. $K = 0$ bands of even parity are easily recognized in many nuclei (S55).

The Hamiltonian relevant to the rotational collective motion is

$$H_{\text{rot}} = H_p + (\hbar^2/2\mathcal{I})(\mathbf{I}^2 + \mathbf{J}^2 - 2\mathbf{I}\cdot\mathbf{J})\,, \qquad \text{(VII.30)}$$

where the vibrational terms are neglected and we have already noted that for a spheroidal deformation, $I'_z - J'_z$ may be equated to zero. Clearly the term in $\mathbf{J}^2$ affects only the intrinsic structure, and therefore, may be incorporated into H_p

$$H_{\text{rot}} = H'_p + (\hbar^2/2\mathcal{I})(\mathbf{I}^2 - 2\mathbf{I}\cdot\mathbf{J})\,. \qquad \text{(VII.30a)}$$

The $\mathbf{I}\cdot\mathbf{J}$ term is not diagonal in the $\psi(IKM)$ representation, and thus, it can perturb the energy levels within a rotational band by mixing K values. If J lies along the z' axis, then $\mathbf{I}\cdot\mathbf{J} = \mathbf{J}^2$ and then this perturbation becomes independent of I. For this reason

$$(\hbar^2/\mathcal{I})\mathbf{I}\cdot\mathbf{J} \qquad \text{(VII.30b)}$$

measures the decoupling of the intrinsic motion from the z' axis, and is generally referred to as the decoupling term.

At this point, it is usually assumed that the mixing of K bands can be neglected as a first approximation. Diagonal matrix elements of H_{rot} are then bound to be

$$\langle IKM|H_{\text{rot}}|IKM\rangle = \langle H'_p\rangle + (\hbar^2/2\mathcal{I})\{I(I+1) + a(-1)^{I+1/2}(I+\tfrac{1}{2})\delta(K\tfrac{1}{2})\}, \qquad \text{(VII.30c)}$$

where a is given by

$$a = \textstyle\sum_J |c_J|^2(-1)^{J-1/2}(J+\tfrac{1}{2})\,, \qquad \text{(VII.30d)}$$

and c_J is the amplitude of the function χ_{JK} in the deformed intrinsic wave function χ_K. This latter term in $\delta(K\tfrac{1}{2})$ is needed, since $\mathbf{I}\cdot\mathbf{J}$ has a nonvanishing diagonal matrix element for $K = 1/2$ (D53b).

The lowest state of a rotational band is now clearly seen to have angular momentum $I = K$. The reason why three like nucleons in j-orbitals are frequently observed to couple in such a way as to give an $I = j - 1$ ground state can now be readily understood. If the surface-to-particle coupling dominates over particle–particle coupling (as it will for strongly deformed nuclei) two of the nucleons will pair off by antialigning along the z' axis (that is, one will have $\Omega_p = +j$, while the other has $\Omega_p = -j$). The third nucleon will then go into a state with $\Omega_p = j - 1$, so that $\Omega = \sum\Omega_p = j - 1$, and $I = K = \Omega = j - 1$. On the other hand, if particle–particle coupling dominates, three nucleons in an unfilled shell will first couple to $J = j$ with projection $\Omega = j$ on the z' axis. One then has a ground state angular momentum $I = K = \Omega = j$, as predicted by the simple single-particle model. The occurrence of $I = j - 1$ ground states may therefore be considered symptomatic of a strongly deformed nucleus.

A detailed model of the intrinsic structure has been worked out by Nilsson (N55). The Hamiltonian for a single particle in the deformed potential of the nucleons comprising the core is given by

$$H = \tfrac{1}{2}M\omega_0^2(\delta)[(x^2+y^2)(1+\tfrac{2}{3}\delta) + z^2(1-\tfrac{4}{3}\delta)] - (\hbar^2/2M)\nabla^2 + a\mathbf{l}\cdot\mathbf{s} + D\mathbf{l}^2\,. \qquad \text{(VII.31)}$$

The term $D\mathbf{l}^2$ is introduced to compensate for the tendency of the oscillator well to raise the energy of states with high l. For small deformations δ is related to the previous parameter β by

$$\delta = \tfrac{3}{4}(5/\pi)^{1/2}\beta \cong 0.95\,\beta\,, \qquad \text{(VII.31a)}$$

and $\omega_0(\delta)$ is fixed by the condition that equipotential surfaces have constant volume:

$$\omega_x\omega_y\omega_z = \text{const}\,,$$
$$\omega_0(\delta) = (1 - \tfrac{4}{3}\delta^2 - (16/27)\delta^3)^{-1/6}\omega_0^0\,, \qquad \text{(VII.31b)}$$

[32] If $K = 0$, one has
$$\psi(IOM) = [(2I+1)/8\pi^2]^{1/2}D^I_{MO}\chi_O\,.$$

where ω_0^0 is the value of ω_0 when there is no deformation. a and D are determined by requiring a good fit to the observed single-particle level order for a vanishing deformation.

The Hamiltonian is then written as

$$H = H_0 + H_\delta + a\mathbf{l}\cdot\mathbf{s} + D\mathbf{l}^2 , \quad \text{(VII.31c)}$$

where

$$H_0 = -(\hbar^2/2M)\nabla^2 + \tfrac{1}{2} M\omega_0^2 r^2 , \quad \text{(VII.31d)}$$

and

$$H_\delta = -M\omega_0^2\delta r^2 \tfrac{4}{3} (\pi/5)^{1/2} Y_{20}(\theta,\varphi) . \quad \text{(VII.31e)}$$

The basis functions may then be taken to be $|Nl\Lambda\sum\rangle$, where,

$$\begin{aligned} H_0|Nl\Lambda\Sigma\rangle &= (N + \tfrac{3}{2})\hbar\omega_0|Nl\Lambda\Sigma\rangle \\ \mathbf{l}^2|Nl\Lambda\Sigma\rangle &= l(l+1)|Nl\Lambda\Sigma\rangle \\ l_z|Nl\Lambda\Sigma\rangle &= \Lambda|Nl\Lambda\Sigma\rangle \\ s_z|Nl\Lambda\Sigma\rangle &= \Sigma|Nl\Lambda\Sigma\rangle . \end{aligned} \quad \text{(VII.31f)}$$

H_0 and $\mathbf{l}^2$ are obviously diagonal in this representation, while $\mathbf{l}\cdot\mathbf{s}$ will couple basis functions in which $\Lambda \rightarrow \Lambda \pm 1$, $\sum \rightarrow \sum \pm 1$, and H_δ connects functions

$$\langle Nl\Lambda\Sigma|H_\delta|N \pm 2, l \pm 2, \Lambda\Sigma\rangle \neq 0 . \quad \text{(VII.31g)}$$

No operator in the Hamiltonian may couple N to $N \pm 1$, due to invariance of the wave function under reflections. The coupling of different N values is thereby neglected as a first approximation, owing to the fact that $2\hbar\omega$ is much larger than the nondiagonal coupling energies in most of the applications. It is expected that the coupling between shells with different N is accounted for by modifying the parameters in the Hamiltonian.

N is thereby regarded as a good quantum number, and of course so is $\Omega = \Lambda + \sum$ due to the axial symmetry of the problem. The single-particle functions are then of the form

$$|N\Omega\rangle = \sum_{l,\Lambda+\Sigma=\Omega} a_{l\Lambda}|Nl\Lambda\Sigma\rangle , \quad \text{(VII.31h)}$$

and for each $|N\Omega\rangle$ there is a matrix to be diagonalized. The largest matrix to appear in the Nilsson calculation is 7×7. The parameters of the Hamiltonian turn out to be a function of A with

$$\begin{aligned} \hbar\omega_0^0 &\cong 41A^{-1/3}\ \text{MeV} \\ \mu = 2D/a &= 0 \quad \text{if} \quad N = 0,1,2 , \\ &= 0.35 \quad \text{if} \quad N = 3 . \end{aligned} \quad \text{(VII.31i)}$$

Single-particle level positions are then calculated in terms of the deformation parameter

$$\eta = (\delta/\chi)[1 - \tfrac{4}{3}\delta^2 - \tfrac{16}{27}\delta^3]^{-1/6} \quad \text{(VII.31j)}$$

where $\chi = -\frac{1}{2}(a/\hbar\omega_0^0)$. The decoupling factor and the gyromagnetic ratio for the intrinsic motion may then be computed from the derived $|N\Omega\rangle$.

A great many applications of the collective model have been made to specific nuclei throughout the periodic table. We shall describe only a few examples in light nuclei ($A < 40$) here.

There is definite evidence for the existence of very weak collective effects in O^{17} (W59a). The electric-quadrupole moment is

$$Q(O^{17}) = (-0.026 \pm 0.009)(10^{-24})\ \text{cm}^2 . \quad \text{(VII.32)}$$

The motion of the last neutron about the doubly magic core gives rise to a reactive motion of the protons in the core, producing a quadrupole moment $\sim$1/20 the observed value. Fallieros and Ferrell (F59b) have demonstrated that a consistent treatment of this effect gives rise to a quantum mechanical contribution from the exchange of the last neutron with neutrons in the $1p$ shell. Using oscillator functions it was then found that the quadrupole moment just exactly vanished, laying to rest any hope of interpreting the quadrupole moment of O^{17} without distorting the core. Raz (R57b) finds a quadrupole moment of

$$Q_{\text{theory}} = (-0.030 \pm 0.006) \times 10^{-24}\ \text{cm}^2 \quad \text{(VII.32a)}$$

by a straightforward application of the collective model in weak coupling, while Amado (A57a) has related the collective effect to configuration mixing in the nuclear core.

F^{19} has been treated both by a strong (P57) and an intermediate coupling (A58a) procedure. Intermediate coupling yields amplitudes for states involving two quanta of surface oscillation which are a bit large (about 30% of the wave function for $I = 3/2+$), so that this procedure does not appear trustworthy, even though the results agree well with experiment. Paul finds very good agreement with the observed properties of F^{19} by applying the Nilsson model. The even parity states are assigned to the two rotational bands with $K = 1/2$, and $K = 3/2$, which lie much lower than the others, if $\delta > 0$. The $I = 1/2+$ ground state is then immediately interpreted as the lowest state of the $K = 1/2$ band. The level order is correct, but the level spacing is greater than that observed. To remedy this, Paul introduces mixing between the $K = 1/2$ and $K = 3/2$ bands through the term $\mathbf{I}\cdot\mathbf{J}$ in the Hamiltonian. The observed values are then quite well reproduced. Table VII-1 compares a few results with the calculations by Elliott and Flowers (E55a). Chi (C62) mixes in the $K = 5/2$ band as well as those with $K = 1/2$ and $K = 3/2$,

improving agreement with the position of the $I = 5/2+$ state which Paul found to be about 20% too high.

Evidence for the deformation of the core persists through the $1d$–$2s$ shell (B60d, R57a). Paul and Montague (P58b) successfully treat Na^{23} in a manner similar to Paul's analysis of F^{19}, while Litherland *et al.* (L58a, G57e) find strong coupling applicable at $A = 25$.

TABLE VII-1. Predictions of the shell model in intermediate coupling (E55a), with those of the Nilsson model (P57) in F^{19}.

Property	P57	E55a	Experiment
γ-ray branching ratio $\frac{3}{2} \rightarrow \frac{1}{2}+$ to $\frac{3}{2} \rightarrow \frac{5}{2}+$	0.8%	0.6%	4%
Magnetic moments (nuclear magnetons)			
$\frac{1}{2}+$	2.75	2.80	2.63
$\frac{5}{2}+$	3.70	3.30	3.50 ± 0.24
Lifetime (sec) $\frac{5}{2}+$ state	0.9×10^{-7}	5×10^{-7}	$(1.25 \pm 0.025) \times 10^{-7}$
log ft Ne^{19} decay	3.52	3.2	3.3

Si^{29} and P^{29} provide excellent examples for investigation due to the fact that one may regard $_{14}Si_{14}$ as a doubly magic core, and, consequently, a single-particle treatment is appropriate for the odd nucleon. In addition the $A = 29$ systems have been treated by several different procedures, and thereby we have an opportunity to compare these procedures.

Bromley *et al.* (B57b) and Chi (C62) adopt strong coupling at $A = 29$. Chi obtains remarkably close agreement with the position of the first four levels (1/2+, 3/2+, 5/2+, and 3/2+) in Si^{29}–P^{29} with a relatively small distortion parameter ($\beta = 0.150$ in P^{29}, and $\beta = 0.162$ in Si^{29}).

The intermediate coupling procedure has also been applied to the low levels of Si^{29} and P^{29} (F55, G56c). The wave function

$$\psi(II_z) = \sum_{\nu jk} p_{\nu jk} \Phi(\nu jK; II_z) \qquad \text{(VII.33)}$$

includes all possible states with $\nu = 0$, 1, and 2 coupling $2s_{1/2}$ and $1d_{3/2}$ single-particle orbitals. Amplitude coefficients for the lowest $I = 1/2$ and $I = 3/2$ states are displayed in Tables VII-2 and VII-3. We note that the $I = 1/2$ state contains a 34% admixture of the $1d_{3/2}$ orbital, while the $I = 3/2$ state contains a 17% admixture of the $2s_{1/2}$ orbital. The two surfon components amount to only 4.1% of the total wave function for both cases, which would seem to justify the intermediate coupling procedure and cast some doubt on the validity of strong coupling.

The magnetic-dipole moment of Si^{29} provides an interesting comparison of several methods. The experimental value is

$$\mu_{\text{exp}}(Si^{29}) = -0.555 \text{ nm} . \qquad \text{(VII.34)}$$

which deviates strongly from the single-particle (Schmidt) value

$$\mu(\tfrac{1}{2}+) = -1.91 \text{ nm} . \qquad \text{(VII.34a)}$$

TABLE VII-2. Amplitude coefficients for Si^{29} (or P^{29}) with $I = \frac{1}{2}$ ($E = -0.622\ \hbar\omega$).

ν	K	j	$p^2_{\nu jk}$
0	0	$\frac{1}{2}$	0.645
1	2	$\frac{3}{2}$	0.314
2	0	$\frac{1}{2}$	0.015
2	2	$\frac{3}{2}$	0.026

The strong coupling procedure yields a small improvement to the Schmidt value (C62)

$$\mu_{\text{strong}} = -1.779 \text{ nm} , \qquad \text{(VII.34b)}$$

while for intermediate coupling we obtain (F55, G56c)

$$\mu_{\text{int}} = -1.211 \text{ nm} . \qquad \text{(VII.34c)}$$

Intermediate coupling appears to yield a correction much closer to experiment, lying about halfway between the single-particle moment and the observed value. Neither correction, however, comes as close

TABLE VII-3. Amplitude coefficients for Si^{29} (or P^{29}) with $I = \frac{3}{2}$ ($E = -0.622\ \hbar\omega$).

ν	K	j	$p^2_{\nu jk}$
0	0	$\frac{3}{2}$	0.645
1	2	$\frac{3}{2}$	0.157
1	2	$\frac{1}{2}$	0.157
2	0	$\frac{3}{2}$	0.015
2	2	$\frac{3}{2}$	0.013
2	2	$\frac{1}{2}$	0.013

to the observed moment as the results obtained by configuration mixing (N59a, reviewed in Sec. IV.5), where one gets

$$\mu_{\text{c.m.}} = -0.93 \text{ nm} , \qquad \text{(VII.34d)}$$

by mixing configurations in which one nucleon in a $1d_{5/2}$ orbit is promoted to a $1d_{3/2}$ orbit.

This comparison indicates that, when the core is not strongly deformed, configuration mixing proba-

bly provides a clearer interpretation of magnetic-dipole moments than the collective model, which presumably takes a large number of mixed configurations into account (A59a). The reasons for this were explained in detail in Sec. IV.5. The situation is quite different in calculating electric-quadrupole moments, however, where the effect of a small admixture in which many particles are excited may provide a substantial correction to the result.

In the intermediate coupling approach it is necessary to introduce the single-particle energies as additional parameters (T60c), so that this method throws little light on the level order. If the deformation is not strong, however, it would appear to be reasonable to take these parameters out of a simple shell-model interpretation.

In P^{31} the intermediate coupling procedure breaks down (G56c), and one must refer to the Nilsson model in order to understand the level order (B58f). One notes, in fact, that in the $1d$–$2s$ shell the parameters needed to interpret levels in terms of the collective model are subject to rapid fluctuations in going from one isobar to the next (C62).

This general problem persists in heavier nuclei as well. While many nuclei exhibit a collective behavior, the parameters needed for quantitative comparisons are never near the hydrodynamic values which we have discussed so far. The observed moments of inertia, in particular, are generally much larger than those given by Eq. (VII.27c). The maximum value that the moment of inertia may take is, obviously, the rigid body value $\mathcal{I}_{\text{rig}}$. When one assumes irrotational flow for the nuclear fluid, one finds a moment of inertia

$$\mathcal{I}_{\text{irrot}} \backsim \mathcal{I}_{\text{rig}}\beta^2 , \qquad \text{(VII.35)}$$

which is usually found to be about half the observed value. The assumption of irrotational flow is clearly in error, and at the same time $\mathcal{I}_{\text{rig}}$ turns out to be larger than the observed moments of inertia by a factor of 2

$$\mathcal{I}_{\text{irrot}} < \mathcal{I}_{\text{obs}} < \mathcal{I}_{\text{rig}} , \qquad \text{(VII.35a)}$$

so that one must seek a method of computing $\mathcal{I}$ more sophisticated than either of these two simple extremes.

Inglis (I54, I55, I56) has devised a simple model (the "cranking model") which estimates the collective response of nucleons to a rotation of the binding field. In the cranking model the nucleons are assumed to move in a spheroidal potential well which rotates with constant angular velocity ω about an axis perpendicular to the symmetry axis. Let x, y, z be the Cartesian coordinates of a nucleon in the laboratory system, and x', y', z' the corresponding coordinates in the body-fixed system. If z' is the symmetry axis and x' the axis of rotation then

$$\begin{aligned} x' &= x \\ y' &= y\cos\omega t + z\sin\omega t \\ z' &= -y\sin\omega t + z\cos\omega t . \end{aligned} \qquad \text{(VII.36)}$$

The Hamiltonian for a single nucleon in the laboratory frame is

$$H = T_p + V(xyz,t) , \qquad \text{(VII.36a)}$$

while the wave function will be denoted as $\psi(xyz,t)$ in the laboratory and $\varphi(x'y'z',t)$ in the body-fixed frame of reference. In order to be able to deal with a static potential in solving

$$H\psi = i\hbar(\partial\psi/\partial t) , \qquad \text{(VII.36b)}$$

we transform to the body-fixed frame and note that

$$V(xyz,t) = V(x'y'z',\text{o}) . \qquad \text{(VII.36c)}$$

Now differentiate

$$\psi(xyz,t) = \varphi(x'y'z',t) \qquad \text{(VII.36d)}$$

totally with respect to t (x is fixed)

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} &= i\hbar\left\{\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial y'}\frac{\partial y'}{\partial t} + \frac{\partial\varphi}{\partial z'}\frac{\partial z'}{\partial t}\right\} \\ &= i\hbar\left\{\frac{\partial}{\partial t} + \omega\left(z'\frac{\partial}{\partial y'} - y'\frac{\partial}{\partial z'}\right)\right\}\varphi \\ &= i\hbar\frac{\partial\varphi}{\partial t} + \omega l_x\varphi . \end{aligned} \qquad \text{(VII.36e)}$$

Schrödinger's equation in the body-fixed frame now becomes

$$i\hbar(\partial\varphi/\partial t) = H'\varphi , \qquad \text{(VII.37)}$$

where H' has only static terms

$$H' = T_p + V(x'y'z') - \omega l_x . \qquad \text{(VII.37a)}$$

If ω is small we may solve Eq. (VII.37) as a simple stationary-state equation and disregard the slow time dependence of the energy eigenvalue and the eigenfunction.

We seek the solution up to terms in ω^2, which are most easily obtained by perturbation theory. The zero-order Hamiltonian is then

$$(T_p + V(x'y'z'))|i\rangle = E_i^{(0)}|i\rangle \qquad \text{(VII.37b)}$$

and ωl_x is the perturbation. Obviously,

$$\langle i|l_x|i\rangle = 0 , \qquad \text{(VII.37c)}$$

and hence the energy shift in second order is given by

$$E_n'(\omega) = E_n^{(0)} + \omega^2 \sum_i \frac{|\langle i|l_x|n\rangle|^2}{E_n^{(0)} - E_i^{(0)}} \quad \text{(VII.37d)}$$

with corresponding eigenfunctions

$$\varphi_n = |n\rangle + \omega \sum \frac{\langle n|l_x|i\rangle|i\rangle}{E_i^{(0)} - E_n^{(0)}} . \quad \text{(VII.37e)}$$

$E_n'(\omega)$ is not quite the correction we seek, since

$$H = H' + \omega l_x . \quad \text{(VII.37f)}$$

The perturbed energy in the laboratory system is

$$\begin{aligned} E_n(\omega) &= E_n'(\omega) + \omega\langle\varphi_n|l_x|\varphi_n\rangle \\ &= E_n^{(0)} + \omega^2 \sum_i \frac{|\langle i|l_x|n\rangle|^2}{E_i^{(0)} - E_n^{(0)}} . \end{aligned} \quad \text{(VII.37g)}$$

If several particles are present in the well we must sum the above expressions over all occupied orbitals. One interprets Eq. (VII.37g) by recalling that classically

$$E(\omega) = E(0) + \tfrac{1}{2}\mathcal{I}\omega^2 \quad \text{(VII.38)}$$

where $\mathcal{I}$ is the effective moment of inertia of the system

$$\mathcal{I} = 2 \sum_n \sum_i \frac{|\langle i|l_x|n\rangle|^2}{E_i^{(0)} - E_n^{(0)}} . \quad \text{(VII.38a)}$$

Straightforward application of Eq. (VII.38a) to a system of independent nucleons leads to an effective moment of inertia very close to the rigid-body value, and consequently too large to coincide with the observed values. Bohr and Mottelson (B55) suggest that the independent-particle picture is too naive and that residual interactions among the nucleons must be taken into account. In particular, one might expect that the pairing effect would tend to lower the ground state energy $E_n^{(0)}$ in an even-even nucleus, so that the energy denominators in Eq. (VII.38a) are much larger than an independent-particle treatment would indicate, reducing the calculated moment of inertia. Consistent perturbation treatments (A59, R59a) of two-body interactions in many-fermion systems show, however, that the modifications of the wave functions is such as to exactly cancel the effect of energy shifts on the moment of inertia.

Bohr, Mottelson, and Pines (B58g) propose that there is an "energy gap" in nuclei that display the properties of a strongly deformed core. In the regions $155 < A < 180$ and $230 < A < 250$, where such strong deformations are found, the first excited states of even–even nuclei lie about one MeV above the ground state, while the first excited states of odd A nuclei have excitation energies of only about 0.2 MeV. This effect may be traced to the pairing energy. In the first region ($155 < A < 180$) the protons are filling the $1h_{11/2}$ orbital as the neutrons fill the $1h_{9/2}$ orbital, while in the second region ($230 < A < 250$) protons fill the $1h_{9/2}$ orbital while the neutrons go into the $2g_{9/2}$ orbital. The point is now clear. In odd A nuclei the first excited state may arise simply from promoting the odd nucleon to a higher orbital, but an even–even nucleus in these regions cannot be excited without breaking up a zero-coupled pair of particles. Since the pairing energy is roughly proportional to $2j + 1$ the fact that both the protons and the neutrons in the regions of strong deformation fill orbitals with very high j values means that the excitation of a pair requires a great deal of energy.

The resulting energy gap is reminiscent of that found in superconductors[33] due to the pairing of electrons with opposite momenta. Belyaev (B59f) has applied the formalism of the theory of superconductivity to the problem of nuclear matter. The energy gap is shown to develop, if the two-body interactions are such as to create a self-consistent field plus a residual attractive interaction whose principal effect is to provide the pairing energy. One then obtains an effective moment of inertia for the system given by

$$\mathcal{I} = \sum_{i,k} \frac{|\langle i|j_x|k\rangle|^2}{E_i + E_k}\left[1 - \frac{(\epsilon_i - \lambda)(\epsilon_k - \lambda) + \Delta^2}{E_i E_k}\right], \quad \text{VII.38b}$$

where the sum on i goes over all unfilled orbitals, and the sum on k goes over all filled orbitals. λ is a chemical potential approximately equal to the Fermi energy of the system, the ϵ_i are the single-particle eigenvalues of the self-consistent field, Δ is the "superconducting" energy gap, and

$$E_i = [(\epsilon_i - \lambda) + \Delta^2]^{1/2} . \quad \text{(VII.38c)}$$

Griffin and Rich (G59b) have applied this theory to nuclei[34] in the rare-earth region, obtaining excellent agreement with experiment (% error varies from 20.1% to 0.8%). This work has recently given rise to a refinement of the shell model sometimes called the "quasi-particle" model (K60b, K61b). No applications of this work to light nuclei have as yet appeared in the literature.

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[33] For a thorough bibliography on the formalism of the theory of superconductivity, relevant to the present problem, we refer the reader to Belyaev (B59f).

[34] Solutions developed by Belyaev have the defect that the number of nucleons is not conserved and thus spurious states are mixed into the wave function. Consequently, these results pertain to an average behavior among neighboring isobars. Baranger (B60e) has since devised a formalism in which such spurious states are eliminated.

It is curious that both the shell model and the collective model offer descriptions of nuclear properties which are in good agreement with those observed, even though basic assumptions of the two models are quite different. This question was brought into sharp focus when it was found that the anomalous nucleus F^{19} (the ground state having $I = 1/2 +$ even though F^{19} is in the $1d_{5/2}$-shell) could be well understood in terms of either model (P57, E55a).

In Sec. V we saw that it is possible to put the shell model on a firm theoretical foundation, in terms of Schrödinger's equation for A interacting particles. This status has not yet been reached by the collective model. The main problem lies in the fact that the collective-model wave function is never written in terms of all the nucleon coordinates. In connecting the solutions of the A-body Schrödinger equation with the collective model one must somehow associate the coordinates describing the collective motion (α_i) with the particle coordinates $(\mathbf{r}_1\mathbf{r}_2\cdots\mathbf{r}_A)$. Villars (V57, V58) has approached the problem by making a canonical transformation from the laboratory system to an intrinsic set of axes within the nucleus, in the hope that the Hamiltonian will split into clearly distinguishable rotational and intrinsic parts. Unfortunately the coupling term between these parts turns out to be complicated and quite large, and no one has as yet devised a successful scheme in which it can be made small. Thus the method of canonical transformation does not appear to be a promising approach to the problem.

Hill and Wheeler (H53) originally suggested writing the nuclear wave function in the form

$$\psi(\mathbf{r}_i) = \int \chi(\alpha_i)\varphi(\alpha_i\mathbf{r}_i)d\alpha_i \qquad \text{(VII.39)}$$

where φ is referred to as the construction potential, and χ is a generating function. One might then take φ to be an independent-particle wave function referred to the intrinsic axes while the collective properties of the system are contained in χ. The method has the advantage that the collective coordinates (α_i) appear only as variables of integration, and thus the problem of redundant coordinates if avoided. The method has been extended by Peierls and Yoccoz (P57a, Y57), who treat φ by a Hartree–Fock approach and vary χ to minimize the energy eigenvalue

$$E = \int \psi^* H\psi d\tau \Big/ \int \psi^*\psi d\tau\,. \qquad \text{(VII.40)}$$

A simple illustration is found by treating the center-of-mass motion in this manner. We note that

$$\varphi = (A!)^{-1/2}\,|u_1u_2\cdots u_A| \qquad \text{(VII.41)}$$

does not have the proper translational symmetry and choose χ to be a function of the center-of-mass coordinate $\mathbf{R} = \sum\mathbf{r}_i/A$, replacing $\mathbf{r}_i$ by $\mathbf{r}_i - \mathbf{R}$ in φ. Variation of χ in Eq. (VII.40) then yields

$$\chi \backsim e^{i\mathbf{k}\cdot\mathbf{R}}\,, \qquad \text{(VII.42)}$$

so that

$$\psi \backsim \int e^{i\mathbf{k}\cdot\mathbf{R}}\varphi(\mathbf{r}_i - \mathbf{R})d\mathbf{R}\,. \qquad \text{(VII.43)}$$

It is evident that the method will yield

$$\chi \backsim D^I_{MK} \qquad \text{(VII.44)}$$

when applied to the rotational problem where ψ is required to be an eigenfunction of $\mathbf{I}^2$ and I_z.

The method does indeed provide generating functions which are intuitively of the right form, and one may then compare the derived inertial parameters with those found in nature. When the procedure is applied to the harmonic oscillator (in its ground state), the mass parameter is given precisely. This is not surprising since, as we have seen in Sec. IV.1, the center-of-mass motion may be separated out exactly by elementary means for this case. A similar calculation for the square well leads to an inertial constant which is substantially different from the mass of the system. Application of the method to calculating nuclear moments of inertial (Y57) yield results of the right order of magnitude, but no detailed agreement is attained. A part of the difficulty may be attributed to the fact that the $\varphi(\mathbf{r}_i)$ used were not actual solutions of the Hartree–Fock, self-consistent field problem.

Elliott (E58a, E58b) has made a careful study of the similarity between the shell model and the collective model, particularly for nuclei in the $1d$–$2s$ shell. The calculations on F^{19} demonstrate that the shell model is applicable when the mixing of configurations involving orbitals in the same oscillator shell (s^3, s^2d, $s\,d^2$, and d^3) are taken into account[35] (E55a, R55), while interpretation in terms of the collective model requires the mixing of K bands (P57). Now it must certainly be possible to find an alternate description of the collective motion in terms of the mixing of shell-model configurations, since all such configurations form a complete set of basis functions. The usefulness of such a description depends upon the number of configurations that must be taken into account, and the difficulty in finding the mixing coefficients. Elliott (E58a) demonstrates

[35] Some distortion of the nuclear core is needed in the shell-model calculations, however, to obtain agreement with the electromagnetic transitions.

a classification for oscillator orbitals of the same shell, such that the basis functions so defined display a band structure similar to the Nilsson levels.

The classification is based on the unitary unimodular group in three dimensions (SU_3), and is defined in terms of the linear transformations in three dimensions among the oscillator quanta contained in the wave function for a single particle in the oscillator well. One may define operators a_1^*, a_0^*, and a_{-1}^* which create one oscillator quantum with angular momentum projection 1, 0, and -1 along the z axis. Likewise one has the destruction operators a_1, a_0, and a_{-1}, obeying the usual commutation relations

$$[a_i,a_l] = [a_i^*,a_l^*] = 0\,, \quad [a_i,a_l^*] = \delta_{il}\,. \tag{VII.45}$$

The generators of SU_3 will then be the nine operators which annihilate a quantum i and replace it with a quantum l

$$a_l^* a_i\,, \tag{VII.46}$$

which have commutation relations

$$[a_i^* a_j, a_k^* a_l] = a_i^* a_l \delta_{kj} - a_k^* a_j \delta_{il}\,. \tag{VII.46a}$$

Irreducible representations of this group (for a given oscillator shell) may be designated by two numbers $(\lambda\mu)$, related to the usual partition function $[f]$ by $\lambda = f_1 - f_2$, $\mu = f_2$. The problem of which irreducible representations D^L of the three-dimensional rotation group R_3 are contained in an irreducible representation $(\lambda\mu)$ of SU_3 is then solved by group theoretical methods yielding

$$L = K, K+1, K+2, \cdots, [K + \max\{\lambda,\mu\}]\,, \tag{VII.47}$$

where

$$K = \min\{\lambda,\mu\}, \min\{\lambda,\mu\} - 2, \cdots, 1 \text{ or } 0\,, \tag{VII.47a}$$

unless $K = 0$, in which case one has

$$L = \max\{\lambda,\mu\}, \max\{\lambda,\mu\} - 2, \cdots, 1 \text{ or } 0\,. \tag{VII.47b}$$

Thus, this classification leads to a band structure quite similar to that found in the collective model. The fact that the bands found in SU_3 are cut off at a maximum value of L does not constitute a significant difference between the two theories, since this cutoff is generally high enough to be beyond experimental observation. A comparison of the wave functions for F^{19}, found by Elliott and Flowers (E55a), and those given by the SU_3 representations show an overlap greater than 90%.

Basis functions in the SU_3 scheme are constructed by replacing a_1^* a_1^*, a_0^* a_0, and a_{-1}^* a_{-1} by the commuting operators

$$\begin{aligned} H_0 &= a_1^* a_1 + a_0^* a_0 + a_{-1}^* a_{-1}\,, \\ Q_0 &= 3a_0^* a_0 - H_0\,, \\ L_0 &= a_1^* a_1 - a_{-1}^* a_{-1}\,, \end{aligned} \tag{VII.48}$$

which are to be diagonal. H_0 is just proportional to the oscillator Hamiltonian, and the final intrinsic functions

$$\chi([f](\lambda\mu)\epsilon\Lambda K) \tag{VII.49}$$

are designated by the particle symmetry $[f]$[36], the SU_3 classification $(\lambda\mu)$,

$$\begin{aligned} \epsilon &= 3N_0 - N \\ \Lambda &= \tfrac{1}{2}(N_1 + N_{-1}) \end{aligned} \tag{VII.49a}$$

and the projection of the angular momentum along the z-axis K. The χ may be expanded in terms of basis functions in which the total orbital angular momentum L is a good quantum number

$$\chi_\Omega([f](\lambda\mu)\epsilon\Lambda K) = \sum_{\alpha,L'} c(\lambda\mu\epsilon\Lambda K;\alpha L') \times \psi_\Omega([f](\lambda\mu)\alpha L'K)\,, \tag{VII.50}$$

where α distinguishes states with redundant L, and Ω indicates that the functions are referred to an intrinsic frame of reference where ϵ and Λ are good quantum numbers. The ψ_Ω may be transformed into the laboratory frame in the usual manner

$$\psi_\Omega([f](\lambda\mu)\alpha L'K) = \sum_{K'} D^{L'}_{K'K}(\Omega)^* \psi([f](\lambda\mu)\alpha L'K') \tag{VII.51}$$

so that by multiplying Eq. (VII.50) by $D^L_{MK}(\Omega)$ and integrating we obtain

$$\int \chi_\Omega([f](\lambda\mu)\epsilon\Lambda K) D^L_{MK}(\Omega)\,d\Omega = \sum_\alpha \frac{c(\lambda\mu\epsilon\Lambda K;\alpha L)}{(2L+1)} \psi([f](\lambda\mu)\alpha LM)\,. \tag{VII.52}$$

If we redefine the ψ so that states are distinguished by K rather than α one can write:

$$\psi([f](\lambda\mu)KLM) = \frac{2L+1}{c(\lambda\mu,KL)} \int \chi_\Omega([f](\lambda\mu)K) D^L_{MK}(\Omega)\,d\Omega\,, \tag{VII.53}$$

which express the laboratory wave functions in terms of Hill–Wheeler integrals over the wave functions in

[36] In the $1p$ shell the particle symmetry is identical with the SU_3 partition function since each particle is associated with one oscillator quantum. Koltun (K61) has applied an interesting computational technique to intermediate coupling problems in the $1p$ shell based on this method.

the intrinsic frame. The intrinsic states χ_Ω are redundant in that there are several sets of ϵ and Λ which generate the same ψ. Elliott chooses $\epsilon = \epsilon_{\max} = 2\lambda + \mu$, $\Lambda = \frac{1}{2}\mu$ if $\lambda \geqslant \mu$ and $\epsilon = \epsilon_{\min} = -(2\lambda + \mu)$, $\Lambda = \frac{1}{2}\lambda$ if $\lambda < \mu$, in order to ensure that all states of a given representation $(\lambda\mu)$ are generated by as few intrinsic functions as possible.

Thus the similarity of the states classified by the irreducible representations of SU_3 to a rotational band [found in Eq. (VII.47)], is not merely coincidental since Eq. (VII.53) displays the relation between the intrinsic states and a rotational wave function referred to the laboratory system. Goshen and Lipkin (G59c) and Pilkuhn (P59) have devised simplified models of this scheme (in one and two dimensions) in which the oscillator Hamiltonian is shown to split into a collective and intrinsic part.

An interesting comparison of intermediate coupling wave functions and wave functions generated by applying Elliott's method to Nilsson's intrinsic functions in the $1p$ shell has been made by Kurath and Picman (K59c). The Nilsson scheme yields three single-particle levels with $K = 3/2,(1/2)^2$ which may be expanded in terms of eigenstates $\psi(\alpha IK)$ of the total angular momentum I and its projection on the z axis K

$$|NK\rangle_\Omega = \sum_{\alpha,I} C_{\alpha IK}(\eta)\Phi_\Omega(\alpha IK)\,, \quad \text{(VII.54)}$$

where η is the deformation parameter [defined in Eq. (VII.31j)], and the subscript Ω again denotes the fact that these functions are referred to an intrinsic set of axes. The wave functions in the laboratory system are then obtained by Elliott's generating procedure

$$\psi(\alpha IM) = N_{IK}(\eta)(2I+1)\int [D^I_{KM}(\Omega)]^* |NK\rangle_\Omega d\Omega\,, \quad \text{(VII.55)}$$

where N_{IK} normalizes ψ. The $\psi(\alpha IM)$ are then compared with the functions

$$\Phi_{IM} = \sum_\alpha d_{I\alpha}\phi_{\alpha IM} \quad \text{(VII.56)}$$

obtained in Kurath's (K56) intermediate coupling calculations (discussed in Sec. IV.3). Overlaps of 99% or better are obtained by choosing $\eta\, a/K$ roughly proportional to the number of particles or holes in the shell, and the sign as positive if the shell is less than half full, and negative if the shell is more than half full. The spin-orbit coupling strength a cancels out in the product $\eta\ a/K$, thus this parameter is a measure of the amount of deformation needed to simulate the effect of the two-body interaction. Kurath (K59a) has used the generating procedure to compute the mixing of higher configurations ($2\hbar\omega$ above the ground configuration) in the wave functions of $1p$-shell nuclei, and finds considerably improved agreement in the estimates of $E2$ transition rates.

VIII. CLUSTER MODELS

The observation of α decay in radioactive nuclei led to the speculation that α particles might exist as stable structures within the nucleus very early in the development of nuclear theory. The resulting α-particle model had a strong intuitive appeal. The number of particles to be dealt with in a complex nucleus such as Ne^{20} would be reduced from 20 fermions with spin $\frac{1}{2}$ and isobaric spin $\frac{1}{2}$ to only 5 bosons with spin zero and isobaric spin zero.

The most striking feature of this model was its interpretation of binding energies. The α particles were assumed to be arranged in the ground state configuration so that they were as closely packed as possible. In O^{16}, for example, the four α substructures should form a tetrahedron, so that each of the six α pairs are close together. Now the binding energy of a nucleon in the α particle is about 7 MeV, so that nearly 90% of the nuclear binding energy is already accounted for. The remainder, denoted as the "inter-alpha binding energy," will for a nucleus comprising n α-particles be given by

$$B_\alpha = E - nE_\alpha\,, \quad \text{(VIII.1)}$$

where E is the total binding energy of the nucleus, and E_α is the binding energy of the α particle. Table VIII-1 displays values of B_α for a few light nuclei. We note that the inter-alpha binding per α bond is nearly constant[37] at $\sim$2.4 MeV.

This simple observation appears to give considerable weight to the α-particle model, but upon closer examination two major defects become obvious. The interalpha binding in Be^8 should be a cornerstone of the theory, but this nucleus is not even dynamically stable. In addition, the constant value of the α bond is simply related to the fact that the average binding energy per nucleon is roughly constant in these nuclei.

Another difficulty of the model is the fact that the α–α interaction is quite weak, while at the same time the α particles must be closely packed in order to reproduce the observed nuclear densities. Therefore the amplitude for zero-point oscillations of the α particles could not be small compared to their spac-

[37] In Ne^{20} and Mg^{24} there is some doubt about how many bonds one should count for the chosen configuration. We have simply taken the number which is in best accord with the other nuclei.

TABLE VIII-1. Inter-alpha binding energies B_α in light $A = 4n$ nuclei.

Nucleus	Number of α particles	Configuration	Number of bonds	B_α(MeV)	B_α/bond
Be^8	2	Dumbbell	1	−0.12	−0.12
C^{12}	3	Triangle	3	7.33	2.45
O^{16}	4	Tetrahedron	6	14.4	2.42
Ne^{20}	5	Trigonal Bipyramid	8	19.3	2.41
Mg^{24}	6	Octahedron	12	28.8	2.40
Si^{28}	7	Pentagonal Bipyramid	16	37.8	2.36
S^{32}	8	Hexagonal Bipyramid	19	46.8	2.47

ing. This must create a strong overlap between the internal wave functions of the α-particles, so that a constant interchange of nucleons will occur, and it is hard to believe that individual α particles retain their identity.[38] To remedy the situation the model was revised so that the α particles were pictured as dissolving into nuclear matter and then reforming (F37b, W37, W37c). It is found, however, that the "lifetime" of an α particle is not long in nuclear matter, so that the model does not appear to be useful in the description of complex nuclei.

It is possible that configurations in which several nucleons are closely correlated (or "clustered") may have large amplitudes in the wave functions of a few light nuclei. As we related in Sec. V, clustering is entirely negligible in saturated nuclear matter, but near the surface where the density is below its saturated value, clusters very likely will form. A reasonable approach for determining the significance of clustering in a given nucleus is to examine the energy needed to separate the nucleus into a given cluster configuration. If we break a nucleus of A nucleons up into k clusters, each composed of n_k nucleons so that

$$\sum_k n_k = A\ , \qquad \text{(VIII.2)}$$

then the "intercluster binding energy" may be defined as

$$B(n_1,n_2,\cdots) = E - \sum_k E_k\ , \qquad \text{(VIII.3)}$$

where E is the binding energy of the nucleus in question and E_k the binding energy of the (free) kth cluster.

Intercluster binding energies for a few nuclei in the $1p$ shell are displayed in Table VIII-2. If the intercluster binding is small compared to the binding energy of the least strongly bound cluster of a given configuration, then the formation of this configuration is favored energetically and may strongly overlap the actual nuclear wave function. On this basis the cluster model should be applicable to Be^8 and Be^9 if it is ever applicable at all. It could not possibly be realistic in B^{10} since it requires 6.1 MeV to break this nucleus up into smaller groups of particles ($He^4 + He^4 + H^2$), and the most weakly bound system (H^2) in the favored group is only bound by 2.2 MeV.

TABLE VIII-2. Inter-cluster binding energies for some $1p$-shell nuclei.

Nucleus	Cluster configuration	Inter-cluster binding energy (MeV)
Li^6	$He^4 + H^2$	1.5
Li^7	$He^4 + H^3$	2.5
Be^8	$He^4 + He^4$	−0.1
Be^9	$He^4 + He^4 + n$	1.6
B^{10}	$He^4 + He^4 + H^2$	6.1
	$Li^6 + He^4$	4.5
C^{12}	$(He^4)^3$	7.2
O^{16}	$(He^4)^4$	14.4

O^{16} and C^{12} display enough stability against break up into four and three α particles that application of the cluster model is very doubtful in the ground states. The excited states of these nuclei lie fairly high above ground, however, and may overlap certain α-configurations strongly. Li^6 and Li^7 may be considered as borderline cases subject to more intensive investigation (K61c).

By any standards Be^8 is the most favorable nucleus for application of the cluster model. Wildermuth and Kanellopoulos (W58) employ the separability of the harmonic oscillator Hamiltonian

$$\frac{1}{2M}\left\{\sum_{i=1}^{A}\mathbf{p}_i^2 + a^2\hbar^2\sum_{i=1}^{A}\mathbf{r}^2\right\}\psi = E\psi \qquad \text{(VIII.4)}$$

to write a cluster-model wave equation

$$\left\{\sum_{i=1}^{k}H_i + \frac{1}{2M}\sum_{i=1}^{k}\frac{1}{n_i}\mathbf{p}_i^2 + \frac{a^2\hbar^2}{2M}\sum_{i=1}^{k}n_i\mathbf{R}_i^2\right\}\psi = E\psi\ , \qquad \text{(VIII.5)}$$

where $a = M\omega/\hbar$, and

$$\mathbf{P}_i = \sum_{s=1}^{n_i}\mathbf{p}_s\ , \quad R_i = \frac{1}{n_i}\sum_{s=1}^{n_i}\mathbf{r}_i\ . \qquad \text{(VIII.6)}$$

[38] Modern estimates place the mean free path of an α particle within the nucleus at about 2×10^{-13} cm (R58a).

The H_i now depend only on the relative coordinates between the particles of the ith cluster. In Be^8 the coordinates of the two α clusters are

$$\mathbf{R}_{\alpha 1} = \tfrac{1}{4}\sum_{i=1}^{4}\mathbf{r}_i \qquad \text{(VIII.7a)}$$

$$\mathbf{R}_{\alpha 2} = \tfrac{1}{4}\sum_{i=5}^{8}\mathbf{r}_i \qquad \text{(VIII.7b)}$$

with internal coordinates for the $\alpha 1$ cluster

$$\boldsymbol{\varrho}_{12} = \mathbf{r}_1 - \mathbf{r}_2\,, \quad \boldsymbol{\varrho}_{34} = \mathbf{r}_3 - \mathbf{r}_4\,,$$
$$\boldsymbol{\varrho}_{1234} = (\mathbf{r}_1 + \mathbf{r}_2) - (\mathbf{r}_3 + \mathbf{r}_4)\,, \qquad \text{(VIII.7c)}$$

and a similar set for the $\alpha 2$ cluster. The wave function for Be^8 may then be written in the form

$$\Psi = \mathcal{A}\{\psi(\alpha 1)\psi(\alpha 2)X(\mathbf{R}_{\alpha 1} - \mathbf{R}_{\alpha 2})\}\,, \qquad \text{(VIII.8)}$$

where $\mathcal{A}$ denotes the fact that one must antisymmetrize the wave function with respect to all of the nucleons. Because of the Pauli principle $X(\mathbf{R}_{\alpha 1} - \mathbf{R}_{\alpha 2})$ must be an oscillator function of at least fourth degree so that four nucleons are forced into the $1p$ shell in the corresponding independent-particle wave function. Oscillator functions of the fourth degree involve the $4s$, $2d$, and $1g$ states.

Wildermuth and Kanellopoulos (W58) then use the two-body interaction (which reproduces two-nucleon scattering data up to 50 MeV)

$$V_{12} = -V_0 \exp -(\mathbf{r}_{12}/\beta)^2\{w[1 - \tfrac{1}{4}(1 + \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) \times (1 + \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)] + b[\tfrac{1}{2}(1 + \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) - \tfrac{1}{2}(1 + \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)]\}\,, \qquad \text{(VIII.9)}$$

where $V_0 = 68.6$ MeV, $\beta = 1.55 \times 10^{-13}$ cm, $w = 0.41$, and $b = 0.09$. A $^1S_0$ ground state is obtained for Be^8 with excited states $^1D_2$ and $^1G_4$. The excitation of the ^1D and ^1G states is proportional to $L(L+1)$ so that one has a rotational band cut off at $L = 4$. The oscillator parameter used for Be^8 was

$$a = 4.7 \times 10^{25}\,\text{cm}^{-2}\,, \qquad \text{(VIII.10)}$$

which would yield a binding energy for He^4 of 28.5 MeV with rms radius 1.55×10^{-13} cm, very near the experimental values of 28.2 MeV and 1.44×10^{-13} cm.

One then has a very simple picture of the low-lying levels of Be^8 in terms of two α particles, whose relative rotations provide the observed spectrum (a large energy gap exists between the second and third excited states, indicating the break up of an α particle). The shell-model interpretation of these levels is equally simple. Table IV-1 tells us that the states of the $(1p)^4$ configuration with maximum orbital symmetry ($[\lambda] = [4]$) are the $^1S_0$, $^1D_2$, and $^1G_4$ states. These will then be the low-lying levels of Be^8 and it is well known that their spectrum resembles a rotational band (F37a). There is therefore no distinction between the predictions of the shell model and the α-particle model.

It is found, in fact, that after the wave function in Eq. (VIII.8) is properly antisymmetrized the usual shell-model wave function for the system is obtained (P56a, K60). This result is quite a general one. Perring and Skyrme (P56a) have demonstrated that a similar situation attains for any nucleus which can be written as a configuration of α particles, and their results have been generalized to other nuclei by Kanellapoulos and Wildermuth (K60). The reason for this is easily understood. The particles are not really uncorrelated in the independent-particle model. In Be^8, for example, due to the fact that the four $1p$ nucleons are in a relative state of maximum orbital symmetry, there is a high probability that they will be found clustered together on one side of the closed $1s$ shell.

What then is the distinction between the independent-particle and cluster models? If one consistently includes configuration interaction in both models, certainly there is none. On the other hand, if we try to represent nuclei by a single configuration we find that certain types of collective motion may be incorporated into the cluster model in a simple way. For Be^8 the mean separation between the two α particles may be treated as an additional parameter. If this separation is much larger than the shell model indicates, then the α-particle model does indeed describe a collective distortion of the Be^8 wave function which could only be described in shell theory by including configuration mixing. The cluster model should then be looked upon as an alternate method of investigating distortions of nuclear wave functions, and we have the problem of comparing it with the model reviewed in Sec. VII.

Blair and Henley (B58h) propose a "strong coupling" version of the α-particle model for Be^9. The system is composed of two α particles separated to form a dumbbell, and a neutron which is strongly coupled to the motion of this dumbbell so that the projection of its angular momentum (Ω) along the symmetry axis is a good quantum number. The wave function for the system is then taken to be

$$\Phi_{IKM} = \left(\frac{2I+1}{16\pi^2}\right)^{1/2} X(|\mathbf{R}_{1\alpha} - \mathbf{R}_{2\alpha}|) \times \{\psi_{j\Omega}(\mathbf{r})D^I_{MK}(\theta_i) + (-1)^{I-j}\psi_{j-\Omega}(r)D^I_{M-K}(\theta_i)\}\,, \qquad \text{(VIII.11)}$$

where $\mathbf{r}$ is the coordinate of the odd neutron, θ_i represents the three Euler angles which specify the orientation of the symmetry axis of the dumbbell, K is the projection of the total angular momentum $\mathbf{I}$ on the symmetry axis, and M its projection on the space-fixed axis. The model is motivated by the observation that the energy levels in Be^8 and Be^9 are strongly reminiscent of a rotational band structure. Kunz (K60) has performed detailed calculations on Be^9, which are formally similar in structure to Nilsson's calculations (described in Sec. VII) for the strong coupling collective model.

The rotational eigenvalues

$$E_I = (\hbar^2/2\mathcal{I})[I(I+1) - I_0(I_0+1)] \quad \text{(VIII.12)}$$

with moment of inertia

$$\mathcal{I} = 2M_\alpha R^2 \quad \text{(VIII.12a)}$$

(M_α is the mass of an α particle and $2R$ the equilibrium separation of the two α particles), yield a reasonable fit to the $K = 0$ band of Be^8 and the $K = 3/2$ band of Be^9 for an equilibrium separation of

$$2R \cong 4.6 \times 10^{-13} \text{ cm} . \quad \text{(VIII.12b)}$$

Kunz relates the energy levels of Be^8 (E_L) to those of Be^9 (E_J) by

$$E_J = \sum_L [1 + (-1)^L]|C^{JjL}_{m-m0}|^2 E_L - E_{3/2} \quad \text{(VIII.13)}$$

reproducing the K = 3/2 band of Be^9 (3/2 − , 5/2 − , 7/2 − , 9/2 −) to an accuracy of about 10%.

The calculated electric–quadrupole moment of Be^9 is 7.9×10^{-26} cm², nearly three times the observed value of 2.9×10^{-26}. Thus it seems that this nucleus is not as distorted as the α-particle model suggests. Kurath's (K56) intermediate coupling calculations reproduce the level sequence in Be^9 quite as well as the α-particle model, but with a spin-orbit coupling strength which is surprisingly weak near the middle of the $1p$ shell (see Table IV-6). The magnetic moment of Be^9 (−1.18 nuclear magnetons) is closely reproduced by both models.

The need for introducing some collective effects into Be^9 is evident from the large quadrupole moment, but the α-particle model appears to be somewhat too extreme. Probably this nucleus is best described by some compromise between the two models involving considerable configuration interaction, perhaps the Nilsson picture.

Dennison (D54) succeeds in correlating the positions, angular momenta, and parities of sixteen levels with excitation energies up to 13.25 MeV in O^{16}. Perring and Skyrme (P56a) find, however, that once the α-particle wave functions are antisymmetrized they become combinations of a few shell-model states involving the excitation of one, two, or three nucleons out of the $(1s)^4$ $(1p)^{12}$ configuration. It is hard to decide whether or not the agreement found in the α-particle model is just coincidental. Dennison raises an interesting argument. The ground state of O^{16} is certainly spherically symmetric, while the α-particle model describes it as a tetrahedron. If the nuclear volume is divided into four equal parts, each part will on the average contain two protons and two neutrons, the spins in each pair being oppositely oriented. The probability of finding a fifth nucleon in a small volume is greatly reduced by the fact that the orbital part of the wave function for two like nucleons with spins oriented in the same direction is necessarily antisymmetric. Thus even a weak tendency to form four-nucleon clusters in the nucleus may be amplified. It is conceivable that even small cluster admixtures play a deciding role in determining the order of closely competing shell-model states.

The status of Li^6 in the cluster model is an enigma. The 1.5 MeV required to break this nucleus up into an α particle and a deuteron is not very small compared to the 2.2 MeV binding energy of the deuteron. Hence, while the $(1s)^4$ core is most likely stable, a deuteronlike cluster description for the two nucleons in the $1p$ shell does not seem favorable. On the other hand the separation of the 1+ ground state relative to the first excited 3+ state (favored as the ground state by Nordheim's rule) at 2.189 MeV is frequently attributed to admixtures of deuteron correlations in the ground-state wave function (S53), and analysis of the scattering of deuterons by α particles indicates that the α–d model is worthy of further study (G60a).

Wackman and Austern (W62) examine a three-body model of Li^6, in which the α particle is kept intact and the two extra nucleons are treated in quite a general manner. The interaction between a nucleon and the α particle is adopted from an analysis of n–α scattering, while the n–p interactions considered are those of Gammel and Thaler (with hard core) and Pease and Feshbach (without a repulsive core). Basis functions are of the form

$$\psi^J_{LS}(\mathbf{r}_1,\mathbf{r}_2) = \phi^J_{LS}(\Omega_1,\Omega_2,s_1,s_2) f(r_1,r_2,\rho) , \quad \text{(VIII.14)}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the respective distances of the two nucleons from the α particle and ρ is the distance between the two nucleons. The spin-orbit functions ϕ^J_{LS} are denoted by the orientations (Ω_i) of the vectors $\mathbf{r}_i$ and the spins of the nucleons (s_1, s_2). The ground state ($J\pi = 1+$) will then be written as the linear combination

$$\Psi = a_0\psi^1_{01} + a_1\psi^1_{10} + a_2\psi^1_{21} . \quad \text{(VIII.15)}$$

The radial functions are simple products

$$f(r_1,r_2,\rho) = h(r_1)h(r_2)g(\rho)\,, \quad \text{(VIII.16)}$$

where the $h(r)$ are chosen to be

$$h(r) = re^{-\mu r/2} \quad \text{(VIII.17)}$$

and μ is a variational parameter.

The $g(\rho)$ express the correlation between the two nucleons, and must be chosen very carefully. If the n–p interaction has no repulsive core, the choice

$$g(\rho) = e^{-\nu\rho/2} \quad \text{(VIII.18)}$$

is reasonable. If a hard core is present the function $g(\rho)$ must vanish inside the core radius. It was then found that the function

$$\begin{aligned} g(\rho) &= e^{-\nu\rho/2}[1 - e^{-\lambda(\rho-c)}]\,, \quad c < \rho \\ &= 0 \qquad\qquad\qquad\quad , \quad c > \rho\,, \end{aligned} \quad \text{(VIII.19)}$$

which is often suggested for such correlations (see Sec. V), will not even produce binding. The difficulty is attributed to the fact that both the potential and kinetic energies are very large just outside the repulsive core, and, consequently, small errors in the wave function for this region can have large effects on the energy eigenvalue (A60b). This error is avoided by solving the two-body Schrödinger equation with zero-energy eigenvalue in the region near the core, and using this solution in that part of the variational trial function from the core out to some cutoff ρ_c, where the effect of the core becomes small.

A complication in the three-body model arises from spurious states which violate the Pauli principle. These states are introduced because there is nothing in the formalism to prevent the two $1p$ nucleons from being demoted to the $1s$ shell. Such spurious states are projected out of the final wave functions. The ordering of levels with zero isobaric spin (J = 1,3,2) is then given correctly, but the $T = 1$ level with $J = 0$ is found to lie below the $J = 3$ level, contrary to experiment. The level spacing is quite reasonable, but all levels are displaced upwards in energy by about 3.5 MeV above the experimental values. Since this is a variational calculation, such a discrepancy is to be expected. Its source may be that configuration mixing due to the tensor interaction is not fully accounted for in the calculation. We note that the second-order term

$$\langle {}^3S_1|S_{ij}(E - H_0)^{-1}S_{ij}|{}^3S_1\rangle\,, \quad \text{(VIII.20)}$$

which provides the triplet-singlet splitting in H^2, will not appear in this problem. This term should displace the $T = 0$ levels downward by two or three MeV and affect the $T = 1$ levels hardly at all, so that it may provide the needed corrections both for the absolute energies and the relative positions of the $T = 1$ and $T = 0$ levels.

The magnetic moment of Li^6 is

$$\mu(Li^6) = 0.822 \text{ nm} \quad \text{(VIII.21)}$$

compared to a magnetic moment for a p^2 configuration in the ${}^3S_1$ state of

$$\mu(p^2;{}^3S_1) = 0.879 \text{ nm}\,. \quad \text{(VIII.22)}$$

The above calculation yields

$$\mu = 0.866 \text{ nm}\,, \quad \text{(VIII.23)}$$

so that one does improve upon the naive estimate of Eq. (VIII.22). Since relativistic corrections are being neglected and we are dealing with small differences, this does not present a critical test of the wave function.

A problem arises with regard to the electric-quadrupole moment of Li^6. The experimental value is uncertain, but estimates place it near (W62)

$$Q(Li^6) \approx -11 \times 10^{-28} \text{ cm}^2\,. \quad \text{(VIII.24)}$$

Wackman and Austern obtain

$$Q_{\text{calc.}} = 3.3 \times 10^{-28} \text{ cm}^2\,. \quad \text{(VIII.25)}$$

It is quite possible that the tensor effects mentioned previously will provide the indicated negative moment in Li^6 as they do in H^2 [one should point out, however, that even the sign in Eq. (VIII.24) is not definite].

Bayman and Bohr (B59g) have shown that the cluster model provides an alternate description of certain states in Elliott's SU_3 coupling scheme. Possibly the SU_3 scheme represents a proper description of the mixing of shell-model states in nature, manifesting itself in the cluster model for light nuclei with only a few clusters, and in the Bohr–Mottelson model for heavier nuclei. This, however, implies a significance for the SU_3 scheme which is not yet on a firm theoretical foundation. For the present, it seems safe to conclude that cluster configurations are mixed into the wave functions of light nuclei with sufficient amplitude to have some influence on the properties of a few favored nuclei. In particular, such cluster configurations may play a dominant role in determining branching ratios and angular distributions of nuclear reactions due to the strong overlap that these configurations will have with the unbound states yielded by the reaction (T60d).

ACKNOWLEDGMENTS

It is a pleasure to thank Professor H. S. Valk for several interesting discussions concerning the determination of nuclear size parameters and Professor J. P. Davidson for providing access to work on the collective model prior to its publication. The author would also like to take this opportunity to express his gratitude to Professor Eugene Feenberg for introducing him to the methods of nuclear spectroscopy.

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