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The Theory of Small-Angle Multiple Scattering of Fast Charged Particles*

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CONTENTS

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SUMMARY

A systematic unified summary and review is given of the basic statistical theory of the multiple scattering of fast charged particles in the small-angle range. The approximation considered is that of the Snyder–Scott–Molière theory, and only slight attention is given to the less accurate Gaussian approximation. The single-scattering formulas of Molière are derived, along with the modifications of them given by Nigam, Sundaresan, and Wu. Molière's multiple-scattering calculation is presented by an improvement of Bethe's method, and the work of Nigam *et al.* is given by the same method with newly computed tables. Snyder's calculations are outlined, and previously unpublished work on spatial-angle scattering is reported with tables. Calculations by Keil, Zeitler, and Zinn for very thin films are given, as well as a detailed discussion following Lenz on scattering at very small angles. The work of Mühlschlegel and Koppe on the multiple scattering of polarized electrons is included, with the important

correction that no depolarization appears in the approximation to which they worked. The distributions of lateral deflections and other characteristics are considered, but the details of applications to emulsions, cloud and bubble chambers, etc. are not entered into, nor are the electron-penetration and path-length problems handled. Asymptotic formulas for relatively large angles are treated, as are various types of mean values.

I. INTRODUCTION

MULTIPLE scattering occurs whenever traveling particles or waves undergo successions of similar processes that change the direction of motion and the successive scatterings are statistically independent or almost independent. Successive scatterings thus are considered, for a multiple-scattering process, to be incoherent in the quantum-mechanical sense; the occurrence of interference and diffraction in the scattering by a crystalline medium is treated as a correction, rather than as a principal effect.

Systems undergoing multiple scattering may be classified in two ways: The scatterers may be nucleons in a single nucleus, individual nuclei or atoms, successive crystallites, dust particles or other aggregates of matter; and the particles scattered may be neutral (neutrons or photons), charged (with or without strong interactions with nuclei), or primarily treatable as waves (electromagnetic radiation of relatively low frequency).

Multiple scattering within nuclear matter has been treated by Watson (1957) and others. Lax (1951) has reviewed the case of electromagnetic waves. A great deal of work has been done with isotropic and nearly-isotropic scattering, for instance of neutrons and X rays. Grosjean (1951, 1952, 1954, 1956a, 1956b, 1957) has turned out many papers on this subject; Richards (1955) has made a valuable contribution, and the whole question of neutron transport theory is treated in a book by Davison and Sykes (1957).

For charged particles, the multiple scattering is dominated by electrostatic (Coulomb) forces; little work has been done on the inclusion of strong-interaction (nuclear-force) effects with Coulomb scattering. Deviations from the Coulomb field caused by the screening and inelastic scattering effects of the atomic electrons and by the finite size of nuclei have been taken into account and will be discussed (the last only briefly) in this review article.

Charged-particle multiple scattering can be divided into three realms. Low-energy electron scattering is important in solid-state physics, plasma physics, etc. Large-angle scattering at moderate energies involves complex boundary conditions and path-length problems which are only partially soluble at the present time; high-energy, small-angle multiple scattering theory, on the other hand, is well advanced, and the time seems ripe for the review attempted in this article.

Large-angle scattering can be formulated rigorously for the case in which angular and spatial distributions after exactly n scatterings are sought [Wigner (1954), Grosjean (1951)], but experimental conditions for making observations of such scatterings are difficult to arrange. A formulation for the scattering distribution after a given path has been traversed in an infinite medium without boundaries has been obtained by Goudsmit and Saunderson (1940a, b), but related observations are only possible in track-visualization devices. The prediction of scattering in thin foils or other geometrical arrangements requires the knowledge of distributions at fixed points in space; in addition, the presence of boundaries involves albedo or absence-of-albedo problems as compared with infinite media.

These difficulties disappear in the small-angle approximation, and in addition, computations with the latter are very much simplified. Since most of the scattering by Coulomb fields on high-energy particles is concentrated in the forward direction, the small-angle calculation covers many useful applications and makes a good first approximation for studies at larger angles.

Multiple scattering first became relevant to particle physics in connection with Rutherford's (1911) discovery of the nucleus by means of alpha-particle scattering experiments. If the Thomson picture of the atom were correct, only small angular deflections could occur at each scattering, and any large angles observed must necessarily be caused by multiple scattering, under conditions for which a normal or Gaussian distribution was to be expected [Thomson (1910)]. The observations were in extreme contradiction to such a distribution, and were explained quantitatively on the assumption of the now well-known Rutherford single-scattering law and the neglect of any multiple-scattering effects.

Wentzel (1922) recognized that multiple scattering must have played a role in some of the later experiments of the Rutherford group, especially on beta particles, and gave formulas for plural scattering, involving up to seven scattering events. Wentzel also gave a criterion for the conditions under which single scattering can be assumed to hold. Bothe (1921, a, b, c) in a general discussion of the circumstances under which a Gaussian distribution law will hold for errors or fluctuations, showed that such distributions do not hold when the elementary events being combined have probability distributions with long "tails." Each of the last named authors made use of folding-integrals for successive events; Bothe applied Fourier and Hankel transforms, and Wentzel wrote down a summation formula (for 0,1,2,... scatterings) without evaluating it or using transforms.

Williams (1939, 1940) devised a moderately successful theory of multiple scattering based on a method of fitting together a Gaussian curve for the central part of the distribution and a single-scattering tail. Goudsmit and Saunderson (1940, a, b) exploited the addition theorem for spherical harmonics, and evaluated the sum over the orders of scattering for arbitrarily large angles, using Legendre polynomial expansions. This solution, exact except for the difficulties noted above, is essentially the same as the later developments of Molière (1948) and Snyder and Scott (1949) in the small angle approximation; Molière briefly indicated the relation of his theory to that of Goudsmit and Saunderson; Lewis (1950) and Bethe (1953) discussed this relation in some detail.

The Snyder development proceeded from a solution of the Boltzmann transport equation, but it is equivalent to the Wentzel–Molière summation method. Scott (1952) gave an explicit statement of the relation between the two small-angle developments and evaluated numerous mean-value quantities for the combined theory.

Many applications of this theory have been made to emulsion, cloud- and bubble-chamber, and foil-scattering experiments, but we shall not list them here. Some further improvements in calculational methods [e.g., Butler (1950)] have been developed without modifying the basic theory. Basic changes that have occurred are those of Fano (1954) on the inclusion of inelastic scattering of the atomic electrons; of Cooper and Rainwater (1955), and later Ter-Mikayelian (1959), on the inclusion of finite nuclear size effects; of Nigam, Sundaresan, and Wu (1959) on the use of the improved single-scattering cross-section; of Dalitz (1951), and of Mühlschlegel and Koppe (1958), on the multiple scattering of polarized particles. An extension of the theory to include lateral deflections and other scattered-track "characteristics" was made by Scott and Snyder (1950) and much more elegantly by Molière (1955).

It is the purpose of this review article to give a connected account of the Molière theory and the various modifications above-mentioned.

Numerous brief reviews or summaries of small-angle multiple scattering have been published—several with formulas, graphs and tables—but none have attempted to unify and clarify the basic theory. Among these we mention Bohr (1948), Goldschmidt-Clermont, King, Muirhead and Ritson (1948), Maier-Leibnitz (1950), Paul and Frank (1950), Camerini, Lock, and Perkins (1951), Beiser (1952), Voyvodic and Pickup (1952), Rossi (1952), Saletan (1952), Lawson (1952), Bethe and Ashkin (1953), Goldschmidt-Clermont (1953), Mayer (1953), Gottstein (1953), Voyvodic (1954), and Birkhoff (1958).

II. BASIC STATISTICAL THEORY

A. Distribution Functions

The basic statistical theory for small-angle multiple scattering involves the calculation of either the spatial-angle distribution function $F(\theta,\beta,t)$ or the projected-angle functions[1] $F_p(\phi,t)$ and $f(\phi,t) = F_p(\phi,t) + F_p(-\phi,t)$, when the single-scattering function $W(\theta,t)$ is known. We use θ and β to indicate the polar angle and azimuth of the track of a scattered particle, measured with respect to the initial direction, and ϕ is the angle of the track when projected on a given plane containing the original direction of the particle's motion. The distributions are taken to be functions of the thickness t of scattering material, measured along the initial direction; they are considered to be averaged over the space coordinates normal to t. Distributions in angle as functions of these coordinates, and distributions of tracks with respect to these coordinates, will be considered later.

The small-angle approximation consists in: (a) replacing $\sin\theta$ by θ, and $\cos\theta$ by 1; (b) replacing the relations for the two projected angles $\phi = \phi_x$ and ϕ_y (Fig. 1),

$$\tan\phi_x = \tan\theta\cos\beta$$
$$\tan\phi_y = \tan\theta\sin\beta \qquad (2.1)$$

by

$$\phi = \phi_x = \theta\cos\beta$$
$$\phi_y = \theta\sin\beta\,; \qquad (2.2)$$

and (c) replacing the upper limit π for θ and the limits $\pm\pi$ for ϕ by the values ∞ and $\pm\infty$, respectively. This last substitution involves the assumption that all the functions of θ and ϕ, over which integrals are taken, fall off sufficiently rapidly for large argu-

[1] The function f is the distribution of the absolute values of the projected angle.

ments. We shall see later (Sec. IX) that this substitution must in certain cases be modified to avoid error. The relation (2.2) above can be summarized by describing the deflection (θ,β) as a vector $\boldsymbol{\theta}$ which is the projection in a plane normal to the original beam direction of a line segment of unit length lying in the direction (θ,β). The element of solid angle $\sin\theta\, d\theta d\beta$ becomes $\theta d\theta d\beta$.

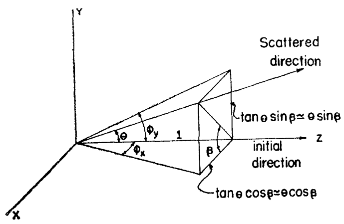

FIG. 1. Illustrating the spatial angle θ and the projected angles ϕ_x and ϕ_y.

The functions F, F_p, and f are normalized according to

$$\int_0^\infty \theta d\theta \int_0^{2\pi} d\beta F(\theta,\beta,t) = 1 \tag{2.3a}$$

$$\int_{-\infty}^\infty d\phi F_p(\phi,t) = 1 \tag{2.3b}$$

$$\int_0^\infty d\phi f(\phi,t) = \int_0^\infty d\phi F_p(\phi,t) + \int_0^\infty d\phi F_p(-\phi,t) = 1\,. \tag{2.3c}$$

In most cases of interest, F will be independent of the azimuth β, F_p will be even in ϕ, and we have

$$2\pi \int_0^\infty \theta d\theta F(\theta,t) = 1 \tag{2.3d}$$

$$\int_0^\infty d\phi f(\phi,t) = 2\int_0^\infty d\phi F_p(\phi,t) = 1\,. \tag{2.3e}$$

The relation between F, F_p, and f is that

$$F_p(\phi_x,t)d\phi_x = d\phi_x \int_{-\infty}^\infty d\phi_y F[(\phi_x^2+\phi_y^2)^{1/2}, \tan^{-1}\phi_y/\phi_x, t] \tag{2.4a}$$

or, when cylindrical symmetry holds,

$$f(\phi,t) = 2F_p(\phi,t) = 2\int_{-\infty}^\infty d\phi_y F[(\phi^2+\phi_y^2)^{1/2},t]\,. \tag{2.4b}$$

We have used here the "rectangular-coordinate" element of area $d\phi_x d\phi_y$ in place of its "polar-coordinate" equal $\theta d\theta d\beta$, in accordance with 2.2.

We shall use the quantity $2\pi\theta d\theta W(\theta,t)dt$ to represent the probability of one scattering occurring in dt at t through an angle between θ and $\theta+d\theta$. If the scattering is in a medium that consists, in the neighborhood of the thickness t, of $N(t)$ independent scattering atoms per unit volume, each with a differential scattering cross-section $2\pi\sigma(\theta,t)\theta d\theta$, we have

$$W(\theta,t)dt = N(t)\sigma(\theta,t)dt\,. \tag{2.5}$$

The single-scattering law will have an azimuthal dependence—$\sigma(\theta,t)$ will become $\sigma(\theta,\beta,t)$—if either the scattered particles or the scattering centers are polarized. Unlike the successive angular deflections of unpolarized particles, the successive scatterings when polarization is present are not incoherent in the quantum-mechanical sense, and density matrix techniques must be used in place of the ordinary classical probability theory that is applicable in the absence of polarization. These techniques are introduced in Sec. V below.

B. Transforms

The use of Fourier and Hankel transforms are essential for the development of our theory. We define the Fourier transform $\tilde{F}_p(\xi,t)$ of the projected distribution by

$$\tilde{F}_p(\xi,t) = \int_{-\infty}^\infty d\phi e^{i\xi\phi} F_p(\phi,t) \tag{2.6}$$

with its inverse

$$F_p(\phi,t) = \frac{1}{2\pi}\int_{-\infty}^\infty d\xi e^{-i\xi\phi}\tilde{F}_p(\xi,t)\,. \tag{2.7}$$

For the distribution of absolute values

$$\tilde{f}(\xi,t) = \int_0^\infty d\phi \cos\xi\phi\, f(\phi,t) = \mathrm{Re}\,\tilde{F}_p(\xi,t) \tag{2.8}$$

with its inverse

$$f(\phi,t) = \frac{2}{\pi}\int_0^\infty d\xi \cos\xi\phi\, \tilde{f}(\xi,t)\,. \tag{2.9}$$

When $F_p(\phi,t) = F_p(-\phi,t)$, then $\tilde{F}_p(\xi,t)$ is real and is itself equal to $\tilde{f}(\xi,t)$.

For the spatial-angle distributions, we introduce a double Fourier transform in terms of the two projected angles ϕ_x and ϕ_y:

$$\tilde{F}[(\xi_x^2+\xi_y^2)^{1/2}, \tan^{-1}\xi_y/\xi_x] = \int_{-\infty}^\infty d\phi_x \int_{-\infty}^\infty d\phi_y \times \exp(i\xi_x\phi_x + i\xi_y\phi_y) F[(\phi_x^2+\phi_y^2)^{1/2}; \tan^{-1}\phi_y/\phi_x] \tag{2.10a}$$

where we have indicated that in the space of the transform variables ξ_x and ξ_y we can also use either "Cartesian" or "polar" coordinates:

$$\xi = (\xi_x^2+\xi_y^2)^{1/2},\quad \alpha = \tan^{-1}\xi_y/\xi_x\,, \tag{2.11}$$

$$\xi_x = \xi\cos\alpha\,,\quad \xi_y = \xi\sin\alpha\,. \tag{2.12}$$

Note that placing $\xi_y = 0$ in (2.10a) yields the transform of the function that has been integrated over ϕ_y.

The inverse of (2.10a) is obviously

$$F[(\phi_x^2 + \phi_y^2)^{1/2}; \tan^{-1} \phi_y/\phi_x] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\xi_x \int_{-\infty}^{\infty} d\xi_y \times \exp(-i\xi_x\phi_x - i\xi_y\phi_y)\tilde{F}[(\xi_x^2 + \xi_y^2)^{1/2}, \tan^{-1} \xi_y/\xi_x] . \tag{2.13a}$$

These formulas can be written in terms of "polar" coordinates in both the original and transform spaces as follows:

$$\tilde{F}(\xi,\alpha,t) = \int_0^{\infty} \theta d\theta \int_0^{2\pi} d\beta \exp[i\xi\theta \cos(\beta - \alpha)]F(\theta,\beta,t), \tag{2.10b}$$

$$F(\theta,\beta,t) = \frac{1}{(2\pi)^2} \int_0^{\infty} \xi d\xi \int_0^{2\pi} d\alpha \times \exp[-i\xi\theta \cos(\beta - \alpha)]\tilde{F}(\xi,\alpha,t) . \tag{2.13b}$$

If $F(\theta,\beta,t)$ is expanded in a Fourier series in β,

$$F(\theta,\beta,t) = \sum_{n=-\infty}^{\infty} F_n(\theta,t)e^{in\beta} , \tag{2.14}$$

and use is made of a modification of Bessel's integral [see Jahnke–Emde (1943), p. 149] for the Bessel function $J_n(z)$:

$$J_n(z) = \frac{i^{-n}}{2\pi} \int_0^{2\pi} d\beta e^{in\beta + iz \cos \beta} = \frac{i^{-n}}{2\pi} \int_0^{2\pi} d\beta e^{in(\beta-\alpha) + iz \cos(\beta-\alpha)} . \tag{2.15}$$

we can then write

$$\tilde{F}(\xi,\alpha,t) = 2\pi \sum_{n=-\infty}^{\infty} i^{-n} e^{in\alpha} \int_0^{\infty} \theta d\theta J_n(\xi\theta)F_n(\theta,t) = 2\pi \sum_{n=-\infty}^{\infty} i^{-n} \tilde{F}_n^{(n)}(\xi,t)e^{in\alpha} , \tag{2.16}$$

so that we have the transform expanded as a Fourier series each of whose coefficients is $2\pi i^{-n}$ times the Hankel transform of order n [signified by the tilde followed by (n)] of the nth coefficient in the original development.

In particular, if F is independent of β, the transform is 2π times a Hankel transform of order zero:

$$\tilde{F}(\xi,t) = \tilde{F}[(\xi_x^2 + \xi_y^2)^{1/2},t] = 2\pi \int_0^{\infty} \theta d\theta J_0(\xi\theta)F(\theta,t) \tag{2.17a}$$

$$= \int_{-\infty}^{\infty} d\phi_x \int_{-\infty}^{\infty} d\phi_y \exp(i\xi_x\phi_x + i\xi_y\phi_y)F[(\phi_x^2 + \phi_y^2)^{1/2},t] \tag{2.17b}$$

with the inverse

$$F(\theta,t) = \frac{1}{2\pi} \int_0^{\infty} \xi d\xi J_0(\xi\theta)\tilde{F}(\xi,t) . \tag{2.17c}$$

If (2.17b) is applied to the case $\alpha = 0$, or $\xi_y = 0$ and $\xi_x = \xi$, we see by use of (2.4), (2.6), and (2.8) that

$$\tilde{F}(\xi,t) = \tilde{F}_p(\xi,t) = \tilde{f}(\xi,t) . \tag{2.18}$$

Thus, only a single transform is required in the case of cylindrical symmetry to determine both spatial-angle and projected distributions.

The most important property of Fourier and Hankel transforms for our purposes is their application to folding integrals. If two successive, independent, projected scatterings occur through angles ϕ_1 and ϕ_2, and the probabilities of ϕ_1 to $\phi_1 + d\phi_1$ and ϕ_2 to $\phi_2 + d\phi_2$ occurring are respectively $F_{1p}(\phi_1)d\phi_1$ and $F_{2p}(\phi_2)d\phi_2$, the probability of getting an over-all deflection $\phi = \phi_1 + \phi_2$ between the prescribed values ϕ and $\phi + d\phi$ is, by the ordinary rules of probability,

$$F_p(\phi)d\phi = d\phi \int_{-\infty}^{\infty} d\phi_1 F_{1p}(\phi_1)F_{2p}(\phi - \phi_1) \tag{2.19}$$

whose transform may be calculated as follows:

$$\tilde{F}_p(\xi) = \int_{-\infty}^{\infty} d\phi e^{i\xi\phi} \int_{-\infty}^{\infty} d\phi_1 F_{1p}(\phi_1)F_{2p}(\phi - \phi_1) = \int_{-\infty}^{\infty} d\phi_2 \int_{-\infty}^{\infty} d\phi_1 e^{i\xi(\phi_1+\phi_2)}F_{1p}(\phi_1)F_{2p}(\phi_2) = \tilde{F}_{1p}(\xi)\tilde{F}_{2p}(\xi) . \tag{2.20}$$

The transform of a folding integral is the product of the transforms of the individual functions. The theorem is readily extended to $(n - 1)$-fold integrals of n distribution functions over all values of their arguments that add to a predetermined sum.

If two spatial scatterings follow each other, we can use "Cartesian" coordinates, Eq. (2.2), for each, and find a double folding integral, which reduces to products of Hankel transforms. Let $\boldsymbol{\theta}$ be the resultant of $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$:

$$\theta \cos \beta = \phi_x = \varphi_{1x} + \phi_{2x} = \theta_1 \cos \beta_1 + \theta_2 \cos \beta_2 , \quad \theta \sin \beta = \phi_y = \varphi_{1y} + \varphi_{2y} = \theta_1 \sin \beta_1 + \theta_2 \sin \beta_2 , \tag{2.21}$$

or in vector form

$$\boldsymbol{\theta} = \boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 . \tag{2.22}$$

Then we have

$$F(\theta,\beta) = \int_{-\infty}^{\infty} d\phi_{1x} \int_{-\infty}^{\infty} d\phi_{1y} F_1[(\phi_{1x}^2 + \phi_{1y}^2)^{1/2}, \tan^{-1}\phi_{1y}/\phi_{1x}] \times F_2\{[(\phi_x - \phi_{1x})^2 + (\phi_y - \phi_{1y})^2]^{1/2}, \times \tan^{-1}[(\phi_y - \phi_{1y})/(\phi_x - \phi_{1x})]\} , \tag{2.23}$$

and

$$\begin{aligned}\tilde{F}(\xi,\alpha) &= \int_{-\infty}^{\infty} d\phi_x \int_{-\infty}^{\infty} d\phi_y \ \exp\,[i\xi_x\phi_x + i\xi_y\phi_y]F(\theta,\beta) \\ &= \int_{-\infty}^{\infty} d\phi_{2x} \int_{-\infty}^{\infty} d\phi_{2y} \int_{-\infty}^{\infty} d\phi_{1x} \int_{-\infty}^{\infty} d\phi_{1y} \\ &\quad \times \exp\,[i\xi_x(\phi_{1x}+\phi_{2x}) + i\xi_y(\phi_{1y}+\phi_{2y})] \\ &\quad \times F_1[(\phi_{1x}^2+\phi_{1y}^2)^{1/2}, \tan^{-1}(\phi_{1y}/\phi_{1x})] \\ &\quad \times F_2[(\phi_{2x}^2+\phi_{2y}^2)^{1/2}, \tan^{-1}(\phi_{2y}/\phi_{2x})] \\ &= \int_0^{\infty} \theta d\theta \int_0^{2\pi} d\beta e^{i\xi\cdot\theta} F(\theta,\beta) \\ &= \int_0^{\infty} \theta_2 d\theta_2 \int_0^{2\pi} d\beta_2 e^{i\xi\cdot\theta_2} \int_0^{\infty} \theta_1 d\theta_1 \int_0^{2\pi} d\beta_1 e^{i\xi\cdot\theta_1} \\ &\quad \times F_1(\theta_1,\beta_1)F_2(\theta_2,\beta_2) \\ &= \tilde{F}_1(\xi,\alpha)\tilde{F}_2(\xi,\alpha)\,, \end{aligned} \tag{2.24}$$

so the transform of the distribution of the vector sum $\boldsymbol{\theta} = \boldsymbol{\theta}_1 + \boldsymbol{\theta}_2$ is the product of the transforms of the distributions of the separate vectors $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$. This relation can be iterated, so that if $\boldsymbol{\theta} = \boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 + \cdots + \boldsymbol{\theta}_n$, the transform $\tilde{F}(\xi,\alpha)$ is given by

$$\tilde{F}(\xi,\alpha) = \tilde{F}_1(\xi,\alpha)\tilde{F}_2(\xi,\alpha)\cdots\tilde{F}_n(\xi,\alpha)\,. \tag{2.25}$$

It will be noted from the normalization rules (2.3) and the definitions of the transforms (2.6), (2.8), (2.10b), and (2.17a), that

$$\tilde{F}_p(0,t) = \tilde{f}(0,t) = \tilde{F}(0,t) = \tilde{F}(0,\alpha,t) = 1\,. \tag{2.26}$$

C. The Wentzel Summation Method

Assume a beam of like particles to pass through a layer of thickness t of homogeneous matter, with a given single-scattering function $W(\theta)$. (We suppress temporarily the variation with t.) Since (θ,β) are measured from the original direction, we have the initial distribution

$$F(\theta,\beta,o) = \delta_s(\theta)\,, \tag{2.27}$$

where $\delta_s(\theta)$ is a spatial Dirac function:

$$\delta_s(\theta) = \delta_s[(\phi_x^2+\phi_y^2)^{1/2}] = \delta(\phi_x)\delta(\phi_y)\,, \tag{2.28}$$

$$\begin{aligned}\int_0^{\infty} \theta d\theta \int_0^{2\pi} d\beta g(\phi_x,\phi_y)\delta_s(\theta) &= \int_{-\infty}^{\infty} d\phi_x \int_{-\infty}^{\infty} d\phi_y g(\phi_x,\phi_y)\delta(\phi_x)\delta(\phi_y)\,, \\ &= g(0,0)\,, \end{aligned} \tag{2.29a}$$

and

$$2\pi \int_0^{\infty} \theta d\theta \delta_s(\theta) = 1\,. \tag{2.29b}$$

We shall find the probability distribution for the beam after it has passed through a thickness t by summing over the distributions resulting from exactly 0, 1, 2, 3, $\cdots$ scatterings. The expression

$$2\pi dt \int_0^{\infty} \theta d\theta W(\theta) = \omega_0 dt \tag{2.30}$$

gives the probability that one scattering through any angle whatsoever will occur in dt. The probability of no scattering is then $1 - \omega_0 dt$, and it is well known that for a finite thickness Δt, the probability of no scattering is

$$P_0(\Delta t) = e^{-\omega_0 \Delta t}\,. \tag{2.31}$$

The probability that n scatterings occur, of deflections $\boldsymbol{\theta}_1$ in $\theta_1 d\beta_1 d\beta_1$, $\boldsymbol{\theta}_2$ in $\theta_2 d\theta_2 d\beta_2, \cdots \boldsymbol{\theta}_n$ in $\theta_n d\theta_n d\beta_n$, at the positions t_1 to $t_1 + dt_1$, t_2 to $t_2 + dt_2 \cdots$, t_n to $t_n + dt_n$, is the product

$$W(\theta_1)\theta_1 d\theta_1 d\beta_1 dt_1 W(\theta_2)\theta_2 d\theta_2 d\beta_2 dt_2 \cdots W(\theta_n)\theta_n d\theta_n d\beta_n dt_n\,. \tag{2.32}$$

The probability that no other scatterings occur is given by 2.31, with Δt equal to the total space between the dt's, namely $t - dt_1 - dt_2 - \cdots - dt_n$. In the limits taken in the n integrals over thickness, Δt becomes t. Thus, the probability that *exactly* n scatterings occur, as specified, is (2.32) multiplied by $e^{-\omega_0 t}$. To find the probability that scatterings through these specified angles occur anywhere in t, we integrate over the t's. If we keep $0 < t_1 < t_2 \cdots < t_n < t$, we shall count each scattering once, but if we let each t_i range from 0 to t, we shall count all the permutations, and merely need divide the result by $n!$

Thus we have $t^n/n!$ times the product of n W's, for which $\boldsymbol{\theta} = \boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 + \cdots + \boldsymbol{\theta}_n$. The distribution of $\boldsymbol{\theta}$ for this case is, by (2.25), the nth power of the transform of W. Let us write

$$\omega(\xi) = 2\pi \int_0^{\infty} \chi d\chi J_0(\xi\chi)W(\chi)\,, \tag{2.33}$$

so the transform of the distribution in θ produced by exactly n scatterings is

$$\tilde{F}_n(\xi,t) = e^{-\omega_0 t}[\omega(\xi)t]^n/n!\,. \tag{2.34a}$$

The distribution for no scatterings at all is $e^{-\omega_0 t}$ times the transform of $\delta_s(\theta)$, namely unity, so (2.34a) holds for $n = 0$.

Using our temporary assumption of the constancy of ω_0 in t, we can write $\tilde{F}_n(\xi,t)$ as the product of the probability of the occurrence of exactly n scatterings (Poisson distribution) and the transform of the normalized distribution in θ after n scatterings have occurred (nth power of $\omega(\xi)/\omega_0$):

$$\tilde{F}_n(\xi,t) = [e^{-\omega_0 t}(\omega_0 t)^n/n!][\omega(\xi)/\omega_0]^n . \quad (2.34b)$$

The complete distribution in $\boldsymbol{\theta}$ is the sum over all n, so we have

$$\tilde{F}(\xi,t) = \sum_{n=0}^{\infty} \tilde{F}_n(\xi,t) = e^{\omega(\xi)t-\omega_0 t} . \quad (2.35)$$

Since $\omega_0 = \omega(0)$, we see that $\tilde{F}$ obeys the rule (2.26).

If the original beam lies at an angle $\boldsymbol{\theta}_0$ to some chosen direction, the same distribution results if $\boldsymbol{\theta}$ is measured with respect to $\boldsymbol{\theta}_0$. If a beam scattered through a thickness t' enters a second layer of thickness t'', we can use the folding-integral rule to find the resulting distribution, since we can write $\boldsymbol{\theta} = \boldsymbol{\theta}' + \boldsymbol{\theta}''$. We simply take the product of two exponentials:

$$\tilde{F}(\xi,t'+t'') = \exp\{[\omega(\xi) - \omega_0](t'+t'')\} . \quad (2.36)$$

If, now, we have a series of layers of thicknesses $\Delta t'$, $\Delta t''$, $\cdots$ and $W(\chi,t)$ is a function of t, so that it is a different approximately-constant-in-t function in each layer, we can write the transform with an integral in the exponent, which we shall call $\Omega(\xi,t) - \Omega_0$:

$$\tilde{F}(\xi,t) = \exp[\Omega(\xi,t) - \Omega_0] , \quad (2.37a)$$

$$\Omega(\xi,t) = \int_0^t \omega(\xi,t')dt' = 2\pi\int_0^\infty \chi d\chi \int_0^t dt' J_0(\xi\chi)W(\chi,t'), \quad (2.37b)$$

$$\Omega_0(t) = \Omega(0,t) = \int_0^t \omega_0(t')dt' = \int_0^t \omega(0,t')dt'$$
$$= 2\pi\int_0^\infty \chi d\chi \int_0^t dt' W(\chi,t') . \quad (2.37c)$$

When $W(\chi,t)$ is independent of t, we write

$$\Omega(\xi,t) = \omega(\xi)t = 2\pi t\int_0^\infty \chi d\chi J_0(\xi\chi)W(\chi) \quad (2.38a)$$

$$\Omega_0(t) = \Omega(0,t) = \omega_0 t = 2\pi t\int_0^\infty \chi d\chi W(\chi) , \quad (2.38b)$$

so that (2.37a) is the general expression for the transform for small-angle multiple scattering in a thickness t of scattering material. In accordance with the usual properties of a cross section, and the relation (2.5), Ω_0 is seen to represent the mean number of scatterings occurring in thickness t.

Exactly the same result can be obtained if we calculate the projected scattering in a given plane. We find a projected single-scattering distribution $w(\phi,t)d\phi dt$ by use of (2.4a), and exploit (2.20) just as we did (2.24). Equation (2.18) shows that the result is exactly the same transform, and that the same function $\Omega(\xi,t)$ will be involved.

The property represented by (2.36) for homogeneous materials can be generalized for any successive thicknesses t_1 and t_2 in any materials. We simply write

$$\Omega(\xi,t_1+t_2) = \int_0^{t_1+t_2} \omega(\xi,t')dt' = \int_0^{t_1} \omega(\xi,t')dt' + \int_{t_1}^{t_1+t_2} \omega(\xi,t')dt' \quad (2.39a)$$

or, since we have not explicitly indicated that Ω depends on the material traversed, we can write

$$\Omega(\xi,t_1+t_2) = \Omega(\xi,t_1) + \Omega(\xi,t_2) , \quad (2.39b)$$

where the terms on the right-hand side are understood to refer to successive thicknesses t_1 and t_2.

The inclusion of the term for $n = 0$ in the sum (2.35) leads to a mathematical error, for this term is a constant independent of ξ, whereas every valid Hankel or Fourier transform has the property (necessary for convergence of the inversion integrals) that it goes to zero as $\xi \to \infty$.

The expression (2.37) does not obey this property, for $\omega(\xi) \to 0$ as $\xi \to \infty$ and $\tilde{F} \to e^{-\Omega}$. A proper treatment would be to separate out the term for $n = 0$ and not calculate its transform. Then we would have, using the inverse transform relation,

$$F(\theta,t) = e^{-\Omega_0(t)}\delta_s(\theta) + \int_0^\infty \xi d\xi J_0(\xi\theta)[e^{\Omega-\Omega_0} - e^{-\Omega_0}] . \quad (2.40)$$

However, since $\delta_s(\theta)$ may be "represented" by

$$\int_0^\infty \xi d\xi J_0(\xi\theta) ,$$

Eq. (2.40) becomes the same as (2.37). Except in Sec. X, we shall assume that Ω_0 ranges from 20 to 10^5 or more, and, hence, the $n = 0$ term makes no appreciable contribution for either way of writing $F(\theta,t)$. To neglect this term in evaluation without encountering convergence difficulties, it is necessary to carry out the ξ integral by an approximate method that does not involve values of ξ larger than a value at which the first term just becomes of the order of the second. Let us assume that angles below some minimum value $\theta_{\min}$ are not of interest (because of the difficulty of measuring scattering angles near zero). The Bessel function will oscillate rapidly for values of ξ larger than say $10/\theta_{\min}$, and the contribution of the integral beyond this value of ξ will be negligible for this reason.

A further consequence of the limit beyond which values of $\tilde{F}$ and $\Omega(\xi,t)$ are not needed is that in the expression

$$\Omega(\xi,t) - \Omega_0 = 2\pi \int_0^t dt' \int_0^\infty \chi d\chi [J_0(\xi\chi) - 1] W(\chi,t') \tag{2.41}$$

the values of W for χ below a certain limit are irrelevant. For small $\xi\chi$, $J_0(\xi\chi) - 1$ is approximately equal to $-\xi^2\chi^2/4$. If $\chi < 1/5\xi_{max}$ where $\xi_{max} \simeq 10/\theta_{min}$, i.e., if $\chi < \theta_{min}/50$, the value of $\frac{1}{4}\xi^2\chi^2$ is less than 1/100 and the contribution of W to $\exp[\Omega - \Omega_0]$ for such values of χ is negligible.

Now, the dependence of W on χ for small χ is determined largely by the structure of the outer parts of the scattering atom, and in condensed materials, by the overlapping parts of the wave functions of adjacent atoms. We see then that, unless extremely small deviations are to be measured, the *details* of the screening influence of outer atomic electrons and of molecular and crystalline combination, do not influence the multiple-scattering distribution. The screening itself cannot be overlooked, but as we shall see in Sec. VII, only a single parameter characterizing the screening will be relevant.

D. The Transport Equation Method

The form of the Boltzmann transport equation appropriate to the determination of $F(\theta,t)$ is

$$\frac{\partial F(\theta,t)}{\partial t} = -\omega_0(t)F(\theta,t) + \int_0^\infty \theta' d\theta' \int_0^{2\pi} d\beta' W(\chi,t) F(\theta',t) \tag{2.42a}$$

where the first term on the right represents the scattering out of an angular range around θ per unit path length, and the second term represents the scattering into the given range from another range θ' to $\theta' + d\theta'$ at an azimuthal angle β' (which may be measured from the same plane as is the deflection θ, or directly from the plane of θ itself). The single-scattering angle χ is that which when combined with θ' yields θ:

$$\boldsymbol{\theta}' + \boldsymbol{\chi} = \boldsymbol{\theta} \quad \text{or} \quad \chi^2 = \theta^2 + \theta'^2 - 2\theta\theta' \cos\beta' . \tag{2.43}$$

An alternate form for (2.42a) is

$$\frac{\partial F(\theta,t)}{\partial t} = \int_0^\infty \chi d\chi \int_0^{2\pi} d\beta'' W(\chi,t)[F(\theta',t) - F(\theta,t)] , \tag{2.42b}$$

where β'' is the azimuth of χ, and we have replaced $\theta' d\theta' d\beta'$ by $\chi d\chi d\beta''$; the Jacobian $\partial(\theta',\beta')/\partial(\chi,\beta'')$ is χ/θ' as can easily be found from Eq. (2.21), written for θ',β',χ, and β''.

In either form, the transport equation is immediately reducible to an ordinary differential equation by multiplying both sides by $\theta d\theta d\beta\, J_0(\xi\theta)$ and integrating. We obtain

$$\partial\tilde{F}(\xi,t)/\partial t = [\omega(\xi,t) - \omega_0(t)]\tilde{F}(\xi,t) , \tag{2.44}$$

whose solution, subject to $\tilde{F}(\xi,0) = 1$, is clearly (2.37) with the definition (2.38).

Again, the same solution can be obtained by writing a transport equation for the projected scattering distribution, using the projected single-scattering law in place of W.

E. The Fokker-Planck Equation and the Gaussian Approximation

Although it is outside the scope of this article to deal with the Gaussian approximation to multiple scattering, we shall show here how it may be derived from the transport equation by the method of Fokker (1914) and Planck (1917). Let us assume that $W(\chi,t)$ is sufficiently sharply peaked at $\chi = 0$ that it possesses a finite mean square $\langle\chi^2\rangle_{av}$ and that the only values of $F(\theta',t) - F(\theta,t)$ that contribute appreciably to the integral in Eq. (2.42b) are those for which a Taylor expansion in $\theta' - \theta$ taken to the second order is sufficiently accurate.

Furthermore, we can write the relation between θ' and θ using β'' and approximate it for small χ:

$$\theta'^2 = \chi^2 + \theta^2 + 2\chi\theta\cos\beta'' , \tag{2.45a}$$

$$\theta' \simeq \theta + \chi\cos\beta'' + (\chi^2 \sin^2\beta'')/2\theta + \cdots . \tag{2.45b}$$

Then (2.42b) becomes

$$\begin{aligned}\frac{\partial F}{\partial t} &= \int_0^\infty \chi d\chi \int_0^{2\pi} d\beta'' W(\chi,t) \left[\left(\chi\cos\beta'' + \frac{\chi^2\sin^2\beta''}{2\theta} \right) \times \frac{\partial F}{\partial\theta} + \tfrac{1}{2}\chi^2\cos^2\beta'' \frac{\partial^2 F}{\partial\theta^2} + \cdots \right] \\ &= \tfrac{1}{4}\left[\frac{\partial^2 F(\theta,t)}{\partial\theta^2} + \frac{1}{\theta}\frac{\partial F(\theta,t)}{\partial\theta} \right] \times 2\pi \int_0^\infty \chi^3 d\chi W(\chi,t) + \cdots \end{aligned} \tag{2.46}$$

[Butler (1950) has included the next higher term].

By use of (2.5) we see that the mean square angle of scattering after a thickness dt of material is

$$\langle\chi^2\rangle_{dt} = 2\pi \int_0^\infty \chi^3 d\chi W(\chi,t) \tag{2.47}$$

so that the equation for F reads

$$\frac{\partial F}{\partial t} = \tfrac{1}{4}\langle\chi^2\rangle_{dt} \left[\frac{\partial^2 F}{\partial\theta^2} + \frac{1}{\theta}\frac{\partial F}{\partial\theta} \right] . \tag{2.48}$$

It is easily shown that if the "Cartesian" angles ϕ_x and ϕ_y are used, the bracket in (2.48) becomes $\partial^2F/\partial\phi_x^2 + \partial^2F/\partial\phi_y^2$.

The solution of (2.48), normalized according to (2.3d) and satisfying (2.27), is

$$F(\theta,t) = [\pi\langle\chi^2\rangle_t]^{-1} \exp[-\theta^2/\langle\chi^2\rangle_t], \quad (2.49)$$

where we have denoted by $\langle\chi^2\rangle_t$ the integrated mean square

$$\langle\chi^2\rangle_t = \int_0^t dt'\langle\chi^2\rangle_{dt'} = \int_0^t dt'\cdot 2\pi\int_0^\infty \chi^2 d\chi W(\chi,t'). \quad (2.50)$$

The results for projected scattering follow immediately when we write $\theta^2 = \phi_x^2 + \phi_y^2$, for then F becomes a product,

$$F(\theta,t) = f(\phi_x,t)f(\phi_y,t) \quad (2.51)$$

with

$$f(\phi_x,t) = [\pi\langle\chi^2\rangle_t]^{-1/2} \exp[-\phi_x^2/\langle\chi^2\rangle_t]. \quad (2.52)$$

We see that (2.50) is equal to the mean square multiple-scattering spatial angle $\langle\theta^2\rangle_{av}$ and by (2.51), twice the mean square projected angle $2\langle\phi_x^2\rangle_{av}$ in accordance with the standard theorem about the mean of the sum of squares of a set of independent events.

We shall see later that the Gaussian approximation is not very accurate for fast charged particles. However, it can make a good first approximation if the divisor of θ^2 in the exponent is suitably chosen. For the cross-sections considered in small-angle approximation in Sec. VI, $\langle\chi^2\rangle_t$ does not exist; other methods of finding a suitable "Gaussian width" are mentioned in sec. VIII.

There are a number of useful applications of multiple scattering with regards to boundary problems [Øveräs (1960)], joint distributions [Scott (1949)], path-length calculations [Yang (1951)], emulsion applications [Molière (1955)], etc. which have only been done in Gaussian approximation and are beyond the scope of this article.

III. LATERAL DEFLECTIONS AND OTHER CHARACTERISTICS

A. Basic Theorems

The distribution of spatial angles $\boldsymbol{\theta}$ could be described as a type of joint distribution of ϕ_x and ϕ_y, where independent contributions to each of these are made at each scattering event. It is also possible to find the joint distribution of $\phi_x \equiv \phi$ and x, where x represents the lateral deflection in the x direction, or could be taken as any other observable characteristic of the track that is additive with respect to the contributions made by each scattering event. Other characteristics besides the lateral deflection include the angle made by a chord drawn to the track, (essentially proportional to x), the sum or difference of chord angle and tangent angle ϕ_x, one of the coordinates of a line fitted to the track by a least-squares-deviation method, etc.

Let us suppose that the probability distribution of deflections ϕ and x in a single-scattering event is $w(\phi,x,t)d\phi dxdt$, with the projected-angle distribution being given by the integral over x:

$$w(\phi,t) = \int_{-\infty}^\infty dx w(\phi,x,t). \quad (3.1)$$

The transform of $w(\phi,x,t)$ will be written

$$\tilde{w}(\xi,\zeta,t) = \int_{-\infty}^\infty d\phi\int_{-\infty}^\infty dx e^{i\xi\phi+i\zeta x} w(\phi,x,t) \quad (3.2)$$

with, as a consequence,

$$\tilde{w}(\xi,0,t) = \int_{-\infty}^\infty d\phi e^{i\xi\phi} w(\phi,t) = \omega(\xi,t). \quad (3.3)$$

The folding theorem can be used here as before. If

$$g(\phi,x,t) = \int_{-\infty}^\infty d\phi_1\int_{-\infty}^\infty dx_1 g_1(\phi_1,x_1,t') \times g_2(\phi-\phi_1,x-x_1,t''), \quad (3.4a)$$

then

$$\tilde{g}(\xi,\zeta,t) = \tilde{g}_1(\xi,\zeta,t')\tilde{g}_2(\xi,\zeta,t''). \quad (3.4b)$$

Now we can follow the summation method of Sec. II-C by combining the successive events (ϕ_1,x_1), $(\phi_2,x_2)\cdots(\phi_n,x_n)$ in place of the vectors $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \cdots \boldsymbol{\theta}_n$. Let us call the resulting joint distribution $g(\phi,x,t)$ as above. Then we have[2]

$$g(\phi,x,t) = \exp\{\Omega(\xi,\zeta,t) - \Omega(0,0,t)\}, \quad (3.5a)$$

where

$$\Omega(\xi,\zeta,t) = \int_0^t dt'\tilde{w}(\xi,\zeta,t'), \quad (3.5b)$$

and

$$\Omega(0,0,t) = \int_0^t dt'\omega(0,t') = \Omega_0 \quad (3.5c)$$

by (3.3) and (2.38).

This formalism can easily be extended to more variables, for instance to the combination of ϕ_x, ϕ_y, x, and y. A fourfold Fourier transform will be needed in this case.

[2] We use the symbol Ω for the exponent of any transform, although in its different uses it refers to different actual functions. The context will always make clear which function is intended.

So far we have said nothing about the relation, if any, between ϕ and x in each scattering event. There is no need that they be independent. In fact, if x represents the contribution to the net lateral deflection in the x–t plane, produced by a given scattering event, and ϕ represents the contribution to the over-all angular deflection, ϕ and x are strictly linked. If a deflection ϕ_i occurs at a distance $t - t_i$ from the end of the track being considered, the lateral deflection resulting at the end of the track will be $x_i = (t - t_i)\phi_i$ (Fig. 2). In the small-angle approximation, the total deflection will be a sum of all such individual deflections. Thus x is additive as required,

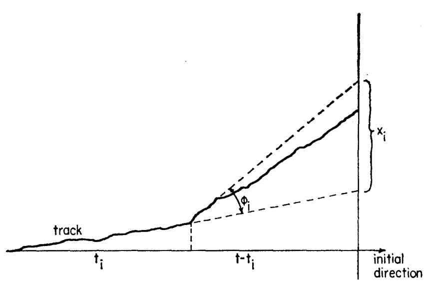

FIG. 2. A scattered track showing the contribution of a single scattering at depth t_i to the lateral deflection at depth t.

and the direct relation between x_i and ϕ_i can be introduced into $w(\phi,x,t')$ by writing

$$w(\varphi,x,t') = w(\phi,t')\delta[x - (t - t')\phi] , \quad (3.6)$$

where the δ is the ordinary Dirac function, and t refers to the end of the track. With this relation, which obviously satisfies (3.1), (3.2) becomes

$$\tilde{w}(\xi,\zeta,t') = \int_{-\infty}^{\infty} d\phi \exp[i\xi\phi + i\zeta(t - t')\phi]w(\phi,t')$$
$$= \omega[\xi + \zeta(t - t'),t'] , \quad (3.7)$$

and we have

$$\Omega(\xi,\zeta,t) = \int_0^t dt'\omega[\xi + \zeta(t - t',t'] . \quad (3.8)$$

The chord angle ψ of a projected track is equal to x/t. We can find the transform of the joint distribution of ϕ and ψ by replacing the argument of the delta function in 3.6 by $\psi - (t - t')\phi/t$. Thus the joint distribution of ϕ and ψ may be found from a transform with the exponent

$$\Omega(\xi,\eta,t) = \int_0^t dt'\omega[\xi + \eta(t - t')/t,t'] , \quad (3.9)$$

where η is the transform variable corresponding to ψ.

We have noted above in Eq. (2.39) that we can add the Ω's if we wish to find the distribution after two or more successive thicknesses $t_1, t_2 \cdots$. If we observe characteristics $X_1^{(j)}, X_2^{(j)} \cdots$ of the tracks at $t_1, t_2 \cdots$ we can find the corresponding transform exponent in the same way as before if we know the probability of a contribution $X_i^{(j)}$ to the jth characteristic produced at the ith scattering. Insofar as the characteristics are the direct result of transport resulting from angular deflections, we will have a delta-function multiplying $w(\phi,t)$ of the form

$$\delta[X_i^{(j)} - a^{(j)}(t_i)\phi_i] , \quad (3.10)$$

where the $a^{(j)}(t_i)$ are called "coupling constants" and in general will depend on some of the $t_1, t_2 \cdots t$ at which observations are to be made, in addition to the t_i at which a scattering is considered to occur.

The result will be, if we let $\zeta^{(j)}$ be the transform variable corresponding to $X^{(j)}$,

$$\Omega(\xi,\zeta^{(1)},\zeta^{(2)},\cdots;t) = \int_0^t dt'\omega[\xi + \zeta^{(1)}a^{(1)}(t')$$
$$+ \zeta^{(2)}a^{(2)}(t') + \cdots;t'] . \quad (3.11)$$

A more symmetrical form can be written, if we define a coupling constant for ϕ itself as being unity, counting ϕ as one of the $X^{(j)}$ and omitting the special term with ξ in (3.11).

Linear combinations of the $X^{(j)}$ may sometimes be of interest. The joint distribution of any number of such combinations can be found from the following general theorem relating to Fourier transforms. Suppose we have given a distribution $F(X^{(1)},X^{(2)} \cdots X^{(n)})$ and its transform $\tilde{F}(\zeta^{(1)}, \zeta^{(2)}, \cdots \zeta^{(n)})$. We wish to find the joint distribution of the m linear combinations

$$Y_k = \sum_{j=1}^{n} a_{jk}X^{(j)}; k = 1,2,\cdots m . \quad (3.12)$$

This distribution may be written, by use of delta functions, as

$$G(Y_1,Y_2,\cdots Y_m) = \int_{-\infty}^{\infty} dX^{(1)} \int_{-\infty}^{\infty} dX^{(2)} \cdots \int_{-\infty}^{\infty} dX^{(n)}$$
$$\times F(X^{(1)},X^{(2)},\cdots X^{(n)})\delta(Y_1 - \Sigma a_{j1}X^{(j)})$$
$$\times \delta(Y_2 - \Sigma a_{j2}X^{(j)}) \cdots \delta(Y_m - \Sigma a_{jm}X^{(j)}) . \quad (3.13)$$

If we multiply this equation by

$$dY_1 \cdots dY_m \exp\left[i \sum_{k=1}^{m} \eta_k Y_k\right]$$

and integrate, and simultaneously replace F by the n-fold integral of its transform, we will have a $2n$-fold integral over all the $X^{(j)}$ and $\zeta^{(j)}$. The jth pair of integrals reads

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}dX^{(j)}\int_{-\infty}^{\infty}d\zeta^{(j)}\exp\left[-i\zeta^{(j)}X^{(j)}+i\sum_{k=1}^{m}X^{(j)}a_{jk}\eta_k\right]\times\tilde{F}(\zeta^{(1)},\cdots\zeta^{(n)}) \quad (3.14)$$

which operation, by the two Eqs. (2.6) and (2.7), amounts to replacing the variable $\zeta^{(j)}$ by the fixed value $\sum_{k=1}^{m} a_{jk}\eta_k$ in the function $\tilde{F}$. Thus we find the general result

$$\tilde{G}(\eta_1,\eta_2\cdots\eta_m)=\tilde{F}(\Sigma a_{1k}\eta_k,\Sigma a_{2k}\eta_k,\cdots\Sigma a_{nk}\eta_k)\,. \quad (3.15)$$

The theorem about the distribution of the sum of independent variables is a special case of (3.15) in which $\tilde{F}$ is a product of functions of the separate $\zeta^{(j)}$, $m=1$, and $a_{j1}=1$ for each j.

If we add to the above the remark made already below Eq. (2.12) that when the distribution function is integrated over one or more of its variables, the corresponding transform variables are set equal to zero, we have a set of rules adequate to determine the transforms for the distributions of a number of quantities of interest.

B. One or Two Segments of Track

Figures 3 and 4 show some quantities of interest, respectively for one and for two segments of track, and Table I gives the expressions for Ω of some of these quantities.

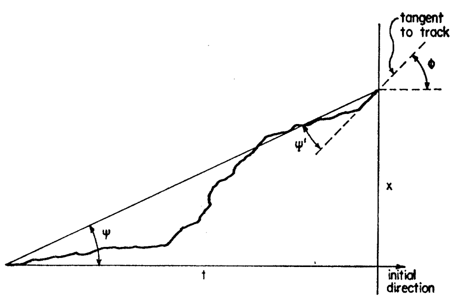

FIG. 3. A scattered track showing the tangent angle ϕ, the chord angles ψ and ψ', and the lateral deflection x, for a depth t.

The last entry of Table I is of particular interest. It can be derived from the previous line by setting $\eta_1=\eta_2'=0$, $\eta_1'=\eta_2=\eta$, $t'=t'''t_1/(t_1+t_2)$ and $t''=t_2-t'''t_2/(t_1+t_2)$; finally t''' is relabeled t'. It is evident that the distribution of the angle between two chords drawn to two successive projected track segments depends, in the case that the elementary scattering law does not vary explicitly with t, only on the total length t_1+t_2 and not on t_1 and t_2, separately. Thus the distribution is unchanged if $t_1=0$, or in other words $\alpha=\psi_2$.

This theorem was given by Scott and Snyder (1950) and applied to the calculation of scattering-produced curvatures by means of a calculation of the distribution of x. A number of further examples of coupling coefficients and resulting distributions are given by Solntseff (1957) and Molière (1955). The above-mentioned paper by Snyder and Scott derives

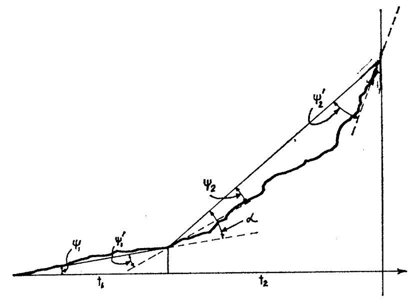

FIG. 4. A scattered track showing chord angles and the deflection α between two chords, for two segments of track.

the transform for the joint distribution function $g(\phi,x,t)$ from a transport equation, namely

$$\frac{\partial g(\phi,x,t)}{\partial t}+\phi\frac{\partial g(\phi,x,t)}{\partial x}=\int_{-\infty}^{\infty}d\phi' w(\phi-\phi')\times[g(\phi',x,t)-g(\phi,x,t)]\,, \quad (3.16)$$

which was solved, with a result equivalent to that given in Table I, by the use of Fourier transforms in ϕ and x and a Laplace transform in t.

For spatial-angle scattering, lateral deflections and other characteristics can be included in a similar way. We illustrate with the lateral deflections x and y, whose coupling coefficients are $a_x=(t-t_i)\phi_x$ and $a_y=(t-t_i)\phi_y$, but the extension to other, and more, characteristics is straightforward.

Each individual scattering generates deflections ϕ_x and ϕ_y that are independent of each other, in the sense that we can write $W(\theta,t)=W[(\phi_x^2+\phi_y^2)^{1/2},t]$ as a function $W_c(\phi_x,\phi_y,t)$ of ϕ_x,ϕ_y, and t. Thus the i'th scattering event contributes a factor $W_c(\phi_{xi},\phi_{yi},t_i)$ $\delta[x_i-\phi_{xi}(t-t_i)]\delta[y_i-\phi_{yi}(t-t_i)]$ to the product involved in the summation method. The transform Ω will be of the form

$$\Omega(\xi_x,\xi_y,\zeta_x,\zeta_y,t)=\int_0^t dt'\tilde{W}_c[\xi_x+\zeta_x(t-t'),\ \xi_y+\zeta_y(t-t'),t']\,, \quad (3.17)$$

TABLE I. Expressions for the exponent Ω in the transforms of the distributions of several variables and combinations of variables. The symbols are explained in Figures 3 and 4.

Independent Variables	Transform Variables	Ω
ϕ	ξ	$\int_0^t dt'\omega(\xi,t')$
x	ζ	$\int_0^t dt'\omega[\zeta(t-t'),t']$
ψ	η	$\int_0^t dt'\omega\left[\eta\left(1-\frac{t'}{t}\right),t'\right]$
ϕ,x	ξ,ζ	$\int_0^t dt'\omega[\xi+\zeta(t-t'),t']$
ϕ,ψ	ξ,η	$\int_0^t dt'\omega\left[\xi+\eta\left(1-\frac{t'}{t}\right),t'\right]$
ψ,ψ'	η,η'	$\int_0^t dt'\omega\left[\eta\left(1-\frac{t'}{t}\right)+\frac{\eta't'}{t},t'\right]$
$\psi_1,\psi_2,\psi_1',\psi_2'$	$\eta_1,\eta_1',\eta_2,\eta_2'$	$\int_0^{t_1} dt'\omega\left[\eta_1\left(1-\frac{t'}{t_1}\right)+\frac{\eta_1't'}{t_1},t'\right]$ $+\int_0^{t_2} dt''\omega\left[\eta_2\left(1-\frac{t''}{t_2}\right)+\frac{\eta_2't''}{t_2},t_1+t''\right]$
$\alpha=\psi_1'+\psi_2$ [when $\omega=\omega(\xi)$]	η	$\int_0^{t_1+t_2} dt'\omega\left[\frac{\eta t'}{(t_1+t_2)}\right]$

where $\tilde{W}_c(\xi_x,\xi_y,t')$ is the "Cartesian" form of the transform of the single scattering law $W_c(\phi_x,\phi_y,t)$.

In the event of cylindrical symmetry, which we are assuming at present, $\tilde{W}_c=\omega[(\xi_x^2+\xi_y^2)^{1/2},t]$ is a function only of $\xi=(\xi_x^2+\xi_y^2)^{1/2}$. Thus Ω becomes

$$\Omega(\xi_x,\xi_y,\zeta_x,\zeta_y,t)=\int_0^t dt'\omega\{([\xi_x+\zeta_x(t-t')]^2+[\xi_y+\zeta_y(t-t')]^2)^{1/2}\}\,. \tag{3.18}$$

If we ask only for the distribution in x and y, we integrate over ϕ_x and ϕ_y, i.e., set $\xi_x=\xi_y=0$. Then the result simplifies to

$$\Omega(\zeta_x,\zeta_y,t)=\Omega[(\zeta_x^2+\zeta_y^2)^{1/2},t]=\int_0^t dt'\omega[(\zeta_x^2+\zeta_y^2)^{1/2}(t-t'),t']\,, \tag{3.19}$$

so that the spatial lateral deflection has the same transform as the projected one (see Table I). The combined distribution as given by (3.18) does not have a simple relation to the projected distribution, owing to the correlation between $\boldsymbol{\theta}$ and the vector displacement $\mathbf{r}$, whose components are x and y.

Berger (1952) has treated lateral deflections by a moment method based on the theory of Goudsmit and Saunderson (1940, a, b).

C. Several Segments of Track

Following Molière (1955), we shall now consider several different measurable quantities or characteristics associated with the observation of the track of a scattered particle at several successive points, which may or may not be equally spaced. As before, we shall consider only the projection of the scattering in a plane. The quantities we shall consider are:

1. The slopes of tangents measured at several points along the track or the angles between them (Molière's case Ia);
2. The lateral displacements at several points along the track, measured from some arbitrary straight line that is roughly parallel to the track, or equivalently the angles between the chords drawn between the points where the displacements are measured (Molière's case IIa);
3. The slopes of a set of least-squares-fitted straight lines, fitted to the track at several locations. The lines may be contiguous, or they may be shorter

than the distances between their centers (Molière's case Ib for contiguous lines);

4. The lateral displacements of the centers of the least-squares-fitted lines referred to above (Molière's cases IIb and IIc); and

5. The use of linear combinations of these primary quantities, such as second and higher differences, or combinations of chord and tangent angles.

The inclusion of grain and measuring "noise," for emulsion tracks, will not be taken up here. However, the folding theorem would allow the calculation of the distribution of the sum of deflections caused by scattering and by noise by multiplying together the Molière type of Fourier transform for the scattering (Sec. VII) and a Gaussian type of transform for the noise.

Suppose that tangents to the particle track are constructed at points whose coordinates on a straight line approximately parallel to the track are t_0, t_1, $t_2 \cdots t_n$. The lengths of the track segments we shall call $s_1 = t_1 - t_0$, $s_2 = t_2 - t_1, \cdots s_n = t_n - t_{n-1}$. Let us then take as measured quantities the angles between successive tangents—i.e., let $X^{(j)} = \phi_j$ be the angle between the tangent at $t = t_j$ and that at $t = t_{j-1}$. This angle is just the projected multiple scattering angle ϕ_j for the segment of track or "cell" s_j. The individual scattering angles that occur in s_j contribute to ϕ_j, but scattering angles in other cells do not. Thus, we have

For angles between successive tangents
$$a_\phi^{(j)}(t) \begin{cases} = 0 \quad ; \quad 0 \leqslant t \leqslant t_{j-1} \\ = 1 \quad ; \quad t_{j-1} \leqslant t \leqslant t_j \\ = 0 \quad ; \quad t_j \leqslant t \,. \end{cases} \tag{3.20}$$

In order to handle the cases involving the lateral displacement of a track, it is convenient to suppose that the lateral position of any particular track is given by a function

$$x = x(t) \,. \tag{3.21a}$$

This function can be written in terms of the actual single-scattering angles ϕ_i by the method illustrated in Fig. 2. If the initial direction and lateral position of the track are, respectively, ϕ_0 and x_0, we have

$$x(t) = x_0 + \phi_0(t - t_0) + \sum_{ti=0}^{t} (t - t_i)\phi_i \,, \tag{3.21b}$$

where the summation is over all the scattering events that have occurred for that particular track between its beginning and the point t; t_i is the coordinate of the ith single scattering.

It is then a straightforward matter to derive coupling constants for the lateral displacements x_0, $x_1, \cdots x_n$ measured at the points 0, $t_1 \cdots t_n$ on a track. Since the initial displacement x_0 and direction ϕ_0 are arbitrarily determined by the conditions of observation, the simplest meaningful characteristics derived from the x_j are the angles α_j between successive chords. In the small-angle approximation we have

$$\alpha_j = \frac{x_{j+1} - x_j}{t_{j+1} - t_j} - \frac{x_j - x_{j-1}}{t_j - t_{j-1}} = \frac{x_{j+1} - x_j}{s_{j+1}} - \frac{x_j - x_{j-1}}{s_j} \,. \tag{3.22a}$$

If the cell lengths are of equal length s, we have

$$\begin{aligned} \alpha_j &= (x_{j+1} - 2x_j + x_{j-1})/s \\ &= \Delta^2 x_{j-1}/s \,, \end{aligned} \tag{3.22b}$$

where $\Delta^2 x_{j-1}$ signifies the second difference of the lateral displacements.

Writing (3.21b) successively for t equal to t_{j+1}, t_j, and t_{j-1}, we can substitute into (3.22a) and pick out the coefficient of ϕ_i to get expressions for the coupling constant. It is convenient to drop the subscript i, writing $t_i = t$, and to introduce the auxiliary variable $\nu_j = \nu_j(t)$, defined only in the region $t_j \leqslant t \leqslant t_{j+1}$ by

$$\nu_j(t) = \frac{t - t_j}{t_{j+1} - t_j} = \frac{t - t_j}{s_{j+1}} \quad ; \quad t_j \leqslant t \leqslant t_{j+1} \,, \tag{3.23a}$$

which represents a linear function of t having the properties

$$\nu_j(t_j) = 0 \quad ; \quad \nu_j(t_{j+1}) = 1 \,. \tag{3.23b}$$

If all the cells are of equal length s, $\nu_j(t)$ has the same shape in every cell. It can be written in this case in the simple form

$$\nu_j(t) = \frac{t - t_j}{s} = \frac{t - js}{s} = \frac{t}{s} - j \,. \tag{3.23c}$$

Then we find

For angles α_j between successive chords
$$a_\alpha^{(j)}(t) \begin{cases} = \nu_{j-1} & t_{j-1} \leqslant t \leqslant t_j \\ = 1 - \nu_j & t_j \leqslant t \leqslant t_{j+1} \\ = 0 & t \leqslant t_{j-1} \quad \text{and} \quad t \geqslant t_{j+1} \end{cases} \tag{3.24}$$

This coupling constant can also be obtained from inspection of Table I.

The coupling constants for the second differences (in the case of equal cell lengths) are obtained by merely multiplying $a^{(j)}(t)$ in (3.24) by s.

The third and fourth differences of the lateral displacements are sometimes useful in efforts to eliminate distortion and spurious scattering errors. Their coupling factors are easily found by differencing Eq. (3.24) [Solntseff (1957)]. Including the second-difference result for completeness, we have

For second differences, $\Delta^2 x_{j-1} = x_{j+1} - 2x_j + x_{j-1}$

$$a_{2\Delta}^{(j)}(t)\begin{cases} = s\nu_{j-1} & (j-1)s \leqslant t \leqslant js \\ = s(1-\nu_j) & js \leqslant t \leqslant (j+1)s \\ = 0 \text{ otherwise}. \end{cases} \tag{3.25}$$

For third differences, $\Delta^3 x_{j-1} = x_{j+2} - 3x_{j+1} + 3x_j - -x_{j-1}$

$$a_{3\Delta}^{(j)}(t)\begin{cases} = -s\nu_{j-1} & (j-1)s \leqslant t \leqslant js \\ = (2\nu_j - 1)s & js \leqslant t \leqslant (j+1)s \\ = (1-\nu_{j+1})s & (j+1)s \leqslant t \leqslant (j+2)s \\ = 0 \text{ otherwise}. \end{cases} \tag{3.26}$$

For fourth differences, $\Delta^4 x_{j-1} = x_{j+3} - 4x_{j+2} + 6x_{j+1} - 4x_j + x_{j-1}$

$$a_{4\Delta}^{(j)}(t)\begin{cases} = \nu_{j-1}s & (j-1)s \leqslant t \leqslant js \\ = (1-3\nu_j)s & js \leqslant t \leqslant (j+1)s \\ = (3\nu_{j+1} - 2)s & (j+1))s \leqslant t \leqslant (j+2)s \\ = (1-\nu_{j+2})s & (j+2)s \leqslant t \leqslant (j+3)s \\ = 0 & \text{otherwise}. \end{cases} \tag{3.27}$$

Suppose now that a track is observed by the locations of photographic emulsion grains, or droplets in a cloud-chamber photograph. In order to make measurements of track direction, it is necessary to fit segments of straight lines to portions of the track. Although this process may generally be performed visually, it is convenient to assume that the fit is made by the method of least squares, for in this case an analytical procedure is possible. In fact, we shall assume that we fit lines directly to the continuous (but zigzag) function $x(t)$, Eq. (3.21).[3] We shall divide the track up as before into cells, and construct the straight lines at the centers of the cells, using cell lengths $\hat{s}_j$ for the lines that may or may not be equal to the full lengths s_j. Let us describe the jth line by its lateral displacement $\hat{x}_j$ and slope $\hat{\beta}_j$ at its midpoint $\hat{t}_j$. Then the sum of the squares of the displacements of this line from the curve $x = x(t)$ becomes the integral

$$\Delta_j = \int_{\hat{t}_j-\hat{s}_j/2}^{\hat{t}_j+\hat{s}_j/2} [\hat{x}_j + \hat{\beta}_j(t-\hat{t}_j) - x(t)]^2 dt\,, \tag{3.28}$$

[3] This assumes the presence of sufficiently many emulsion grains or cloud-chamber droplets. Alternatively we may say merely that the grain noise and reading error are not included at this stage of our calculation.

and the conditions on $\hat{x}_j$ and $\hat{\beta}_j$ that make $\hat{\Delta}_j$ a minimum are found by setting $\partial\hat{\Delta}_j/\partial\hat{x}_j$ and $\partial\hat{\Delta}_j/\partial\hat{\beta}_j$ equal to zero. We obtain

$$\hat{x}_j\hat{s}_j = \int_{\hat{t}_j-\hat{s}_j/2}^{\hat{t}_j+\hat{s}_j/2} x(t)dt\,, \tag{3.29a}$$

and

$$\tfrac{1}{12}\,\hat{\beta}_j\hat{s}_j^3 = \int_{\hat{t}_j-\hat{s}_j/2}^{\hat{t}_j+\hat{s}_j/2} (t-\hat{t}_j)x(t)dt \tag{3.29b}$$

Note that

$$\int_{\hat{t}_j-\hat{s}_j/2}^{\hat{t}_j+\hat{s}_j/2} (t-\hat{t}_j)dt = 0\,.$$

Using (3.21b) in (3.29), we have

$$\hat{x}_j\hat{s}_j = x_0\hat{s}_j + \phi_0(\hat{t}_j - t_0) + \sum_i \phi_i \int_{\text{interval}} (t'-t_i)dt' \tag{3.30a}$$

$$\tfrac{1}{12}\,\hat{\beta}_j\hat{s}_j^3 = \tfrac{1}{12}\,\phi_0\hat{s}_j^3 + \sum_i \phi_i \int_{\text{interval}} (t-\hat{t}_j)(t-t_i)dt'\,, \tag{3.30b}$$

where the interval over which the integrals are calculated is

$$\begin{aligned}\text{interval} &= \hat{t}_j - \hat{s}_{j/2} \quad \text{to} \quad \hat{t}_j + \hat{s}_{j/2} && \text{if} \quad t_i \leqslant \hat{t}_j - \hat{s}_{j/2} \\ &= t_i \quad \text{to} \quad \hat{t}_j + \hat{s}_{j/2} && \text{if} \quad \hat{t}_j - \hat{s}_{j/2} \leqslant t_i \leqslant \hat{t}_j + \hat{s}_{j/2} \\ &= 0 && \text{if} \quad t_i > \hat{t}_j + \hat{s}_{j/2} \,. \end{aligned} \tag{3.30c}$$

We can read directly from (3.30a) the coupling factor for the quantity $\hat{x}_j - x_0 - \phi_0(\hat{t}_j - t_0)$. We shall, as before, drop the subscript i, setting $t_i = t$.

For $\hat{x}_j - x_0 - \phi_0(\hat{t}_j - t_0)$

$$\hat{a}_x^{(j)}(t) \begin{cases} = (\hat{t}_j - t) & ; \quad t < \hat{t}_j - \tfrac{1}{2}\hat{s}_j \\ = \dfrac{1}{2\hat{s}_j}(\hat{t}_j - t + \hat{s}_{j/2})^2 & ; \quad \hat{t}_j - \hat{s}_{j/2} < t < \hat{t}_j + \hat{s}_{j/2} \\ = 0 & ; \quad \hat{t}_j + \hat{s}_{j/2} < t \quad j = 1,2,3\cdots n-1 \,. \end{cases} \tag{3.31}$$

If all the $\hat{s}_j$ are allowed to go to zero, the second range disappears and we get the result expressed by Eq. (3.6).

The coupling factor for $\hat{\beta}_j - \phi_0$ can be obtained from (3.30b).

For slope of fitted line, $\hat{\beta}_j - \phi_0$

$$\hat{a}_\beta^{(j)}(t) \begin{cases} = 1 & ; \quad t < \hat{t}_j - \hat{s}_{j/2} \\ = \tfrac{1}{2} - \dfrac{2}{\hat{s}_j^3}(\hat{t}_j - t)^3 + \dfrac{3}{2\hat{s}_j}(\hat{t}_j - t) & ; \quad \hat{t}_j - \hat{s}_{j/2} \leqslant t \leqslant \hat{t}_j + \hat{s}_{j/2} \\ = 0 & \quad\;\; \hat{t}_j + \hat{s}_{j/2} \leqslant t \quad j = 1,2,\cdots n-1 \,. \end{cases} \tag{3.32}$$

The result for $\hat{s}_j = 0$ is the expected one for the angle between tangents at t_0 and t_j.

Now we can find the coupling factors for the simplest measurable quantities. The angles between successive lines, $\hat{\phi}_j = \hat{\beta}_j - \hat{\beta}_{j-1}$ have coupling factors derived by calculating $\hat{a}_\beta^j - \hat{a}_\beta^{j-1}$ from (3.32):

for $\hat{\phi}_j = \hat{\beta}_j - \hat{\beta}_{j-1}$, $\hat{s}_j < s_j$,

$$\hat{a}_\phi^{(j)}(t) \begin{cases} = 0 & ; \quad t \leqslant \hat{t}_{j-1} - \hat{s}_{j-1/2} \\ = \tfrac{1}{2} + \dfrac{2}{\hat{s}_{j-1}^3}(\hat{t}_{j-1} - t)^3 - \dfrac{3}{2\hat{s}_{j-1}}(\hat{t}_{j-1} - t) & ; \quad \hat{t}_{j-1} - \hat{s}_{j-1/2} \leqslant t \leqslant \hat{t}_{j-1} + \hat{s}_{j-1/2} \\ = 1 & ; \quad \hat{t}_{j-1} + \hat{s}_{j-1/2} \leqslant t \leqslant \hat{t}_j - \hat{s}_{j/2} \\ = \tfrac{1}{2} - \dfrac{2}{\hat{s}_j^3}(\hat{t}_j - t)^3 + \dfrac{3}{2\hat{s}_j}(\hat{t}_j - t) & ; \quad \hat{t}_j - \hat{s}_{j/2} \leqslant t \leqslant \hat{t}_j + \hat{s}_{j/2} \\ = 0 & : \quad \hat{t}_j + \hat{s}_{j/2} \leqslant t \,. \end{cases} \tag{3.33}$$

Eq. (3.20) results from (3.33) when $\hat{s}_j \to 0$.

When the $\hat{s}_j$ are equal to the full cell lengths we have

$$\hat{t}_{j-1} + \hat{s}_{j-1/2} = t_{j-1} = \hat{t}_j - \hat{s}_{j/2} \,, \tag{3.34}$$

and we can write a simpler formula for the coupling constants (if all the $s_j = s$, we have Molière's case IIa).

for $\hat{\phi}_j = \hat{\beta}_j - \hat{\beta}_{j-1}$

$\hat{s}_j = s_j$

$$\hat{a}_\phi^{(j)}(t)\Big|_{\hat{s}_j = s_j} \begin{cases} = 0 & ; \quad t < t_{j-2} = \hat{t}_{j-1} - s_{j-1/2} \\ = 1 + \dfrac{2}{s_{j-1}^3}(t_{j-1} - t)^3 - \dfrac{3}{s_{j-1}^2}(t_{j-1} - t)^2 & ; \quad t_{j-2} < t < t_{j-1} = t_{j-2} + s_{j-1} \\ = 1 - \dfrac{2}{s_j^3}(t_{j-1} - t)^3 - \dfrac{3}{s_j^2}(t_{j-1} - t)^2 & ; \quad t_{j-1} < t < t_j = \hat{t}_j + s_{j/2} \\ = 0 & \quad\;\; t_j \prec t \quad j = 2,\cdots n-1 \,. \end{cases} \tag{3.35}$$

For the slopes of lines joining the midpoints of successive least-squares-fitted lines, we have from (3.31):

$$\text{for } \hat{\psi} = \frac{\hat{x}_j - \hat{x}_{j-1}}{\hat{t}_j - \hat{t}_{j-1}} - \phi_0$$

$$\hat{a}_{\psi}^{(j)}(t) \begin{cases} = 1 & ; \quad t \leqslant \hat{t}_{j-1} - \hat{s}_{j-1/2} \\ = \left\{(\hat{t}_j - t) - \dfrac{1}{2\hat{s}_{j-1}}(\hat{t}_{j-1} - t + \tfrac{1}{2}\hat{s}_{j-1})^2\right\} / (\hat{t}_j - \hat{t}_{j-1}) & ; \quad \hat{t}_{j-1} - \hat{s}_{j-1/2} \leqslant t \leqslant \hat{t}_{j-1} + \hat{s}_{j-1/2} \\ = \dfrac{\hat{t}_j - t}{\hat{t}_j - \hat{t}_{j-1}} & ; \quad \hat{t}_{j-1} + \hat{s}_{j-1/2} \leqslant t \leqslant \hat{t}_j - \hat{s}_{j/2} \\ = \dfrac{(\hat{t}_j - t + \frac{1}{2}\hat{s}_j)^2}{2\hat{s}_j(\hat{t}_j - \hat{t}_{j-1})} & ; \quad \hat{t}_j - \hat{s}_{j/2} \leqslant t \leqslant \hat{t}_j + \hat{s}_{j/2} \\ = 0 & ; \quad \hat{t}_j + \hat{s}_{j/2} \leqslant t \quad , \quad j = 2,3 \cdots n-1 \, . \end{cases} \tag{3.36}$$

The angles $\hat{\alpha}_j$ of actual interest are the differences of the slopes just calculated. We define $\hat{\alpha}_j$ by

$$\hat{\alpha}_j = \frac{\hat{x}_{j+1} - \hat{x}_j}{\hat{t}_{j+1} - \hat{t}_j} - \frac{\hat{x}_j - \hat{x}_{j-1}}{\hat{t}_j - \hat{t}_{j-1}} \, , \tag{3.37a}$$

and we have for the coupling factor:

for $\hat{\alpha}_j$

$\hat{s}_j < s_j$

$$\hat{a}_{\alpha}^{(j)}(t) \begin{cases} = 0 & ; \quad t \leqslant \hat{t}_{j-1} - \tfrac{1}{2}\hat{s}_{j-1} \\ = 1 - \dfrac{\left\{\hat{t}_j - t - \dfrac{1}{2\hat{s}_{j-1}}(\hat{t}_{j-1} - t + \tfrac{1}{2}\hat{s}_{j-1})^2\right\}}{\hat{t}_j - \hat{t}_{j-1}} & ; \quad \hat{t}_{j-1} - \tfrac{1}{2}\hat{s}_{j-1} \leqslant t \leqslant \hat{t}_{j-1} + \tfrac{1}{2}\hat{s}_{j-1} \\ = \dfrac{t - \hat{t}_{j-1}}{\hat{t}_j - \hat{t}_{j-1}} & ; \quad \hat{t}_{j-1} + \hat{s}_{j-1/2} \leqslant t \leqslant \hat{t}_j - \hat{s}_{j/2} \\ = \dfrac{\left\{\hat{t}_{j+1} - t - \dfrac{1}{2\hat{s}_j}(\hat{t}_j - t + \tfrac{1}{2}\hat{s}_j)^2\right\}}{\hat{t}_{j+1} - \hat{t}_j} - \dfrac{(\hat{t}_j - t + \frac{1}{2}\hat{s}_j)^2}{2\hat{s}_j(\hat{t}_j - \hat{t}_{j-1})} & ; \quad \hat{t}_j - \tfrac{1}{2}\hat{s}_j \leqslant t \leqslant \hat{t}_j + \tfrac{1}{2}\hat{s}_j \\ = \dfrac{\hat{t}_{j+1} - t}{\hat{t}_{j+1} - \hat{t}_j} & ; \quad \hat{t}_j + \tfrac{1}{2}\hat{s}_j \leqslant t \leqslant \hat{t}_{j+1} - \tfrac{1}{2}\hat{s}_{j+1} \\ = \dfrac{(\hat{t}_{j+1} - t + \frac{1}{2}\hat{s}_{j+1})^2}{2\hat{s}_{j+1}(\hat{t}_{j+1} - \hat{t}_j)} & ; \quad \hat{t}_{j+1} - \tfrac{1}{2}\hat{s}_{j+1} \leqslant t \leqslant \hat{t}_{j+1} + \tfrac{1}{2}\hat{s}_{j+1} \\ = 0 & ; \quad \hat{t}_{j+1} + \tfrac{1}{2}\hat{s}_{j+1} \leqslant t \quad j = 2,3,\cdots n-2 \, . \end{cases} \tag{3.37b}$$

The result of setting the $\hat{s}_j = 0$ in this expression is Eq. (3.24b). If all the s_j are equal, and the $\hat{s}_j$ are all taken as the same fraction of the s_j, so that

$$\left.\begin{aligned} s_j &= s \\ \hat{s}_j &= rs_j = rs \end{aligned}\right\} j = 1,2,\cdots n \quad ; \quad 0 < r < 1 \, ,$$

we have the somewhat simpler result (Molière's case IIc);

$$\text{for } \hat{\alpha}_j \quad \hat{s}_j = rs_j = rs$$

$$\hat{a}_\alpha^{(j)}(t)\Big|_{\hat{s}_j = rs} \begin{cases} = 0 & ; \quad t \leqslant \hat{t}_j - (1 + \frac{1}{2} r)s \\ = \dfrac{1}{2rs^2}[\hat{t}_j - t - (1 + \frac{1}{2} r)s]^2 & ; \quad \hat{t}_j - (1 + \frac{1}{2} r)s \leqslant t \leqslant \hat{t}_j - (1 - \frac{1}{2} r)s \\ = 1 + \dfrac{t - \hat{t}_j}{s} & ; \quad \hat{t}_j - (1 - \frac{1}{2} r)s \leqslant t \leqslant \hat{t}_j - \frac{1}{2} rs \\ = 1 - \frac{1}{4} r - \dfrac{1}{r}\left(\dfrac{\hat{t}_j - t}{s}\right)^2 & ; \quad \hat{t}_j - \frac{1}{2} rs \leqslant t \leqslant \hat{t}_j + \frac{1}{2} rs \\ = \dfrac{\hat{t}_j - t}{s} + 1 & ; \quad \hat{t}_j + \frac{1}{2} rs \leqslant t \leqslant \hat{t}_j + (1 - \frac{1}{2} r)s \\ = \dfrac{1}{2rs^2}[\hat{t}_j - t + (1 + \frac{1}{2} r)s]^2 & ; \quad \hat{t}_j + (1 - \frac{1}{2} r)s \leqslant t \leqslant \hat{t}_j + (1 + \frac{1}{2} r)s \\ = 0 & ; \quad \hat{t}_j + (1 + \frac{1}{2} r)s \leqslant t \quad , \quad j = 2,3 \cdots n - 2 . \end{cases} \tag{3.38}$$

The result is still simpler when $r = 1$ (Molière's case IIb):

$$\text{for } \hat{\alpha}_j \quad \hat{s}_j = s_j = s$$

$$\hat{a}_\alpha^{(j)}(t)\Big|_{\hat{s}_j = s} \begin{cases} = 0 & ; \quad t \leqslant \hat{t}_j - \frac{3}{2} s \\ = \dfrac{1}{2s^2}[\hat{t}_j - t - \frac{3}{2} s]^2 & ; \quad \hat{t}_j - \frac{3}{2} s \leqslant t \leqslant \hat{t}_j - \frac{1}{2} s \\ = \frac{3}{4} - \dfrac{1}{s^2}(\hat{t}_j - t)^2 & ; \quad \hat{t}_j - \frac{1}{2} s \leqslant t \leqslant \hat{t}_j + \frac{1}{2} s \\ = \dfrac{1}{2s^2}[t_j - t + \frac{3}{2} s]^2 & ; \quad \hat{t}_j + \frac{1}{2} s \leqslant t \leqslant \hat{t}_j + \frac{3}{2} s \\ = 0 & ; \quad \hat{t}_j + \frac{3}{2} s \leqslant t . \end{cases} \tag{3.39}$$

IV. ASYMPTOTIC EXPANSIONS

Although a detailed study of asymptotic expansions will be given later after explicit expressions for $\Omega(\xi,t)$ have been introduced, some general properties of these expansions can usefully be introduced at this point. We use the method of Snyder and Scott (1949), applied primarily to the spatial case.

The method consists in rewriting the integral for $F(\theta)$,[4]

$$F(\theta) = \frac{1}{2\pi}\int_0^\infty \xi d\xi J_0(\xi\theta) \exp[\Omega(\xi) - \Omega_0] , \tag{4.1}$$

in such a way that the path of integration can legitimately be bent away from the real axis into the upper half of the complex plane; we then approximate its entire value by the integral of a portion along the imaginary axis. We first observe that the Hankel function of zero order, $H_0{}^{(1)}(z)$ can be written in terms of two real functions $J_0(z)$ and $N_0(z)$, and also in terms of the Bessel function of the second kind of purely imaginary argument, $K_0(-iz)$, as follows:

$$H_0^{(1)}(z) = J_0(z) + iN_0(z) = \frac{2}{\pi i} K_0(-iz) . \tag{4.2}$$

The function $N_0(z)$ is the Neumann function [see Jahnke–Emde (1943), Jahnke–Emde–Lösch (1960), and Watson (1952); the latter author uses the symbol Y_0 in place of N_0].

Now, when τ is large, $K_0(\tau)$ behaves like $e^{-\tau}$. Thus, if we set $z = i\tau$ on part of an integration path, the combination $J_0(z) + iN_0(z)$ behaves like $e^{-\tau}$, although neither of them separately decreases.

We write, therefore, before changing the integration path,

$$F(\theta) = \frac{1}{2\pi} \operatorname{Re} \int_0^\infty \xi d\xi H_0^{(1)}(\xi\theta) \exp[\Omega(\xi) - \Omega_0]$$
$$= -\frac{1}{\pi^2} \operatorname{Re} i\int_0^\infty \xi d\xi K_0(-i\xi\theta) \exp[\Omega(\xi) - \Omega_0] . \tag{4.3}$$

[4] We suppress the variable t in this section.

Now let us deform the path of integration from C in Fig. 5 to C', where τ_1 is to be chosen later.

Along the vertical portion of the path, we have

$$\frac{1}{\pi^2}\,\mathrm{Re}\, i\int_0^{\tau_1} \tau d\tau K_0(\tau\theta)\exp\left[\Omega(i\tau)-\Omega_0\right]. \qquad (4.4)$$

We shall shortly see that $\Omega(\xi)-\Omega_0$ behaves for small ξ as $-\xi^2$ or as $+\tau^2$. Hence, the exponential factor will increase with τ. However, if θ is large enough, the $e^{-\theta\tau}$ behavior of K_0 will produce a rapid decrease before the increase of the other factor is important. The value τ_1 is to be chosen so as to give an approximate minimum of the product of the two factors. The integral on the horizontal part of C' will decrease because of the (presumed) convergence of the exponential factor and will contribute in its entirety only a small amount to the result.

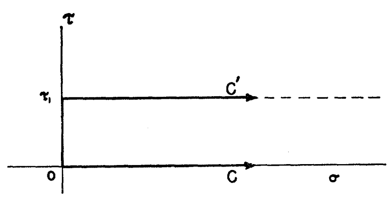

FIG. 5. The ξ plane; $\xi=\sigma+i\tau$.

If now the exponential factor is approximated by the first few terms of its expansion in powers of τ, which should be adequate under the assumption made about the size of θ, the integral of each separate term will be very nearly the same as if τ_1 were infinite. We can then use the integral formula [Erdelyi *et al.* T. I. T. (1954) 10.2(1); and Watson (1952), p. 388]:

$$\int_0^\infty d\tau K_0(\tau\theta)\tau^\nu = \frac{2^{\nu-1}}{\theta^{\nu+1}}\Gamma^2\left(\frac{\nu+1}{2}\right). \qquad (4.5)$$

Since we shall find that terms involving $\ln\tau$ appear in the expansion, we need the formula obtained from (4.5) by differentiating with respect to ν:

$$\int_0^\infty d\tau K_0(\tau\theta)\tau^\nu \ln\tau = \frac{2^{\nu-1}}{\theta^{\nu+1}}\Gamma^2\left(\frac{\nu+1}{2}\right) \times\left[\ln\frac{2}{\theta}+\psi\left(\frac{\nu-1}{2}\right)\right] \qquad (4.6)$$

where $\psi(x)$ is the logarithmic derivative of the gamma function, defined by $\psi(x)=\Gamma'(x+1)/\Gamma(x+1)$.

In accordance with what has been said, we need an expansion of $\Omega(\xi)-\Omega_0$ for relatively small values of ξ, which will in turn be partly but not entirely determined by $W(\chi)$ for large values of χ.

Let us assume then that we have an asymptotic expansion of $W(\chi)$ that is valid above some angle χ_1. We write

$$2\pi t W(\chi) = \frac{b_3}{\chi^3}+\frac{b_4}{\chi^4}+\frac{b_5}{\chi^5}+\frac{b_6}{\chi^6}+\cdots;\ \chi>\chi_1, \qquad (4.7)$$

where we have chosen the lowest inverse power for which the integral for Ω_0, (2.38b), will converge,[5] and carried it to χ^{-6} for purposes of illustration.

We can then write, using (2.38a) and (2.38b),

$$\begin{aligned}\Omega(\xi)-\Omega_0 &= 2\pi t\int_0^\infty \chi d\chi W(\chi)[J_0(\xi\chi)-1]\\ &= 2\pi t\int_0^{\chi_1} \chi d\chi W(\chi)[J_0(\xi\chi)-1]\\ &\quad+\int_{\chi_1}^\infty \chi d\chi[J_0(\xi\chi)-1]\left[\frac{b_3}{\chi^3}+\frac{b_4}{\chi^4}+\frac{b_5}{\chi^5}+\frac{b_6}{\chi^6}\cdots\right].\end{aligned} \qquad (4.8)$$

Let us now assume that the necessary values of ξ are small enough that we can use only three terms in the expansion $J_0(\xi\chi)=1-\chi^2\xi^2/4+\chi^4\xi^4/64\cdots$. This will be true if $|\xi\chi_1|$ is of the order 1 or less for the largest ξ's required—e.g., if $\chi_1\tau_1$ is of the order 1 or less. This in turn requires that $\theta\tau_1$ be enough greater than 1 for the $e^{-\theta\tau}$ behavior of K_0 to have reduced the integrand for $F(\theta)$ to quite a small number—e.g., $\theta\tau_1$ of the order of 4 to 6 or more. The asymptotic expression will begin to be valid for θ roughly 4 to 6 times larger than χ_1.

The integral $2\pi t\int_0^{\chi_1}\chi^3 d\chi W(\chi)$ can be interpreted as the mean-square angle of scattering produced in a thickness t by the cutoff distribution

$$\begin{aligned} W &= W(\chi)\quad ;\quad \chi<\chi_1\\ W &= 0\quad ;\quad \chi>\chi_1, \end{aligned} \qquad (4.9)$$

since for any distribution whatsoever that has a mean square, the mean-square, multiple-scattering angle is just the mean-square, single-scattering angle multiplied by the mean number of scatterings. Hence, we shall write

$$2\pi t\int_0^{\chi_1}\chi^3 d\chi W(\chi) = \langle\theta^2\rangle_1 \qquad (4.10)$$

and similarly

$$2\pi t\int_0^{\chi_1}\chi^5 d\chi W(\chi) = \langle\theta^4\rangle_1, \qquad (4.11)$$

where the subscript refers to the cutoff chosen. (The

[5] We shall see in Sec. IX that the inverse second power appears using the Dalitz cross-section, and a modification of the small-angle approximation is necessary.

last integral will turn out to disappear when we only calculate to terms in θ^{-6}.)

The final result should be independent of this cutoff; this can be seen by calculating $\langle\theta^2\rangle_2$ for a larger cutoff χ_2:

$$\begin{aligned}\langle\theta^2\rangle_2 &= 2\pi t\int_0^{\chi_1}\chi^3 d\chi W(\chi) + 2\pi t\int_{\chi_1}^{\chi_2}\chi^3 d\chi W(\chi)\\ &= \langle\theta^2\rangle_1 + b_3(\chi_2-\chi_1) + b_4\ln(\chi_2/\chi_1)\\ &\quad - b_5\left(\frac{1}{\chi_2}-\frac{1}{\chi_1}\right) - b_6\left(\frac{1}{2\chi_2^2}-\frac{1}{2\chi_1^2}\right), \qquad (4.12)\end{aligned}$$

from which we see that

$$\begin{aligned}\langle\theta^2\rangle_2 - b_3\chi_2 - b_4\ln\chi_2 + b_5/\chi_2 + b_6/2\chi_2^2 &= \langle\theta^2\rangle_1 - b_3\chi_1 - b_4\ln\chi_1 + b_5/\chi_1 + b_6/2\chi_1^2\\ &= -b_4(\ln\chi_\alpha + \tfrac{1}{2}) \qquad (4.13)\end{aligned}$$

is a constant independent of the cutoff, as long as the cutoff is in the region in which (4.7) is valid. The last line of (4.13) defines the screening constant χ_α, for use in later calculation.

Now we can use the results of Appendix I, and write out an expression for $\Omega(\xi) - \Omega_0$:

$$\begin{aligned}\Omega(\xi)-\Omega_0 = &-b_3\xi - [\langle\theta^2\rangle_1 - b_3\chi_1 - b_4\ln(\gamma\chi_1/2e)\\ &+ b_5/\chi_1 - b_6/2\chi_1^2](\xi^2/4) + \tfrac{1}{9}b_5\xi^3 + [\langle\theta^4\rangle_1 - \tfrac{1}{3}b_3\chi_1^3\\ &- \tfrac{1}{2}b_4\chi_1^2 - b_5\chi_1 - b_6\ln(\gamma\chi_1/2e^{3/2})](\xi^4/64)\\ &+ \tfrac{1}{4}b_4\xi^2\ln\xi - (b_6/64)\xi^4\ln\xi\,. \qquad (4.14)\end{aligned}$$

The coefficient of ξ^2 simplifies by use of (4.13); the coefficient of ξ^4 also simplifies if the corresponding calculation is made for $\langle\theta^4\rangle_1$, but since this coefficient disappears in what follows, we shall simply call it A_4. We have

$$\begin{aligned}\Omega(\xi)-\Omega_0 = &-b_3\xi + \tfrac{1}{4}b_4\xi^2\ln(\gamma\chi_\alpha\xi/2e^{1/2})\\ &+ \tfrac{1}{9}b_5\xi^3 + A_4\xi^4 - (b_6/64)\xi^4\ln\xi\,. \qquad (4.15)\end{aligned}$$

It is tempting to seek an expression for $F(\theta)$ which would be a folding integral of a Gaussian multiple-scattering distribution and a single-scattering distribution corresponding to the "tail" given by (4.7). This could be achieved if $\exp[\Omega(\xi)-\Omega_0]$ were written as a product of a factor like $\exp[-\frac{1}{4}\xi^2\langle\theta^2\rangle_1]$ and a factor like $[1+2\pi t\tilde{W}_{\text{tail}}(\xi)]$, hopefully resulting from an expansion of the part of $\Omega(\xi)-\Omega_0$ for $\chi > \chi_1$ (the unity here will produce a pure Gaussian that is negligible in the asymptotic regions). The trouble with this procedure is that the terms to be expanded are of the same order as the term left in the exponent, and the choice of the multiple-scattering "width" is considerably dependent on χ_1, so the procedure is quite arbitrary; the expansion will be of doubtful validity just for those values of ξ that are supposed to establish the main part of the Gaussian.

If we proceed to use (4.14) with $\xi = i\tau$ in (4.4), and use the integrals (4.5) and (4.6), we obtain after some reduction

$$\begin{aligned}2\pi F(\theta) \approx \frac{b_3}{\theta^3} + \frac{b_4}{\theta^4} + \left[b_5 + \frac{9}{4}b_3b_4\ln\frac{4\theta}{\chi_\alpha} - \frac{39}{8}b_3b_4\right.\\ \left. - \frac{3}{2}b_3^3\right]\frac{1}{\theta^5} + \left[b_6 + 4b_4^2\ln\frac{\theta}{\chi_\alpha} - 4b_4^2 - 8b_3^2b_4\right]\frac{1}{\theta^6}\cdots.\end{aligned} \qquad (4.16)$$

We note that to order θ^{-4} the asymptotic formula is identical with the single-scattering tail expected for the given thickness, and that if b_3 and b_4 were each zero, the formula would agree with the θ^{-5} and θ^{-6} terms in W. Furthermore, if b_4 is the dominant term, multiplying the W by a correction factor (e.g., for spin effects) will multiply $F(\theta)$ by nearly the same factor, as speculated by Bethe (1953) and approximately verified by Spencer and Blanchard (1954).

A similar procedure can be used for the projected scattering. We write $\cos\xi\phi = \operatorname{Re}(e^{i\xi\phi})$ and proceed similarly.

We can also find directly a relation between the asymptotic formulas for $F(\theta)$ and $f(\phi)$. Using (2.4b), we find that if

$$F(\theta) \simeq A_n/\theta^n\,, \qquad (4.17a)$$

then

$$f(\phi) \simeq 2\pi^{1/2}\,\Gamma\left(\frac{n-1}{2}\right)A_n/\Gamma(n/2)\phi^{n-1}\,, \qquad (4.17b)$$

and by differentiating under the integral sign, treating n as a continuous variable, if

$$F(\theta) \simeq B_n\ln\theta/\theta^n\,, \qquad (4.18a)$$

then

$$\begin{aligned}f(\phi) \simeq 2\pi^{1/2}\,\Gamma\left(\frac{n-1}{2}\right)\\ \times B_n\left\{\ln\phi - \tfrac{1}{2}\psi\left(\frac{n-3}{2}\right)\right.\\ \left. + \tfrac{1}{2}\psi\left(\frac{n}{2}-1\right)\right\}\Big/\Gamma\left(\frac{n}{2}\right)\phi^{n-1}\,. \qquad (4.18b)\end{aligned}$$

For later use, we shall need the result of another differentiation with respect to n. If

$$F(\theta) \simeq C_n\ln^2\theta/\theta^n \qquad (4.19a)$$

then

$$f(\phi) \simeq \frac{2\pi^{1/2}\,\Gamma\left(\frac{n-1}{2}\right) C_n}{\Gamma\left(\frac{n}{2}\right)\phi^{n-1}} \times \left\{\left[\ln\phi - \tfrac{1}{2}\psi\left(\frac{n-3}{2}\right) + \tfrac{1}{2}\psi\left(\frac{n}{2}-1\right)\right]^2 + \tfrac{1}{4}\psi'\left(\frac{n-3}{2}\right) - \tfrac{1}{4}\psi'\left(\frac{n}{2}-1\right)\right\}. \quad (4.19b)$$

V. MULTIPLE SCATTERING OF POLARIZED PARTICLES

A. Density-Matrix Treatment of Spin

As we stated in Sec. II-A, polarization must be treated quantum-mechanically even while angular deflection probabilities are combined in a classical way. Specifically, $\sigma(\theta,\beta) = |u(\theta,\beta)|^2$ represents the absolute square of the scattering amplitude in which the different components of spin, if any are averaged over.

When we consider spin (and the only important cases are for spin $\frac{1}{2}$) we have to consider two spin states along with $u(\theta,\beta)$. Since in nearly all cases the beam of particles under consideration will not be in a pure spin state—i.e., the beam will be at least partially unpolarized—we must consider mixtures of states. The appropriate technique for dealing with mixtures is that of the density matrix [Fano (1957), Tolhoek (1956)] which we shall now introduce.

The spin of a traveling particle is detected by resolving the beam in which the particle is contained into two opposite spin directions or channels, and determining into which channel the particle is deflected. A pure state, resolved in this way, could be represented by the two-column matrix

$$\left\|\begin{matrix} c^+\psi \\ c^-\psi \end{matrix}\right\| = \left\|\begin{matrix} c^+ \\ c^- \end{matrix}\right\| \psi\,, \quad (5.1)$$

where ψ is a function of position or linear momentum of the particle. We use here the Pauli spin formalism; if for relativistic particles we define the spin direction to be the one that would be observed if the particles were brought to rest by a purely longitudinal decelerating electric field, we can treat the spin by the Pauli formalism applied to the two "large" components of the Dirac spinor —ψ_3 and ψ_4 in the usual notation when electrons are under consideration [Tolhoek (1956), Fradkin and Good (1961)].

If, for instance, the detector is set to count particles with positive spins, $c^{+*}c^+\psi^*\psi$ will be the probability (per unit solid angle, etc.) of getting a count at (θ,β), and $c^{-*}c^-\psi^*\psi = (1 - c^{+*}c^+)\psi^*\psi$ will be the probability of a particle passing and not registering a count.

It is more convenient and general to use an arbitrary pair of spin directions as basis for representation instead of those for which the detector is set. In this case, the $\pm$ states for the detector would be linear combinations

$$\psi_\pm = a_1^\pm\psi_1 + a_2^\pm\psi_2 = a_1^\pm\left\|\begin{matrix}1\\0\end{matrix}\right\|\psi + a_2^\pm\left\|\begin{matrix}0\\1\end{matrix}\right\|\psi = \left\|\begin{matrix}a_1^\pm\\a_2^\pm\end{matrix}\right\|\psi \quad (5.2a)$$

with orthogonality and normalization given by

$$a_1^{\pm *}a_1^\pm + a_2^{\pm *}a_2^\pm = 1\,,$$
$$a_1^{\pm *}a_1^\mp + a_2^{\pm *}a_2^\mp = 0\,, \quad (5.2b)$$

together with

$$\int \psi^*\psi d\tau = 1\,. \quad (5.3)$$

Since ψ_1 and ψ_2 are also normalized and orthogonal, we have the further conditions

$$a_1^{+*}a_1^+ + a_1^{-*}a_1^- = 1\,, \quad a_2^{+*}a_2^+ + a_2^{-*}a_2^- = 1\,,$$
$$a_1^{+*}a_2^+ + a_1^{-*}a_2^- = a_1^+a_2^{+*} + a_1^-a_2^{-*} = 0\,. \quad (5.4)$$

Now, if we have an arbitrary state given by

$$\psi_S = c_1\psi_1 + c_2\psi_2 = \left\|\begin{matrix}c_1\\c_2\end{matrix}\right\|\psi \quad (5.5a)$$

with

$$c_1^*c_1 + c_2^*c_2 = 1\,, \quad (5.5b)$$

we can expand it in terms of the $\psi_\pm$:

$$\psi_S = A_+\psi_+ + A_-\psi_-\,, \quad (5.6)$$

where A_+ is given by

$$\int \psi_+^*\psi_S d\tau = \left\|\begin{matrix}a_1^{+*} & a_2^{+*}\end{matrix}\right\| \left\|\begin{matrix}c_1\\c_2\end{matrix}\right\| \int \psi^*\psi d\tau = a_1^{+*}c_1 + a_2^{+*}c_2\,. \quad (5.7)$$

The probability of getting a count in the detector is $A_+^*A_+$ which we can write in summation notation as

$$P(+) = a_i^+c_i^*a_j^{+*}c_j\,. \quad (5.8)$$

If we have a mixture M of states, by which we mean an incoherent superposition, the probability of a count is just the ordinary classical weighted sum of terms like (5.8). Let us describe the mixture in terms of two orthogonal states $\psi^{(1)} = c^{(1)}{}_1\psi_1 + c^{(1)}{}_2\psi_2$ and

$\psi^{(2)} = c^{(2)}{}_1\psi_1 + \psi^{(2)}{}_2\psi_2$. The probabilities of the two states will be denoted by $p^{(1)}$ and $p^{(2)}$.

Then the probability $P_M(+)$ can be written

$$P_M(+) = \sum_k p^{(k)} a_i^+ c_i^{(k)*} a_j^{+*} c_j^{(k)} . \quad (5.9)$$

This sum can be written as the trace of the product of two matrices, defined by

$$\rho_{\text{det}} = \begin{Vmatrix} a_1^+ a_1^{+*} & a_1^+ a_2^{+*} \\ a_2^+ a_1^{+*} & a_2^+ a_2^{+*} \end{Vmatrix} ; \; (\rho_{\text{det}})_{ij} = a_i^+ a_j^{+*} \quad (5.10)$$

and

$$\rho_M = \begin{Vmatrix} \sum_k p^{(k)} c_1^{(k)} c_1^{(k)*} & \sum_k p^{(k)} c_1^{(k)} c_2^{(k)*} \\ \sum_k p^{(k)} c_2^{(k)} c_1^{(k)*} & \sum_k p^{(k)} c_2^{(k)} c_2^{(k)*} \end{Vmatrix} ;$$

$$(\rho_M)_{ij} = \sum_k p^{(k)} c_i^{(k)} c_j^{(k)*} . \quad (5.11)$$

Then we have

$$P_M(+) = \text{Tr}\,(\rho_{\text{det}}\rho_M) = \text{Tr}\,(\rho_M\rho_{\text{det}}) . \quad (5.12)$$

Note also that

$$\text{Tr}\,(\rho_{\text{det}}) = \text{Tr}\,(\rho_M) = 1 . \quad (5.13)$$

The matrix ρ_M is called the statistical matrix of the mixture M. A pure state S would have a statistical matrix ρ_S without the sum over $p^{(k)}$; in fact, ρ_{det} is the statistical matrix of the pure state ψ_+. The formula (5.12) allows us to obtain all information obtainable from the given mixture of states by use of all possible detector states (orientations) ψ_+. It is a special case of the general formula for the mean value of an operator Q, as given for instance by Fano [1957, Eq. (3.5)]. The operator represented by ρ_{det} has the value 1 when the spin is plus and 0 when it is minus, so $P(+)$ is its mean value.

For any state S, as given by (5.5), we can find the expectation values of the Pauli spin matrices[6] σ_x, σ_y, σ_z in the representation based on ψ_1 and ψ_2. We find using P_x, P_y, and P_z to represent these values,

$$P_x = \langle\sigma_x\rangle = c_1c_2^* + c_2c_1^* , \quad (5.14a)$$

$$P_y = \langle\sigma_y\rangle = i(c_1c_2^* - c_2c_1^*) , \quad (5.14b)$$

$$P_z = \langle\sigma_z\rangle = c_1c_1^* - c_2c_2^* , \quad (5.14c)$$

with the further relation added for completeness

$$1 = c_1c_1^* + c_2c_2^* . \quad (5.14d)$$

Note that the length of the vector **P** is unity:

$$P_x^2 + P_y^2 + P_z^2 = (c_1c_1^* + c_2c_2^*)^2 = 1 . \quad (5.15)$$

Using these results and the Pauli matrices, it is easy to write the matrix of the state S in vector-matrix form

$$\rho_S = \tfrac{1}{2}\,(\mathbf{I} + \mathbf{P}\cdot\boldsymbol{\sigma}) , \quad (5.16)$$

where $\mathbf{I}$ is the unit matrix, and $\boldsymbol{\sigma}$ is the vector whose three components are the Pauli matrices.

For relativistic particles when four-component Dirac matrices are required, the mean values of the components of $\boldsymbol{\sigma}$ are no longer given by the c's of (5.13), which belong to the two "large" components only. Tolhoek (1956) has shown that the matrix whose components give the direction of the magnetic moment, which is the observable quantity, and which do yield the formulas in (5.13), may be written [Mühlschlegel and Koppe (1958)], taking the direction of the linear momentum as the z-axis

$$\boldsymbol{\Sigma} = \sigma_z\mathbf{1}_z + \frac{E}{mc^2}\,[\boldsymbol{\sigma} - \sigma_z\mathbf{1}_z] , \quad (5.17)$$

where $\mathbf{1}_z$ is the unit vector in the direction of z, and E and mc^2 are the total energy and rest energy of the particle, respectively. When the particle is at rest, $E = mc^2$ and the longitudinal and transverse components of $\boldsymbol{\sigma}$ are weighted equally.

For a mixture, we take a summation over each P_x, P_y, and P_z and can write (we continue to use the Pauli $\boldsymbol{\sigma}$)

$$\rho_M = \tfrac{1}{2}\,[\mathbf{I} + \mathbf{P}\cdot\boldsymbol{\sigma}] ;$$
$$\mathbf{P} = \sum_k p^{(k)}\mathbf{P}^{(k)} = p^{(1)}\mathbf{P}^{(1)} + p^{(2)}\mathbf{P}^{(2)} . \quad (5.18)$$

Note that knowledge of **P** as given by (5.14) is enough to determine P_S, since the four equations can be solved for the four matrix elements; the c_i themselves can be determined to within a common phase factor. Similarly knowledge of **P** as given by (5.18) is sufficient to determine ρ_M.

If now we choose a representation (orientation of axes) in which the two states of the mixture are just ψ_1 and ψ_2, we have $c^{(1)}{}_1 = 1$, $c^{(1)}{}_2 = 0$, $c^{(2)}{}_1 = 0$, and $c^{(2)}{}_2 = 1$, and

$$\rho_M = \begin{Vmatrix} p^{(1)} & 0 \\ 0 & p^{(2)} \end{Vmatrix} . \quad (5.19)$$

$$P_x = P_y = 0$$
$$P_z = p^{(1)} - p^{(2)} \quad (5.20)$$

Since $p^{(1)} + p^{(2)} = 1$, we see that for a pure state **P** is a unit vector, whereas for a mixture, $|\mathbf{P}|$ is less than unity. In the case that $p^{(1)} > p^{(2)}$ we could describe the original mixture as a combination of two mixtures: one actually the pure state $\psi^{(1)}$ with the probability $p^{(1)} - p^{(2)}$, and the other a completely unpolarized mixture with equal probabilities $p^{(2)}$ of

[6] We use x,y,z for three spatial coordinates, and only when we deal with the small-angle approximation will we specify that z coincides with the thickness t.

either $\psi^{(1)}$ or $\psi^{(2)}$. Thus $p^{(1)} - p^{(2)} = |\mathbf{P}| = P$ is the degree of polarization of the mixture. The vector $\mathbf{P}$ itself characterizes the polarization state completely, just as does the density matrix ρ_M, for any mixture. However, the density matrix will be of more direct use.

The detector matrix can also be written in the form (5.18), by introducing $\mathbf{P}^+$ whose components are formed from the $a_i^\dagger$'s instead of the c_i's:

$$\rho_{\text{det}} = \tfrac{1}{2}\,[\mathbf{I} + \mathbf{P}^+\cdot\boldsymbol{\sigma}]\,. \tag{5.21}$$

For detection of the opposite sign, we have merely to write

$$(\rho_{\text{det}})_- = \tfrac{1}{2}\,[\mathbf{I} + \mathbf{P}^-\cdot\boldsymbol{\sigma}]\,, \tag{5.22}$$

where $\mathbf{P}^- = -\mathbf{P}^+$. [That $\mathbf{P}^- = -\mathbf{P}^+$ follows by use of (5.4)].

We can then compute the trace of $\rho_{\text{det}}\rho_M$ by use of the rules of spin matrix calculus: $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbf{I}$; $\sigma_x\sigma_y = i\sigma_z = -\sigma_y\sigma_x$, etc; $\text{Tr}\ (\sigma_x) = \text{Tr}\ (\sigma_y) = \text{Tr}\ (\sigma_z) = 0$; $\text{Tr}\ (\mathbf{I}) = 2$. The following corollary is useful, where $\mathbf{P}$ and $\mathbf{P}'$ are any two vectors:

$$(\mathbf{P}\cdot\boldsymbol{\sigma})(\mathbf{P}'\cdot\boldsymbol{\sigma}) = (\mathbf{P}\cdot\mathbf{P}')\mathbf{I} + i\boldsymbol{\sigma}\cdot(\mathbf{P}\times\mathbf{P}')\,. \tag{5.23}$$

We readily obtain the result

$$\text{Tr}\ (\rho_{\text{det}}\rho_M) = \tfrac{1}{2}\,(1 + \mathbf{P}^+\cdot\mathbf{P})\,. \tag{5.24}$$

Now if we have an unpolarized detector that detects both signs of spin, the resulting counting rate will be the sum of the traces (5.24) for $\mathbf{P}^+$ and $\mathbf{P}^-$, namely unity.

B. The Transport Equation

Now consider a scattering event. Scattering theory describes the relation between initial and final states, before and after scattering. In the ordinary case without polarization, the amplitude $u(\theta,\beta)$ multiplies the final outgoing wave and may be considered a multiplying operator that converts an initial plane wave at angle (0,0) into a final wave, plane within an infinitesimal solid angle, at angle (θ,β).

The appropriate way to describe the state of a beam of particles will be by means of a product of the density matrix for the spins and a probability function for the angular distribution. We write the product as

$$F(\theta,\beta)\rho(\mathbf{P}) = \tfrac{1}{2}\,[F(\theta,\beta)\mathbf{I} + \boldsymbol{\Pi}(\theta,\beta)\cdot\boldsymbol{\sigma}]\,, \tag{5.25}$$

where the vector $\boldsymbol{\Pi}$ is given by

$$\boldsymbol{\Pi} = F(\theta,\beta)\mathbf{P}\,. \tag{5.26}$$

The elementary scattering probability must be derived from an operator acting on an initial two-component wave function. If the initial wave is a plane wave in the $\theta = 0$ direction, we can write the resulting wave after scattering in terms of a matrix

$$\begin{Vmatrix} c_1' \\ c_2' \end{Vmatrix} \psi' = \begin{Vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{Vmatrix} \cdot \begin{Vmatrix} c_1 \\ c_2 \end{Vmatrix} \psi\,, \tag{5.27}$$

where $\psi' = u(\theta,\beta)$ as above mentioned. The new density matrix can then be found from the old as follows. We have from (5.27),

$$\begin{aligned} c_i'\psi' &= A_{ik}c_k\psi\,, \\ c_i'c_j'^*\psi'^*\psi' &= A_{ik}c_kA_{jl}^*c_l^*\psi^*\psi \\ &= A_{ik}c_kc_l^*A_{lj}^\dagger\psi^*\psi \end{aligned} \tag{5.28}$$

in matrix form. The symbol $A^\dagger$ denotes the Hermitian conjugate of the matrix A. In terms of the cross section $\sigma = u^*u$, we have

$$\sigma(\theta,\beta)\rho' = A\rho A^\dagger\,, \tag{5.29}$$

where the components of A are functions of θ and β. Although we have not indicated it explicitly, u itself must depend on the vector $\mathbf{P}$.

If a particle before a scattering event has the distribution matrix $F_1(\theta,\beta)\rho(\mathbf{P})$, then the distribution after a single scattering (in thickness t, to use the definitions of Sec. II) will be

$$\begin{aligned} F_2(\theta,\beta)\rho' = Nt\int_0^\infty \theta' d\theta' \int_0^{2\pi} d\beta' A(\theta'',\beta'')\rho(\mathbf{P}) \\ \times A^\dagger(\theta'',\beta'')F_1(\theta',\beta')\,, \end{aligned} \tag{5.30}$$

where as before the vector relation $\boldsymbol{\theta} = \boldsymbol{\theta}' + \boldsymbol{\theta}''$ holds. The matrix ρ' is a function of $\mathbf{P}$, and, by writing it in the standard form $\frac{1}{2}(\mathbf{I} + \mathbf{P}'\cdot\boldsymbol{\sigma})$, it is possible to determine the new polarization vector $\mathbf{P}'$ in terms of $\mathbf{P}$. We can obtain an equation for F_2 by itself by taking the trace of both sides of (5.30),

$$\begin{aligned} F_2(\theta,\beta) = Nt\int_0^\infty \theta'\, d\theta' \int_0^{2\pi} d\beta'\ \text{Tr}\,[A(\theta'',\beta'')\rho(\mathbf{P}) \\ \times A^\dagger(\theta'',\beta'')]F_1(\theta',\beta')\,. \end{aligned} \tag{5.31}$$

Waldmann (1958) has given a derivation of the Boltzmann equation for scattering on fixed centers by polarized particles. In the special case for which the forward scattering does not alter the polarization (isotropic scattering centers), and the small-angle approximation is valid, the equation reads [Mühlschlegel and Koppe (1958)][7]:

[7] Waldmann's Eq. (91.12) is modified by the use of his Eq. (89.16), the "optical" or "shadow" theorem, on the assumption that $A(0, 0)$ is a multiple of the unit matrix.

$$\frac{1}{2}\frac{\partial}{\partial t}[F(\theta,\beta)\mathsf{I} + \mathbf{\Pi}(\theta,\beta)\cdot\boldsymbol{\sigma}] = \frac{N}{2}\int_0^\infty \theta' d\theta' \int_0^{2\pi} d\beta'$$
$$\times \{A(\theta'',\beta'')[F(\theta',\beta')\mathsf{I} + \mathbf{\Pi}(\theta',\beta')\cdot\boldsymbol{\sigma}]A^\dagger(\theta'',\beta'')$$
$$- A(\theta'',\beta'')A^\dagger(\theta'',\beta'')[F(\theta,\beta)\mathsf{I} + \mathbf{\Pi}(\theta,\beta)\cdot\boldsymbol{\sigma}]\} . \quad (5.32)$$

This equation is the generalization of (2.42) to the case of spin-$\frac{1}{2}$ particles.

C. General Solution for Small Angles

We must now specify the matrix A. Any 2×2 matrix can be written as a linear sum of I and the three Pauli matrices. The matrix is a scalar, and to preserve rotational invariance (in the general, large-angle case), A must clearly be a linear combination of a multiple of I and a scalar product of $\boldsymbol{\sigma}$ with a vector. The only relevant vector direction associated with a deflection from an original direction along unit vector $\mathbf{1}_z$ to one along $\mathbf{1}_{z'}$ is the orbital-angular-momentum direction along $\mathbf{1}_z \times \mathbf{1}_{z'} = \mathbf{1}_0 \sin\theta$, where θ is the angular deflection. Thus we expect to find [Mühlschlegel and Koppe (1958)] that A is of the form

$$A(\theta,\beta) = f(\theta)\mathsf{I} + ig(\theta)\sin\theta \mathbf{1}_0\cdot\boldsymbol{\sigma}$$
$$= f(\theta)\mathsf{I} + ig(\theta)\sin\theta[-\sigma_x\sin\beta + \sigma_y\cos\beta] . \quad (5.33)$$

In fact, application of Dirac theory for a central force gives just such a form [Mott and Massey (1949) Sec. 4; Tolhoek (1956)]. We use here the definition of the functions f and g given by Mühlschlegel and Koppe (1958); those of Mott and Massey, and Tolhoek, are equivalent to our f and $-g\sin\theta$, respectively.

A lengthy, but straightforward matrix calculation leads to the following properties of A:

$$AA^\dagger = J(\theta)\mathsf{I} + D(\theta)\sin\theta\mathbf{1}_0\cdot\boldsymbol{\sigma} \quad (5.34)$$

$$A\boldsymbol{\sigma}A^\dagger = D(\theta)\sin\theta\mathbf{1}_0\mathsf{I} - E(\theta)\sin\theta\mathbf{1}_0\times\boldsymbol{\sigma}$$
$$+ [J(\theta) - \sin^2\theta G(\theta)]\boldsymbol{\sigma} + \sin^2\theta G(\theta)(\mathbf{1}_0\cdot\boldsymbol{\sigma})\mathbf{1}_0 , \quad (5.35)$$

where

$$J(\theta) = f^*(\theta)f(\theta) + \sin^2\theta g^*(\theta)g(\theta) ,$$
$$D(\theta) = i[f^*(\theta)g(\theta) - f(\theta)g^*(\theta)] ,$$
$$E(\theta) = -[f^*(\theta)g(\theta) + f(\theta)g^*(\theta)] ,$$
$$G(\theta) = 2g^*(\theta)g(\theta) . \quad (5.36)$$

With these results, (5.29) becomes

$$\tfrac{1}{2}\sigma(\theta,\beta)(\mathsf{I} + \mathbf{P}'\cdot\boldsymbol{\sigma}) = \tfrac{1}{2}\{[J(\theta) + \sin\theta D(\theta)\mathbf{1}_0\cdot\mathbf{P}]\mathsf{I}$$
$$+ [J(\theta)\mathbf{P} - \sin^2\theta G(\theta)\mathbf{P} + \sin\theta D(\theta)\mathbf{1}_0$$
$$+ \sin^2\theta(\mathbf{1}_0\cdot\mathbf{P})G(\theta)\mathbf{1}_0 + \sin\theta E(\theta)(\mathbf{1}_0\times\mathbf{P})]\cdot\boldsymbol{\sigma} \quad (5.37)$$

from which we see that

$$\sigma(\theta,\beta) = J(\theta) + \sin\theta D(\theta)\mathbf{1}_0\cdot\mathbf{P} \quad (5.38a)$$

and

$$[J(\theta) + \sin\theta D(\theta)(\mathbf{1}_0\cdot\mathbf{P})]\mathbf{P}' = [J(\theta) - \sin^2\theta G(\theta)]\mathbf{P}$$
$$+ [\sin\theta D(\theta) + \sin^2\theta(\mathbf{1}_0\cdot\mathbf{P})G(\theta)]\mathbf{1}_0$$
$$+ [\sin\theta E(\theta)](\mathbf{1}_0\times\mathbf{P}) . \quad (5.38b)$$

It will be observed that if $\mathbf{P} = 0$ (originally unpolarized beam)

$$\mathbf{P}' = \sin\theta D(\theta)\mathbf{1}_0/J(\theta) , \quad (5.39)$$

and the resulting beam is polarized along the orbital-angular-momentum direction. This is Mott polarization [Mott and Massey (1949), Sec. 4.1]; $\sin\theta D(\theta)/J(\theta)$ is the usual asymmetry factor $S(\theta)$ [Sherman (1956)]. Further, when $\theta = 0$, $\mathbf{P}' = \mathbf{P}$. If $\mathbf{P}$ is not parallel to $\mathbf{1}_0$, the term $\mathbf{1}_0 \times \mathbf{P}$ shows that $\mathbf{P}'$ cannot have the same direction as $\mathbf{P}$. The polarization effects, being due to spin-orbit coupling, are a relativistic phenomenon. For nonrelativistic particles, $g(\theta) = 0 = D = E = G$, A is a multiple of the unit matrix, and $\rho' = \rho$. Other consequences of (5.37) are given by Mühlschlegel and Koppe (1958).

To apply these results to the transport equation, we replace θ by θ'' and evaluate $\mathbf{1}_0$ as follows. Let the direction of the beam before scattering at angles (θ',β') be given by $\mathbf{1}_{z'}$ and that after at angles (θ,β) be given by $\mathbf{1}_z$. In the small angle approximation, with $\mathbf{1}_t$ the original beam direction and $\boldsymbol{\theta}'$, $\boldsymbol{\theta}''$, and $\boldsymbol{\theta}$ vectors normal to $\mathbf{1}_t$, we have, to sufficient accuracy[8],

$$\left.\begin{aligned}\mathbf{1}_{z'} &= \mathbf{1}_t + \boldsymbol{\theta}' ,\\ \mathbf{1}_z &= \mathbf{1}_t + \boldsymbol{\theta} ,\end{aligned}\right\} \quad (5.40)$$

$$\theta''\mathbf{1}_0 = \mathbf{1}_{z'}\times\mathbf{1}_z = \mathbf{1}_t\times(\boldsymbol{\theta} - \boldsymbol{\theta}') = \mathbf{1}_t\times\boldsymbol{\theta}''$$
$$= \theta''\mathbf{1}_t\times\mathbf{1}_{\theta''} . \quad (5.41)$$

Now if we multiply (5.34) by $\frac{1}{2}[F(\theta,\beta)\mathsf{I} + \mathbf{\Pi}(\theta,\beta)\cdot\boldsymbol{\sigma}]$ and (5.37) by $F(\theta',\beta')$, replace θ and $\sin\theta$ in each of these equations by θ'', and use (5.41), we can write the transport Eq. (5.32) in a form suitable for analysis. After writing the two integrals in terms of

[8] Note interchange of z and z' in $\mathbf{1}_0$.

I and $\boldsymbol{\sigma}$, we can equate the coefficients of each matrix, and obtain the set of equations:

$$\frac{\partial F(\theta,\beta)}{\partial t} = N\int_0^\infty \theta' d\theta' \int_0^{2\pi} d\beta' \{J(\theta'')[F(\theta',\beta') - F(\theta,\beta)] + \theta'' D(\theta'')[\mathbf{1}_t \times \mathbf{1}_{\theta''} \cdot \mathbf{\Pi}(\theta')]\}, \quad (5.42)$$

$$\frac{\partial \mathbf{\Pi}(\theta,\beta)}{\partial t} = N\int_0^\infty \theta' d\theta' \int_0^{2\pi} d\beta' \{J(\theta'')[\mathbf{\Pi}(\theta',\beta') - \mathbf{\Pi}(\theta,\beta)] + \theta'' D(\theta'') F(\theta',\beta')[\mathbf{1}_t \times \mathbf{1}_{\theta''}] + \theta'' E(\theta'') \times [(\mathbf{1}_t \times \mathbf{1}_{\theta''}) \times \mathbf{\Pi}(\theta',\beta')] - \theta''^2 G(\theta'') \mathbf{\Pi}(\theta',\beta') + \theta''^2 G(\theta'')[(\mathbf{1}_t \times \mathbf{1}_{\theta''}) \cdot \mathbf{\Pi}(\theta',\beta')](\mathbf{1}_t \times \mathbf{1}_{\theta''})\}. \quad (5.43)$$

In deriving (5.42), and (5.43), terms arise from the scattering-out expression (5.34) that involve a function of θ times the integral over θ' and β' of a function of θ'' multiplied by a vector expression in $\mathbf{1}_{\theta''}$. Since θ' does not enter the integrand, we change $\theta' d\theta' d\beta'$ to $\theta'' d\theta'' d\beta''$ and find that for each component of $\mathbf{1}_{\theta''}$, the β'' integral is over $\cos(\beta' - \beta_0)$, where β_0 is some constant, yielding a zero result for each such term.

Now let us take the double Fourier transform of Eqs. (5.42) and (5.43), in the "polar coordinate" form (2.10b). We multiply both sides by $\theta d\theta d\beta \exp(i\boldsymbol{\xi}\cdot\boldsymbol{\theta})$ and integrate. In the combined fourfold integral, we can change variables from $(\theta,\beta,\theta',\beta')$ to $(\theta'',\beta'',\theta',\beta')$, except in the terms with $F(\theta,\beta)$ and $\mathbf{\Pi}(\theta,\beta)$ where we change to $(\theta,\beta,\theta'',\beta'')$. In the former case, we can write $\exp(i\boldsymbol{\xi}\cdot\boldsymbol{\theta}) = \exp(i\boldsymbol{\xi}\cdot\boldsymbol{\theta}' + i\boldsymbol{\xi}\cdot\boldsymbol{\theta}'') = \exp[i\xi\theta'\cos(\beta' - \alpha) + i\xi\theta''\cos(\beta'' - \alpha)]$, where α is the azimuth of the vector $\boldsymbol{\xi}$.

With the new choice of variables, the angle of $\mathbf{1}_{\theta''}$ no longer enters into θ'. We can then use the theorems (A.19) and (A.20) of Appendix II. When $\mathbf{1}_{\theta''}$ appears once in a term, the transform will have $\mathbf{1}_\alpha$ in its place, where $\mathbf{1}_\alpha$ is a unit vector in the $x - y$ plane at azimuth α. When $\mathbf{1}_{\theta''}$ appears twice, the transform will have $\mathbf{1}_\alpha$ twice, and another term with $\mathbf{1}_{\theta''}$ replaced twice by the normal $\mathbf{1}_t \times \mathbf{1}_\alpha$, or what is equivalent, $\mathbf{1}_t \times \mathbf{1}_{\theta}''$ replaced twice by $\mathbf{1}_\alpha$. We find

$$\frac{\partial \tilde{F}(\xi,\alpha)}{\partial t} = [\tilde{\jmath}(\xi) - \tilde{\jmath}(0)]\tilde{F}(\xi,\alpha) + i\tilde{d}(\xi)(\mathbf{1}_t \times \mathbf{1}_\alpha) \cdot \tilde{\mathbf{\Pi}}(\xi,\alpha) \quad (5.44)$$

$$\frac{\partial \tilde{\mathbf{\Pi}}(\xi,\alpha)}{\partial t} = [\tilde{\jmath}(\xi) - \tilde{\jmath}(0)]\tilde{\mathbf{\Pi}}(\xi,\alpha) + i\tilde{d}(\xi)(\mathbf{1}_t \times \mathbf{1}_\alpha) \times \tilde{F}(\xi,\alpha) + i\tilde{e}(\xi)[(\mathbf{1}_t \times \mathbf{1}_\alpha) \times \tilde{\mathbf{\Pi}}(\xi,\alpha)] - \tilde{g}(\xi)\tilde{\mathbf{\Pi}}(\xi,\alpha) + [\tilde{g}(\xi) - 2\tilde{h}(\xi)][\mathbf{1}_t \times \mathbf{1}_\alpha \cdot \tilde{\mathbf{\Pi}}(\xi,\alpha)] \times (\mathbf{1}_t \times \mathbf{1}_\alpha) + 2\tilde{h}(\xi)[\mathbf{1}_\alpha \cdot \tilde{\mathbf{\Pi}}(\xi,\alpha)]\mathbf{1}_\alpha, \quad (5.45)$$

where the various transforms are

$$\tilde{\jmath}(\xi) = 2\pi N \int_0^\infty \theta d\theta J(\theta) J_0(\xi\theta); \; \tilde{\jmath}(0) = \tilde{\jmath}_0,$$
$$\tilde{d}(\xi) = 2\pi N \int_0^\infty \theta d\theta \cdot \theta D(\theta) J_1(\xi\theta),$$
$$\tilde{e}(\xi) = 2\pi N \int_0^\infty \theta d\theta \cdot \theta E(\theta) J_1(\xi\theta),$$
$$\tilde{g}(\xi) = 2\pi N \int_0^\infty \theta d\theta \cdot \theta^2 G(\theta) J_0(\xi\theta),$$
$$\tilde{h}(\xi) = \tfrac{1}{2}\pi N \int_0^\infty \theta d\theta \cdot \theta^2 G(\theta)[J_0(\xi\theta) + J_2(\xi\theta)]. \quad (5.46)$$

Note that we have used $-i$ in place of i as used in Mühlschlegel and Koppe (1958).

Following the same authors, we can separate this set of four coupled equations into two independent pairs, if we choose components of $\tilde{\mathbf{\Pi}}$ along $\mathbf{1}_\alpha, \mathbf{1}_t \times \mathbf{1}_\alpha$, and $\mathbf{1}_t$, respectively. Set

$$\tilde{\Pi}_1 = \mathbf{1}_\alpha \cdot \tilde{\mathbf{\Pi}}; \; \tilde{\Pi}_2 = \mathbf{1}_t \times \mathbf{1}_\alpha \cdot \tilde{\mathbf{\Pi}}; \; \tilde{\Pi}_3 = \mathbf{1}_t \cdot \tilde{\mathbf{\Pi}}, \quad (5.47)$$

and we have

$$\partial\tilde{F}/\partial t = (\tilde{\jmath} - \tilde{\jmath}_0)\tilde{F} + i\tilde{d}\tilde{\Pi}_2,$$
$$\partial\tilde{\Pi}_2/\partial t = i\tilde{d}\tilde{F} + (\tilde{\jmath} - \tilde{\jmath}_0 - 2\tilde{h})\tilde{\Pi}_2, \quad (5.48)$$

$$\partial\tilde{\Pi}_1/\partial t = (\tilde{\jmath} - \tilde{\jmath}_0 - \tilde{g} + 2\tilde{h})\tilde{\Pi}_1 + i\tilde{e}\tilde{\Pi}_3,$$
$$\partial\tilde{\Pi}_3/\partial t = -i\tilde{e}\tilde{\Pi}_1 + (\tilde{\jmath} - \tilde{\jmath}_0 - \tilde{g})\tilde{\Pi}_3. \quad (5.49)$$

The initial conditions for these equations are determined by (2.27); $F(\theta,\beta,0) = \delta_S(\theta)$ and $\mathbf{\Pi}(\theta,\beta,0) = \delta_S(\theta)\, \mathbf{P}_0$ where $\mathbf{P}_0$ is the initial value of the polarization vector. Hence, we have

$$\tilde{F}(\xi,\alpha,0) = 1, \quad \tilde{\mathbf{\Pi}}(\xi,\alpha,0) = \mathbf{P}_0. \quad (5.50)$$

The axes chosen for $\tilde{\mathbf{\Pi}}_1$ and $\tilde{\mathbf{\Pi}}_2$ rotate with the angle α, so we must write the initial conditions in terms of the components of $\mathbf{P}_0$ with respect to fixed axes. We have then:

$$\tilde{\Pi}_1(\xi,\alpha,0) = \tilde{\Pi}_{10} = P_{0x}\cos\alpha + P_{0y}\sin\alpha,$$
$$\tilde{\Pi}_2(\xi,\alpha,0) = \tilde{\Pi}_{20} = -P_{0x}\sin\alpha + P_{0y}\cos\alpha,$$
$$\tilde{\Pi}_3(\xi,\alpha,0) = \tilde{\Pi}_{30} = P_{0z}. \quad (5.51)$$

Standard methods readily yield the solutions of the two sets of differential equations. We have

$$\tilde{F}(\xi,\alpha,t) = \left[\frac{i\tilde{d}(\xi)\tilde{\Pi}_{20} + \tilde{h}(\xi)}{(\tilde{h}^2 - \tilde{d}^2)^{1/2}} \sinh t(\tilde{h}^2 - \tilde{d}^2)^{1/2} + \cosh t(\tilde{h}^2 - \tilde{d}^2)^{1/2}\right] \exp t[\tilde{\jmath}(\xi) - \tilde{\jmath}_0 - h(\xi)], \quad (5.52)$$

$$\Pi_2(\xi,\alpha,t) = \left[\frac{i\tilde{d}(\xi) - \tilde{h}(\xi)\tilde{\Pi}_{20}}{(\tilde{h}^2 - \tilde{d}^2)^{1/2}} \sinh t(\tilde{h}^2 - \tilde{d}^2)^{1/2} + \tilde{\Pi}_{20}\cosh t(\tilde{h}^2 - \tilde{d}^2)^{1/2}\right] \exp t[\tilde{\jmath}(\xi) - \tilde{\jmath}_0 - \tilde{h}(\xi)], \quad (5.53)$$

$$\tilde{\Pi}_1(\xi,\alpha,t) = \left[\frac{i\tilde{e}(\xi)\tilde{\Pi}_{30}+\tilde{h}(\xi)\tilde{\Pi}_{10}}{(\tilde{h}^2+\tilde{e}^2)^{1/2}}\sinh t(\tilde{h}^2+\tilde{e}^2)^{1/2} + \tilde{\Pi}_{10}\cosh t(\tilde{h}^2+\tilde{e}^2)^{1/2}\right] \times \exp t[\tilde{j}(\xi)-\tilde{j}_0-\tilde{g}(\xi)+\tilde{h}(\xi)], \quad (5.54)$$

$$\tilde{\Pi}_3(\xi,\alpha,t) = \left[-\frac{(i\tilde{e}(\xi)\tilde{\Pi}_{10}+h(\xi)\tilde{\Pi}_{30})}{(\tilde{h}^2+\tilde{e}^2)^{1/2}}\sinh t(\tilde{h}^2+\tilde{e}^2)^{1/2} + \tilde{\Pi}_{30}\cosh t(\tilde{h}^2+\tilde{e}^2)^{1/2}\right] \times \exp t[\tilde{j}(\xi)-\tilde{j}_0-\tilde{g}(\xi)+\tilde{h}(\xi)]. \quad (5.55)$$

The α dependence of these functions is given by use of (5.51). Before inverting the transforms, we must express $\tilde{\mathbf{\Pi}}$ in components with respect to the fixed axes x, y, and z, namely by the combinations

$$\tilde{\Pi}_x = \tilde{\Pi}_1\cos\alpha - \tilde{\Pi}_2\sin\alpha, \quad \tilde{\Pi}_y = \tilde{\Pi}_1\sin\alpha + \tilde{\Pi}_2\cos\alpha, \quad \tilde{\Pi}_z = \tilde{\Pi}_3. \quad (5.56)$$

Making these linear combinations of (5.53), (5.54), and (5.55), using (5.51), multiplying the resulting equations, along with (5.52), by $e^{-i\xi\cdot\theta}\xi d\xi d\alpha/4\pi^2$, and integrating over α by means of Appendix II, we find

$$F(\theta,\beta,t) = \frac{1}{2\pi}\int_0^\infty \xi d\xi \exp t[\tilde{j}(\xi)-\tilde{j}_0-\tilde{h}(\xi)]\cdot\left\{\left[\frac{\tilde{h}\xi}{\tilde{\mu}(\xi)}\sinh t\tilde{\mu}(\xi)+\cosh t\tilde{\mu}(\xi)\right]J_0(\xi\theta) + [-P_{0x}\sin\beta+P_{0y}\cos\beta] \times \frac{\tilde{d}(\xi)}{\tilde{\mu}(\xi)}\sinh t\tilde{\mu}(\xi)J_1(\xi\theta)\right\}, \quad (5.57)$$

$$\begin{aligned}\Pi_x(\theta,\beta,t) = \frac{1}{2\pi}\int_0^\infty \xi d\xi \Big\{ &\tfrac{1}{2}\exp t[\tilde{j}(\xi)-\tilde{j}_0-\tilde{h}(\xi)]\left[-\frac{\tilde{h}(\xi)}{\tilde{\mu}(\xi)}\sinh t\tilde{\mu}(\xi)+\cosh t\tilde{\mu}(\xi)\right] \\ &\times [P_{0x}J_0(\xi\theta)+(P_{0x}\cos 2\beta+P_{0y}\sin 2\beta)J_2(\xi\theta)] + \tfrac{1}{2}\exp t[\tilde{j}(\xi)-\tilde{j}_0-\tilde{g}(\xi)+\tilde{h}(\xi)] \\ &\times\left[\frac{\tilde{h}(\xi)}{\tilde{\nu}(\xi)}\sinh t\tilde{\nu}(\xi)+\cosh t\tilde{\nu}(\xi)\right][P_{0x}J_0(\xi\theta)-(P_{0x}\cos 2\beta+P_{0y}\sin 2\beta)J_2(\xi\theta)] \\ &+\exp t[\tilde{j}(\xi)-\tilde{j}_0-\tilde{g}(\xi)+\tilde{h}(\xi)]\frac{\tilde{e}(\xi)}{\tilde{\nu}(\xi)}\sinh t\tilde{\nu}(\xi)P_{0z}\cos\beta J_1(\xi\theta) - \exp t[\tilde{j}(\xi)-\tilde{j}_0-\tilde{h}(\xi)] \\ &\times\frac{\tilde{d}(\xi)}{\tilde{\mu}(\xi)}\sinh t\tilde{\mu}(\xi)J_1(\xi\theta)\sin\beta\Big\}, \end{aligned} \quad (5.58)$$

$$\begin{aligned}\Pi_y(\theta,\beta,t) = \frac{1}{2\pi}\int_0^\infty \xi d\xi \Big\{ &\tfrac{1}{2}\exp t[\tilde{j}(\xi)-\tilde{j}_0-\tilde{h}(\xi)]\left[-\frac{\tilde{h}(\xi)}{\tilde{\mu}(\xi)}\sinh t\tilde{\mu}(\xi)+\cosh t\tilde{\mu}(\xi)\right] \\ &\times [P_{0y}J_0(\xi\theta)+(P_{0x}\sin 2\beta-P_{0y}\cos 2\beta)J_2(\xi\theta)] + \tfrac{1}{2}\exp t[\tilde{j}(\xi)-\tilde{j}_0-\tilde{g}(\xi)+\tilde{h}(\xi)] \\ &\times\left[\frac{\tilde{h}(\xi)}{\tilde{\nu}(\xi)}\sinh t\tilde{\nu}(\xi)+\cosh t\tilde{\nu}(\xi)\right][P_{0y}J_0(\xi\theta)-(P_{0x}\sin 2\beta-P_{0y}\cos 2\beta)J_2(\xi\theta)] \\ &+\exp t[\tilde{j}(\xi)-\tilde{j}_0-\tilde{g}(\xi)+\tilde{h}(\xi)]\frac{\tilde{e}(\xi)}{\tilde{\nu}(\xi)}\sinh t\tilde{\nu}(\xi)P_{0z}\sin\beta J_1(\xi\theta) + \exp t[\tilde{j}(\xi)-\tilde{j}_0-\tilde{h}(\xi)] \\ &\times\frac{\tilde{d}(\xi)}{\tilde{\mu}(\xi)}\sinh t\tilde{\mu}(\xi)\cos\beta J_1(\xi\theta)\Big\}, \end{aligned} \quad (5.59)$$

$$\Pi_z(\theta,\beta,t) = \frac{1}{2\pi}\int_0^\infty \xi d\xi \exp t[\tilde{j}(\xi)-\tilde{j}_0-\tilde{g}(\xi)+\tilde{h}(\xi)]\left\{-\frac{\tilde{e}(\xi)}{\tilde{\nu}(\xi)}\sinh t\tilde{\nu}(\xi)(P_{0x}\cos\beta+P_{0y}\sin\beta)J_1(\xi\theta) + \left[-\frac{\tilde{h}(\xi)}{\tilde{\nu}(\xi)}\sinh t\tilde{\nu}(\xi)+\cosh t\tilde{\nu}(\xi)\right]P_{0z}J_0(\xi\theta)\right\}. \quad (5.60)$$

In these equations, we use the abbreviations

$$\tilde{\mu}(\xi) = [\tilde{h}^2(\xi)-\tilde{d}^2(\xi)]^{1/2} \quad (5.61a)$$

$$\tilde{\nu}(\xi) = [\tilde{h}^2(\xi)+\tilde{e}^2(\xi)]^{1/2}. \quad (5.61b)$$

These results agree with those of Mühlschlegel and Koppe (1958) if the angle β is set equal to zero, and a mistake in the exponent in the last part of their formula for Π_y is corrected. Further evaluation of these results requires a knowledge of the scattering matrix A, which will be taken up later (Sec. XIV).

VI. THE SINGLE-SCATTERING LAW

A. Rutherford Law

The scattering of fast charged particles by atoms is determined by a modified form of the basic Rutherford law. This law yields, for the scattering of a nonrelativistic particle of charge ze by a bare nucleus of charge Ze into the angular range χ to $\chi + d\chi$, the cross section

$$\sigma_{\mathrm{Ru}}(\chi)2\pi \sin \chi d\chi = \frac{(2zZe^2/mv^2)^2}{[2 \sin (\chi/2)]^4} 2\pi \sin \chi d\chi , \quad (6.1)$$

where $\frac{1}{2} mv^2$ is the kinetic energy of the particle, and the nucleus is assumed to have infinite mass.

When there are $N(t)$ scattering centers per unit volume, relativistic as well as nonrelativistic particles are considered, and only small angles are involved, we can write the single-scattering probability $W(\chi,t)$ as [Mott and Massey (1949)]

$$W(\chi,t) = N(t)\sigma_{\mathrm{Ru}}(\chi) = 4N(t)\alpha^2/k^2\chi^4 , \quad (6.2)$$

where the so-called Born parameter is given by

$$\alpha = zZ/137\beta = zZe^2/\hbar v , \quad (6.3)$$

and

$$1/k = \lambda\!\!\!^{-}_0 = \hbar/p \quad (6.4)$$

is the reduced incident wavelength of the scattered particle; k is the reduced wavenumber.

This law requires modification because of several effects:

1. The effects of the screening of the nuclear Coulomb field by the atomic electrons. This leads to the most important modification, which is carried out by one or another degree of Born approximation.
2. Spin and relativity effects when Born approximations higher than the first are considered.
3. The contribution to the total scattering of scattering by the atomic electrons.
4. Effects of the finite size and the structure of the scattering nuclei.
5. Recoil effects when the scattered particle has a mass comparable to that of the scatterer.
6. Effects of crystalline structure in condensed material (and for light elements, modification of the screening resulting from close packing).
7. Effects of mixtures, included by replacing NZ^2 in (6.2) by $\sum_i N_i Z_i^2$ summed over the different atomic species, and by a corresponding replacement in the screening-correction calculation.

Considerable attention will be given in this section to the first two effects, with brief remarks at the end on the others.

B. The First Born Approximation with Exponential Screening

The simplest way in which the screening of the atomic electrons may be taken into account is by use of a pure exponential factor, yielding the so-called Wentzel (1927) or Yukawa potential

$$V(r) = \pm(zZe^2/r)e^{-\mu r/r_0} , \quad (6.5)$$

where r_0 is a screening radius which is usually taken to be the Thomas–Fermi (T–F) radius,

$$r_0 = 0.885a_0Z^{-1/3} = 0.468 \times 10^{-8}Z^{1/3}\ \mathrm{cm} \quad (6.6)$$

with a_0 the Bohr radius (m_e is the electron mass)

$$a_0 = \hbar^2/m_e e^2 = 5.292 \times 10^{-9}\ \mathrm{cm} , \quad (6.7)$$

and μ is a factor of order unity used by Nigam, Sundaresan, and Wu (1959). With this potential, and the standard first Born approximation method, one finds

$$W(\chi,t) = \frac{4N(t)\alpha^2}{k^2[2 \sin^2 (\chi/2) + \mu^2/k^2r_0^2]^2} \simeq \frac{4N(t)\alpha^2}{k^2(\chi^2 + \chi_\mu^2)^2}, \quad (6.8)$$

where the Born screening angle χ_μ is given by

$$\chi_\mu = \mu\chi_0 = \frac{\mu}{kr_0} = \frac{\mu\hbar}{pr_0} = \frac{\mu\lambda\!\!\!^{-}_0}{r_0} . \quad (6.9a)$$

Using 6.6 and 6.7, we find for χ_0

$$\chi_0 = \frac{\lambda\!\!\!^{-}_0}{r_0} = \frac{1.13}{137} Z^{1/3}(m_ec/p)\ \text{radians}$$
$$= 0.472\ Z^{1/3}(m_ec/p)\ \text{degrees} , \quad (6.9b)$$

which becomes less than 2° even for the largest values of Z when $p \simeq m_ec$, corresponding to an energy of 210 keV for electrons and $255(m_e/m)$ keV for heavier particles of mass m. We shall assume that χ_0 is always less than 4° or 1/15 radian. The function (6.8), with $\mu \equiv 1$, was used by Snyder and Scott (1949).

The result can be written as a product of the basic Rutherford formula with a screening factor $q(\chi)$:

$$W(\chi,t) = [4N(t)\alpha^2/k^2\chi^4]q(\chi) \quad (6.10)$$

with

$$q(\chi) = \chi^4/(\chi^2 + \chi_\mu^2)^2 . \quad (6.11)$$

The screening factor goes to zero as $\chi \to 0$ (small angles of scattering occur classically for passage of the scattering particle far from the nucleus where the screening is most effective) and goes to 1 for large angles where the screening effect is negligible (in Sec. IX this upper limit will be seen to differ somewhat from 1 for large α).

It is to be seen in Sec. VII that we can use (6.10) without needing (6.11), obtaining a good approximation to the multiple-scattering distribution with knowledge only of the general behavior of q as characterized in the previous paragraph, and a single parameter calculated from its actual form. It will be seen later, however, that the small-angle approximation used in (6.10) must be applied with caution for more complicated types of screening factor than (6.11).

C. Improved Screening Potentials

Molière in his detailed study of single scattering (1947) proposed a useful fit to the Thomas–Fermi function for heavy atoms. If we write the potential as

$$V(r) = \pm (zZe^2/r)\omega_M(r/r_0) \qquad (6.12)$$

and set $r/r_0 = r'$, we can write Molière's fit to ω_M as

$$\omega_M(r') = 0.10e^{-6r'} + 0.55e^{-1.2r'} + 0.35e^{-0.3r'}. \qquad (6.13a)$$

According to Molière, this expression fits the more exact T–F function within 0.002 for $0 \leq r' \leq 6$. Rozental (1935), gives a similar but different sum of three exponentials valid in the region $1 \leq r' \leq 10$, namely

$$\omega_R(r') = 0.164\, e^{-4.356r'} + 0.581\, e^{-0.947r'} + 0.255\, e^{-0.246r'}. \qquad (6.13b)$$

The exponential falloff of these expressions is more realistic than the well-known r'^{-3} asymptotic behavior of the T–F function, this latter behavior being one of the chief defects of the T–F method. Furthermore, Molière actually calculates the result of using the usual T–F function involving fractional powers of r' for a certain range of scattering angles, and shows that only a small discrepancy is made by the use of (6.13).

Nigam, Sundaresan, and Wu (1959) use the form (6.5) and adjust μ to the value 1.80 to fit experimental data for gold, and also to fit calculations using the approximate analytic wave functions of Fock and Petrashen (1935) for beryllium (see Sec. IX-D below).

Fleischman (1960) fits the Hartree–Fock calculations of Hartree and Hartree (1935a, b) for beryllium with the formula, adjusted to include exchange effects,

$$V(r) = \frac{4Ze^2}{r}\{e^{-2.45r/a_0} + 1.43(r/a_0)^2 e^{-2.00r/a_0} + 0.0010(r/a_0)^4 e^{-r/a_0}\}. \qquad (6.14)$$

D. The Dalitz Formula

Nigam, Sundaresan, and Wu (1959) use the cross section for electron scattering calculated by Dalitz (1951) in which a relativistic procedure that is correct and complete to the second power of α (second Born approximation) is used. (In particular, certain mistakes in evaluating integrals in earlier calculations were corrected by Dalitz.) The method uses the rules of Feynman and Dyson for the S matrix for a static potential of the form (6.5). The result for the screening factor is [Nigam, Sundaresan, and Wu (1959), Eq. (48)].

$$q(\chi) = \left[\frac{4\sin^2\chi/2}{\chi_\mu^2 + 4\sin^2(\chi/2)}\right]^2 \Big\{1 - \beta^2\sin^2(\chi/2) + 2\alpha\left[\left(\frac{\chi_\mu}{\sin(\chi/2)}\right)^2 + 4\right] X\tan^{-1}\chi_\mu X + \alpha\beta^2[\chi_\mu^2 + 4\sin^2(\chi/2)]\Big(\left[\frac{\chi_\mu^2}{2\sin^2(\chi/2)} - 1\right] \times X\tan^{-1}\chi_\mu X - \tfrac{1}{2}\tan^{-1}(2/\chi_\mu) + \frac{1}{2\sin(\chi/2)}\tan^{-1}\left[\frac{\sin(\chi/2)}{\chi_\mu}\right]\Big)\Big\} \qquad (6.15)$$

with the abbreviation

$$X = \sin(\chi/2)\{\chi_\mu^4 + 4[\chi_\mu^2 + \sin^2(\chi/2)]\}^{-1/2}. \qquad (6.16)$$

The small-angle approximation can be used for most of the terms in this expression, but care is needed in handling the terms with $\sin(\chi/2)$ and $\sin^2(\chi/2)$ in the numerator (see Sec. IX). It can be seen that for small α and χ, and fairly small β (6.15) reduces to (6.11). Dalitz' result has been corroborated by Mitter and Urban (1953), Lewis (1956), Kacser (1959), and Mitra (1961). [But see the different result obtained by Biswas (1952)]. Our final considerations will be based on this formula, but its limitation in using only a single exponential will have to be discussed.

E. Molière's Method for Single Scattering

Molière attempted to calculate a scattering formula valid for large α (i.e., not restricted to first Born approximation) and for large angles χ, (up to 90°). Although the validity of his results is open to some question, they are important enough to be summarized here.

Molière writes the nonrelativistic first Born approximation result in the form

$$\sigma_{\text{Born}}(\chi) = k^2\left|\int_0^\infty \rho d\rho J_0\left(2k\rho\sin\frac{\chi}{2}\right)\Phi(\rho)\right|^2 \qquad (6.17)$$

with J_0 the usual Bessel function and $\Phi(\rho)$ given by

$$\Phi(\rho) = -\frac{2}{\hbar v}\int_\rho^\infty \frac{V(r)rdr}{(r^2-\rho^2)^{1/2}} = -\frac{1}{\hbar v}\int_{-\infty}^\infty V[(x^2+\rho^2)^{1/2}]dx \qquad (6.18)$$

If (6.5) is used for $V(r)$, $\Phi(\rho)$ becomes $2\alpha K_0(\mu\rho/r_0)$, where K_0 is a Bessel function, and (6.17) yields (6.8). (Molière takes $\mu = 1$.)

Molière then calculates a cross section for arbitrary α and small χ, by a combination of WKB-type ray optics for the passage of the particle through the atom, and wave optics for the spreading of the particle's wave function between the vicinity of the scattering atom and the point of observation. The ray-optics calculation proceeds by finding the phase shift of the particle along a (nearly) straight trajectory through the atom. This shift is given by

$$\Phi(\rho) = \int_{-\infty}^{\infty} \{k_r[(x^2+\rho^2)^{1/2}] - k\}dx\,, \quad (6.19)$$

where $k_r(r)$ is the relativistic wavenumber for the particle at a distance r from the nucleus:

$$\hbar c k_r(r) = \{[E - V(r)]^2 - m^2c^4\}^{1/2}. \quad (6.20)$$

The quantity ρ is seen to be the impact parameter of the trajectory or "ray." As before, k is the initial or asymptotic value of the wavenumber.

If k_r is expanded as a series of powers of $V(r)/\hbar k\beta c$, the first-degree term yields the same expression for $\Phi(\rho)$ as Eq. (6.18). It is readily seen that this expansion of $k_r(r)$ is essentially in powers of $\alpha\chi_0$, assuming that the values of r important in the integral in (6.19) are of the order of r_0. Since $2K_0(1) = 0.84$, it is seen that $\Phi(\rho)$ when expanded to the first order in $\alpha\chi_0$, is itself of order χ.

This phase shift is used to establish the wavefunction at a plane just beyond the influence of the atomic field. A Green's function calculation is then used to find the wavefunction at a distant region of observation.

The final result is

$$\sigma(\chi) = k^2\left|\int_0^{\infty} \rho d\rho J_0(k\rho\chi)[e^{i\Phi(\rho)} - 1]\right|^2, \quad (6.21)$$

where the Bessel function arises out of the assumption of small scattering angles and the fact of observation at a point far from the scatterer in comparison to atomic radii. The first Born approximation for small angles is obtained by expanding $\Phi(\rho)$ to first order in $\alpha\chi_0$, and expanding the exponential in (6.21) to first order in α.

Let us digress at this point to show that the small-angle theory requires for its validity the general condition

$$\alpha\chi_0 \ll 1\,. \quad (6.22)$$

In the first place, when $\alpha \ll 1$, the first Born approximation is valid; by (6.11) we see that the region of values of χ for which screening effects are noticeable is that for $\chi \lesssim \chi_0$. There will be little scattering for angles appreciably greater than, say, $10\chi_0$. Hence, we must have $\chi_0 \ll 1$ if the small-angle approximation is to be valid over the entire effective range of scattering, and (6.22) must hold. (Fig. 6 shows that the effects of screening extend out to $\chi \sim 10\chi_0$ for

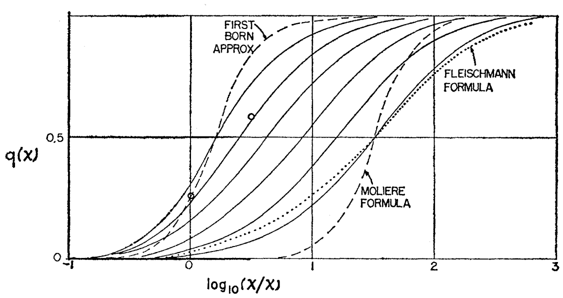

FIG. 6. The screening factor $q(\chi)$ as a function of $\log_{10}(\chi/\chi_0)$. Solid lines represent the calculations of Molière as described in the text. The two circles represent Molière's two numerical checks.

small α, and to $\chi \sim 100\chi_0$ for large α.) For general α, we can write

$$\alpha\chi_0 \simeq V(r_0)/p\beta c\,,$$

and this ratio of energies will give approximately the angle of scattering on a classical picture for particles just at the inside of the region of screening. By a similar argument to that given in the previous paragraph, we again find (6.22), which applies therefore to all cases of interest.

Using (6.3), (6.6), and (6.9), we can write $\alpha\chi_0$ as

$$\alpha\chi_0 = [(0.885mzZ^{2/3}\chi_0^2/m_e)^2 + (e^2zZ\chi_0/\hbar c)^2]^{1/2}$$

so that the condition $\alpha\chi_0 < \epsilon$ implies that the square bracket and *a fortiori* the first term is less than ϵ^2. We find an upper limit of χ_0 itself, dependent on the choice of ϵ:

$$\chi_0 < [\epsilon m_e/mzZ^{2/3}]^{1/2}\,, \quad (6.23a)$$

which constitutes an additional restriction to that given just below Eq. (6.9). Using the nonrelativistic expression for kinetic energy, we find that this upper limit on χ_0 corresponds to a lower limit on the kinetic energy of the scattered particle:

$$E \gtrsim \frac{20\,zZ^{4/3}}{\epsilon}\,\text{eV} \simeq 100\,zZ^{4/3}\,\text{eV for } \epsilon \simeq 1/5\,. \quad (6.23b)$$

In order to get a still more accurate result, Molière proceeds from the exact [Faxen–Holtsmark (1927)] phase-shift analysis. According to this method

$$\sigma(\chi) = \left| \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1)(e^{2i\delta_k} - 1)P_l(\cos\chi) \right|^2, \quad (6.24)$$

where P_l (cos x) is a Legendre function, and the phase-shift δ_l is given exactly by the integral equation [Nigam *et al.* (1959), Morse and Feshbach (1953), p. 1072]

$$\sin\delta_l = -\frac{2m}{\hbar_2} \int_0^\infty r dr \left(\frac{\pi kr}{2}\right)^{1/2} J_{l+1/2}(kr)V(r)R_l(r) , \quad (6.25)$$

where $R_l(r)$ is usually taken as satisfying the nonrelativistic wave equation

$$\left[\frac{d^2}{dr^2} + \left\{k^2 - \frac{l(l+1)}{r^2} - \frac{2mV(r)}{\hbar^2}\right\}\right] rR_l(r) = 0 \quad (6.26)$$

and behaves asymptotically like

$$R_l(r) \backsim \sin(k_r r - l\pi/2 + \delta_l)/kr \quad (6.27)$$

The first Born approximation is obtained for (6.24) and (6.25) by replacing $e^{2i\delta_l} - 1$ by $2i\delta_l$, and replacing (6.25) by

$$\delta_l = -\frac{\pi m}{\hbar^2} \int_0^\infty r dr J^2_{l+1/2}(kr)V(r) , \quad (6.28)$$

i.e., by using for R_l under the integral sign the appropriate solution for $V = 0$. Higher approximations would involve terms with δ_l under the integral sign, so that the final value for δ_l would be a series in powers of α. In accordance with the behavior of the expansion of $\Phi(\rho)$, we would expect this series to be actually in powers of $\alpha\chi_0 \simeq \alpha\mu/kr_0$.

That this is so may be made plausible by the following argument. The first Born approximation leads in the limit of no screening to the exact Rutherford cross section. Higher Born approximations will thus vanish in this limit, and the corrections they produce with screening will depend largely on the values of the integral in (6.25) for r in the screening region. If we describe the carrying out of the next approximation as roughly a matter of replacing one factor $J_{l+1/2}(kr)$ in (6.28) by $J_{l+1/2}(kr + \delta_l) \simeq J_{l+1/2}(kr) + \delta_l J'_{l+1/2}(kr)$, the correction term will be of order δ_l/kr_0 compared to (6.25), multiplied by $\alpha = zZe^2m/\hbar^2k$ which gives the order of magnitude of that integral, so we have, very roughly,

$$\delta_{l\,(\mathrm{corr})} \approx \delta_l/(1 - \alpha/kr_0) .$$

It is not hard to see from the Dalitz formula that for nonrelativistic particles ($0.1 < \beta \ll 1$) the correction terms introduced by the correct application of the second Born approximations are all of the order $\alpha\chi_0$, both for small and large χ, which constitutes an indirect verification of the above conjecture. However, the Dalitz formula shows that this conjecture cannot be correct for the relativistic region, where only for $\chi < \chi_0$ are the corrections negligible, being of order α for large χ.

Molière calculates the phase shift by an approximate, WKB-type, method that allows for a ready expansion in powers of $\alpha\chi_0$. He replaces $k^2 - 2mV(r)/\hbar^2$ in (6.26) by the relativistic value $k_r^2(r)$ given by (6.20) and rewrites the now relativistic Schrödinger equation for R_l in the form

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(l+\frac{1}{2})^2}{r^2} + k_r^2(r)\right] r^{1/2}R_l(r) = 0 . \quad (6.29)$$

Assuming the asymptotic form of $R_l(r)$ to have a phase factor $e^{i\gamma_l(r)}$, he finds by WKB methods

$$\gamma_l(r) = -\frac{\pi}{4} + \int_{r''}^{r} dr'[k_r^2(r') - (l+\tfrac{1}{2})^2/r'^2]^{1/2} \quad (6.30)$$

(r'' is the value of r' for which the integrand vanishes).

The unscattered wave, with $k_r(r)$ replaced by k, yields a Bessel function of order $l + \frac{1}{2}$ (actually $H^{(1)}_{l+1/2}$ for the outgoing wave) whose phase shift can be written [cf. Jahnke–Emde (1943), p. 140; and Jahnke–Emde–Lösch (1960), p. 148 (4.26)].

$$\gamma_l^{(0)}(r) = -\frac{\pi}{4} + \int_{(l+1/2)/k}^{r} dr'\{k^2 - (l+\tfrac{1}{2})^2/r'^2\}^{1/2} . \quad (6.31)$$

The phase shift δ_l is then given by

$$\delta_l = \lim_{r\to\infty} [\gamma_l(r) - \gamma_l^{(0)}(r)] , \quad (6.32)$$

and if k_r^2 is expanded as above in powers of $V(r)/\hbar k\beta c$, we find

$$2\delta_l = \Phi\left(\frac{l+\frac{1}{2}}{k}\right) , \quad (6.33)$$

using (6.18). Molière conjectured that this calculation should yield a quite accurate value of δ_l, and Fleischmann (1960) considered that the smallness of the correction for $\chi < \chi_0$ as indicated in the Dalitz formula is an indication of the correctness of this conjecture.

The next step in the calculation is to replace the sum in (6.24) by an integral. Molière uses the rather accurate asymptotic formula for P_l (cos χ):

$$P_l(\cos\chi) \approx (\chi/\sin\chi)^{1/2} J_0[(l+\tfrac{1}{2})\chi] , \quad (6.34)$$

which is not only very good for large l, but even for $l = 0$ involves only a 7% error at $\chi = \pi/2$.

Then the Euler sum formula is used, according to which

$$\sum_{l=0}^{\infty} f(l + \tfrac{1}{2}) = \int_0^\infty f(x)dx + \frac{1}{24} f'(0) + \cdots . \quad (6.35)$$

The result obtained is (setting $x = k\rho$)

$$\sigma(\chi) = k^2(\chi/\sin \chi)\left|\int_0^\infty \rho d\rho J_0(k\chi\rho)[e^{i\Phi(\rho)} - 1]\right|^2 . \quad (6.36)$$

Molière makes one further correction to this formula, with a view toward improving its behavior at large angles. He observes that for χ large compared to the screening range, but small in the absolute sense, and for small α, the small-angle Rutherford result should ensue, so that the angular behavior of (6.36) is $\sigma(\chi) \sim \chi^{-4}$. For angles wherein $\chi/\sin \chi$ is appreciably different from 1, i.e., when k is small, the integral will behave as χ^{-4}, and the overall result will be proportional to

$$(\chi^3 \sin \chi)^{-1} \simeq \chi^{-4}\left(1 + \frac{\chi^2}{6} + \frac{7\chi^4}{360} + \cdots\right) . \quad (6.37)$$

The exact Rutherford result can be written as proportional to

$$(2 \sin \chi/2)^4 \simeq \chi^{-4}\left(1 + \frac{\chi^2}{6} + \frac{11}{720}\chi^4 + \cdots\right), \quad (6.38)$$

which agrees within 0.5% up to 60° with (6.37). Hence, in order to obtain a result that agrees as it should with the Rutherford result to 90° and beyond, Molière replaces $(\chi/\sin \chi)$ with $\chi^4/(2 \sin \chi/2)^4$ and writes

$$q(\chi) = \frac{(k\chi)^4}{4\alpha^2}\left|\int_0^\infty \rho d\rho J_0(k\chi\rho)[e^{i\Phi(\rho)} - 1]\right|^2 \quad (6.39)$$

where as defined in (6.10), $q(\chi)$ is the ratio $\sigma(\chi)/\sigma_{\mathrm{Ruth}}(\chi)$.

The problems of evaluating multiple-scattering results at angles for which this correction is important are complex and difficult (involving especially the "detour factor") and are beyond the scope of this article. Hence, we shall not try to distinguish (6.21), (6.36), and (6.39).

F. Molière's Scattering Formula

To evaluate (6.39) with (6.12) and (6.13a), we need to calculate $\Phi(\rho)$, using Eq. (6.18) which as indicated below (6.20), is good to terms of order $\alpha\chi_0$. We have, in fact,

$$\Phi(\rho) = \mp\frac{2zZe^2}{\hbar v}\int_\rho^\infty \frac{dr\omega(r/r_0)}{(r^2 - \rho^2)^{1/2}}$$
$$= \mp 2\alpha\int_y^\infty \frac{dr'\omega(r')}{(r'^2 - y^2)^{1/2}} \; ; y = \frac{\rho}{r_0} . \quad (6.40)$$

Using (6.13) and the integral [13.2(17)] of Erdelyi *et al.* (T. I. T., 1954), [see also Watson (1952), p. 170] we have

$$\Phi(\rho) = \mp 2\alpha[0.10K_0(6y) + 0.55K_0(1.2y) + 0.35K_0(0.3y)] , \quad (6.41)$$

where K_0 is the usual Bessel function of the third kind:

$$K_0(z) = \tfrac{1}{2}\pi i H_0^{(1)}(iz) . \quad (6.42a)$$

For small z, we have

$$K_0(z) \simeq -\ln(\gamma z/2) + \tfrac{1}{4}z^2[1 - \ln(\gamma z/2)] + \cdots , \quad (6.42b)$$

and for large z,

$$K_0(z) \approx (\pi/2z)^{1/2}e^{-z}(1 - 1/8z + \cdots) , \quad (6.42c)$$

where $\ln \gamma$ (often denoted by γ) is Euler's constant

$$\ln \gamma = 0.5772 \; ; \gamma = 1.7811 . \quad (6.43)$$

The derivatives of $K_0(z)$ are given by

$$K_0'(z) = -K_1(z) = \tfrac{1}{2}\pi H_1^{(1)}(iz) , \quad (6.44a)$$
$$K_0''(z) = K_0(z) + K_1(z)/z . \quad (6.44b)$$

The functions K_0 and K_1 (or $H_0^{(1)}$ and $H_1^{(1)}$) are tabulated in Jahnke–Emde (1943), Jahnke–Emde–Lösch (1960), and Watson (1952).

For sufficiently small α, exp $[i\Phi(\rho)]$ can be replaced by $1 + i\Phi(\rho)$ in (6.39). Using entry [8.13(2)] of Erdelyi *et al.* (T. I. T. 1954), we find, using (6.9) with $\mu = 1$, that the limiting value of $q(\chi)$ as $\alpha \to 0$ is

$$q(\chi) = \left(\frac{\chi}{\chi_0}\right)^4\left[\frac{0.1}{(\chi/\chi_0)^2 + 36} + \frac{0.55}{(\chi/\chi_0)^2 + 1.44} + \frac{0.35}{(\chi/\chi_0)^2 + 0.09}\right] . \quad (6.45)$$

When χ/χ_0 is large—that is, considerably larger than 6, we can write (6.45) as

$$[0.10(1 - 36\chi_0^2/\chi^2 + \cdots) + 0.55(1 - 1.44\chi_0^2/\chi^2 + \cdots) + 0.35(1 - 0.09\chi_0^2/\chi^2 + \cdots)]^2$$
$$= [1 - 4.424\chi_0^2/\chi^2 + \cdots]^2 \simeq 1 - 8.85\chi_0^2/\chi^2 . \quad (6.46)$$

[The number 8.85 would be replaced by 7.32 if the expression (6.13b) were used].

For large values of α, we cannot expand the exponential in (6.39). In fact, this exponential function oscillates rapidly as ρ varies with α large. The Bessel function also oscillates, and, in fact, the term with -1 contributes nothing for $\chi \to 0$ since it yields a (two-dimensional) delta function. Strictly, its effect appears for χ near zero, for which large values of ρ are important, $\Phi \to 0$[cf. Eq. (6.42c)], and exp $[i\Phi(\rho)] \to 1$. Hence, we shall assume the χ is not too near zero.[9]

The product of two oscillating functions will only make a substantial contribution to the integral if the oscillations are approximately in phase. Since one "half-cycle" of the Bessel function occurs for a change $\Delta\rho$ in ρ given by $k\chi\Delta\rho \simeq \pi$, we must have $|\Delta\Phi(\rho)| \simeq |\Delta\rho\Phi'(\rho)| \sim \pi$ or

$$k\chi \simeq |\Phi'(\rho)| \,. \tag{6.47}$$

Since $|\Phi|$ behaves for small ρ as $-2\alpha \ln \rho$, $\Phi'(\rho) = d\Phi/d\rho$ varies inversely with ρ and (6.47) gives the value of ρ at which the oscillations are most in phase. For values of ρ that make J_0 oscillate, we expect that we can use the asymptotic expression for J_0

$$J_0(k\chi\rho) \approx (\pi k\chi\rho/2)^{-1/2} \cos(k\chi\rho - \pi/4) \,. \tag{6.48}$$

In fact, from (6.47) we see that $k\chi\rho \simeq |\rho d\Phi/d\rho| \simeq 2\alpha$ and if $\alpha \gg 1$, we are in the region where the asymptotic expression is valid.[10] Thus, we write

$$\int_0^\infty \rho d\rho J_0(k\chi\rho)[e^{i\Phi(\rho)} - 1] \simeq (2\pi k\chi)^{-1/2} \int_0^\infty \rho^{1/2} d\rho \times \{\exp(i[\Phi(\rho) + k\chi\rho - \pi/4]) + \exp(i[\Phi(\rho) - k\chi\rho + \pi/4])\} \tag{6.49}$$

The phase agreement involves a particular maximum in the integrand, where the phase of one of the two terms is stationary. If Φ is positive, (and $d\Phi/d\rho$ negative), it will be the first term, and the second term will not contribute appreciably to the integral. In saddle-point calculus fashion, we expand the contributing phase about its extreme value.

Specifically, we determine ρ_0 by the equation

$$k\chi = |\Phi'(\rho_0)| \tag{6.50}$$

and write for the exponent in the contributing term

$$i[\Phi(\rho_0) \pm k\chi\rho_0 \mp \pi/4 + \tfrac{1}{2}(\rho - \rho_0)^2\Phi''(\rho_0)] \,.$$

The $\rho^{1/2}$ we replace by $\rho_0^{1/2}$, which will make little error if ρ is large enough.[11]

When a complex function such as the exponential we are considering has an extreme value, it will show saddle-point behavior in the neighborhood. The method of steepest descents for calculating an integral through such a point involves distorting the integration path so that the point is a maximum for the new path. In this case, we let $\rho - \rho_0 = \pm i^{1/2}x$ where the sign corresponds to that of Φ and Φ''. The integral to be evaluated is then

$$\int_{-\infty}^{\infty} dx \exp\{-\tfrac{1}{2}x^2|\Phi''(\rho_0)|\} = \{2\pi/|\Phi''(\rho_0)|\}^{1/2}$$

where further approximation enters in assuming that this expression can be integrated between the indicated limits.

We finally arrive at the result for large α

$$q(\chi) = \rho_0 k^3\chi^3/4\alpha^2|\Phi''(\rho_0)| \,. \tag{6.51}$$

This expression may be evaluated by choosing a set of values of $y_0 = \rho_0/r_0$ which with (6.50) and (6.44a) lead to a set of values of $k\chi r_0 = \chi/\chi_0$ and with (6.51) and (6.44b) to the corresponding values of $q(\chi)$.

For large values of χ/χ_0, as we have indicated before, only small values of ρ are important in the integral. Let us then consider the expansion of (6.41) for small $y = \rho/r_0$. Using (6.42b), we find

$$\Phi(\rho) = \mp\alpha[0.516 - 2\ln y - 0.81y^2 - 2.21y^2 \ln y + \cdots] \,. \tag{6.52}$$

Using the prescription just given for large α, with small y_0, we can take, using (6.52) and (6.50),

$$\chi/\alpha\chi_0 \simeq 2/y_0$$

and, keeping only terms in $\ln y$ and $1/y^2$,

$$q(\chi) \simeq 1 + y_0^2(9.86 + 8.85 \ln y_0) \simeq 1 - [35.4\alpha^2/(\chi/\chi_0)^2]\left[\ln\left(\frac{\chi}{2\chi_0}\right) - \ln\alpha - 1.12\right]. \tag{6.53}$$

For general α and large χ/χ_0, we again throw away the -1 in the bracket of (6.39) and expand that part of exp $\{i\Phi(\rho)\}$ that does not involve $0.516 - 2 \ln y$.

[9] If we assume $\alpha = 10$ and take $\Phi(\rho) = 2\alpha k_0(\rho/r_0)$, then inspection of (6.42c) will show that oscillations are unimportant for $\rho/r_0 > 3$, or $k\chi\rho > 3\chi/\chi_0$. This range of ρ will be important in the integration if $3\chi/\chi_0$ is considerably less than 1. Hence we shall assume $\chi > \chi_0/3$.

[10] Even for $\alpha = 1$, it is not too bad. The first two roots of J_0 occur at 2.405 and 5.520, with a minimum at 3.83 of value -0.4028. The asymptotic expression gives roots at 2.36 and 5.52, with a minimum at 3.79 of value -0.406.

[11] Saddle-point calculations showing the relative effect of ignoring more drastic variations than this one are reported by Scott and Uhlenbeck (1942).

Molière carries this only to first order in α. Using the integration variable $x = k\rho\chi = y\chi/\chi_0$, we obtain

$$q(\chi) = \frac{1}{4\alpha^2}\left|\int_0^\infty x^{1-2i\alpha}dxJ_0(x) \times\left[1-\frac{i\alpha x^2}{(\chi/\chi_0)^2}\left(0.81+2.21\ln\frac{x\chi_0}{\chi}\right)\right]\right|^2. \quad (6.54)$$

This integral is not convergent for real α, but if we temporarily allow the condition $-1 < \mathrm{Re}\,(1-2i\alpha) < 0$, we can use formulas [8.5(7)] and [8.6(25)] of Erdelyi *et al.* (T. I. T. 1954) together with the identity

$$\left|\frac{\Gamma(1-i\alpha)}{\Gamma(i\alpha)}\right|^2 = \frac{\Gamma(1-i\alpha)\Gamma(1+i\alpha)}{\Gamma(-i\alpha)\Gamma(i\alpha)} = (-i\alpha)(i\alpha) = \alpha^2$$

and find

$$q(\chi) = \left|1+\frac{4i\alpha(1-i\alpha)^2}{(\chi/\chi_0)^2}\left\{0.81+2.21\left[\tfrac{1}{2}\psi(i\alpha)+\tfrac{1}{2}\psi(-i\alpha)+\frac{1}{1-i\alpha}-\frac{1}{2i\alpha}-\ln\frac{\chi}{2\chi_0}\right]\right\}\right|^2. \quad (6.55)$$

The negative imaginary part of α can now be allowed to go to zero, by analytic continuation.

Molière has fitted a simple formula to the sum of ψ functions in (6.55):

$$\tfrac{1}{2}\psi(i\alpha)+\tfrac{1}{2}\psi(-i\alpha) = \mathrm{Re}\,\psi(i\alpha) \simeq \tfrac{1}{4}\ln\left(\alpha+\frac{\alpha^2}{3}+0.13\right). \quad (6.56)$$

Using this expression in (6.55), and expanding with neglect of higher orders in α^2 and $(\chi/\chi_0)^{-2}$, we obtain

$$q(\chi) \simeq 1-\frac{8.85}{(\chi/\chi_0)^2} \times\left[1+\alpha^2\ln\frac{7.1\times10^{-4}(\chi/\chi_0)^4}{(\alpha^4+\frac{1}{3}\alpha^2+0.13)}\right], \quad (6.57)$$

where 7.1×10^{-4} signifies $2^{-4}e^{-4.48}$. This expression reduces to (6.53) for large α, and to (6.45) for $\alpha = 0$.

Nigam, Sundaresan, and Wu (1959) have criticized this development on two grounds. They suggest, incorrectly, that failure to expand $x^{-2i\alpha}$ in powers of α leads to inconsistencies to order of α^2. As we have seen above, this factor is kept in this form to provide convergence, and a different method of including it would still not change the expansion to this order.[12]

[12] Fleischmann (1960) points out that the errors made by Molière in this step of the approximation at the most amount only to a few percent.

On the other hand, they are right in saying that the first Born approximation has been used for $\Phi(\rho)$, omitting terms of higher order in α than the first, and that therefore the term in α^2 in (6.57), and the absence of a term in α, cannot be correct.

Molière has calculated q for all χ for $\alpha = 0$ by (6.45) and for $\alpha = 9.6$ by (6.50) and (6.51), and for large χ/χ_0 by (6.57). From these results, he devised an interpolation scheme, based on a linear relation between $(\chi/\chi_0)^2$ and α^2 for fixed q. The results were tested by direct numerical integration of (6.39) with $\alpha = 0.6$ and $(\chi/\chi_0)^2 = 1$ and 10, and the error in χ for a given q is estimated to be less than 10%.

The results are given in Table II, for the coefficients in the relation

$$(\chi/\chi_0)^2 = A_q+\alpha^2 B_q\,. \quad (6.58)$$

TABLE II. Coefficients for Eq. (6.58)

q	A_q	B_q	q	A_q	B_q
0.05	0.102	0.059	0.5	2.75	10.85
0.1	0.209	0.214	0.6	4.68	22.8
0.2	0.525	0.891	0.7	8.71	50.8
0.3	0.977	2.31	0.8	19.5	128.8
0.4	1.675	5.20	0.9	61.7	421

Figure 6, from Molière, shows the various results indicated above.

It will be seen in the next section that the only property of the single-scattering law that is needed for Molière's method of calculating multiple scattering is the Molière screening angle χ_α defined by

$$\ln\chi_\alpha = -\tfrac{1}{2}-\lim_{\chi_m\to\infty}\left\{\int_0^{\chi_m}d\chi\frac{q(\chi)}{\chi}-\ln\chi_m\right\} \quad (6.59a)$$

$$= -\tfrac{1}{2}-\lim_{\chi_m\to\infty}\left\{\left[q(\chi)\ln\left(\frac{\chi}{\chi_0}\right)\right]_0^{\chi_m} -\int_0^{q(\chi_m)}dq\ln\left(\frac{\chi}{\chi_0}\right)-\ln\chi_m\right\} \quad (6.59b)$$

$$= -\tfrac{1}{2}+\ln\chi_0+\int_0^1 dq\ln\left(\frac{\chi}{\chi_0}\right),$$

where we have used the fact that $q(0) = 0$ and

$$\lim_{\chi_m\to\infty} q(\chi_m) = 1\,.$$

This screening angle was used in Eqs. (4.13), (4.15), and (4.16).

Equation (6.59a) can be evaluated directly for $\alpha = 0$, using (6.45). The result is

$$\ln\chi_\alpha = \ln\chi_0+0.0793\,;$$

$$\chi_\alpha = 1.0825\chi_0 = (1.174)^{1/2}\chi_0\,,\quad \text{for } \alpha = 0\,. \quad (6.60a)$$

If we use (6.58) for $\alpha = 0$, and perform a numerical integration, assuming the area between the curve of $\ln A(q)$ and the $q = 1.0$ axis above the value of $\ln A$ for $q = 0.9$ to be equal to that between the curve and the $q = 0$ axis below the value for $q = 0.1$, we find

$$\ln \chi_\alpha = \ln \chi_0 + 0.0679 ;$$
$$\chi_\alpha = 1.0700\chi_0 = (1.145)^{1/2}\chi_0 , \quad \text{for } \alpha = 0 . \quad (6.60b)$$

For large α, substitution of (6.50) and (6.51) in (6.59a) yields (the use of $\chi = 0$ as a lower limit in place of $\chi_0/3$ as suggested earlier will make a very small error)

$$\int_0^{\chi_m} d\chi \frac{q(\chi)}{\chi} - \ln \chi_m = \frac{1}{4\alpha^2}\int_{\rho m}^{\infty} \rho d\rho \Phi'^2(\rho) - \ln |\Phi'(\rho_m)| + \ln k ,$$

where ρ_m is the value of ρ corresponding to χ_m. As $\chi_m \to \infty$, $|\Phi'(\rho)| \to 2\alpha/\rho = 2\alpha/yr_0$ by (6.52), so that we get [using (6.41) and (6.44a)]

$$\ln \chi_\alpha = -\tfrac{1}{2} + \ln 2\alpha\chi_0 - \lim_{ym\to 0}\left\{\tfrac{1}{4}\int_{ym}^{\infty} ydy[1.2K_1(6y) + 1.32K_1(1.2y) + 0.21K_1(0.3y)]^2 + \ln y_m\right\} .$$

The Bessel-function integrals can be done in closed form [Watson (1952), p. 134],[13] yielding a result that differs from the $\alpha = 0$ result by exactly $\ln \alpha\gamma$

$$\ln \chi_\alpha = \ln \chi_0 + 0.0793 + \ln \alpha\gamma ;$$
$$\chi_\alpha = 1.93\alpha\chi_0 = \chi_0(3.72\alpha^2)^{1/2} . \quad (6.61a)$$

If in Eq. 6.58, we neglect $A(q)$, which is justifiable for $\alpha > 2$, we can again perform a single numerical integration and obtain

$$\ln \chi_\alpha = \ln \alpha\chi_0 + 0.6729 ;$$
$$\chi_\alpha = 1.96\alpha\chi_0 = \chi_0(3.84\alpha^2)^{1/2} . \quad (6.61b)$$

Assuming a linear relation between χ_α^2 and α^2, Molière writes the following interpolating formula based on the above information

$$\chi_\alpha^2 = \chi_0^2(1.13 + 3.76\alpha^2) . \quad (6.62)$$

In view of the two-figure accuracy in the initial Fermi–Thomas function (6.13), the uncertain character of that function, and Molière's incorrect method of including the α^2 terms, the numerical discrepancies among (6.60), (6.61), and (6.62) are completely negligible. Furthermore, as we shall see in the next section, the multiple-scattering distribution is insensitive to the exact value of χ_α. (See, for example, the relatively small effects of changing χ_α by the factor 1.80/1.12 in the calculations in Table IX.)

Molière has also proposed a simple functional form for $q(\chi)$, namely a form similar to (6.11)

$$q(\chi) = \chi^4/(\chi^2 + \chi_\alpha^2)^2 , \quad (6.63)$$

which satisfies (6.59a). However, Fleischmann (1960) has suggested that a better fit, especially for large α, can be made with

$$q(\chi) = \chi/(\chi + \chi_1) ; \quad \chi_1 = \chi_\alpha e^{1/2} \quad (6.64)$$

(cf. Fig. 6). This latter form will be useful for one method of estimating higher-order corrections to the Molière theory of the next section.

We have indicated in reference to the Dalitz formula that for nonrelativistic scattering, and also for small α in general, the first Born approximation result (6.11) appears to be a good approximation. To compensate the use of only a single exponential in $V(r)$, Nigam, Sundaresan, and Wu (1959) propose to introduce the coefficient μ in (6.5) and (6.9), which leads to a result not greatly different from that obtained by use of Molière's formula (6.63) with a different determination of χ_α than that given by (6.62). In fact, χ_α is in all cases very close to χ_μ as long as $\alpha\chi_0 \ll 1$ [see Eq. (9.19) below]. Nigam *et al.* quote the result of a numerical integration, based on work of Goudsmit and Saunderson (1940a,b) and Mott and Massey (1949, pp. 188–90, 196–8), which yields

$$\chi_\alpha \simeq \chi_\mu \simeq 1.12\chi_0 ; \quad \mu = 1.12 , \quad (6.65a)$$

or, to compare with (6.60),

$$\ln \chi_\alpha \simeq \ln \chi_0 + 0.113 . \quad (6.65b)$$

The justification of Molière's conjecture that his calculation method is essentially correct is upheld [Fleischmann (1960)] by the smallness of the correction terms in Dalitz' formula, but invalidated by the use of only the first Born term in the exponent. Further discussion of the relative merits of the two results will have to be postponed to Sec. IX.

G. Special Calculations for Beryllium

Because of discrepancies between theory and experiment for beryllium [Hanson, Lanzl, Lyman, and Scott (1951)] special calculations for this case have been made by Nigam, Sundaresan, and Wu (1959) and Fleischmann (1960). The former authors use the analytic functions of Fock and Petrashen (1935) which allows a new first Born approximation

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[13] The reader may verify from (6.44) that $(b^2 - a^2) \int zdzK_1(az)K_1(bz) = z[aK_0(az)K_1(bz) - bK_1(az)K_0(bz)]$, and by a limiting process may find the result for $a = b$.

with its value of χ_0 to be evaluated. The resulting value of μ was 2.18, larger than, but not in serious disagreement with, these authors' empirical value 1.80 (cf. Sec. IX-D).

Fleischmann, on the other hand, uses the Hartree–Fock formula given in (6.14) with the small-α development that let up to (6.45). The calculation involves similar but much lengthier operations with

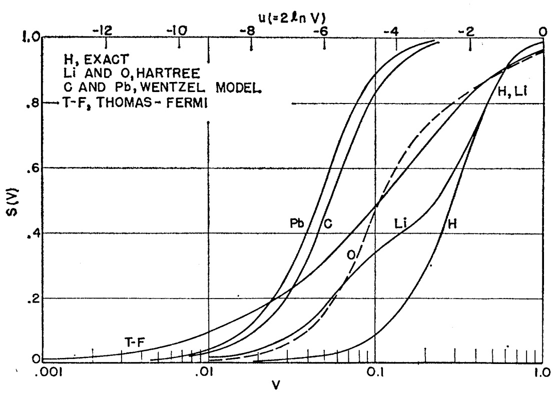


FIG. 7. Incoherent scattering function $S(v)$, for H, Li, C, O, and Pb [Grodstein (1957), Fig. 6].

Bessel functions, which lead to an algebraic function for $q(\chi)$ of considerable complexity. The resulting value of χ_α is $\chi_\alpha = (1.13 \pm 0.01)\chi_0$ which compares favorably with the values in (6.60b) and (6.65a). Fleischmann then estimates the effect of the modification of the screening by virtue of the close approach of beryllium nuclei in the crystalline state as modifying the value of $\chi_\alpha \simeq \chi_\mu$ to $1.6\chi_0$, which also agrees reasonably with the empirical value of $1.8\chi_0$.

H. Corrections for Scattering by Atomic Electrons

The scattering of fast particles, electrons or others, by the electrons of the scattering material has been treated by Fano (1954). For sufficiently large angles, the scattering by each electron is given by the Rutherford formula with $Z = 1$. Since there are Z electrons per atom and each scatters independently of the others in the energy range we are considering, the effect will be that a term of order Z is to be added to that of order Z^2 as given by 6.1—i.e., we should replace Z^2 by $Z(Z+1)$.

However, this does not hold down to the smallest angles, for the cross section goes to a constant value at zero angle instead of increasing indefinitely. The effect is given by Fano in terms of multiplying the Rutherford cross section by $1/Z$ times the incoherent scattering function $S(v_{1,2})$ where $v_{1,2}$ is a convenient parameter related to the angle of scattering, and S itself is defined as $(1/Z)$ times a sum over all electron states of the generalized atomic form factor. Grodstein (1957) has given a survey of available calculations of $S(v_{1,2})$; the results are given in Figs. 7 and 8. Just as for Molière's $q(\chi)$, S approaches zero for small χ and unity for large χ.

For electron–electron scattering, we use the variable v_1 given by

$$v_1 = (3\pi/256)^{1/3}Z^{-2/3}|\mathbf{q}|a_0/\hbar \simeq 0.333Z^{-2/3}p\chi a_0/\hbar$$
$$= 0.333Z^{-2/3}\chi a_0/\chi_0 = 0.376Z^{-2/3}\mu(\chi/\chi_0)\,, \qquad (6.66)$$

where $\mathbf{q}$ is the momentum transferred to the scattering electron; for small angles, $|\mathbf{q}|$ is equal to $p\chi$, to a good approximation. (Strictly, $|\mathbf{q}|^2 = (\epsilon/\beta c)^2 + p^2\chi^2$, where ϵ is the energy loss and βc the speed of the scattered particle. The discrepancy is only important in the region where screening effects reduce the scattering anyway.) Most of the variation of S occurs for values of v_1 between 0.1 and 1.0, so that S becomes important for angles smaller than approximately $3Z^{1/3}\chi_0$.

The effect of electron–electron scattering is thus included in the total by replacing $q(\chi)$ as given above by

$$q_{el}(\chi) = q(\chi) + Z^{-1}S(v_1)\,. \qquad (6.67)$$

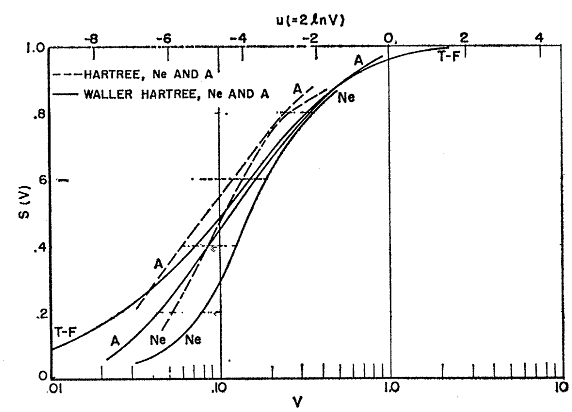


FIG. 8. Incoherent scattering function $S(v)$ for Ne and A [Grodstein (1957), Fig. 7].

It will be noted that $q_{el}(\chi) \to 1 + Z^{-1} = (Z+1)/Z$ as $\chi \to \infty$.

For heavy incident particles, a simple correction to $q(\chi)$ is not possible, because the incoherent scattering no longer resembles Rutherford scattering. The recoiling electron may take a substantial amount of energy E_e, which varies from a minimum value, essentially zero (low-energy recoils contribute negligibly to multiple scattering), to a maximum value given by

$$E_{e\,\max} = \frac{2m_e\beta^2c^2E^2}{m^2c^4 + 2m_ec^2E} \simeq \frac{2m_e\beta^2c^2}{1-\beta^2} = \frac{2m_ep^2}{m^2}, \quad (6.68)$$

where E is the total energy (including rest-energy), p is the momentum and βc the velocity, of the incident particle, and the approximation assumes that $E/mc^2 \ll m/m_e$. The electron recoil energy and the angle of scattering are related to good approximation by

$$\chi^2 = \frac{2m_eE_e}{p^2}\left[1 - \frac{E_e}{E_{e\,\max}S(v_2)}\right], \quad (6.69)$$

where the argument v_2 of the incoherent scattering function is given by

$$v_2 = 0.333Z^{-2/3}(a_0/\hbar)(2m_eE_e)^{1/2}. \quad (6.70)$$

For this type of scattering, there is an upper limit for χ^2, which can be estimated from (6.69) by assuming that $S(v_2)$ is constant. Then the bracket is $\frac{1}{2}$ for the maximum value of χ^2, and we have

$$\chi^2_{\max} \simeq m_eE_{e\,\max}S(v_2)/2p^2 \approx (m_e/m)^2. \quad (6.71)$$

Unless $p/me \lesssim 1/137$ and the incident kinetic energy is therefore less than $mc^2/2(137)^2$, we have $\chi_{\max}{}^2 \gg \chi_0^2$. [See Eq. (6.9b).]

The scattering cross section is most conveniently written in terms of the variable E_e [which is a double-valued function of χ according to (6.69)]. Fano gives the formula corresponding to (6.10) as

$$W_{el,h}(\chi)\chi d\chi = \frac{4N(t)\alpha^2}{k^2}\frac{S(v_2)}{Z}\frac{p^2}{4m_eE_e^2}\left(1 - \frac{E_e\beta^2}{E_{e\,\max}}\right)dE_e. \quad (6.72)$$

A method of including this cross section in multiple-scattering theory will be given after Molière's calculation is presented in Sec. VII.

I. Other Corrections

i. Nuclear size effects. These become important for scattering angles of the order of $\lambda\!\!\!^{-}_0/r_N$, and larger, where r_N is an appropriate nuclear radius. Since r_0 is at least 10^3 times r_N, the angles at which nuclear effects are important are very much larger than those where screening is effective. Nuclear size effects can be described by multiplying the already-modified Rutherford cross section by a nuclear form factor $\mathfrak{F}_N(\chi)$, which goes to 1 for angles considerably less than χ_0/r_N, and to some small value for large χ. If we can assume that for very large angles at high energy, the incident particle passes through the nucleus and is scattered independently by Z point protons, the limit of $\mathfrak{F}_N(\chi)$ will be just $1/Z$, by analogy with the electron–electron scattering. If the finite extension of each proton is taken into account then $\mathfrak{F}_N(\chi)$ may hover around $1/Z$ for intermediate values, but will go to zero in the limit. A survey of available calculations of this effect is given in Sec. XV.

ii. Correction for massive scattered particles. If the mass of the scattered particle is comparable to the mass of the scattering nucleus, recoil of the latter must be considered [Bethe and Ashkin (1953)]. The effect is negligible for very small angles, and may be treated along with $\mathfrak{F}_N(\chi)$ for larger angles (see Sec. XV).

iii. Crystal-structure effects. The effects of crystal structure will become important when the crystalline or microcrystalline character of the scattering material is such that there is an appreciable probability of two or more coherent scatterings with one crystal. If this is so, a crystalline form factor similar to those used in electron-diffraction theory must be included. In fact, if we assume the material to be composed of numerous very small randomly oriented identical crystals, each containing N atoms, the usual X-ray and electron-diffraction structure factor becomes, in small-angle approximation,

$$S_{\rm cryst}(\chi) = 1 + \frac{1}{N}\sum_{i,j}\frac{\sin(\chi r_{ij}/\lambda\!\!\!^{-}_0)}{\chi r_{ij}/\lambda\!\!\!^{-}_0}. \quad (6.73)$$

This is the type of factor used in X-ray powder pattern photographs. No adequate application to the case of fast charged particles has yet been made.[14] However, if the individual crystals of the scattering material are sufficiently microscopic and random, its neglect is probably quite well justified. Reference to two experiments in which small crystal-structure effects may have been observed is given in Sec. XI [Lenz (1954)].

iv. The effects of mixtures and energy loss. These effects are readily included by suitable averages over the different atomic species that may by present, and suitable integrals along the path involving the variable momentum of the particle. They are taken up in the next section.

VII. THE MOLIÈRE CALCULATION

A. The Transform

We proceed to calculate the transform $\exp[\Omega(\xi,t) - \Omega_0]$ as given in Eq. (2.37), using the form (6.10)

[14] See, however, the discussion by Hoerni, (1956 a,b). Several references are given there to other work on electron diffraction. Shinohara (1949) has made some calculations of this effect by the Williams method.

for $W(\chi,t)$, but assuming that $q(\chi)$ is given by Molière's result as given in the last section, modified if necessary because of the incorrectness of his expansion in powers of α. The correction (6.67) is assumed to be made if the scattering of electrons is under consideration.

If a mixture of scatterers is present, (6.10) must be summed over the different atomic species. In general, we have for the t' integral in (2.37b) to use

$$\int_0^t dt' W(\chi,t') = \frac{4}{\chi^4} \times \int_0^t dt' \frac{\sum_i N_i(t')\alpha_i^2(t')q_i(\chi,t')}{k^2(t')}, \quad (7.1)$$

since the density, the value of α (in its dependence on β), the value of q (in its dependence on α and β) and the value of k (directly dependent on the energy), may vary both with the atomic species (denoted by the subscript i) and the distance t' along the path. Each $q_i(\chi,t')$ will be essentially unity for χ greater than some χ_{0i} (except for electron scattering, when it is $(Z+1)/Z$). For χ greater than the greatest of these, say χ_{om}, we have

$$\int_0^t dt' W(\chi,t') \approx \frac{4}{\chi^4} \times \int_0^t dt' \frac{\sum_i N_i(t')\alpha_i^2(t')}{k^2(t')}; \quad \chi > \chi_{0m}, \quad (7.2a)$$

which we write in simple form as

$$2\pi\chi d\chi \overline{W}(\chi)t = (2\chi_c^2/\chi^3)d\chi; \quad \chi > \chi_{0m}, \quad (7.2b)$$

where the average W is given by

$$t\overline{W}(\chi) = \int_0^t W(\chi,t')dt' \quad (7.3)$$

and the characteristic angle χ_c is defined by

$$\chi_c^2 = 4\pi \int dt' \frac{\sum_i N_i(t')\alpha_i^2(t')}{k^2(t')} = 4\pi e^4 z^2 \int_0^t dt' \frac{\sum_i N_i Z_i^2}{p^2 v^2}. \quad (7.4a)$$

For electron scattering, we introduce the factor $(Z_i+1)/Z_i$ and obtain

$$\chi_c^2 = 4\pi e^4 z^2 \int_0^t \frac{dt'}{p^2v^2} \sum_i N_i(t) Z_i(Z_i+1). \quad (7.4b)$$

For a homogeneous scatterer with no energy loss, we have

$$\chi_c^2 = 4\pi e^4 z^2 Z^2 Nt/p^2v^2 \text{ radians}^2 \text{ (heavy particles)}$$
$$= 4\pi e^4 z^2 Z(Z+1)Nt/p^2v^2 \text{ radians}^2 \text{ (electrons)}. \quad (7.4c)$$

Let us write $N = N_0\rho/A$, where N_0 is the Avogadro number, ρ is the density of the scatterer in g/cm³, and A is the atomic weight, and put pv in MeV, t in cm, and χ_c in degrees. Then we have

$$\chi_c = 22.7(\rho t Z^2/A)^{1/2}(z/pv) \text{ deg (heavy particles)}$$
$$= 22.7(\rho t Z(Z+1)/A)^{1/2}(z/pv) \text{ deg (electrons)}, \quad (7.4d)$$

written with one factor involving the scatterer and one, the scatteree.

It will be seen, according to (7.2b), that the probability of getting one scattering in length t of angle χ_c or greater is just unity,

$$2\pi t \int_{\chi_c}^{\infty} \chi d\chi \overline{W}(\chi) = 2\chi_c^2 \int_{\chi_c}^{\infty} \frac{d\chi}{\chi^3} = 1, \quad (7.5)$$

provided that χ_c is sufficiently larger than χ_{om}. We shall see presently that Molière's method only works for χ_c greater than about 5 times χ_{om}.

Using (7.5b) and (6.9b), we have (setting $\mu = 1$)

$$\frac{\chi_c}{\chi_0} \approx \frac{22.1(\rho t Z^2/A)^{1/2} z}{0.472 Z^{1/3} m_e c^2 \beta} = \frac{94.0 z(\rho t Z^2/A)^{1/2}}{Z^{1/3}\beta}. \quad (7.6)$$

The smallest values of this ratio at a given t occur for small Z, $z = 1$, and $\beta = 1$. For carbon, ρt for $\chi_c/\chi_0 = 5$ is 0.078 g/cm², and for gold, the value is 0.041 g/cm².

Having defined χ_c^2, we can define a mean screening function $\bar{q}(\chi)$ by writing, for all values of χ

$$2\pi\chi d\chi t\overline{W}(\chi) = \frac{2\chi_c^2}{\chi^3}\bar{q}(\chi)d\chi = \frac{8\pi d\chi}{\chi^3}\int_0^t \frac{dt'}{k^2(t')} \times \sum_i N_i(t')\alpha_i^2(t')q_i(\chi,t'). \quad (7.7)$$

Using (2.37) and (7.7), we can now write a general formula for $\Omega(\xi) - \Omega_0$:

$$\Omega(\xi) - \Omega_0 = 2\chi_c^2 \int_0^{\infty} \frac{d\chi}{\chi^3}\bar{q}(\chi)[J_0(\xi\chi) - 1]. \quad (7.8)$$

We now give a calculation of $\Omega - \Omega_0$ following Bethe (1953) with minor modifications. We shall omit the bar over q in (7.7), with the understanding that the mean is taken whenever relevant.

We first estimate the value of Ω_0, by assuming that q is 1 for $\chi > \chi_0$ and 0 for $\chi < \chi_0$, a procedure that will certainly give us the correct order of magnitude. We find from (2.37)

$$\Omega_0 = 2\chi_c^2 \int_0^{\infty} \frac{q(\chi)d\chi}{\chi^3} \approx 2\chi_c^2 \int_{\chi_0}^{\infty} \frac{d\chi}{\chi^3} = \frac{\chi_c^2}{\chi_0^2}, \quad (7.9)$$

so that Ω_0 is at least 25 if, as stated above, $\chi_c \geq 5\chi_0$.

Since $\Omega(\xi) \to 0$ as $\xi \to \infty$, we see that exp $[\Omega - \Omega_0]$ will go to a very small value, and will only be appreciably different from zero when $|\Omega(\xi) - \Omega_0| \ll \Omega_0$.

Let us use this fact to estimate the range of ξ which will contribute appreciably to the calculation of the inverse transform integral for $F(\theta,t)$ or $f(\phi,t)$. Using the same rough approximation to $q(\chi)$, we have

$$\Omega(\xi)-\Omega_0\approx 2\chi_c^2\int_{\chi_0}^{\infty}\frac{d\chi}{\chi^3}\,[J_0(\xi\chi)-1]$$
$$=2\chi_c^2\xi\int_{\xi\chi_0}^{\infty}\frac{du}{u^3}\,[J_0(u)-1]\,. \tag{7.10a}$$

If $\xi\chi_0=1$, the two terms in the integrand give the values

$$\int_1^{\infty}J_0(u)\,\frac{du}{u^3}=0.213\text{ and }\int_1^{\infty}\frac{du}{u^3}=0.500\,,$$

so for $\xi=\chi_0^{-1}$, the approximate value of $|\Omega-\Omega_0|$ is about 57% of Ω_0 itself. For larger ξ, the ratio will be even higher, because of the oscillations of the Bessel function. For smaller ξ, we can use the expansion of Eq. (A.12) in the Appendix, yielding

$$\Omega(\xi)-\Omega_0\simeq 2\chi_c^2\xi\left[\tfrac{1}{4}\ln\xi\chi_0-0.279-\frac{\xi^2\chi^2}{128}+\cdots\right]. \tag{7.10b}$$

This expression will be of the order of -2 if $\xi\chi_c=1$, and will get much more negative if ξ is much greater than $1/\chi_c$, so we have for the range of ξ,

$$0\leqslant\xi\lesssim 1/\chi_c\ll 1/\chi_0 \tag{7.11}$$

This limitation on the range of ξ that needs to be considered allows us to divide the range of integration over χ in the correct expression (2.41) into two parts that allow of two different good approximations. We choose the dividing point to be χ_d, with

$$\chi_0\ll\chi_d\ll\chi_c\,. \tag{7.12}$$

(Only for the smallest values of χ_c/χ_0 will the double-inequality signs be hard to justify.)

Then for $\chi>\chi_d$, we replace $q(\chi)$ by 1, and for $\chi<\chi_d$, we replace $J_0(\xi\chi)$ by the first two terms in its expansion, since $\xi\chi<\xi\chi_d<\xi\chi_c\lesssim 1$. We then have

$$\Omega(\xi)-\Omega_0\simeq 2\chi_c^2\int_0^{\chi_d}\frac{d\chi}{\chi^3}\,q(\chi)[-\tfrac{1}{4}\,\xi^2\chi^2]$$
$$+2\chi_c^2\int_{\chi_d}^{\infty}\frac{d\chi}{\chi^3}\,[J_0(\xi\chi)-1]\,. \tag{7.13}$$

If we use again the development (A.12), we have

$$\Omega(\xi)-\Omega_0\simeq-\tfrac{1}{2}\,\chi_c^2\xi^2\int_0^{\chi_d}\frac{d\chi}{\chi}\,q(\chi)$$
$$+\tfrac{1}{2}\,\chi_c^2\xi^2\left(\ln\frac{\gamma\chi_d\xi}{2}-1\right). \tag{7.14}$$

If we further assume that χ_d is large enough to give close to the limiting value indicated in the definition of the screening angle χ_α, Eq. (6.58a), we can write

$$\Omega(\xi)-\Omega_0=\tfrac{1}{2}\,\chi_c^2\xi^2\left(\ln\frac{\gamma\chi_\alpha\xi}{2}-\tfrac{1}{2}\right)$$
$$=\tfrac{1}{4}\,\chi_c^2\xi^2\,\ln\left(\frac{\gamma^2\chi_\alpha^2\xi^2}{4e}\right), \tag{7.15}$$

which is Molière's expression for the exponent $\Omega-\Omega_0$ of the Fourier–Hankel transform of the multiple-scattering distribution.

Equation (7.15) is also obtained if (6.63) is used for $q(\chi)$ in (7.13), for then the infinite integral yields exactly $(\xi/2\chi_\alpha)K_1(\xi\chi_\alpha)-(1/2\chi_\alpha^2)$, where K_1 is the modified Bessel function. Expansion up to terms in ξ^2 yields (7.15).

The next approximation to $\Omega-\Omega_0$ can be calculated or estimated for the various expressions for $q(\chi)$ given in Sec. VI.F. The correction terms to (7.13) may be written

$$(\Omega-\Omega_0)_{\text{corr}}=\int_0^{\chi_d}\frac{d\chi}{\chi}\,[q(\chi)-1]-\ln\chi_d-\lim_{\chi_m\to\infty}$$
$$\times\left[\int_0^{\chi_m}\frac{d\chi}{\chi}\,[q(\chi)-1]-\ln\chi_m\right]+\frac{\chi_c^2\xi^4}{32}\int_0^{\chi_d}\chi d\chi q(\chi)$$
$$-\frac{\chi_d^2\xi^4}{128}+2\chi_c^2\int_{\chi_d}^{\infty}\frac{d\chi}{\chi^3}\,[J_0(\xi\chi)-1][q(\chi)-1]+\cdots, \tag{7.16}$$

where the next terms in the expansion of J_0 and of (A.12) have been used. If the asymptotic expression for q when χ is large is written

$$q(\chi)\simeq 1+\frac{A_1}{\chi}+\frac{B_1\ln\chi}{\chi}+\frac{A_2}{\chi^2}+\frac{B_2\ln\chi}{\chi^2}+\cdots \tag{7.17}$$

then (7.16) after some rearrangement can be written

$$(\Omega-\Omega_0)_{\text{corr}}=\tfrac{1}{32}\,\chi_c^2\xi^4\left\{\int_0^{\chi_d}\chi d\chi[q(\chi)-1]-A_1\chi_d\right.$$
$$\left.-B_1\chi_d(\ln\chi_d-1)-A_2\ln\chi_d-\tfrac{1}{2}\,B_2\ln^2\chi_d\right\}+\tfrac{2}{9}\,\chi_c^2A_1\xi^3$$
$$+2\chi_c^2B_1\xi^3(\tfrac{5}{27}-\tfrac{1}{9}\ln 4\xi)-\frac{\chi_c^2A_2\xi^4}{32}\left(\ln\frac{\gamma\xi}{2}-\tfrac{3}{2}\right)$$
$$+\frac{\chi_c^2B_2\xi^4}{64}\left(\ln^2\frac{\xi\gamma}{2e^{3/2}}+\tfrac{5}{8}\right)+\cdots. \tag{7.18}$$

The expression in { } is independent of χ_d for χ_d large enough; in this approximation χ_d may be taken as infinite. In deriving (7.18), using (7.17) in the last term of (7.16), the integrals given in (A.17) and (A.18) are needed, along with other results from Appendix I.

If we use Molière's formula (6.62), we set $A_1=B_1=B_2=0$ and $A_2=-2\chi_\alpha^2$, and find

$$(\Omega-\Omega_0)_{\text{corr}}=\tfrac{1}{16}\,\chi_c^2\chi_0^2\xi^4\left[\ln\frac{\chi_\alpha\xi\gamma}{2}-\tfrac{5}{4}\right] \tag{7.19}$$

For Fleischmann's formula (6.63), we have $A_1=-\chi_1=-\chi_\alpha e^{1/2}$, $A_2=\chi_1^2$, $B_1=B_2=0$, whereupon

$$(\Omega-\Omega_0)_{\text{corr}}=-\tfrac{2}{9}\,\chi_c^2e^{1/2}\chi_\alpha\xi^3-\frac{e}{32}\,\chi_c^2\chi_\alpha^2\xi^4\left(\ln\frac{\chi_\alpha\xi\gamma}{2}-1\right), \tag{7.20}$$

which agrees with Fleischmann's results except for the omission of a factor $e/2$ in the fourth-power term of his Eq. 11 (his previous equations do not show this error). These two results (7.19) and (7.20) can also be obtained by exact integration of (7.13) and expansion of the resulting Bessel functions.

If we use the result for $\alpha = 0$, (6.45), we have $A_1 = B_1 = B_2 = 0$, $A_2 = -8.847\chi_0^2$, and the brace in (7.18) becomes, in the limit of large χ_d, $\chi_0^2[1.613 \ln \chi_0 + 1.859]$. The result is then

$$(\Omega - \Omega_0)_{\text{corr}} = \tfrac{1}{32}\chi_c^2\chi_0^2\xi^4\left[1.613 \ln \chi_0 - 11.41 + 8.85 \ln \frac{\gamma\xi}{2}\right]. \tag{7.21}$$

Molière's calculation for large α, involving (6.50) and (6.51), leads to integrals with products of four K_1 functions, whose evaluation has not been attempted. A result may be obtained, however, using the interpolation formula (6.58). For this purpose, we set χ_d equal to the value given by (6.58) when $q = 0.9$, and use the asymptotic formula (6.57). We have $A_1 = B_1 = 0$, $A_2 = -8.85\chi_0^2\{1 - \alpha^2 \ln(\alpha^4 + \frac{1}{3}\alpha^2 + 0.13) - 4.48\alpha^2 - 4\alpha^2 \ln 2\chi_0\}$ and $B_2 = -35.4\alpha^2\chi_0^2$. Then we perform an integration by parts

$$\int_0^{\chi_d} \chi d\chi[q(\chi) - 1] = \frac{\chi^2}{2}[q(\chi) - 1]\Big|_0^{\chi_d} - \tfrac{1}{2}\int_0^{0.9} \chi^2 dq$$

The value of

$$\int_0^{0.9} \chi^2 dq$$

may be obtained from Table II, and is approximately $\chi_0^2(6.36 + 39.0\alpha^2)$. Letting $\chi_d = \chi_{0.9} = \chi_0(61.7 + 421\alpha^2)^{1/2}$, we have

$$(\Omega - \Omega_0)_{\text{corr}} = \tfrac{1}{32}\chi_c^2\chi_0^2\xi^4\Big\{-3.18 - 19.5\alpha^2 + 8.85[1 - \alpha^2 \times \ln(\alpha^4 + \tfrac{1}{3}\alpha^2 + 0.13) - 4.48\alpha^2 - 4\alpha^2 \ln 2\chi_0] \times\left[\ln \frac{\chi_{0.9}\xi\gamma}{2} - 2\right] + 17.7\alpha^2\left[\ln^2 \chi_{0.9} - \ln \chi_{0.9} - \ln^2 \frac{\xi\gamma}{2e^{3/2}} - \tfrac{5}{8}\right]\Big\}. \tag{7.22}$$

With the exception of the first term of (7.20), all the above calculations give results which are small if $\chi_c^2\chi_0^2\xi^4/32$ is small. If $\chi_c^2\xi^2$ is of the order of 1, and $\chi_c^2/\chi_0^2 \geqslant 25$, this number is of the order of 1/800 or less. The first term of (7.20) can lead to more substantial corrections, as indicated by Fleischmann. Its justification lies in the graphical fit of (6.63) to the Molière result for large α. However, the presence of a term A_1/χ in the expression for $q - 1$ is hard to justify for large χ, so this estimate of the correction term is open to some doubt.

In the event that mixtures and energy loss are under consideration, the quantity χ_α in (7.15) must be found by using χ_c^2, as given by (7.4) and $\chi_c^2\bar{q}$ as given by (7.7). Explicitly, we find for electron scattering, using (7.4a) and (6.67),

$$\Omega(\xi) - \Omega_0 \simeq -2\pi\xi^2 \int_0^t \frac{dt'}{k^2(t')} \sum N_i(t')\alpha_i^2(t') \times\left\{\int_0^{\chi_d} \frac{d\chi}{\chi}\left[q_i(\chi) + \frac{S(v_1)}{Z_i}\right] - \frac{\ln \chi_d}{(1 + 1/Z_i)}\right\} + \tfrac{1}{2}\chi_c^2\xi^2 \ln \gamma\xi/2e$$

so that we may define the screening angle χ_α by

$$-\chi_c^2(\ln \chi_\alpha + \tfrac{1}{2}) = 4\pi \lim_{\chi_m\to\infty} \int_0^t \frac{dt'}{k^2(t')} \sum N_i(t')\alpha_i^2(t') \times\left\{\int_0^{\chi_m} \frac{d\chi}{\chi}\left[q_i(\chi) + \frac{S(v_1)}{Z_i}\right] - (1 + 1/Z_i)\ln \chi_m\right\} \tag{7.23}$$

with χ_c^2 given by (7.4a).

Let us define an electron-scattering screening angle $\chi^{(\text{el})}$ by

$$-\ln \chi^{(\text{el})} = \tfrac{1}{2} + \lim_{\chi_m\to\infty}\left[\int_0^{\chi_m} \frac{d\chi}{\chi} S(v_1) - \ln \chi_m\right] = \tfrac{1}{2} + \lim_{v_m\to\infty}\left[\int_0^{v_m} \frac{dv_1}{v_1} S(v_1) - \ln v_m + \ln\left(\frac{0.376\mu}{Z^{1/3}\chi_0}\right)\right] \tag{7.24}$$

and designate the screening angles for the ith species by $\chi_{\alpha i}$ and $\chi_i^{(\text{el})}$. Then we have

$$\chi_c^2 \ln \chi_\alpha = 4\pi \int_0^t \frac{dt'}{k^2(t')} \sum_i N_i\alpha_i^2 \times [\ln \chi_{\alpha i} + (1/Z_i)\ln \chi_i^{(\text{el})}]. \tag{7.25}$$

In the event of no energy loss and a homogeneous mixture, we have

$$\chi_c^2 = \frac{4\pi e^4 Z^2 t}{p^2 v^2} \sum_i N_i Z_i(Z_i + 1) \text{ (electron scattering)} \tag{7.26}$$

and

$$\ln \chi_\alpha = \frac{\sum_i N_i Z_i^2[\ln \chi_{\alpha i} + (1/Z_i)\ln \chi_i^{(\text{el})}]}{\sum_i N_i Z_i(Z_i + 1)} = \sum_i N_i Z_i(Z_i + 1) \ln \chi_{\alpha i} + (Z_i + 1)^{-1} \times \ln(\chi_i^{(\text{el})}/\chi_{\alpha i})]/\sum_i N_i Z_i(Z_i + 1). \tag{7.27}$$

For the scattering of heavy particles, the contribution of the atomic electrons must be taken into account by using Eq. (6.72). We do not correct χ_c^2 in this case, but use the value given by (7.4a).

In using (7.13) to find $\Omega - \Omega_0$, we note that the second integral will involve no contribution for sufficiently large χ_d, owing to the upper limit of inelastic scattering, related to $E_{e\,\max}$. In addition to the expression $q(\chi)d\chi/\chi$ we must add the product of

$$\frac{S(v_2)p^2}{4Zm_eE_e^2}\left(1-\frac{E_e\beta^2}{E_{e\,\max}}\right)dE_e$$

from (6.72) and χ_c^2 as given by (6.69); the integral goes only to $E_{e\,\max}$. Thus we add to the integral for $-\ln\chi_a-\frac{1}{2}$ the quantity $D_i/2Z_i$ (Fano's notation):

$$\frac{D_i}{2Z_i}=\frac{1}{2Z_i}\int_0^{E_{e\,\max}}\frac{dE_e}{E_e}\left[S(v_2)-\frac{E_e}{E_{e\,\max}}\right]\left[1-\frac{E_e\beta^2}{E_{e\,\max}}\right]$$
$$=\frac{1}{2Z_i}\int_0^{E_{e\,\max}}dE_e\left[\frac{S(v_2)}{E_e}-\frac{S(v_2)\beta^2}{E_{e\,\max}}-\frac{1}{E_{e\,\max}}+\frac{E_e\beta^2}{E_{e\,\max}^2}\right]. \quad (7.28)$$

In the second term we can put $S(v_2)=1$ since most of the integral comes from the upper range of E_e where this is approximately correct. For the first term, we can use v_2 as a variable in accordance with (6.69):

$$\int_0^{E_{e\,\max}}\frac{dE_e}{E_e}S(v_2)=2\int_0^{v_{2m}}\frac{dv_2}{v_2}S(v_2)\,.$$

In accordance with (7.24), if v_{2m} is sufficiently large, we can write this expression as

$$-1-2\ln\chi^{(\mathrm{el})}-2\ln(0.376\mu/Z^{1/3}\chi_0v_{2m})$$
$$=-1-2\ln\chi^{(\mathrm{el})}-2\ln p(2m_eE_{e\,\max})^{-1/2}$$
$$\simeq-1-2\ln\chi^{(\mathrm{el})}-2\ln(m/2m_e)\,. \quad (7.29)$$

We have then, for D_i,

$$D_i=2\ln(2m_e/m)-2\ln\chi^{(\mathrm{el})}-2-\tfrac{1}{2}\beta^2\,. \quad (7.30)$$

We finally have, for $\ln\chi_\alpha$, the expression

$$\chi_c^2\ln\chi_\alpha=4\pi\int_0^t\frac{dt'}{k^2(t')}\sum_i N_i\alpha_i^2\left\{\ln\chi_{\alpha i}+(1/Z_i)\times\left[\ln\frac{m\chi_i^{(\mathrm{el})}}{2m_e}+1+\tfrac{1}{4}\beta^2\right]\right\}, \quad (7.31\mathrm{a})$$

which becomes for the case of homogeneous material and no energy loss

$$\ln\chi_\alpha=\sum_i N_iZ_i^2\{\ln\chi_{\alpha i}+(1/Z_i)\times[\ln m\chi_i^{(\mathrm{el})}/2m_e+1+\tfrac{1}{4}\beta^2]\}/\sum_i N_iZ_i^2\,. \quad (7.31\mathrm{b})$$

It is seen in Figs. 7 and 8 that $S(v)$ gets close to 1 when v is about 1. The condition that v_{2m} is large enough becomes

$$(m_e/m)^{1/2}pa_0/\hbar Z^{2/3}\gtrsim 3 \quad (7.32)$$

which is always satisfied for relativistic heavy particles, and implies, for nonrelativistic particles, merely that the kinetic energy be in excess of 120 eV. According to Fano (1954), the integral over the incoherent scattering function, given in (7.24), is approximately

$$\lim_{v_m\to0}\int_0^{v_m}\frac{dv_1}{v_1}S(v_1)+\tfrac{1}{2}-\ln v_m\simeq2.9\,, \quad (7.33)$$

where the constant ranges from 1.8 for hydrogen to 3.1 for lead, with some discrepancies among the results of using different atomic models.

The theory of inelastic scattering, according to Fano, depends for its validity on the Bethe collision theory which assumes the incident particle to be much faster than the electrons. This condition is reasonably well satisfied for low-Z materials where the corrections are important.

We conclude that with proper definitions of χ_c^2 and χ_α, the Molière form (7.15) for $\Omega-\Omega_0$ may be used in all cases up to the degree of approximation indicated. We shall see in Sec. IX that although χ_α is defined differently in the calculation using the Dalitz expression, the same methods of averaging for χ_c^2 and χ_α may be used in that calculation.

B. The Molière Expansion

The multiple-scattering distribution is obtained from the inverse Hankel or Fourier transform of $\exp[\Omega(\xi)-\Omega_0]$. We shall use (7.15) for values of ξ up to $1/\chi_c$, and assume that there is no contribution to the integrals beyond this value. Equation (7.15) gives a large positive exponent for very large ξ, making the integrals diverge if taken to infinity. We shall see that after the Molière expansion is made, the separate integrals of each term may be taken convergently from $1/\chi_c$ to ∞ without appreciable contribution.

The function $\xi^2\ln(\gamma^2\chi_\alpha^2\xi^2/4e)$ has a minimum when the logarithm is equal to -1, or when $\xi\sim1.12/\chi_\alpha$, considerably larger than $1/\chi_c$. It can be written as the sum of two terms, one a negative multiple of ξ^2 and the other with a minimum within the interesting range of ξ, by writing

$$\tfrac{1}{4}\chi_c^2\xi^2\ln\left(\frac{\gamma^2\chi_\alpha^2\xi^2}{4e}\right)=\tfrac{1}{4}\chi_c^2\xi^2\ln\frac{\gamma^2\chi_\alpha^2}{eB\chi_c^2}+\tfrac{1}{4}\chi_c^2\xi^2\ln\frac{B\chi_c^2\xi^2}{4}\,, \quad (7.34)$$

where B may be arbitrarily chosen. The minimum of the second term occurs when $\xi^2=4/eB\chi_c^2$ which is within the range if eB is greater than 4. Since $\chi_c^2/\chi_\alpha^2\gg1$, the logarithm in the first term will be considerably more negative than -1, so that the second term will always be smaller in magnitude than

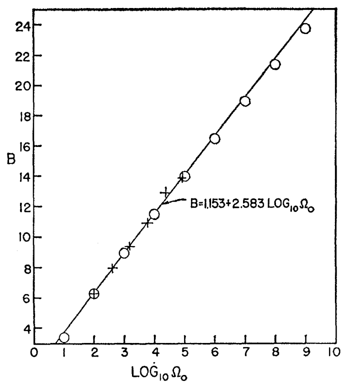


FIG. 9. Graph of Molière's B against Ω_0. The crosses represent Molière's figures, the circled points represent further calculations of the solution of Eq. (7.39), and the straight line is a plot of Eq. (7.40).

the first (when $\chi_c^2\xi^2 = 1$, the second term is still smaller than the first).

Molière's choice of B is governed by the desire to have B appear only as an inverse coefficient of the second term. This is accomplished first by choosing a new integration variable

$$\eta = \xi\chi_c B^{1/2} \tag{7.35}$$

in terms of which $\Omega - \Omega_0$ becomes [writing $\Omega(\eta)$ in place of $\Omega(\xi)$]

$$\Omega(\eta) - \Omega_0 = \frac{\eta^2}{4B}\ln\frac{\gamma^2\chi_\alpha^2}{e\chi_c^2 B} + \frac{\eta^2}{4B}\ln\frac{\eta^2}{4} \tag{7.36}$$

and second by choosing B so that the first term becomes $-\eta^2/4$:

$$\frac{1}{B}\ln\frac{\gamma^2\chi_\alpha^2}{e\chi_c^2 B} = -1 \tag{7.37a}$$

or

$$B - \ln B = \ln\frac{e\chi_c^2}{\chi_\alpha^2\gamma}. \tag{7.37b}$$

The ratio χ_c^2/χ_α^2 is a measure of Ω_0, the mean number of scatterings that occur in thickness t. Actually, Ω_0 is not well-defined, because the uncertainties in the screening function for small χ affect Ω_0 considerably, but have almost no influence on $\Omega - \Omega_0$. For most of the calculations in Sec. VI-F, it is either not possible or not practical to evaluate Ω_0, but for the Molière formula (6.62), it is. Using (7.8), we find exactly the ratio χ_c^2/χ_α^2. Molière calls this ratio Ω_b, but we shall continue to use Ω_0 for it:

$$\Omega_0 = \chi_c^2/\chi_\alpha^2, \tag{7.38}$$

which is taken as a definition of Ω_0 in place of (7.8). In (7.38), the values of χ_c and χ_α are to be taken from (7.4) and (7.25) or (7.31), or their equivalents.

We find, then, for B the transcendental equation

$$B = \ln B + \ln(\Omega_0 e/\gamma^2) = \ln B - 0.1544 + \ln\Omega_0 \tag{7.39}$$

of which the larger of the two roots for $B > 1$ is to be taken. An approximate interpolating formula [Scott (1952)] which is good to 0.5% for Ω_0 from 10^2 to 10^5, and good to 3% out to $\Omega_0 = 10^9$, is (see Fig. 9)

$$B = 1.153 + 2.583\log_{10}\Omega_0. \tag{7.40}$$

Equation (7.36) becomes with this choice of B

$$\Omega(\eta) - \Omega_0 = -\frac{\eta^2}{4} + \frac{\eta^2}{4B}\ln\frac{\eta^2}{4} \tag{7.41}$$

where the interesting range of η is from 0 to $B^{1/2}$.

Molière's expansion method is to consider the second term in (7.37) small enough that its exponential may be expanded to second-order terms:

$$\exp[\Omega(\eta) - \Omega_0] \simeq e^{-\eta^2/4} \times\left[1 + \frac{\eta^2}{4B}\ln\frac{\eta^2}{4} + \tfrac{1}{2}\left(\frac{\eta^2}{4B}\ln\frac{\eta^2}{4}\right)^2 + \cdots\right]. \tag{7.42}$$

The inverse transforms are those of (2.9) and (2.17c). In each case, we can use the variable η in a convenient way if we introduce the reduced angular variables ϑ and φ given by[15]

$$\vartheta = \theta/\chi_c B^{1/2}, \tag{7.43a}$$

$$\varphi = \phi/\chi_c B^{1/2}, \tag{7.43b}$$

together with the normalized "reduced" distribution functions

$$2\pi F_{\text{red}}(\vartheta,t)\vartheta d\vartheta = 2\pi F(\theta,t)\theta d\theta, \tag{7.44a}$$

$$f_{\text{red}}(\varphi,t)d\varphi = f(\phi,t)d\phi. \tag{7.44b}$$

Then we have for the spatial distribution

$$2\pi F_{\text{red}}(\vartheta,t) = \int_0^{B^{1/2}} \eta d\eta J_0(\vartheta\eta)\exp[-\eta^2/4 + (\eta^2/4B) \times\ln(\eta^2/4)]$$
$$\simeq \int_0^\infty \eta d\eta J_0(\vartheta\eta)e^{-\eta^2/4}\left[1 + \frac{\eta^2}{4B}\ln\frac{\eta^2}{4} + \tfrac{1}{2}\left(\frac{\eta^2}{4B}\ln\frac{\eta^2}{4}\right)^2 + \cdots\right] \tag{7.45}$$

or

$$F_{\text{red}}(\vartheta,t) = \frac{1}{2\pi} \times\left[2e^{-\vartheta^2} + \frac{1}{B}F^{(1)}(\vartheta) + \frac{1}{B^2}F^{(2)}(\vartheta) + \cdots\right] \tag{7.46}$$

[15] These reduced variables bear a simple relation to the angular variables $\Theta = pv\theta$ and $\Phi = pv\phi$ sometimes used in cases of no energy loss. From (7.4) we see that

$$\Theta/\vartheta = \Phi/\varphi = (4\pi e^4 z^2 N t\beta\langle Z^2\rangle_{\text{av}})^{1/2}$$

where $\langle Z^2\rangle_{\text{av}}$ must be replaced by $\langle Z(Z+1)\rangle_{\text{av}}$ when the scattering of electrons is under consideration.

with

$$F^{(n)}(\vartheta) = \frac{1}{n!}\int_0^\infty \eta d\eta J_0(\vartheta\eta)\exp(-\eta^2/4)\left(\frac{\eta^2}{4}\ln\frac{\eta^2}{4}\right)^n. \tag{7.47}$$

For the projected distribution, we have

$$\begin{aligned} f_{\text{red}}(\varphi,t) &= \frac{2}{\pi}\int_0^{B^{1/2}} d\eta \cos(\varphi\eta) \\ &\quad \times \exp[-\eta^2/4 + (\eta^2/4B)\ln(\eta^2/4)] \\ &\simeq \frac{2}{\pi}\int_0^\infty d\eta\cos(\varphi\eta)\exp(-\eta^2/4) \\ &\quad \times\left[1+\frac{\eta^2}{4B}\ln\frac{\eta^2}{4} + \tfrac{1}{2}\left(\frac{\eta^2}{4B}\ln\frac{\eta^2}{4}\right)^2+\cdots\right] \end{aligned} \tag{7.48}$$

or

$$f_{\text{red}}(\varphi,t) = \frac{2}{\pi^{1/2}}\exp(-\varphi^2) + \frac{1}{B}f^{(1)}(\varphi) + \frac{1}{B^2}f^{(2)}(\varphi)+\cdots, \tag{7.49}$$

where

$$f^{(n)}(\varphi) = \frac{2}{\pi n!}\int_0^\infty d\eta\cos(\varphi\eta)\exp(-\eta^2/4) \times\left(\frac{\eta^2}{4}\ln\frac{\eta^2}{4}\right)^n. \tag{7.50}$$

Properties of these functions will be discussed in Sec. IX and Appendix III, in connection with the related functions that arise from the use of the Dalitz approximation.

Molière's expansion may be characterized as a particular method of separating the distribution into a Gaussian part and a series of functions having long "tails." However, it must be noted that the functions $F^{(1)}$, $f^{(1)}$, $F^{(2)}$, and $f^{(2)}$ provide substantial contributions for small angles, especially when B is not too large.

To illustrate the relative arbitrariness of the above-mentioned separation, let us show what happens if we replace the right side of (7.35a) by a constant $-a^2$. Eq. (7.39) can then be written

$$a^2B_a = \ln a^2B_a - 0.1544 + \ln(\Omega_0/a^2), \tag{7.51}$$

where B_a is the new value of B. In (7.36) we set $\eta' = a\eta$, and find that

$$\Omega-\Omega_0 = -\frac{\eta'^2}{4}+\frac{\eta'^2}{4a^2B_a}\left[\ln\frac{\eta'^2}{4}-\ln a^2\right]. \tag{7.52}$$

The second term in brackets leads to an expansion with new terms in the integrals (7.45) and (7.48). To terms in B_a^{-1}, we have

$$F_{\text{red}}(\vartheta) = \frac{1}{2\pi a^2}\left\{2e^{-\vartheta^2/a^2}+\frac{1}{a^2B_a}\left[F^{(1)}(\vartheta/a) + 2\ln a^2\left(\frac{\vartheta^2}{a^2}-1\right)e^{-\vartheta^2/a^2}\right]+\cdots\right\} \tag{7.53}$$

$$f_{\text{red}}(\varphi) = \frac{1}{a}\left\{\frac{2}{\pi^{1/2}}e^{-\vartheta^2/a^2}+\frac{1}{a^2B_a}\left[f^{(1)}(\varphi/a) + \frac{2}{\pi^{1/2}}\ln a^2\left(\frac{\varphi^2}{a^2}-\tfrac{1}{2}\right)e^{-\varphi^2/a^2}\right]+\cdots\right\}. \tag{7.54}$$

If a^2 is taken to be small, Ω_0/a^2 and a^2B_a will be large and the relative contribution of the "non-Gaussian" terms may be reduced. However, the presence of $-\ln a^2$, and its square in the next term, makes the errors hard to estimate. The convenience of choosing $a = 1$ is clearly evident.

The over-all error in Molière's method arises from terms of order $1/B^2$ in the expansion (7.42) and from the neglected terms involved in $[\Omega-\Omega_0]_{\text{corr}}$. The latter are [except for the form in (7.20)] of the order of $\chi_c^2\chi_\alpha^2\xi\ln(\chi_\alpha\xi\gamma/2) = \eta^4\ln(\chi_\alpha\xi\gamma/e)/B^2\Omega_0$ by use of (7.35) and (7.37) which in turn gives a correction of the order of $1/B\Omega_0 \simeq e^{-B}$. When $B = 4.5$, the two corrections are of the same order, and for larger B, the $1/B^3$ terms predominate. [For Eq. (7.20), the corresponding correction is of the order $B^{-3/2}\Omega_0^{-1/2} \simeq e^{-B/2}/B$ which is equal to $1/B^3$ for $B = 1.43$]. Molière (1948) considered his method to be good for $B > 4.5$ and $\Omega_0 > 20$.

C. Application to Lateral Deflections and Other Characteristics

Let us now consider how by Molière's method we may calculate the distribution of any single one of the various characteristics of a scattered track whose coupling factors are given in Sec. III-C, or of a single linear combination of characteristics. We use (3.11), setting equal to zero ξ and all the ζ's except the particular $\zeta^{(j)}$ belonging to the quantity $X^{(j)}$ whose distribution we seek. The value of $\Omega-\Omega_0$ is then to be found by substituting $a^{(j)}(t')\zeta^{(j)}$ for ξ in the calculation of part A of this section for either electron or heavy particle scattering, with the factor $a^{(j)}(t')$ placed under the integral sign.

Considerable simplification can be made if energy loss and inhomogeneity of material do not need to be considered. Let us introduce the two coupling factor integrals

$$C_1^{(j)} = \frac{1}{s_a}\int_0^t dt'[a^{(j)}(t')]^2 , \quad (7.55a)$$

$$C_2^{(j)} = \frac{1}{s_a}\int_0^t dt'[a^{(j)}(t')]^2 \ln [a^{(j)}(t')]^2 , \quad (7.55b)$$

where s_a is a path length that may be chosen as equal to the range of t for which $a^{(j)}$ is different from zero, or may be chosen as equal to a cell length s when equal cells are under consideration. Then if χ_c, χ_α, and Ω_0 are defined in the normal way for the path length s_a, we have, using a superscript to indicate that a particular type of characteristic is involved,

$$\chi_c^{(a)2} = \chi_c^2 C_1 , \quad (7.56a)$$

$$C_1 \ln \chi_\alpha^{(a)2} = C_1 \ln \chi_\alpha^2 + C_2 , \quad (7.56b)$$

$$\chi_\alpha^{(a)2} = \chi_\alpha^2 e^{C_2/C_1} , \quad (7.56c)$$

$$\Omega_0^{(a)} = \Omega_0 C_1 e^{-C_2/C_1} . \quad (7.56d)$$

In the case of linear combinations of measured quantities, the integrals C_1 and C_2 contain in place of $a^{(j)}(t')$, the same linear combination of the corresponding a's.

It will be noted that if s_a is changed to some other length s_a', C_1 becomes $C_1(s_a'/s_a)$ and $\Omega_0{}^{(a)}$ is likewise multiplied by the factor s_a'/s_a. Furthermore, if the coupling factor refers to a length instead of an angle, so that it contains as a factor, a unit of length, say l, it is easy to show that $\chi_c^{(a)}$ and $\chi_\alpha^{(a)}$ each contain the factor l, but that $\Omega_0^{(a)}$ does not. For instance, $a_{2\Delta}^{(j)}(t) = a_\alpha{}^{(j)}(t)s$; C_1 and C_2 for the second difference contain s, but $\Omega_0^{(j)}$ is the same for both the second difference and the chord angle.

We can then write for the exponent of the multiple-scattering transform

$$\Omega^{(a)}(\xi_j,t) - \Omega_0^{(a)} = \tfrac{1}{4}\, \xi_j^2 \chi_c^{(a)^2} \ln (\gamma^2 \xi_j^2 \chi_\alpha^{(a)^2}/4e) \quad (7.57a)$$

$$\Omega_0^{(a)} = \chi_c^{(a)^2}/\chi_\alpha^{(a)^2} . \quad (7.57b)$$

The Molière analysis can be applied and the distribution obtained in the usual way. The general rule for calculating the distribution of any such quantity $X^{(j)}$ can be stated as follows. Find Ω_0 for the track length under consideration, and modify its logarithm by adding the correction

$$\log_{10} \Omega_0^{(a)} - \log_{10} \Omega_0 = \Delta \log_{10} \Omega_0$$
$$= \log_{10} (C_1^{(j)} - C_2^{(j)})/2.303C_1^{(j)} . \quad (7.58)$$

Then find B from the modified logarithm; the final distribution function will be (normalized for positive values only)

$$f^{(j)}(X^{(j)})dX^{(j)} = \frac{dX^{(j)}}{(\chi_c^2 C_1^{(j)} B)^{1/2}} \left\{ \frac{2}{\pi^{1/2}} \times \exp(-X^{(j)2}/\chi_c^2 C_1^{(j)} B) + \frac{1}{B} f^{(1)}[X^{(j)}/(\chi_c^2 C_1^{(j)} B)^{1/2}] + \frac{1}{B_2} f^{(2)}[X^{(j)}/(\chi_c^2 C_1^{(j)} B)^{1/2}] + \cdots \right\} . \quad (7.59)$$

Table III gives results, due to Molière (1955), for several quantities, including some linear combinations.

The problem of a satisfactory calculation in the Molière approximation of the joint distribution of two or more variables has not been solved.

TABLE III. Correction coefficients for calculating distributions of several track characteristics by Eqs. (7.58) and (7.59).[a]

Quantity	$C_1{}^{(j)}$	$\Delta \log_{10} \Omega_0$
ϕ	1	0
$(\phi + \psi)/\sqrt{2}$	7/6	−0.030
ψ	1/3	−0.188
$(\psi - \phi)/\sqrt{2}$	1/6	−0.188
$\hat{\phi}_j$	26/35	−0.050
α_j	2/3	+0.113
$\hat{\alpha}_j$	11/20	−0.192
$\phi_j + \phi_{j+1}$	2	+0.301
$\phi_j - \phi_{j+1}$	2	+0.301
$\hat{\phi}_j + \hat{\phi}_{j+1}$	61/35	−0.325
$\hat{\phi}_j - \hat{\phi}_{j+1}$	43/35	−0.280
$\alpha_j + \alpha_{j+1}$	5/3	−0.337
$\alpha_j - \alpha_{j+1}$	1	−0.290
$\hat{\alpha}_j + \hat{\alpha}_{j+1}$	23/15	−0.360
$\hat{\alpha}_j - \hat{\alpha}_{j+1}$	2/3	−0.370
$\Delta^2 x_{j-1}$	$(2/3)s^2$	0.113
$\Delta^3 x_{j-1}$	s^2	0.290
$\Delta^4 x_{j-1}$	$(8/3)s^2$	0.264

[a] All values are calculated with equal cell lengths s, and with $\hat{s} = s = s_a$.

VIII. THE SNYDER CALCULATION

Snyder's method, as used by Snyder and Scott (1949), consisted of a direct numerical integration of the inverse transform for the projected scattering, using the form (6.8), which is mathematically equivalent to the use of (6.63). The results may be taken as applying to the latter, so we shall use the corresponding notation here. (The Snyder–Scott work was done before Molière's (1948) results had become available). With the definition (7.36) for Ω_0, which is a measure of the thickness or mean number of scatterings, we have

$$f(\phi,t) = \frac{2}{\pi}\int_0^\infty d\xi \cos\phi\xi \exp\{\Omega_0[\xi\chi_\alpha K_1(\xi\chi_\alpha) - 1]\} \tag{8.1}$$

[see paragraph following Eq. (7.15)].

Let us introduce a new integration variable s and a new reduced-angle variable φ_s, by

$$s = \xi\chi_\alpha \tag{8.2a}$$

$$\varphi_s = \phi/\chi_\alpha = \varphi(\Omega_0 B)^{1/2}. \tag{8.2b}$$

Then we have, using the Snyder–Scott notation W for the projected distribution normalized from $-\infty$ to ∞,

$$W(\varphi_s,\Omega_0) = \frac{1}{\pi}\int_0^\infty ds \cos s\varphi_s \exp \Omega_0[sK_1(s) - 1]. \tag{8.3}$$

This integral was evaluated by finding $sK_1(s) - 1$ to as many as ten significant figures, from the first three terms of its series expansion [Watson (1952), p. 80), and by using ten-figure cosine tables. Numerical integration with Weddle's rule was then used for $\Omega_0 = 100$, 1500, 3000, and 9000, with a final accuracy of three significant figures. Results for other values of Ω_0 were obtained by folding integration, using (2.20) and (2.39). We have, in fact,

$$W(\varphi_s,\Omega_0+\Omega_0') = \int_{-\infty}^\infty d\varphi_s' W(\varphi_s',\Omega_0)W(\varphi_s - \varphi_s',\Omega_0'). \tag{8.4}$$

In this way, tables of W for 29 values of Ω_0 from 100 to 84 000 were constructed.[16] Each of these tabulated functions satisfies the normalization rule to within 1%, and the values agree with the Molière result within 2% everywhere, and in most cases within 1%.

As we have seen in the last section, the distribution depends primarily on $\varphi = \varphi_s(\Omega_0 B)^{-1/2}$ and only secondarily on B. An even sharper division of dependence can be made if we use $\varphi B^{1/2} = \varphi_s\Omega_0^{-1/2}$ as the angle variable. Let us use the first approximation to $sK_1(s) - 1$, which is just the Molière term $(s^2/4)\ln(\gamma^2 s^2/4e)$, and rewrite (8.3) as follows

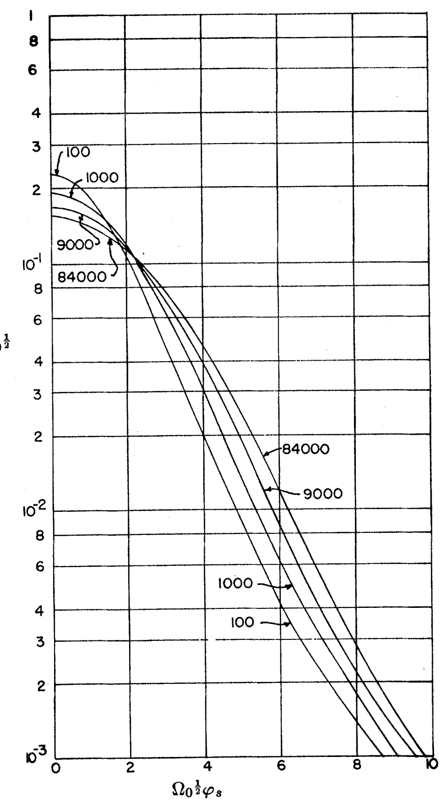


FIG. 11. Semilogarithmic graph of $\Omega_0^{1/2}W$ against $\Omega_0^{-1/2}\varphi_s$ for $\Omega_0 = 100$, 1000, 9000, and 84 000.

$$\Omega_0^{1/2}W(\varphi_s,\Omega_0) = \frac{1}{\pi}\int_0^\infty d(s\Omega_0^{1/2}) \cos[(s\Omega_0^{1/2})(\varphi_s\Omega_0^{-1/2})] \times \exp[\tfrac{1}{4}s^2\Omega_0(\ln s^2\Omega_0 + \ln\gamma^2/4\Omega_0 e)], \tag{8.5}$$

showing that if we use $\Omega_0^{1/2}W$ and $\varphi_s\Omega_0^{-1/2}$ as variables, the dependence on Ω_0 is quite minor. Figs. 10–12 show graphs of these two quantities for several values of Ω_0. Fig. 13 gives a set of graphs for the probability $\mathcal{P}(\delta,\Omega_0)$ of getting a deflection greater than $\varphi_s = \delta\Omega_0^{1/2}$:

$$\mathcal{P}(\delta,\Omega_0) = \int_\delta^\infty d\varphi_s' W(\varphi_s',\Omega_0) \tag{8.6}$$

as a function of Ω_0 for various values of δ.

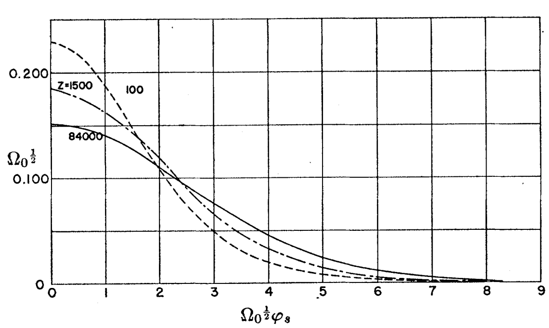


FIG. 10. Linear graph of $\Omega_0^{1/2}W$ against $\Omega_0^{-1/2}\varphi_s$ for $\Omega_0 = 100$, 1500 and 84 000.

[16] Copies of the complete set of tables and also the Snyder functions of Sec. VIII may be obtained from the American Documentation Institute, 1728 N Street, NW, Washington, D. C.

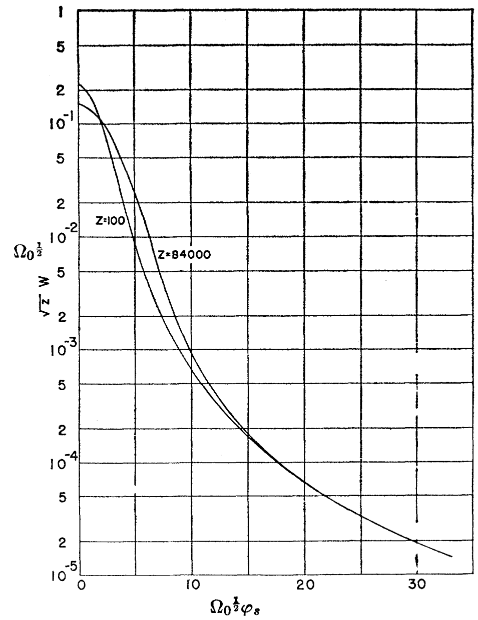


FIG. 12. Semilogarithmic graph of $\Omega_0^{1/2}W$ against $\Omega_0^{-1/2}\varphi_s$ for $\Omega_0 = 100$ and 84 000, showing their agreement for large values of $\Omega_0^{-1/2}\varphi_s$.

These functions may be represented with moderate accuracy by a sum of two Gaussians down to a point at which W is 0.007 of its maximum value, and by a "tail" equal to a sum of φ_s^{-3} and φ_s^{-5} terms for larger angles. The coefficients can be written as simple functions of Ω_0 and $\log_{10}\Omega_0$, so that interpolation is straightforward. We have

$$W(\varphi_s,\Omega_0) \simeq A_1 e^{-a_1\varphi_s^2} + A_2 e^{-a_2\varphi_s^2}\,; \qquad W \geqslant 0.007W(0,\Omega_0)\,, \quad (8.7\text{a})$$

where

$$A_1 = \Omega_0^{-1/2}(-951 + 865\log_{10}\Omega_0)^{-1/2}\,,$$

$$A_2 = \Omega_0^{-1/2}(6.3 + 10.0\log_{10}\Omega_0)^{-1/2}\,,$$

$$a_1 = \Omega_0^{-1}(10.96 + 4.381\log_{10}\Omega_0)^{-1}\,,$$

$$a_2 = \Omega_0^{-1}(0.216 + 2.326\log_{10}\Omega_0)^{-1}\,, \quad (8.7\text{b})$$

$$W(\varphi_s,\Omega_0) \simeq \frac{\Omega_0}{2\varphi_s^3}\left(1 + \frac{11.68\Omega_0\log_{10}(10\Omega_0)}{\varphi_s^2}\right); \qquad W \leqslant 0.007W(0,\Omega_0)\,. \quad (8.7\text{c})$$

These approximations are good to 4% out to angles for which $W = 0.02\,W(0,\Omega_0)$, when $2000 \leq \Omega_0 \leq 42\,000$, and to 8% for such angles when $100 \leq \Omega_0 \leq 2000$ and $42\,000 \leq \Omega_0 \leq 84\,000$. The greatest errors are not more than 10% for any Ω_0; this amount of error occurs over small regions near the "junction" of the two approximations and also near the angles where $W = 0.001\,W(0,\Omega_0)$. Fig. 14 shows the Snyder function and the approximations (8.7a) and (8.7c) for $\Omega_0 = 3000$.

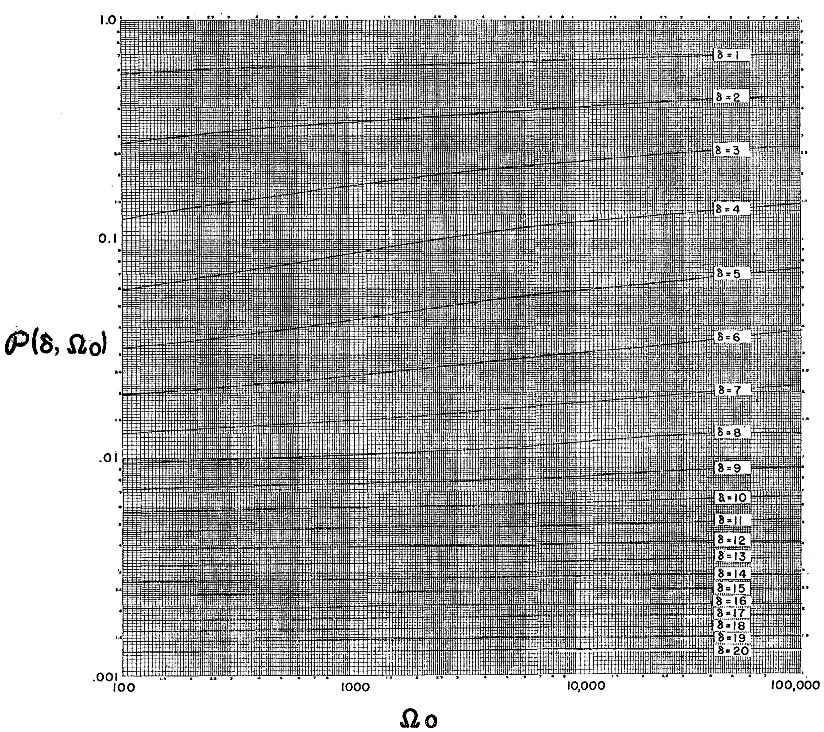


FIG. 13. A set of graphs for the probability $\mathcal{P}(\delta,\Omega_0)$ of getting a deflection greater than $\varphi_s = \delta\Omega_0^{1/2}$, as a function of Ω_0, for twenty values of δ.

H. S. Snyder (private communication) has also made calculations of the spatial-angle distributions, but only mean-value results (see Sec. XIII) have been published (Goldberg, Snyder, and Scott, 1955). Three methods were utilized, a direct calculation for

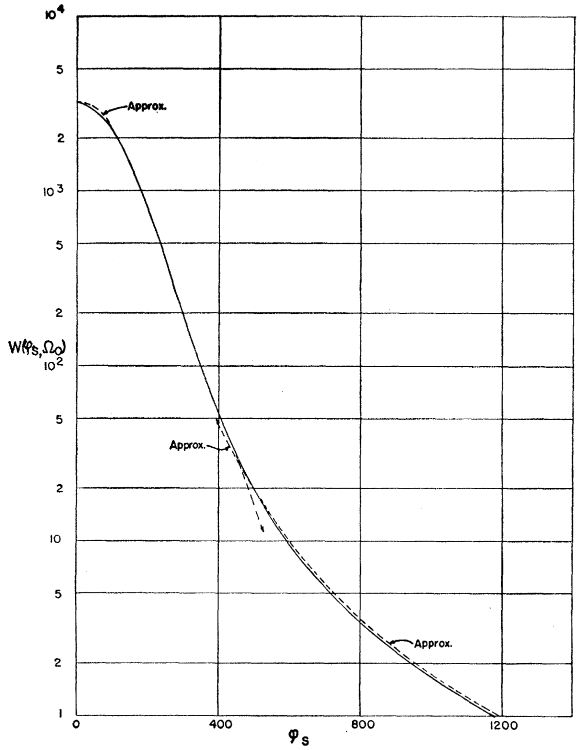


FIG. 14. The Snyder function $W(\varphi_s,\Omega_0)$ and the approximations (8.7a) and (8.7b), as functions of φ_s for $\Omega_0 = 3000$.

$\Omega_0 = 3000$ and an expansion about this value, use of asymptotic formulas, and a numerical inversion method working from the projected distribution.

The spatial distribution given in terms of ϑ_s,

$$\vartheta_s = \theta/\chi_\alpha = \vartheta(\Omega_0 B)^{1/2}, \tag{8.8}$$

may be obtained from (8.3) by replacing cos $s\varphi_s$ by $\frac{1}{2}sJ_0(s\vartheta_s)$. Letting Ω_{00} represent the value 3000, we write the distribution $P(\alpha,\Omega_0)$ for arbitrary Ω_0 in terms of the variable α_s given by

$$\alpha_s = \vartheta_s(\Omega_{00}/\Omega_0)^{1/2} = \vartheta(\Omega_{00}B)^{1/2} = (\theta/\chi_\alpha)(\Omega_{00}/\Omega_0)^{1/2}. \tag{8.9}$$

We also change the integration variable from s to $\sigma = s(\Omega_0/\Omega_{00})^{1/2}$. From (8.5), we readily find

$$P(\alpha_s,\Omega_0) = \frac{1}{2\pi}\int_0^\infty \sigma d\sigma J_0(\sigma\alpha_s) \times \exp\{\tfrac{1}{4}\sigma^2\Omega_{00}[\ln(\gamma^2\sigma^2/4e) - \ln(\Omega_0/\Omega_{00})]\} \tag{8.10a}$$

subject to the normalization

$$2\pi\int_0^\infty \alpha_s d\alpha_s P(\alpha_s,\Omega_0) = 1. \tag{8.10b}$$

We now expand the function exp $[-\frac{1}{4}\sigma^2\Omega_{00}\ln(\Omega_0/\Omega_{00})]$ and obtain a series for $\mathcal{P}$:

$$P(\alpha_s,\Omega_0) = P_0(\alpha_s,\Omega_{00}) + \ln(\Omega_0/\Omega_{00})P_1(\alpha_s,\Omega_{00}) + \ln^2(\Omega_0/\Omega_{00})P_2(\alpha_s,\Omega_{00}) + \ln^3(\Omega_0/\Omega_{00})P_3(\alpha_s,\Omega_{00}), \tag{8.11}$$

where

$$2\pi P_n(\alpha_s,\Omega_{00}) = \frac{1}{n!}\left(-\frac{\Omega_{00}}{4}\right)^n \int_0^\infty \sigma^{2n+1} d\sigma J_0(\sigma\alpha_s) \times \exp[\tfrac{1}{4}\sigma^2\Omega_{00}\ln(\gamma^2\sigma^2/4e)]. \tag{8.12}$$

The integrals for P_n, $n = 0, 1, 2, 3$, were evaluated numerically; the results are given in Table IV and Fig. 15. Table V gives values of $P(\alpha_s,\Omega_0)$ for seven values of Ω_0 (Fig. 16). These values were checked by use of asymptotic formulas for large α_s (see Sec. XII), by the inversion method (below) and finally by comparison with Molière's tables. Agreement with the latter at zero angle is within 0.4% except for $\Omega_0 = 100$

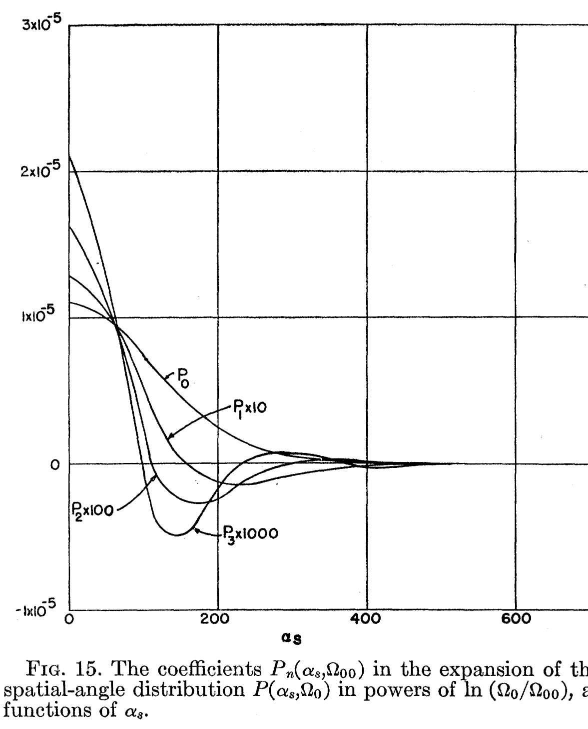


FIG. 15. The coefficients $P_n(\alpha_s,\Omega_{00})$ in the expansion of the spatial-angle distribution $P(\alpha_s,\Omega_0)$ in powers of ln (Ω_0/Ω_{00}), as functions of α_s.

where the error is 2% and at $\Omega_0 = 1\ 024\ 000$ where it is 0.6%. Graphical comparison shows no significant discrepancy between curves of the two results for other angles.

The numerical inversion method made use of Eq. (2.4b), which in terms of ϑ_s and φ_s becomes, using $W_{\mathrm{sp}}(\vartheta_s,\Omega_0)$ to represent the spatial distribution function in ϑ_s:

$$\tfrac{1}{2}W(\varphi_s,\Omega_0)=\int_0^\infty W_{sp}[(\varphi_s^2+\psi^2)^{1/2},\Omega_0]d\psi\,. \quad (8.13)$$

TABLE IV. The Coefficients $P_n(\alpha_s,\Omega_{00})$ for $n=0,1,2,3$ and for several values of α_s. The numbers have been multiplied by 10^9.

α_s	n = 0	1	2	3
0	11030.6	1292.10	161.804	21.2211
50	10007.6	1042.96	114.064	+12.8780
100	7507.05	511.593	+25.4318	−.667869
150	4710.42	+54.0298	−26.5639	−4.87599
200	2534.98	−136.537	−25.7744	−1.79162
250	1207.54	−140.441	−8.91581	+.605793
300	536.252	−85.3825	+1.09597	.786536
350	233.159	−40.9519	2.61506	+.252099
400	106.972	−17.4607	1.69430	−.158029
450	52.7199	−7.24451	.795825	−.034399
500	29.0674	−2.89543	.306759	−.022216
550	17.8319	−1.31679	.118015	−.009293
600	11.4541	−0.649853	.046753	−.003899
700	5.42409	−0.197978	.009752	−.000514

Values of $W(\varphi_s,\Omega_0)$ were available, and also values of W_{sp} for $\vartheta_s \geqslant \vartheta_{sA}$ within the range of validity of the asymptotic formulas. Values of W_{sp} were sought at a series of equally spaced angles $\vartheta_{s1}, \vartheta_{s2},\cdots\vartheta_{sn},\cdots$ up to ϑ_{sA}. Starting with larger angles, the calculation proceeds stepwise. If $W_{\mathrm{sp}}(\vartheta_{sn},\Omega_0)\equiv W_n$ is known, we find $W_{\mathrm{sp}}(\vartheta_{sn-1},\Omega_0)$ by writing

$$\tfrac{1}{2}W(\vartheta_{sn-1},\Omega_0)=\int_0^{(\vartheta_{sn+1^2}-\vartheta_{sn-1^2})^{1/2}} d\psi \times W_{sp}[(\vartheta_{sn-1^2}+\psi^2)^{1/2},\Omega_0] + \int_{(\vartheta_{sn+1^2}-\vartheta_{sn-1^2})^{1/2}}^\infty d\psi \times W_{sp}[(\vartheta_{sn-1^2}+\psi^2)^{1/2},\Omega_0]\,. \quad (8.14)$$

The second integral requires values of W_{sp} for ϑ_s equal to and larger than ϑ_{sn+1}; the first for values between ϑ_{sn-1} and ϑ_{sn+1}. The first integral is then approximated by using central-difference interpolation formulas with the values W_{n-1}, W_n, and W_{n+1}, yielding an equation that can be solved for W_{n-1}. The results agreed quite satisfactorily with the P_n integrals discussed above.

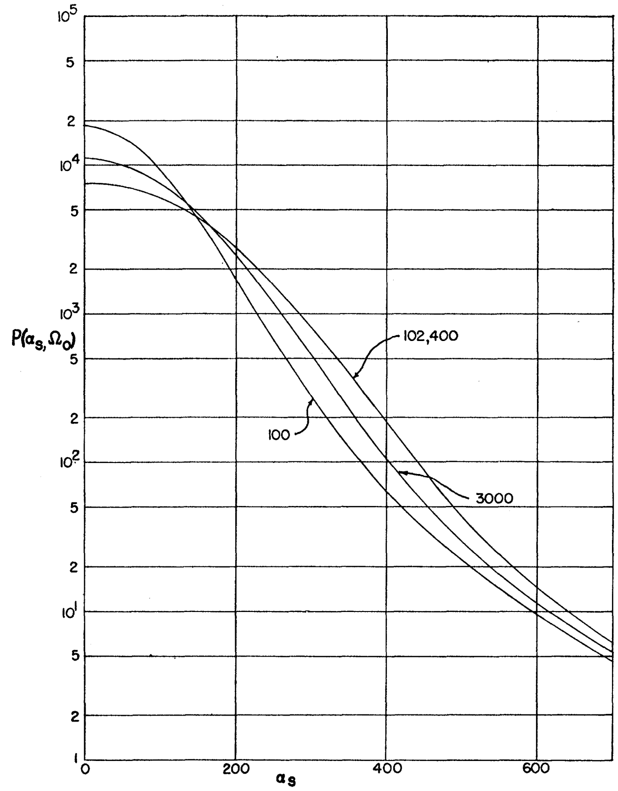


FIG. 16. The spatial-angle distribution $P(\alpha_s,\Omega_0)$ as a function of α_s, for $\Omega_0=100$, 3000, and 102, 400.

TABLE V. The spatial-angle distribution $P(\alpha_s,\Omega_0)$ for seven values of Ω_0. Values have been multiplied by 10^9.

α_s	Ω_0 = 100	400	1600	3000	6400	25 600	102 400
0	18109.3	14452.9	11916.7	11030.6	10134.3	8791.33	7539.99
50	15649.7	12679.9	10711.7	10007.6	9277.33	8168.99	7181.33
100	9513.67	8634.67	7837.99	7507.05	7134.33	6533.67	6047.67
150	4394.67	4671.67	4732.67	4710.42	4686.99	4520.67	4403.33
200	1702.37	2140.79	2438.53	2534.98	2624.47	2727.13	2775.33
250	650.633	893.333	1115.87	1207.54	1308.57	1461.67	1565.67
300	289.559	375.099	482.333	536.252	601.533	716.599	816.733
350	157.939	187.209	232.949	233.159	265.57	330.499	399.267
400	60.9699	77.3767	96.6267	106.972	121.233	153.749	196.683
450	35.9467	41.0867	48.4867	52.7199	58.6933	72.2633	89.7433
500	21.9249	24.3267	27.3919	29.0674	31.4733	36.9299	44.1167
550	14.3529	15.5733	17.0467	17.8319	18.9033	21.2989	24.3599
600	9.62633	10.3233	11.0633	11.4541	11.9733	13.1009	14.5067
700	4.84333	5.05999	5.29999	5.42409	5.57999	5.89867	6.26699

IX. THE NIGAM–SUNDARESAN–WU CALCULATION

A. Evaluation of the Transform

We shall develop in this section the results of Nigam, Sundaresan, and Wu (1959) for spin 1/2 particles, using a method similar to that of Sec. VII. We wish to evaluate (7.8) in the form

$$\Omega(\xi) - \Omega_0 = 2\chi_c^2 \int_0^\pi \frac{\sin \chi d\chi q(\chi)}{[2 \sin (\chi/2)]^4} [J_0(\xi\chi) - 1], \quad (9.1)$$

when $q(\chi)$ is given by (6.15), and we have put back $\sin \chi$ and $\sin (\chi/2)$ respectively in place of χ and $\chi/2$ (cf. 6.1), and put π for the upper limit, in order to proceed with suitable caution in applying the small-angle approximation. (Nigam *et al.* also use the Legendre polynomial sum for which the Hankel transform is an approximation and replace it by the transform at a later stage. The results of the two different methods of approach are the same within the accuracy of either.) We consider first the case of homogeneous material, no energy loss, and no contributions from scattering by electrons.

As indicated in Sec. VI, we take $\chi_\mu \sim \chi_0 < 0.033$ so that we may immediately approximate the X of (6.16) by $\frac{1}{2} \sin (\chi/2)[\chi_\mu^2 + \sin^2 (\chi/2)]^{-1/2}$ which is always less than 1/2, and set $X \tan^{-1}\chi_\mu X = \chi_\mu X^2$. The largest corrections to the Molière result come for large χ in the $\sin^2 (\chi/2)$ terms, and are seen to involve amounts of the order $\pi\alpha\beta$ in the curly bracket. The estimate of Ω_0 made in Sec. VII will not be seriously affected by these corrections, nor will the estimate made there of the important range of ξ. We can therefore repeat the method of Sec. VII by dividing the integration at a value χ_d given by Eq. (7.12).

We shall assume that χ_c itself is a small angle, so that $\sin \chi_c \simeq \chi_c$. Before separating the integral into two parts, however, we can simplify $q(\chi)$. For increasing χ, the combination

$$\tfrac{1}{4} [\chi_\mu^2 + 4 \sin^2 (\chi/2)][\chi_\mu^2 + \sin^2 (\chi/2)]^{-1}$$

becomes closely equal to 1 well before we need to distinguish between $\chi/2$ and $\sin (\chi/2)$. We can therefore replace it for all χ by

$$\text{ratio} = (\chi_\mu^2 + \chi^2)/(4\chi_\mu^2 + \chi^2) . \quad (9.2)$$

This ratio appears twice, the second time with a factor $\beta\chi_\mu^2/4$ in comparison to the first, so we can neglect this second term. We can write $\tan^{-1} (2/\chi_\mu) = \pi/2 - \tan^{-1} (\chi_\mu/2) \simeq \frac{1}{2} (\pi - \chi_\mu)$ and the product of this expression with χ_μ^2 is also negligible. Furthermore, for the term $-\alpha\beta^2[\chi_\mu^2 + 4 \sin^2 (\chi/2)]\chi_\mu X^2$ we can write $-\alpha\beta^2\chi_\mu \sin^2 (\chi/2)\{1 + \frac{3}{4} \chi_\mu^2[\chi_\mu^2 + \sin^2 (\chi/2)]^{-1}\}$ and can neglect second terms in the bracket; the ratio (9.3) differs from 1 only when $\sin^2 (\chi/2)$ is very small. This term will then cancel with the product of $2\alpha\beta \sin^2 (\chi/2)$ and the $\frac{1}{2} \chi_\mu$ from $\tan^{-1} (2/\chi_\mu)$.

The only terms for which the small-angle approximation cannot be used directly are those of the remaining ones that contain $\sin^2 (\chi/2)$ or $\sin (\chi/2) \tan^{-1} [\sin (\chi/2)/\chi_\mu]$. The resulting expression for $q(\chi)$, valid for all χ, is

$$\begin{aligned} q(\chi) = {} & \chi^4(\chi^2 + \chi_\mu^2)^{-2}[1 - \beta^2(1 + \alpha\pi) \sin^2 (\chi/2) \\ & + 2\alpha\chi_\mu(\chi_\mu^2 + \chi^2)/(4\chi_\mu^2 + \chi^2) \\ & + \tfrac{1}{2} \alpha\beta^2(\chi_\mu^2 + 4 \sin^2 (\chi/2)]\csc(\chi/2) \\ & \times \tan^{-1} (\sin (\chi/2)/\chi_\mu) . \end{aligned} \quad (9.3)$$

Figures 17–20 show the separate terms of the bracket in (9.3), the first Born approximation formula $\chi^4(\chi^2 + \chi_\mu^2)^{-2}$ and the resulting $q(\chi)$, as well as Molière's $q(\chi)$, Eq. (6.63), for a relatively extreme case in which the correction terms are large, namely for $\chi_0 = 0.0167$, $\mu = 1.80$, $\chi_\mu = 0.030$, $\alpha = 6$, and $\beta = 0.20$. The difference between the Molière calculations and those of Nigam *et al.* is strikingly evident.

However, for smaller χ_0 and χ_μ the differences tend to disappear. The inverse-tangent term may be approximately written as χ_μ times a function of χ/χ_μ which will shift both to the left and down in the graphs as χ_μ is reduced. It still will in a rough way cancel the $\sin^2 (\chi/2)$ term. The main change as χ_μ is reduced will be to reduce the magnitude of the remaining fractional term and shift the knee of its graph to the left.

Let us now consider the integral for $\Omega - \Omega_0$. The first term in (6.15) will give in small-angle approximation just the Molière result (7.15) with χ_α replaced by χ_μ, and we do not need to treat it separately. We can also carry out exactly the integral with the fractional term. For the other terms, we replace sines by angles for $\chi < \chi_d$, and $J_0(\xi\chi) - 1$ by $-\frac{1}{4} \xi^2\chi^2$; for $\chi > \chi_d$ we need special treatment. Particularly, the last term must be approximated in two different ways for the two parts of the integral. For small χ, we write

$$\frac{\alpha\beta^2(\chi_\mu^2 + 4 \sin^2 (\chi/2)]}{2 \sin (\chi/2)} \tan^{-1} \frac{\sin (\chi/2)}{\chi_\mu} \simeq \frac{\alpha\beta^2(\chi_\mu^2 + \chi^2)}{\chi} \tan^{-1} \left(\frac{\chi}{2\chi_\mu}\right), \quad (9.4a)$$

and for $\chi > \chi_d$, we write

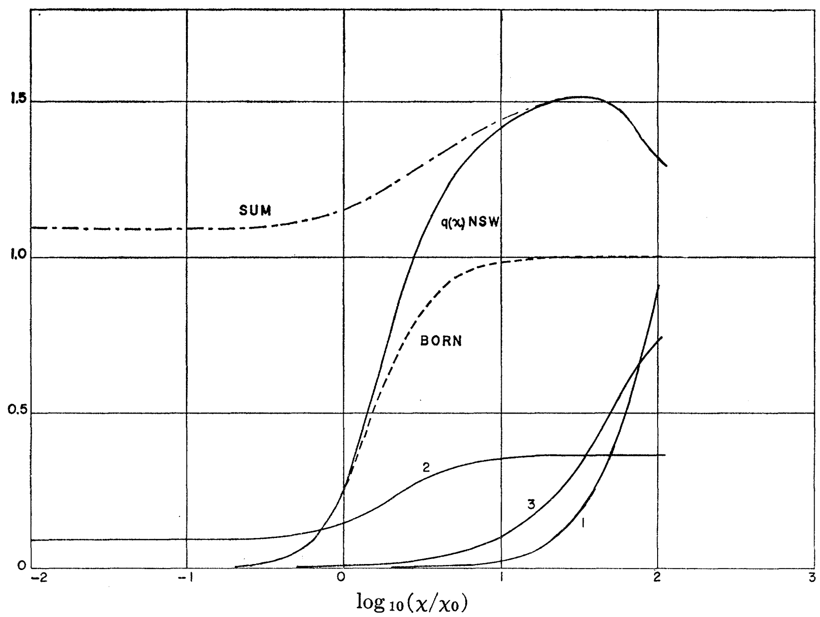


FIG. 17. Graph illustrating the terms in Eq. (9.3), against $\log_{10}(\chi/\chi_0)$. The curves marked 1, 2 and 3 represent in order the three last terms in [ ] in (9.3), and the curve marked "sum" represents the entire bracket. The "Born" curve is for the factor $\chi^4(\chi^2+\chi_\mu{}^2)^{-2}$, and "$q(\chi)$NSW" denotes the entire expression (9.3). Values of parameters chosen: $\chi_0 = 0.0167$, $\mu = 1.80$, $\chi_\mu = 0.030$, $\alpha = 6.0$, $\beta = 0.20$.

$$\frac{\alpha\beta^2[\chi_\mu^2 + 4\sin^2(\chi/2)]}{2\sin(\chi/2)}\tan^{-1}\left[\frac{\sin(\chi/2)}{\chi_\mu}\right]$$
$$\simeq 2\alpha\beta^2\sin(\chi/2)\left[\frac{\pi}{2} - \frac{\chi_\mu}{\sin(\chi/2)}\right]$$
$$= \pi\alpha\beta^2\sin(\chi/2) - 2\alpha\beta^2\chi_\mu\,. \qquad (9.4\text{b})$$

With these considerations in mind, let us write $\Omega - \Omega_0$ as a sum of six integrals:

$$\Omega(\xi) - \Omega_0 = 2\chi_c^2\sum_{n=1}^{6} I_n(\xi)\,, \qquad (9.5)$$

where

$$I_1(\xi) = \int_0^\infty \frac{\chi d\chi[J_0(\xi\chi) - 1]}{(\chi_\mu^2 + \chi^2)^2}\,, \qquad (9.6)$$

$$I_2(\xi) = 2\alpha\chi_\mu\int_0^\infty \frac{\chi d\chi[J_0(\xi\chi) - 1]}{(\chi_\mu^2 + \chi^2)(4\chi_\mu^2 + \chi^2)}\,, \qquad (9.7)$$

$$I_3(\xi) = \tfrac{1}{16}\,\xi^2\beta^2(1 + \alpha\pi)\int_0^{\chi_d} \frac{\chi^5 d\chi}{(\chi_\mu^2 + \chi^2)^2}\,, \qquad (9.8)$$

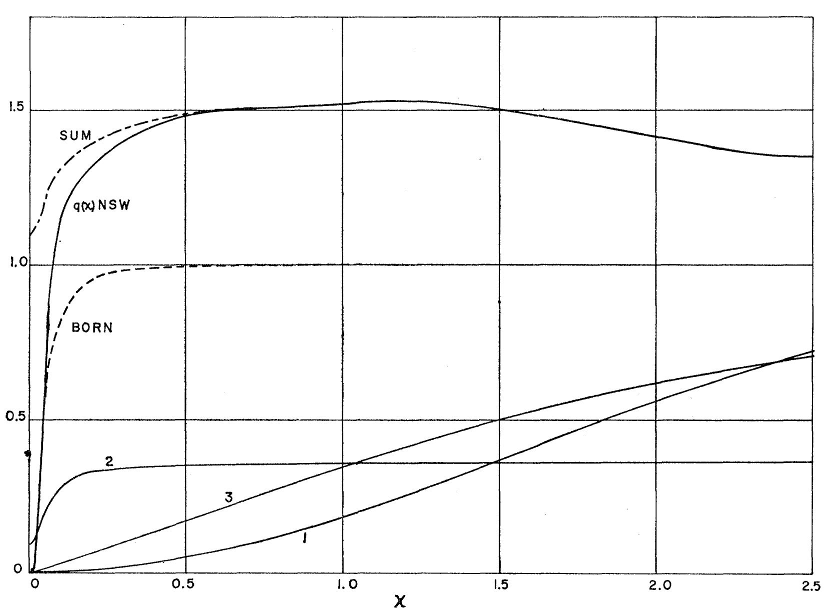


FIG. 18. Same as Fig. 17, plotted against χ on a linear scale.

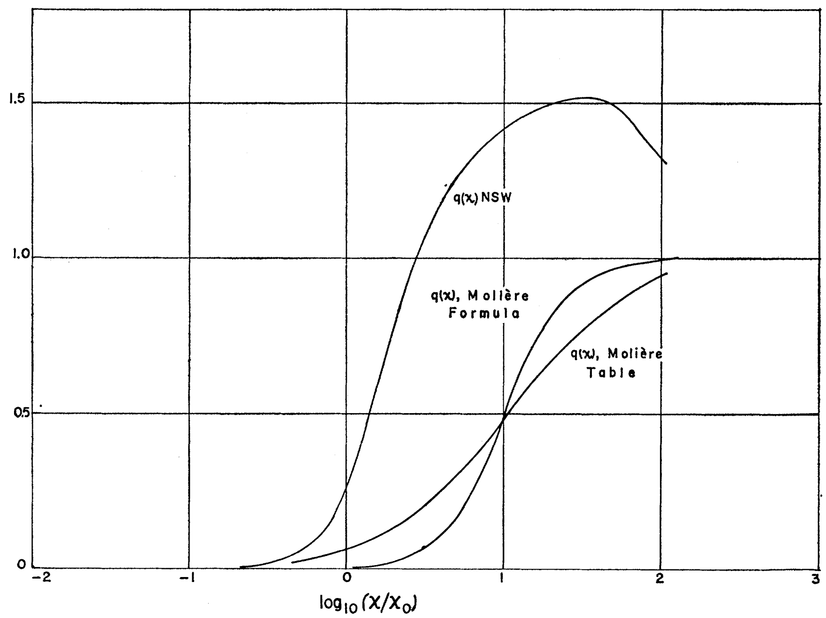


FIG. 19. Graph of $q(\chi)$ by the NSW formula (9.3) (from Fig. 17) along with the Molière formula (6.63) and Molière's numerical results (Table II), plotted against $\log_{10}(\chi/\chi_0)$. Same values of parameters as in Fig. 17.

$$I_4(\xi) = -\beta\,(1+\alpha\pi) \times \int_{\chi_d}^{\pi} \frac{\sin\chi d\chi \sin^2(\chi/2)[J_0(\xi\chi)-1]}{[2\sin(\chi/2)]^4},$$

$$= -\tfrac{1}{8}\,\beta^2(1+\alpha\pi)\int_{\chi_d}^{\pi}\frac{d\chi[J_0(\xi\chi)-1]}{\tan(\chi/2)}, \quad (9.9)$$

$$I_5(\xi) = -\tfrac{1}{4}\,\alpha\beta^2\xi^2\int_0^{\chi_d}\frac{\chi^2 d\chi}{\chi_\mu^2+\chi^2}\tan^{-1}\left(\frac{\chi}{2\chi_\mu}\right), \quad (9.10)$$

$$I_6(\xi) = \int_{\chi_d}^{\pi}\frac{\sin\chi d\chi}{[2\sin(\chi/2)]^4}[J_0(\xi\chi)-1] \times [\alpha\beta^2\pi\sin(\chi/2) - 2\alpha\beta^2\chi_\mu]. \quad (9.11)$$

Let us consider these integrals in order. By (7.15) with χ_α replaced by χ_μ, we have

$$I_1(\xi) = \tfrac{1}{8}\,\xi^2\ln(\gamma^2\chi_\mu^2\xi^2/4e) \quad (9.12)$$

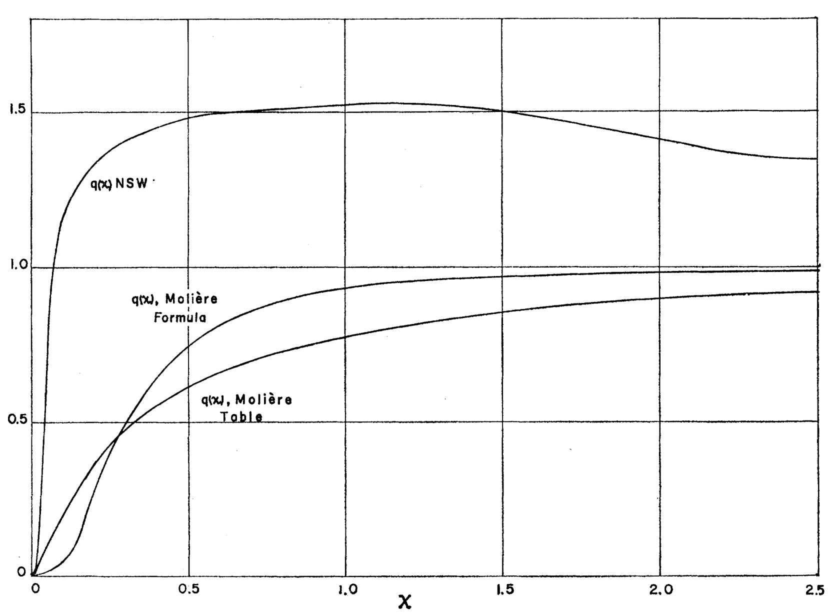


FIG. 20. Same as Fig. 19, but against χ on a linear scale.

For I_2, we write

$$I_2(\xi) = \frac{2\alpha}{3\chi_\mu}\int_0^\infty \chi d\chi[J_0(\xi\chi) - 1] \times\left[\frac{1}{\chi_\mu^2+\chi^2} - \frac{1}{4\chi_\mu^2+\chi^2}\right] = (2\alpha/3\chi_\mu)[K_0(\xi\chi_\mu) - K_0(2\xi\chi_0) - \ln 2]$$

by use of Erdelyi *et al.* (T. I. T. 1954), formula [8.5(20)], and two steps of elementary integration. Using the series for the Bessel functions [Watson (1952), p. 80; Erdelyi *et al.* H.T.F. (1953), (7.2), Eqs. (12) and (37)], we find

$$I_2(\xi) = 2\alpha[\tfrac{1}{4}\xi^2\chi_\mu \ln(\gamma\chi_\mu\xi/e) + \tfrac{1}{12}\xi^2\chi_\mu \ln 2]\,. \quad (9.13)$$

For I_3, it will be observed that the integrand can be written as χ plus terms in χ_μ^2 and χ_μ^4, so, neglecting the latter, we have

$$I_3(\xi) = (\beta^2/32)(1+\alpha\pi)\xi^2\chi_d^2\,. \quad (9.14)$$

For I_4, let us separate the terms in brackets. We have for the second,

$$\int_{\chi_d}^{\pi} d\chi/\tan(\chi/2) = 2\ln\sin(\chi/2)\Big|_{\chi_d}^{\pi} = -2\ln\sin(\chi_d/2) \simeq -2\ln(\chi_d/2) + \chi_d^2/12 + \cdots\,.$$

For the first, we note that $\tan(\chi/2)$ becomes infinite at $\chi = \pi$, so that the upper portion contributes little. In addition, $J_0(\xi\chi)$ will oscillate considerably for $\xi\chi$ between $\xi\chi_d$ which is less than $(\chi_d/\chi_c) \ll 1$, and $\xi\pi$ which will be large compared to 1 except for a very small portion near zero of the range of ξ. We therefore shall set $\tan(\chi/2) = \chi/2$ in the first part, and use (A.7), obtaining finally

$$I_4(\xi) = -\tfrac{1}{4}\beta^2(1+\alpha\pi)[-\ln(\gamma\xi\chi_d/2) + \tfrac{1}{8}\xi^2\chi_d^2 + \cdots] - \tfrac{1}{4}\beta^2(1+\alpha\pi)[\ln(\chi_d/2) - \chi_d^2/24 + \cdots] = \tfrac{1}{4}\beta^2(1+\alpha\pi)[\ln\gamma\xi - \tfrac{1}{8}\xi^2\chi_d^2 + \cdots]\,;\quad \xi\pi \gg 1\,. \quad (9.15a)$$

This formula cannot hold as $\xi \to 0$, for if ξ is small enough, we can write

$$I_4(\xi) \simeq \frac{\xi^2\beta^2}{32}(1+\alpha\pi)\int_{\chi_d}^{\pi}\frac{\chi^2 d\chi}{\tan(\chi/2)} = \tfrac{1}{4}\xi^2\beta^2(1+\alpha\pi)\int_{\chi_d/2}^{\pi/2} u^2 du \cot u = \tfrac{1}{4}\xi^2\beta^2(1+\alpha\pi)[0.658 - \tfrac{1}{8}\chi_d^2 + \cdots]\,;\quad \xi\pi \ll 1\,, \quad (9.15b)$$

where 0.658 is the numerical value of $\int_0^{\pi/2} u^2 du \cot u$. Since the range of ξ extends up to $1/\chi_c$, and since for the small-angle approximation to be valid, we must have χ_c very much less than π, the range for which (9.15b) must be used is a small fraction of that for which (9.15a) is required.

The next integral, $I_5(\xi)$, may be written

$$I_5(\xi) = -\tfrac{1}{4}\alpha\beta^2\xi^2\left[\int_0^{\chi_d} d\chi \tan^{-1}\frac{\chi}{2\chi_\mu} - \chi_\mu^2\int_0^{\chi_d}\frac{d\chi}{\chi_\mu^2+\chi^2}\tan^{-1}\frac{\chi}{2\chi_\mu}\right] = -\tfrac{1}{4}\alpha\beta^2\xi^2\left[\chi_d\tan^{-1}\frac{\chi_d}{2\chi_\mu} - \chi_\mu\ln\left(1+\frac{\chi_d^2}{4\chi_\mu^2}\right) - 2\chi_\mu\int_0^\infty\frac{dx\tan^{-1}x}{1+4x^2} + \frac{\pi\chi_\mu^2}{2\chi_d}\right],$$

where for the last term, we first assumed that $\tan^{-1}x$ is equal to $\pi/2$ when $x > \chi_d/2\chi_\mu$ and then that $\tan^{-1}(\chi_\mu/\chi_d)$, resulting from integration from $\chi_d/2\chi_\mu$ to ∞, is equal to χ_μ/χ_d. The remaining integral has the value 0.448; neglecting the last term and the 1 under the logarithm, and writing $\tan(\chi_d/2\chi_\mu) = \pi/2 - 2\chi_\mu/\chi_d$, we have

$$I_5(\xi) = -\tfrac{1}{4}\alpha\beta^2\xi^2[\tfrac{1}{2}\pi\chi_d - 2\chi_\mu - 2\chi_\mu\ln(\chi_d/2\chi_\mu) - 0.896\chi_\mu]\,. \quad (9.16)$$

Finally, for $I_6(\xi)$, we use a similar method to that for $I_4(\xi)$ writing

$$I_6(\xi) = \tfrac{1}{8}\alpha\beta^2\pi\int_{\chi_d}^{\pi}\frac{d\chi[J_0(\xi\chi)-1]}{\tan(\chi/2)\sin(\chi/2)} - 2\alpha\beta^2\chi_\mu\int_{\chi_d}^{\infty}\frac{d\chi}{\chi^3}[J_0(\xi\chi)-1] = \tfrac{1}{2}\alpha\beta^2\pi\int_{\chi_d}^{\infty}\frac{d\chi J_0(\xi\chi)}{\chi^2} - \tfrac{1}{8}\alpha\beta^2\pi\int_{\chi_d}^{\pi}\frac{d\chi\cos(\chi/2)}{\sin^2(\chi/2)} - 2\alpha\beta^2\chi_\mu\int_{\chi_d}^{\infty}\frac{d\chi}{\chi^3}[J_0(\xi\chi)-1]\,.$$

By means of (A.2), (A.10), and (A.12), we find [setting $\sin(\chi_d/2) = (\chi_d/2)$]

$$I_6(\xi) = \tfrac{1}{2}\alpha\beta^2\pi\xi\left[\frac{1}{\xi\chi_\alpha} - \frac{\xi\chi_d}{4} + \cdots - 1 + \frac{\xi\chi_d}{2} - \cdots\right] + \tfrac{1}{4}\alpha\beta^2\pi(1 - 2/\chi_d) - \tfrac{1}{2}\alpha\beta^2\chi_\mu\xi^2\ln(\gamma\chi_d\xi/2e) + \cdots = \alpha\beta^2\pi(\tfrac{1}{4} - \tfrac{1}{2}\xi + \tfrac{1}{8}\xi^2\chi_d + \cdots) - \tfrac{1}{2}\alpha\beta^2\chi_\mu\xi^2\ln(\gamma\chi_d\xi/2e) + \cdots\,;\quad \xi\pi \gg 1 \quad (9.17a)$$

this formula is also wrong for very small ξ. By a calculation similar to that for (9.15b), we find

$$I_6(\xi) = \alpha\beta^2\pi(-0.2992\xi^2 + \tfrac{1}{8}\xi^2\chi_d + \cdots) - \tfrac{1}{2}\alpha\beta\,\chi_\mu\xi^2 \ln(\gamma\chi_d\xi/2e) + \cdots; \quad \xi\pi \ll 1, \tag{9.17b}$$

where 0.2992 is the value of $\frac{1}{4}\int_0^{\pi/2} u^2 du \cot u \csc u$.

If the six integrals in Eqs. (9.12), (9.13), (9.14), (9.15a), (9.16), and (9.17a) are added and the terms rearranged slightly, we find for the exponent of the transform

$$\Omega - \Omega_0 = \tfrac{1}{4}\xi^2\chi_c^2 \ln(\gamma^2\xi^2/4e) + \tfrac{1}{2}\xi^2\chi_c^2[\ln\chi_\mu + 2\alpha\chi_\mu \ln 2^{1/3}\chi_\mu + 2\alpha\beta^2\chi_\mu(1.448 - \ln\chi_\mu)] + \xi^2\chi_c^2\chi_\mu\alpha(1-\beta^2)\ln(\gamma\xi/e) + \tfrac{1}{2}\chi_c^2\beta^2(1+\alpha\pi) \times \ln\gamma\xi + \chi_c^2\alpha\beta^2\pi(\tfrac{1}{2} - \xi). \tag{9.18}$$

Following Nigam, Sundaresan, and Wu, we define the screening angle χ_α by setting the square bracket equal to $\ln\chi_\alpha$ (our previous definition needs modification to the large-angle case before it can be applied here)[17]:

$$\ln\chi_\alpha = \ln\chi_\mu + 2\alpha\chi_\mu \ln 2^{1/3}\chi_\mu + 2\alpha\beta^2\chi_\mu(1.448 - \ln\chi_\mu). \tag{9.19a}$$

It is the difference between the values of $\ln\chi_\alpha$ given by (9.19a) and (6.62) that provides the main contrast between the present calculation and that of Molière and may be taken as a numerical measure of the contrast between the respective q's in Figs. 17–20.

It should be noticed that, since (9.19a) involves $\alpha\chi_\mu$ in place of α itself, and since $\alpha\chi_\mu$ is small in accordance with (6.22), χ_α and χ_μ cannot differ greatly. In fact an expansion in powers of $\alpha\chi_\mu$ may be taken to first order, yielding the good approximation

$$\chi_\alpha^2 = \chi_\mu^2\{1 + 4\alpha\chi_\mu[(1-\beta^2)\ln\chi_\mu + 0.2310 + 1.448\beta^2]\}. \tag{9.19b}$$

The principle difference between χ_α and χ_0 is thus determined by the value of μ in $\chi_\mu = \mu\chi_0$.

Table VI shows a comparison of screening angles calculated from (6.62) and (9.19b), using for the latter, the values $\mu = 1.12$ and 1.80 as given in the article of Nigam *et al.* (see part D of this section) as well as $\mu = 1$.

[17] Nigam *et al.* assert in their Eq. (56) that $\ln(2/\chi_\alpha) - \frac{1}{2}$ is given by $\int_0^1 dyq(y)/y$ (where $y = \sin\chi/2$), a quantity called $\ln\xi$ by Goudsmit and Saunderson (1940a). Actually, as defined above,
$\ln(2/\chi_\alpha) - \frac{1}{2} = \int_0^1 dyq(y)/y + \frac{1}{2}\beta^2(\alpha\pi - 1) +$ terms in χ_μ^2.

Let us next define a correction constant ζ by

$$\zeta = 2\alpha\chi_\mu(1-\beta^2). \tag{9.20}$$

This constant is small compared to one, particularly for relativistic particles and low-atomic-number materials.

TABLE VI. Screening angles for Molière (1948) and for Nigam *et al.* (1959), for multiple scattering of electrons on beryllium and gold (from Nigam *et al.*, corrected).

β	χ_0	$\chi_\alpha^2/\chi_0^2 - 1.13$, Molière		$\chi_\alpha^2/\chi_\mu^2 - 1$, Nigam *et al.*	
		$\mu = 1$	$\mu = 1$	$\mu = 1.12$	$\mu = 1.80$
Beryllium, $Z = 4$; $z = 1$; $\alpha\beta = 0.0292$					
0.6	0.0174	0.00890	−0.00625	−0.00671	−0.00894
0.7	0.0134	0.00655	−0.00282	−0.00302	−0.00388
0.8	0.00982	0.00501	−0.00073	−0.00075	−0.00076
0.9	0.00634	0.00396	+0.00036	+0.00043	+0.00081
0.99	0.00187	0.00327	0.00034	0.00038	0.00061
0.999	0.00059	0.00321	0.00011	0.00013	0.00021
Gold, $Z = 79$; $z = 1$; $\alpha\beta = 0.576$					
0.6	0.0470	3.48	−0.218	−0.229	−0.269
0.7	0.0360	2.56	−0.895	−0.0927	−0.0970
0.8	0.0265	1.96	−0.0113	−0.0092	+0.0082
0.9	0.0171	1.55	+0.0276	+0.0319	0.0584
0.99	0.0051	1.27	0.0180	0.0202	0.0327
0.999	0.0016	1.25	0.0061	0.0068	0.0109

Using the definitions (9.19) and (9.20), we find

$$\Omega - \Omega_0 = \tfrac{1}{4}\xi^2\chi_c^2 \ln\left[\left(\frac{\gamma^2\xi^2}{4e}\right)^{1+\zeta}\left(\frac{4}{e}\right)^{\zeta}\chi_\alpha^2\right] + \tfrac{1}{2}\chi_c^2\beta^2(1+\alpha\pi)\ln\gamma\xi + \chi_c^2\alpha\beta^2\pi(\tfrac{1}{2} - \xi). \tag{9.21}$$

Again following Nigam *et al.* in essence, we define B_N by the relation

$$B_N = \ln\left[\left(\frac{eB_N\chi_c^2}{\gamma^2}\right)^{1+\zeta}\left(\frac{e}{4}\right)^{\zeta}\frac{1}{\chi_\alpha^2}\right], \tag{9.22}$$

which obeys an equation similar to that for the Molière B, namely,

$$\frac{B_N}{1+\zeta} = \ln[B_N/(1+\zeta)] + \ln(e/\gamma^2) + \ln[(1+\zeta)\chi_c^2\chi_\alpha^{-2/(1+\zeta)}(e/4)^{\zeta/(1+\zeta)}]. \tag{9.23a}$$

In order to use the results in Eqs. (7.39) and (7.40), let us set

$$\Omega_{0N} = \frac{(1+\zeta)\chi_c^2(e/4)^{\zeta/(1+\zeta)}}{\chi_\alpha^{2/(1+\zeta)}}, \tag{9.24a}$$

so that

$$\left(\frac{B_N}{1+\zeta}\right) = \ln\left(\frac{B_N}{1+\zeta}\right) + \ln(\Omega_{0N}e/\gamma^2). \tag{9.23b}$$

When ζ may be set equal to zero, as is frequently the case, Ω_{0N} becomes Ω_0 and B_N becomes B.

For small but not negligible ζ, we can use expansions taken to the first power:

$$B_N \simeq B\{1 + \zeta[0.614 + \ln \chi_\alpha^2/(B-1)]\} \quad (9.23c)$$

$$\Omega_{0N} \simeq \Omega_0\{1 + \zeta[0.614 + \ln \chi_\alpha^2]\}\,, \quad (9.24b)$$

where B and Ω_0 are computed with $\zeta = 0$, i.e., by Eqs. (7.38) and (7.39).

We finally have for the transform exponent as a function of η

$$\begin{aligned}\Omega(\eta) - \Omega_0 = &-\eta^2/4 + (1+\zeta)(\eta^2/4B_N)\ln(\eta^2/4) \\ &+ \tfrac{1}{4}\chi_c^2\beta^2(1+\alpha\pi)\ln(4\gamma^2/\chi_c^2 B_N) \\ &+ \tfrac{1}{4}\chi_c^2\beta^2(1+\alpha\pi)\ln(\eta^2/4) + \tfrac{1}{2}\chi_c^2\alpha\beta^2\pi \\ &- \chi_c\alpha\beta^2\pi\eta/B_N^{1/2} \quad (9.25a)\end{aligned}$$

with

$$\eta = \xi\chi_c B_N^{1/2}\,. \quad (9.25b)$$

This expression agrees with Eq. (62) of Nigam *et al.* with the exception of a term $\frac{1}{16}\chi_c^2[B - (1+\zeta)\ln(\eta^2/4)]$, which they obtain from setting $l + \frac{1}{2}$ equal to the variable we call ξ, and then putting $l(l+1) = (l+\frac{1}{2})^2 - \frac{1}{4}$. The $-1/4$ leads to the extra term; in their final calculations its effect is neglected; furthermore, the errors made by them in using the asymptotic formula $\psi(l) \simeq \ln(l+\frac{1}{2})$ down to $l = 0$ are of the same order. Still another error of this order is involved in the difference between (9.15a) and (9.15b), which we shall consider below.

In Eq. (9.25), there are two variable terms in addition to the Molière-like terms. The coefficient of η, $\chi_c\alpha\beta^2\pi B_N^{-1/2} = \chi_c zZ\beta\pi/137B_N^{1/2}$ will generally be less than 1/4 for singly or doubly charged incident particles, since B_N will be greater than 4.5 and we may assume that the "Gaussian width" $\chi_c B_N^{1/2}$ of the distribution is less than 1/4 of a radian to allow use of the small-angle approximation. The coefficient ι of $\ln(\eta^2/4)$,

$$\iota = \tfrac{1}{4}\chi_c^2\beta^2(1+\alpha\pi) = \tfrac{1}{4}\chi_c^2(\beta^2 + zZ\beta\pi/137) \quad (9.26)$$

will be less than 0.02. These limits will be different for $z > 2$, but the development to be given below may be readily modified for any case in which ι becomes appreciably larger than 0.02.

B. Calculation of the Distribution Functions

We now expand $\exp[\Omega - \Omega_0 + \frac{1}{4}\eta^2]$ in a power series. The terms in α represent contributions of the second Born approximation, so it is not legitimate to carry the expansion to powers of α beyond the first, since the third Born approximation would introduce terms in α^2 in (9.25a). That is to say, $\chi_c\alpha\beta^2\pi B_N^{-1/2}$ must also on this account be small compared to 1. Terms involving $\iota\alpha$ contain α^2; when α is small, the small-angle restriction on χ_c(i.e., $\chi_c^2 \lesssim 1/9B$) will make terms in $\iota(\chi_c\alpha\beta^2\pi)$ generally negligible. One such term, however, has been included for the sake of completeness.

Terms thus neglected can readily be included and evaluated by the methods given below, in the event that it is desired to calculate the errors due to their neglect (which may be comparable to errors due to the neglect of the next higher Born terms).

We find then, for the spatial distribution

$$\begin{aligned}2\pi F_{\text{red}}(\vartheta_N,t) = K\int_0^\infty \eta d\eta J_0(\vartheta_N\eta)e^{-\eta^2/4}\Big[1 + \frac{(1+\zeta)}{B_N} \\ \times\left(\frac{\eta^2}{4}\ln\frac{\eta^2}{4}\right) - \frac{2\chi_c\alpha\beta^2\pi}{B_N^{1/2}}\left(\frac{\eta}{2}\right) + \iota\ln\frac{\eta^2}{4} \\ + \frac{(1+\zeta)^2}{2B_N^2}\left(\frac{\eta^2}{4}\ln\frac{\eta^2}{4}\right)^2 + \frac{\iota(1+\zeta)}{B_N}\left(\frac{\eta^2}{4}\ln^2\frac{\eta^2}{4}\right) \\ - \frac{2\chi_c\alpha\beta^2\pi(1+\zeta)}{B_N^{3/2}}\left(\frac{\eta^3}{8}\ln\frac{\eta^2}{4}\right) - \frac{\chi_c\alpha\beta^2\pi(1+\zeta)^2}{B_N^{5/2}} \\ \times\left(\frac{\eta^5}{32}\ln^2\frac{\eta^2}{4}\right) - \frac{2\iota\chi_c\alpha\beta^2\pi}{B_N^{1/2}}\left(\frac{\eta}{2}\ln\frac{\eta^2}{4}\right)\Big], \quad (9.27a)\end{aligned}$$

and the coefficient K is

$$K = \exp\{\tfrac{1}{2}\chi_c^2[\alpha\beta^2\pi + \beta^2(1+\alpha\pi)\ln 2\gamma/\chi_c B_N^{1/2}]\}\,. \quad (9.27b)$$

The maximum value of the exponent in (9.27b) is seen to be about $1/32 + 0.11 = 0.14$.

The reduced angle ϑ_N is now given by

$$\vartheta_N = \theta/\chi_c B_N^{1/2}\,. \quad (9.28)$$

Nigam *et al.* do not have the terms with ι in (9.27a), nor do they give the next-to-last one. For small B_N, the omission of terms involving the third and higher powers of $B_N^{-1}\ln(\eta^2/4)$ will make a larger error than the omission of the terms containing ι, but we have not included the former because computations relating to them are not available. The inverse transforms of each of these terms can be found from the two integrals {Erdelyi (1954) T.I.T. [8.6(14)] and [1.4(14)]}

$$2\Gamma(a)_1F_1(a;1;-\vartheta^2) = \int_0^\infty \eta d\eta J_0(\eta\vartheta)e^{-\eta^2/4}(\eta^2/4)^{a-1}\,; \quad (9.29a)$$

$$\Gamma(a)_1F_1(a;\tfrac{1}{2};-\varphi^2) = \int_0^\infty d\eta\cos(\eta\varphi)e^{-\eta^2/4}(\eta^2/4)^{a-1/2}\,. \quad (9.29b)$$

The function ${}_1F_1$ is Kummer's confluent hypergeometric function. Differentiating these expressions with respect to a once or twice will give the transforms of terms with $\ln(\eta^2/4)$ and $\ln^2(\eta^2/4)$. Let us define a set of functions $D_n(a;b;z)$ by the relations

$$D_n(a,b,z) = (\partial^n/\partial a^n)\Gamma(a){}_1F_1(a;b;z)\,. \quad (9.29\text{c})$$

The spatial-angle distribution may now be written

$$\begin{aligned}2\pi F_{\text{red}}(\vartheta_N,t) = 2K\Big\{&D_0(1,1,-\vartheta_N^2)+\iota D_1(1,1,-\vartheta_N^2)\\&-\frac{2\chi_c\alpha\beta^2\pi}{B_N^{1/2}}[D_0(\tfrac{3}{2},1,-\vartheta_N^2)+\iota D_1(\tfrac{3}{2},1,-\vartheta_N^2)]\\&+\frac{(1+\zeta)}{B_N}[D_1(2,1,-\vartheta_N^2)+\iota D_2(2,1,-\vartheta_N^2)]\\&-\frac{2\chi_c\alpha\beta^2\pi(1+\zeta)}{B_N^{3/2}}D_1(\tfrac{5}{2},1,-\vartheta_N^2)+\frac{(1+\zeta)^2}{2B_N^2}\\&\times D_2(3,1,-\vartheta_N^2)-\frac{\chi_c\alpha\beta^2\pi(1+\zeta)^2}{B_N^{5/2}}\\&\times D_2(\tfrac{7}{2},1,-\vartheta_N^2)\Big\}, \quad (9.30)\end{aligned}$$

and correspondingly, the distribution in projected angle is

$$\begin{aligned}\frac{\pi}{2}f_{\text{red}}(\varphi_N,t) = K\Big\{&D_0(\tfrac{1}{2},\tfrac{1}{2},-\varphi_N^2)+\iota D_1(\tfrac{1}{2},\tfrac{1}{2},-\varphi_N^2)\\&-\frac{2\chi_c\alpha\beta^2\pi}{B_N^{1/2}}[D_0(1,\tfrac{1}{2},-\varphi_N^2)+\iota D_1(1,\tfrac{1}{2},-\varphi_N^2)]\\&+\frac{(1+\zeta)}{B_N}[D_1(\tfrac{3}{2},\tfrac{1}{2},-\varphi_N^2)+\iota D_2(\tfrac{3}{2},\tfrac{1}{2},-\varphi_N^2)]\\&-\frac{2\chi_c\alpha\beta^2\pi(1+\zeta)}{B_N^{3/2}}D_1(2,\tfrac{1}{2},-\varphi_N^2)\\&+\frac{(1+\zeta)^2}{2B_N^2}D_2(\tfrac{5}{2},\tfrac{1}{2},-\varphi_N^2)\\&-\frac{\chi_c\alpha\beta^2\pi(1+\zeta)^2}{B_N^{5/2}}D_2(3,\tfrac{1}{2},-\varphi_N^2)\Big\}\,. \quad (9.31)\end{aligned}$$

The terms in (9.32) and (9.33) are given in order of increasing inverse powers of B_N, but the actual relative orders of magnitude will depend on the values of B_N, $\chi_c\alpha\beta^2\pi$, ζ, and ι.

The results of the Molière calculations are included in the above, if we set $\vartheta_N=\vartheta$, $\varphi_N=\varphi$, $B_N=B$ and $\iota=\zeta=0$, (all of which equalities are justifiable for nonrelativistic scattering, as indicated in Sec. VI) and ignore the terms with $\chi_c\alpha\beta^2\pi$. In fact, we have the relations [cf. Eqs. (7.47) and (7.50)]

$$D_0(1,1,-\vartheta^2)=e^{-\vartheta^2}, \quad (9.32\text{a})$$

$$D_0(\tfrac{1}{2},\tfrac{1}{2},-\varphi^2)=\pi^{1/2}e^{-\varphi^2}, \quad (9.32\text{b})$$

$$2D_1(2,1,-\vartheta^2)=F^{(1)}(\vartheta), \quad (9.32\text{c})$$

$$(2/\pi)D_1(\tfrac{3}{2},\tfrac{1}{2},-\varphi^2)=f^{(1)}(\varphi), \quad (9.32\text{d})$$

$$D_2(3,1,-\vartheta^2)=F^{(2)}(\varphi), \quad (9.32\text{e})$$

$$(1/\pi)D_2(\tfrac{5}{2},\tfrac{1}{2},-\varphi^2)=f^{(2)}(\varphi)\,. \quad (9.32\text{f})$$

Details concerning the D_n functions are found in Appendix III; Figs. 21–26 and Tables VII and VIII give some numerical values.[17]

Table III of Nigam *et al.* contains a column for their function $f^{(1)\prime}=4\chi_c\alpha\beta^2\pi B^{1/2}D_0(\tfrac{3}{2},1,-\vartheta^2)$, calculated for $\mu=1.80$, $Z=79$, $B=6.98$, $\beta=1$. The results agree with ours for $\vartheta=0$, 1.0, 2.0, 3.0 and 4.0, but disagree considerably for the intermediate values. It is not possible from their article to detect

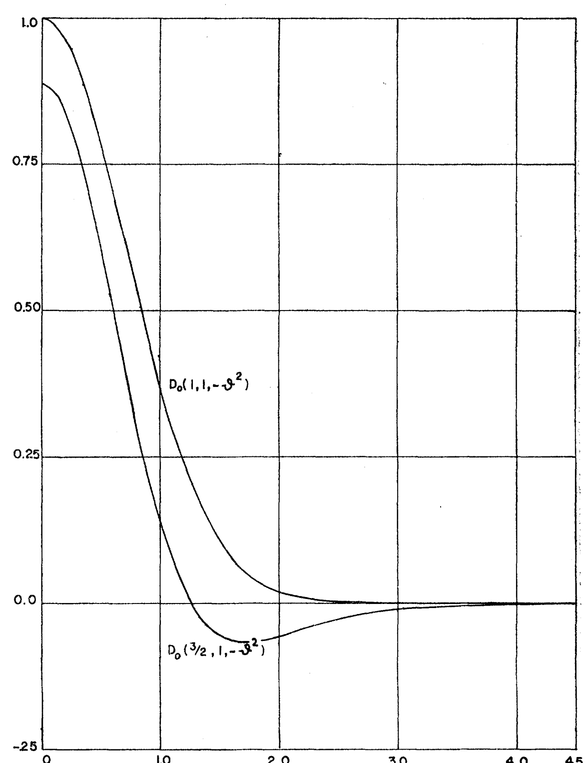


FIG. 21. The functions $D_0(1,1,-\vartheta^2)$ and $D_0(\tfrac{3}{2},1,-\vartheta^2)$ against ϑ.

the source of their mistake.[18] The mistake was continued by Fleischmann (1960), who evidently derived his function $g=4\,D_0(\tfrac{3}{2},1,-\vartheta^2)$ from their

[18] M. K. Sundaresan has indicated (private communication) that he and his coauthors have also detected the existence of mistakes in their numerical values.

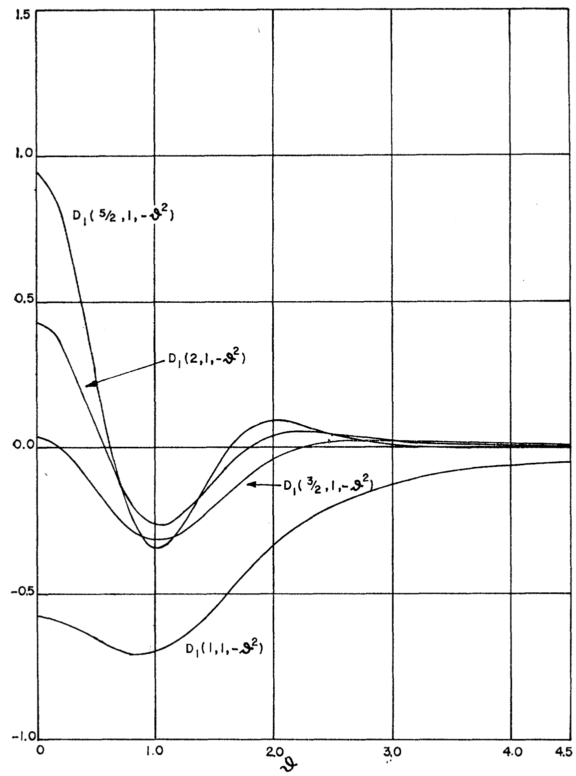


FIG. 22. The functions $D_1(1,1,-\vartheta^2)$, $D_1(\frac{3}{2},1,-\vartheta^2)$, $D_1(2,1,-\vartheta^2)$ and $D_1(\frac{5}{2},1,-\vartheta^2)$ against ϑ.

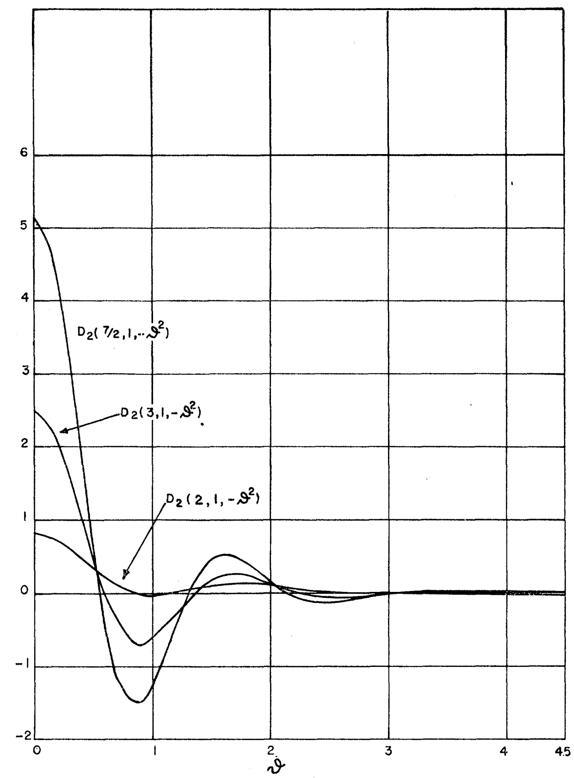


FIG. 23. The functions $D_2(2,1,-\vartheta^2)$, $D_2(3,1,-\vartheta^2)$, and $D_2(\frac{7}{2},1,-\vartheta^2)$ against ϑ.

TABLE VII. Selected values of the D_n functions for spatial-angle scattering.[17] The notation E-02 means that a factor 10^{-2} is to be appended, etc.

ϑ	$D_0(1,1,-\vartheta^2)$	$D_0(3/2,1,-\vartheta^2)$	$D_1(1,1,-\vartheta^2)$	$D_1(3/2,1,-\vartheta^2)$	$D_1(2,1,-\vartheta^2)$	$D_1(5/2,1,-\vartheta^2)$	$D_2(2,1,-\vartheta^2)$	$D_2(3,1,-\vartheta^2)$	$D_2(7/2,1,-\vartheta^2)$
0.	1.0000	0.8862	−0.5772	0.0323	0.4228	0.9347	0.8237	2.4929	5.1423
0.1	0.9901	0.8730	−0.5814	+0.0231	0.4045	0.8985	0.7990	2.3827	4.8926
0.2	0.9608	0.8344	−0.5934	−0.0036	0.3519	0.7944	0.7283	2.0694	4.1857
0.3	0.9139	0.7731	−0.6117	−0.0447	0.2712	0.6358	0.6208	1.6014	3.1399
0.4	0.8521	0.6935	−0.6339	−0.0955	0.1718	0.4425	0.4899	1.0488	1.9229
0.5	0.7788	0.6011	−0.6571	−0.1505	+0.0648	0.2373	0.3514	+0.4896	+0.7191
0.6	0.6977	0.5017	−0.6784	−0.2041	−0.0388	+0.0427	0.2205	−0.0044	−0.3074
0.7	0.6126	0.4010	−0.6950	−0.2509	−0.1292	−0.1224	0.1098	−0.3794	−1.0407
0.8	0.5273	0.3044	−0.7045	−0.2870	−0.1990	−0.2448	+0.0278	−0.6068	−1.4295
0.9	0.4449	0.2159	−0.7055	−0.3097	−0.2443	−0.3186	−0.0220	−0.6852	−1.4873
1.0	0.3679	0.1386	−0.6972	−0.3180	−0.2642	−0.3446	−0.0411	−0.6359	−1.2799
1.1	0.2982	0.7422E-01	−0.6796	−0.3126	−0.2609	−0.3295	−0.0344	−0.4959	−0.9043
1.2	0.2369	+0.2322E-01	−0.6537	−0.2951	−0.2385	−0.2834	−0.0094	−0.3086	−0.4648
1.3	0.1845	−0.1488E-01	−0.6207	−0.2682	−0.2026	−0.2183	+0.0253	−0.1151	−0.0524
1.4	0.1409	−0.4134E-01	−0.5824	−0.2349	−0.1591	−0.1458	0.0619	+0.0525	+0.2692
1.5	0.1054	−0.5789E-01	−0.5407	−0.1983	−0.1134	−0.7559E-01	0.0941	0.1740	0.4685
1.6	0.7730E-01	−0.6648E-01	−0.4972	−0.1612	−0.6978E-01	−0.1481E-01	0.1176	0.2422	0.5440
1.7	0.5558E-01	−0.6905E-01	−0.4536	−0.1256	−0.3149E-01	+0.3251E-01	0.1304	0.2603	0.5164
1.8	0.3916E-01	−0.6739E-01	−0.4113	−0.9335E-01	−0.3050E-03	0.6504E-01	0.1325	0.2387	0.4188
1.9	0.2705E-01	−0.6301E-01	−0.3713	−0.6531E-01	+0.2313E-01	0.8355E-01	0.1253	0.1911	0.2868
2.0	0.1832E-01	−0.5712E-01	−0.3342	−0.4193E-01	0.3911E-01	0.9027E-01	0.1111	0.1316	+0.1513
2.2	0.7907E-02	−0.4414E-01	−0.2700	−0.8731E-02	0.5271E-01	0.7996E-01	0.7182E-01	+0.1964E-01	−0.0526
2.4	0.3151E-02	−0.3261E-01	−0.2193	+0.9518E-02	0.5039E-01	0.5609E-01	0.3245E-01	−0.4672E-01	−0.1291
2.6	0.1159E-02	−0.2380E-01	−0.1804	0.1769E-01	0.4131E-01	0.3313E-01	+0.2828E-02	−0.6489E-01	−0.1141
2.8	0.3937E-03	−0.1756E-01	−0.1507	0.2009E-01	0.3124E-01	0.1678E-01	−0.1496E-01	−0.5460E-01	−0.6668E-01
3.0	0.1234E-03	−0.1326E-01	−0.1278	0.1965E-01	0.2275E-01	0.7167E-02	−0.2320E-01	−0.3569E-01	−0.2519E-01
3.2	0.3571E-04	−0.1028E-01	−0.1100	0.1807E-01	0.1644E-01	0.2310E-02	−0.2536E-01	−0.1923E-01	−0.1440E-02
3.4	0.9540E-05	−0.8177E-02	−0.9583E-01	0.1620E-01	0.1201E-01	+0.1828E-03	−0.2434E-01	−0.8470E-02	+0.7684E-02
3.6	0.2353E-05	−0.6644E-02	−0.8436E-01	0.1439E-01	0.8953E-02	−0.6006E-03	−0.2201E-01	−0.2643E-02	0.8902E-02
3.8	0.5355E-06	−0.5493E-02	−0.7491E-01	0.1276E-01	0.6828E-02	−0.8041E-03	−0.1935E-01	+0.4572E-04	0.7215E-02
4.0	0.1125E-06	−0.4606E-02	−0.6702E-01	0.1134E-01	0.5319E-02	−0.7884E-03	−0.1681E-01	0.1074E-02	0.5111E-02
5.0	0.1389E-10	−0.2207E-02	−0.4175E-01	0.6642E-02	0.1915E-02	−0.3759E-03	−0.8364E-02	0.8326E-03	0.7141E-03
6.0	0.2320E-15	−0.1237E-02	−0.2860E-01	0.4230E-02	0.8700E-03	−0.1766E-03	−0.4548E-02	0.3495E-03	0.1449E-03
7.0	0.5243E-21	−0.7647E-03	−0.2084E-01	0.2870E-02	0.4540E-03	−0.9103E-04	−0.2686E-02	0.1583E-03	0.3840E-04
8.0		−0.5063E-03	−0.1588E-01	0.2044E-02	0.2606E-03	−0.5076E-04	−0.1692E-02	0.7833E-04	0.1197E-04
9.0		−0.3528E-03	−0.1250E-01	0.1511E-02	0.1604E-03	−0.3017E-04	−0.1122E-02	0.4176E-04	0.4131E-05
10.0		−0.2558E-03	−0.1010E-01	0.1151E-02	0.1042E-03	−0.1888E-04	−0.7748E-03	0.2368E-04	0.1511E-05
11.0		−0.1914E-03	−0.8334E-02	0.8992E-03	0.7065E-04	−0.1232E-04	−0.5533E-03	0.1413E-04	0.5604E-06
12.0		−0.1470E-03	−0.6993E-02	0.7166E-03	0.4961E-04	−0.8336E-05	−0.4064E-03	0.8801E-05	0.1963E-06
13.0		−0.1153E-03	−0.5953E-02	0.5812E-03	0.3586E-04	−0.5810E-05	−0.3056E-03	0.5684E-05	0.543E-07

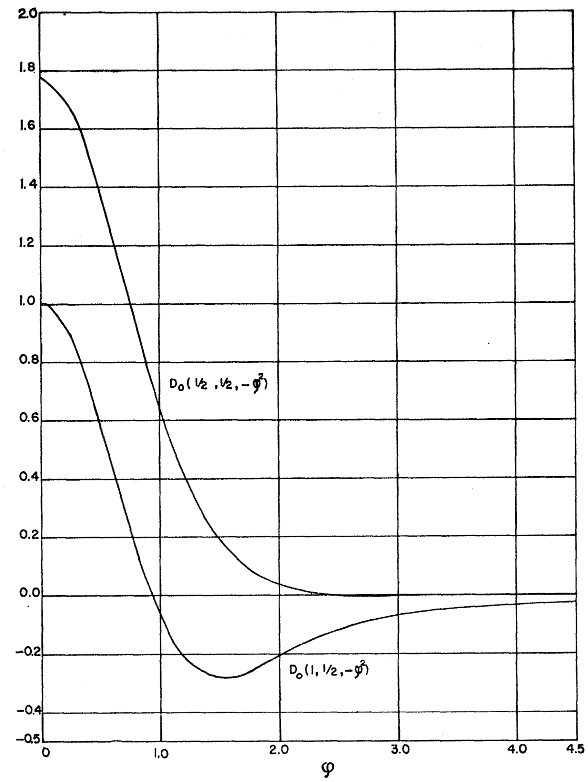


FIG. 24. The functions $D_0(\frac{1}{2},\frac{1}{2}, -\varphi^2)$ and $D_0(1,\frac{1}{2}, -\varphi^2)$ against φ.

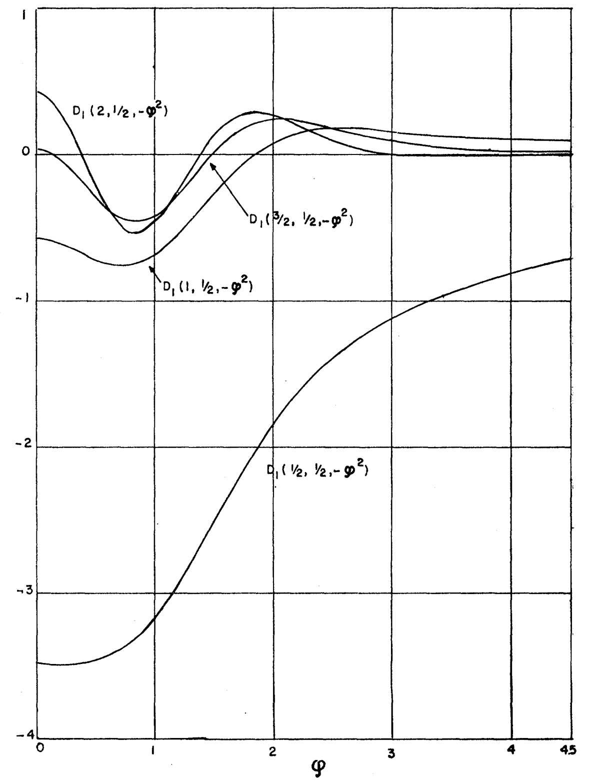


FIG. 25. The functions $D_1(\frac{1}{2},\frac{1}{2}, -\varphi^2)$, $D_1(1,\frac{1}{2}, -\varphi^2)$, $D_1(\frac{3}{2},\frac{1}{2}, -\varphi^2)$ and $D_1(2,\frac{1}{2}, -\varphi^2)$ against φ.

TABLE VIII. Selected values of the D_n functions for projected-angle scattering.[17] Notation same as in Table VII.

φ	$D(1/2,1/2,-\varphi^2)$	$D_0(1,1/2,-\varphi^2)$	$D(1/2,1/2,-\varphi^2)$	$D_1(1,1/2,-\varphi^2)$	$D_1(3/2,1/2,-\varphi^2)$	$D(2,1/2,-\varphi^2)$	$D_2(3/2,1/2,-\varphi^2)$	$D_2(5/2,1/2,-\varphi^2)$	$D_2(3,1/2,-\varphi^2)$
0.	1.7725	1.0000	−3.4802	−0.5772	0.0323	0.4228	0.8296	1.3091	2.4929
0.1	1.7548	0.9801	−3.4808	−0.5855	+0.0139	0.3864	0.8038	1.2080	2.2735
0.2	1.7030	0.9221	−3.4818	−0.6091	−0.0386	0.2830	0.7302	0.9239	1.6610
0.3	1.6199	0.8304	−3.4812	−0.6438	−0.1171	+0.1290	0.6202	0.5112	+0.7820
0.4	1.5104	0.7120	−3.4759	−0.6836	−0.2098	−0.0515	0.4898	+0.0465	−0.1864
0.5	1.3804	0.5756	−3.4622	−0.7213	−0.3028	−0.2314	0.3575	−0.3875	−1.0578
0.6	1.2366	0.4303	−3.4364	−0.7499	−0.3827	−0.3851	0.2404	−0.7192	−1.6773
0.7	1.0859	0.2853	−3.3953	−0.7632	−0.4387	−0.4922	0.1515	−0.9014	−1.9531
0.8	0.9346	0.1486	−3.3369	−0.7570	−0.4633	−0.5412	0.0981	−0.9189	−1.8699
0.9	0.7885	+0.0267	−3.2600	−0.7294	−0.4541	−0.5299	0.0810	−0.7880	−1.4842
1.0	0.6520	−0.0613	−3.1651	−0.6893	−0.4128	−0.4646	0.0950	−0.5495	−0.8945
1.1	0.5285	−0.1577	−3.0537	−0.6136	−0.3451	−0.3587	0.1308	−0.2576	−0.2594
1.2	0.4199	−0.2175	−2.9286	−0.5323	−0.2590	−0.2288	0.1770	+0.0318	+0.3274
1.3	0.3271	−0.2568	−2.7931	−0.4418	−0.1637	−0.0925	0.2220	0.2727	0.7653
1.4	0.2497	−0.2782	−2.6511	−0.3476	−0.0680	+0.0347	0.2564	0.4350	1.0070
1.5	0.1868	−0.2847	−2.5064	−0.2547	+0.0204	0.1412	0.2737	0.5074	1.0499
1.6	0.1370	−0.2798	−2.3624	−0.1673	0.0960	0.2203	0.2710	0.4958	0.9271
1.7	0.9851E-01	−0.2667	−2.2223	−0.0887	0.1556	0.2696	0.2487	0.4182	0.6938
1.8	0.6942E-01	−0.2484	−2.0883	−0.0207	0.1980	0.2910	0.2098	0.2991	0.4112
1.9	0.4795E-01	−0.2273	−1.9621	+0.0357	0.2241	0.2887	0.1589	0.1640	+0.1343
2.0	0.3246E-01	−0.2054	−1.8449	0.0806	0.2357	0.2689	+0.1015	+0.0344	−0.0971
2.2	0.1401E-01	−0.1638	−1.6386	0.1394	0.2269	0.2004	−0.0133	−0.1535	−0.3528
2.4	0.5585E-02	−0.1295	−1.4683	0.1663	0.1942	0.1261	−0.1052	−0.2212	−0.3570
2.6	0.2055E-02	−0.1033	−1.3290	0.1728	0.1554	0.6741E-01	−0.1630	−0.1990	−0.2335
2.8	0.6978E-03	−0.8388E-01	−1.2146	0.1682	0.1203	0.2958E-01	−0.1894	−0.1387	−0.9870E-01
3.0	0.2187E-03	−0.6963E-01	−1.1196	0.1587	0.9240E-01	+0.8772E-02	−0.1936	−0.7854E-01	−0.8383E-02
3.2	0.6330E-04	−0.5896E-01	−1.0395	0.1476	0.7155E-01	−0.1066E-02	−0.1850	−0.3495E-01	+0.3280E-01
3.4	0.1691E-04	−0.5076E-01	−0.9709	0.1366	0.5634E-01	−0.4935E-02	−0.1706	−0.9239E-02	0.4185E-01
3.6	0.4170E-05	−0.4430E-01	−0.9113	0.1264	0.4524E-01	−0.5983E-02	−0.1545	0.3491E-02	0.3641E-01
3.8	0.9492E-05	−0.3908E-01	−0.8590	0.1171	0.3700E-01	−0.5867E-02	−0.1388	0.8610E-02	0.2733E-01
4.0	0.1995E-06	−0.3478E-01	−0.8127	0.1087	0.3074E-01	−0.5351E-02	−0.1245	0.9916E-02	0.1923E-01
5.0	0.2462E-10	−0.2134E-01	−0.6417	0.7786E-01	0.1436E-01	−0.2826E-02	−0.7457E-01	0.5630E-02	0.3312E-02
6.0	0.4111E-15	−0.1451E-01	−0.5312	0.5877E-01	0.7951E-02	−0.1559E-02	−0.4798E-01	0.2718E-02	0.8120E-03
7.0	0.9293E-21	−0.1053E-01	−0.4535	0.4613E-01	0.4884E-02	−0.9281E-03	−0.3278E-01	0.1416E-02	0.2475E-03
8.0		−0.8003E-02	−0.3958	0.3729E-01	0.3221E-02	−0.5878E-03	−0.2346E-01	0.7942E-03	0.8554E-04
9.0		−0.6291E-02	−0.3513	0.3085E-01	0.2238E-02	−0.3912E-03	−0.1742E-01	0.4740E-03	0.3159E-04
10.0		−0.5077E-02	−0.3158	0.2600E-01	0.1620E-02	−0.2710E-03	−0.1331E-01	0.2976E-03	0.1179E-04
11.0		−0.4185E-02	−0.2868	0.2224E-01	0.1210E-02	−0.1941E-03	−0.1043E-01	0.1948E-03	0.4082E-05
12.0		−0.3509E-02	−0.2627	0.1927E-01	0.9285E-03	−0.1428E-03	−0.8330E-02	0.1320E-03	0.9982E-06
13.0		−0.2985E-02	−0.2424	0.1688E-01	0.7280E-03	−0.1076E-03	−0.6769E-02	0.9217E-04	−0.2150E-06

table. The values given by Nigam *et al.* for their $f^{(1)} = D_1(2,\ 1,\ -\vartheta^2)$ which were obtained from Bethe (1953) agree exactly with ours; their table of $f^{(0)} = 2\ D_0(1,\ 1,\ -\vartheta^2) = 2\ \exp\ (-\vartheta^2)$ contains 3 mistakes.

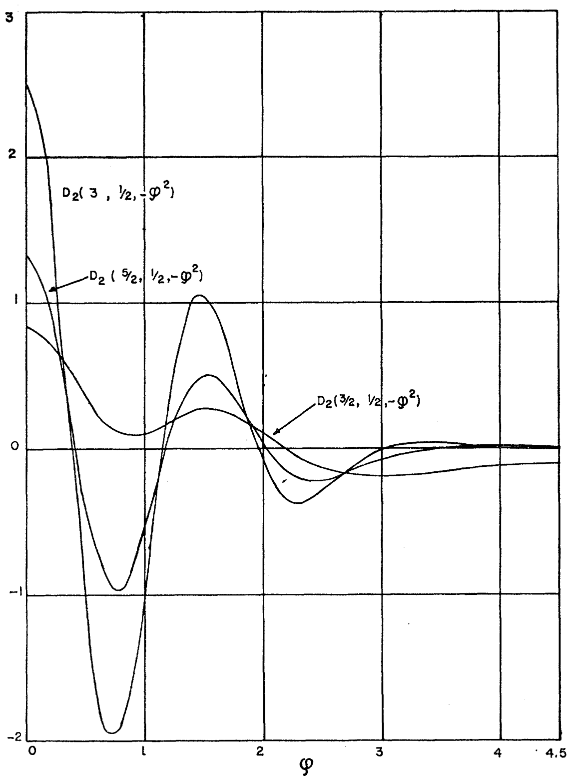


FIG. 26. The functions $D_2(\frac{3}{2},\frac{1}{2},\ -\varphi^2)$, $D_2(\frac{5}{2},\frac{1}{2},\ -\varphi^2)$ and $D_2(3,\frac{1}{2},\ -\varphi^2)$ against φ.

Let us consider the errors made by neglect of the behavior of $\Omega(\xi)-\Omega_0$ at small ξ given by (9.15b) and (9.17b). We note first that normalization of the distribution functions requires by (2.26) the vanishing of $\Omega-\Omega_0$ for $\xi=0$. Equations (9.15a) and (9.17a) do not provide this feature, but (9.15b) and (9.17b) do. Thus, we expect that (9.30) and (9.31) will fail to be normalized; the fact that $K\neq 1$ is an indication of this.

If we denote the approximate exponent used in (9.27a) by $(\Omega-\Omega_0)_{\text{approx.}}$, we can write the difference between (9.27a) or (9.30) and the correct result as

$$\Delta(2\pi F_{\text{red}}) = \int_0^\infty \eta d\eta J_c(\vartheta_N\eta) \times [\exp\ (\Omega-\Omega_0) - \exp\ (\Omega-\Omega_0)_{\text{approx}}]\,.$$

The difference between the integrands is only appreciable for $\xi \leqslant 1/\pi$ or $\eta \leqslant \eta_1 = \chi_c B_N^{1/2}/\pi$, and if the "width" $\chi_c B_N^{1/2}$ of the distribution is as stated above less than 1/4 radian, we have $\eta_1 \leqslant 1/12$. We assume that for $\eta > \eta_1$, the correction vanishes. It is easily seen that for $0<\eta<\eta_1$, the terms beyond the first in the exponent are small enough that only a first-order expansion is needed. Thus we find

$$\Delta(2\pi F_{\text{red}}) = \int_0^{\eta_1} \eta d\eta J_0(\vartheta_N\eta) e^{-\eta^2/4} \left\{ \iota \left[\frac{1.316\eta^2}{\chi_c^2 B_N} - \ln\frac{\eta^2}{4} - \ln\frac{4\gamma^2}{\chi_c^2 B_N} \right] + 2\chi_c^2\alpha\beta^2\pi \left[-\frac{0.299\eta^2}{\chi_c^2 B_N} + -\frac{\eta}{2\chi_c B_N^{1/2}} - \tfrac{1}{4} \right] \right\}. \tag{9.33}$$

We may set $e^{-\eta 2/4}$ equal to 1, and for ϑ_N of the order of 1 or less, also $J_0(\vartheta_N\eta)$. It is easy to see from (9.33) without detailed evaluation that the correction is very small compared to the value $2K$ of expression (9.30) for $\vartheta_N=0$.

For large ϑ_N, we can use the method of Sec. IV to calculate an asymptotic expression for (9.33), assuming that η_1 is large enough that the shift of path is justifiable. Using (4.5) and (4.6), we find

$$\Delta(2\pi F_{\text{red}}) \approx \frac{2\iota}{\vartheta_N^2} - \frac{\chi_c\alpha\beta^2\pi}{B_N^{1/2}\vartheta_N^3}\,, \tag{9.34}$$

which will just cancel the leading terms in the asymptotic expansions of $D_1(1,1,\ -\vartheta_N^2)$ and $D_0(\frac{3}{2},1,\ -\vartheta_N^2)$ in (9.30) (assuming we can take $K=1$ for these terms).

For the projected-angle distribution, we can also show that the correction for small φ_N is negligible, and that asymptotically we have

$$\Delta\left(\frac{\pi}{2}f_{\text{red}}\right) \approx \frac{\pi\iota}{\varphi_N} - \frac{\chi_c\alpha\beta^2\pi}{B_N^{1/2}\varphi_N^2}\,, \tag{9.35}$$

which again cancels the leading terms in $D_1(\frac{1}{2},\frac{1}{2},\ -\varphi_N^2)$ and $D_0(1,\frac{1}{2},\ -\varphi_N^2)$ in (9.31).

However, even for ϑ_N or φ_N up to 13 (the range of our tables), the corrections as estimated directly from (9.34) and (9.35) will remain smaller than the value of the leading terms in the D_1 functions, so no correction of them is actually called for. If the normalization is to be considered, however, such a correction will be necessary, for the functions $D_1(1,1,\ -\vartheta^2)$ and $D_1(\frac{1}{2},\frac{1}{2},\ -\varphi^2)$ as they stand are not normalizable. It is not difficult to show by direct integration of the term in $\ln \eta^2\gamma^2/\chi_c^2 B_N$ that the corrected functions

$$D_{1\text{corr}}(1,1,\ -\vartheta^2) = D_1(1,1,\ -\vartheta^2) + [1-J_0(\vartheta_N\eta_2)]/\vartheta^2 \tag{9.36a}$$

and

$$D_{1\text{corr}}(\tfrac{1}{2},\tfrac{1}{2},\ -\varphi^2) = D_1(\tfrac{1}{2},\tfrac{1}{2},\ -\varphi^2) + \left[\pi/2 - \int_{\varphi\eta_2}^\infty du\sin u/u\right]\Big/\varphi\,, \tag{9.36b}$$

where η_2 is $\chi_c B_N^{1/2}/\gamma$ or any value of that order of magnitude, will represent properly normalized and adjusted functions for the respective terms in (9.30) and (9.31).

C. Inclusion of Atomic–Electron Scattering, Mixtures and Energy Loss

The different behavior of $q(\chi)$ for large χ in the Dalitz formula as compared to that of Molière makes the inclusion of the effects of inelastic scattering and mixtures somewhat more complicated than the corresponding method of Sec. VII.

The correction is basically to be made on Eq.

(9.21), by adding the effects of atomic electrons and of scatterings at different thicknesses and by different atomic species. Thus we expect to use integrals and sums like those in (7.4b) for χ_c^2 and in (7.25) and (7.27) for χ_α, assuming that ζ is negligible or at least constant.

Following the procedure used in Sec. VII, we can write, for the scattering of fast electrons,

$$\Omega(\xi) - \Omega_0 = \tfrac{1}{4}\,\xi^2 \int_0^t dt' \sum\nolimits_i \left\{ \chi_{ci}^2(t') \ln\left[\left(\frac{\gamma^2\xi^2}{4e}\right)^{1+\zeta} \times \left(\frac{4}{e}\right)^{\zeta} \chi_{\alpha i}^2(t') \right] + \frac{1}{Z_i}\, \chi_{ci}^2(t') \ln \frac{\gamma^2\xi^2\chi_i^{(\mathrm{el})2}(t')}{4e} \right\} + \tfrac{1}{2}\int_0^t dt' \sum\nolimits_i \chi_{ci}^2 \{\beta^2(t')[1+\alpha_i(t')\pi] \ln \gamma\xi + \alpha_i(t')\beta^2(t')\pi(\tfrac{1}{2}-\xi)\} \,. \tag{9.37}$$

The small quantity ζ, if it is not entirely negligible, can generally be taken as constant, although its variation can be included in (9.37). If it is negligible, then it becomes legitimate to use (7.4b) or (7.26) for χ_c^2 and 7.27 for $\ln \chi_\alpha$, but of course, (9.19) must be used for each $\chi_{\alpha i}$.

Similarly, the corrections for scattering of heavy particles can be made by use of (7.31a) when ζ is negligible, and by adding $\frac{1}{2}\,\xi^2$ times the terms in (7.31a) with subtraction of $\ln \gamma\xi\chi_{\alpha i}/2e^{1/2}$ in the { }, to (9.21).

D. Application to the Experiments of Hanson *et al.*

As an example of the theory presented in this article, we give its application to the results of Hanson, Lanzl, Lyman, and Scott (1951), which were also treated by Nigam, Sundaresan, and Wu (1959). It was the discrepancy with the results on beryllium whose clearing up was taken by Nigam *et al.* as experimental validation of their calculations.

Table IX gives results for measurements by Hanson *et al.* of spatial-angle electron scattering on two different foil thicknesses for both beryllium and gold. The "$1/e$ width" $\theta_{1/e}$, or angle at which the measured distribution falls to $1/e$ of its value at $\theta = 0$, is used as a measure of scattering.

Hanson *et al.* also calculate an equivalent $1/e$ width θ_0 from the value of the normalized distribution, on the assumption that the curve is Gaussian. We shall show in Sec. XIII (Table XII) that, for the Molière theory, θ_0 is expected to be larger than $\theta_{1/e}$. We can deduce from that section the following results:

$$\theta_{1/e} = \chi_c(1.007B - 1.33)^{1/2}$$
$$\theta_0 = \chi_c(1.012B - 0.64)^{1/2}\,.$$

The expected differences between these values are given in the table for the Molière calculations and show greater differences than found experimentally, probably because the unmeasured "tail" of the distribution affects θ_0 but not $\theta_{1/e}$. The irregular experimental differences between the two angles may, in fact, be a measure of the (unstated) experimental errors. No correction was made for the finite angular divergence of the incident beam; such a correction would reduce the values given by 1 or 2%, which is less than the other errors.

If we first consider the two thicker foils, we see the results quoted by Nigam *et al.* (our numbers are just slightly different than theirs), namely (1) that the Molière results for $\theta_{1/e}$ fit the Au data and are too large, beyond experimental error, for Be; (2) the use of the Thomas–Fermi value $\mu = 1.12$ [see (6.65a)] in the new theory gives results too large for both substances; and (3) the use of $\mu = 1.80$ reduces results for both elements to a good fit. However, these results do not hold for the thinner foils. Although the accuracy is presumably greater for the thicker foils, the data does not appear conclusive. However, we are not attempting in this article to survey the experimental results and only wish to point out the need for more comparisons between theory and experiment.

One important fact must be noted about the calculations in Table IX. The correction terms to the Molière expansions are negligible for Be; the only modification introduced by Nigam *et al.* is in the use of χ_μ in place of Molière's χ_α. However, the correction terms are important for Au, and account for the large value of 4.05° for $\theta_{1/e}$ when $\mu = 1.12$. Thus for the choice $\mu = 1.80$ two corrections for Au cancel each other, and the one important correction for Be allows a fit to be made for this case (considering only the thicker foils for each element).

E. Use of More General Screening Functions

Nigam *et al.* assert that the effect of using a potential of a different form than (6.5) can be included, to the accuracy of their calculations, by merely choosing μ properly in (6.5). However, different functional forms than (6.5) can surely make a considerable difference.

An illustration of the difference that can be made at the smallest angles, where, of course, the screening effect is the largest, is given below in Sec. XI, where a comparison is made between the single-exponential form (6.5) and the sum of three exponentials used by Molière (1947) and by Rozental (1935).

Mitra (1961) following Vachaspati (1954) and

TABLE IX. Multiple scattering of electrons on beryllium and gold [Hanson *et al.* (1951)].

		Be, $Z = 4$, $A = 9.02$, $\alpha = 0.0263$		Au, $Z = 79$, $A = 197$, $\alpha = 0.576$	
Experiment:					
	ρt, g/cm^2	0.257	0.4913	0.01866	0.03728
	E_{kin}, MeV	15.47	15.24	15.69	15.67
	χ_c, degree	1.074°	1.506°	1.087°	1.539°
	χ_0, radian	4.20×10^{-4}	4.25×10^{-4}	1.117×10^{-3}	1.119×10^{-3}
	$\theta_{1/e}$	3.06°	4.25°	2.58°	3.76°
	θ_0	3.01°	4.33°	2.55°	3.78°
Molière Theory:					
$\mu = 1$	${\chi_\alpha}^2/{\chi_0}^2$	1.13_3		2.38_5	
	Ω_0	1760	3390	122	242
	B	9.59	10.30	6.53	7.33
	$\theta_{1/e}$	3.10°	4.53°	2.49°	3.78°
	θ_0	3.23°	4.70°	2.65°	4.01°
Nigam *et al.* Theory:					
	$\chi_c \alpha \beta^2 \pi$	0.00172	0.00241	0.0343	0.0486
	$K - 1$	0.82×10^{-3}	1.45×10^{-3}	2.5×10^{-3}	4.6×10^{-3}
	ι	0.96×10^{-4}	1.88×10^{-4}	2.52×10^{-4}	5.08×10^{-4}
	ζ	$<5 \times 10^{-8}$		$<3 \times 10^{-6}$	
$\mu = 1.12$	$({\chi_\alpha}^2/{\chi_\mu}^2) - 1$	0.83×10^{-4}		4.8×10^{-3}	
	Ω_0	1392	3060	229	457
	B	9.30	10.19	7.26	8.05
	$\theta_{1/e}$	3.04°	4.50°	2.68°	4.05°
$\mu = 1.80$	$({\chi_\alpha}^2/{\chi_\mu}^2) - 1$	1.3×10^{-4}		7.8×10^{-3}	
	Ω_0	616	1184	74.8	177
	B	8.39	9.14	5.94	6.96
	$\theta_{1/e}$	2.87°	4.21°	2.37°	3.71°

Lewis (1956) has given the results of a calculation of Dalitz' type for a screening potential which is the sum of three exponentials. This author, however, only evaluated the scattering cross section at 90° for four special cases, and did not present the results in a form easily amenable to multiple scattering calculations. Use of these results with the methods of this section will be necessary if the assertion of Nigam *et al.* just referred to is to be corroborated or modified.

Other corrections which have not yet been included are those arising from higher terms in the expansion of $\Omega(\xi) - \Omega_0$ as given in Sec. VII and those arising from inverse powers of B beyond the second in the Molière development.

X. MULTIPLE AND PLURAL SCATTERING IN VERY THIN LAYERS

The Molière expansion is not valid for small B, i.e., for small Ω_0. Keil, Zeitler, and Zinn (1960) have made an alternate, numerical calculation for $0.2 \leqslant \Omega_0 \leqslant 20$, which we report in this section. We note first that in accordance with (9.21) and the knowledge that values of $\xi\chi_c$ up to about one are needed, the last two terms of (9.21) (without ξ^2) are negligible when χ_c^2 is very small. Thus, the Molière expression which is derivable from (6.63), modified to read

$$q(\chi) = \chi^4[\chi^2 + (4/e)^{\zeta/(1+\zeta)}{\chi_\alpha}^{2/(1+\zeta)}]^{-2}, \quad (10.1)$$

may be used, along with the use of $(1 + \zeta)\chi_c^2$ in place of χ_c^2. With this assumption, further terms in the expansion of $\Omega - \Omega_0$ may be found, for we will have as in (8.1),

$$\Omega - \Omega_0 = \Omega_0'[\xi\chi_\alpha' K_1(\xi\chi_\alpha') - 1] \quad (10.2a)$$

where we have

$$\Omega_0' = (1 + \zeta)\chi_c^2/\chi_\alpha'^2 \quad (10.2b)$$

$$\chi_\alpha'^2 = (4/e)^{\zeta/(1+\zeta)}\chi_\alpha^{2/(1+\zeta)} . \quad (10.2c)$$

As shown in Sec. IX, χ_α on the theory of Nigam *et al.* is nearly equal to $\mu\chi_0$, and less dependent on α than as indicated by the Molière calculation in Sec. VI-F. Leisegang (1952) found experimentally for thin-foil scattering that χ_α should be taken close to χ_0.

For integral values of $\Omega_0' = m$, say, we can write the spatial-angle distribution function in a manageable form as follows. We use the Snyder variables s and ϑ_s, from Eqs. (8.2a) and (8.8), writing

$$F(m,\vartheta_s)\vartheta_s d\vartheta_s = \vartheta_s d\vartheta_s \int_0^\infty s ds \tilde{F}(s)^m J_0(\vartheta_s s) . \quad (10.3)$$

We follow Keil *et al.* by using the normalization

$$\int_0^\infty F(m,\vartheta_s)\vartheta_s d\vartheta_s = 1 , \quad (10.4)$$

and the transform $\tilde{F}(s)$ for $\Omega_0' = 1$ is given by

$$\tilde{F}(s) = \exp[sK_1(s) - 1] . \quad (10.5)$$

The device used by Keil *et al.*, who follow Leisegang but use a modern computing machine, is to approximate $\tilde{F}(s)$ to within 0.002 for all s by the formula

$$\tilde{F}(s) \simeq e^{-1}(1 + b_1 e^{-c_1 s} + b_2 e^{-c_2 s}) \quad (10.6a)$$

with the coefficients

$$b_1 = 2.10667 \qquad c_1 = 0.935$$
$$b_2 = -0.388388 \qquad c_2 = 5.000 \,. \quad (10.6b)$$

This approximation gives the correct values $\tilde{F}(0) = 1$ and $F(\infty) = e^{-1}$.

The m'th power of (10.6) is then written out:

$$e^{-m}(b_1 e^{-c_1 s} + 1 + b_2 e^{-c_2 s})^m = e^{-m} \sum_{k=0}^{m} \binom{m}{k} b_1^k e^{-kc_1 s}(1 + b_2 e^{-c_2 s})^{m-k} = e^{-m} \sum_{k=0}^{m} \sum_{l=0}^{m-k} \binom{m}{k}\binom{m-k}{l} b_1^k b_2^l e^{-kc_1 s - lc_2 s}, \quad (10.7)$$

and use is made of the Bessel function integral {Erdelyi, T.I.T. (1954), [8.2(20)] and Watson (1952), p. 386, (6)}

$$\left.\begin{aligned} \int_0^\infty s ds e^{-as} J_0(\vartheta s) &= \frac{a}{(a^2 + \vartheta^2)^{3/2}}, \quad a > 0 \\ &= \delta_s(\vartheta), \quad a = 0 \end{aligned}\right\} . \quad (10.8)$$

Thus, we have

$$F(m,\vartheta_s) = e^{-m} \sum_{k=0}^{m} \sum_{l=0}^{m-k} \binom{m}{k}\binom{m-k}{l} b_1^k b_2^l (c_1 l + c_2 k) \times [(c_1 l + c_2 k)^2 + \vartheta_s^2]^{-3/2} . \quad (10.9)$$

The term with $k = l = 0$ gives the contribution of the unscattered particles, $e^{-m}\delta_s(\vartheta_s)$.

Machine computations were made of the nonsingular part of the distribution $G(m,\vartheta_s)$, given by

$$G(m,\vartheta_s) = F(m,\vartheta_s) - e^{-m}\delta_s(\vartheta_s) \quad (10.10)$$

for integral values of m from 1 to 18, and for $m = 20$.

The error in these calculations due to the error remaining in the approximation (10.6) may be estimated by writing

$$\Delta[\tilde{F}(s)]^m = m\tilde{F}(s)^{m-1}\Delta[\tilde{F}(s)]$$

and

$$\Delta F(m,\vartheta_s) = m \int_0^\infty s ds \Delta[\tilde{F}(s)]\tilde{F}(s)^{m-1} J_0(\vartheta_s s) \leqslant 0.002mF(m-1,\vartheta_s) \,. \quad (10.11)$$

The greatest error, of the order of 4%, occurs for $m = 20$ and $\vartheta_s = 0$. Exact numerical integration for $\vartheta = 0$ shows that for $m = 20$ the computations using (10.6) are high by 3.2%, whereas the Molière expansion gives results that are 4.3% too low. For $\vartheta_s > 5$ and $m = 20$, the two methods give very closely the same results. Since the errors in Molière's method decreases as Ω_0' increases, and those of Keil *et al.* increase, we see that $\Omega_0' = m = 20$ is indeed a good division point between the two methods.

For very small Ω_0', less than 1, Keil *et al.* use the direct sum corresponding to (2.34) up to $n = 2$. For $n = 0$ we have, of course, the delta function, and for $n = 1$ the single-scattering law; for $n = 2$, the folding-integral can be calculated directly, so we have

$$F(\Omega_0',\vartheta_s) \approx e^{-\Omega_0'}[\delta_s(\vartheta_s) + \Omega_0' F_1(\vartheta_s) + \tfrac{1}{2}\Omega_0'^2 F_2(\vartheta_s)] \quad (10.12)$$

with

$$F_1(\vartheta) = 2/(1+\vartheta^2)^2 \,. \quad (10.13)$$

$$F_2(\vartheta) = \frac{4}{\vartheta^4(4+\vartheta^2)^2}\left\{\vartheta^2(\vartheta^4 + 2\vartheta^2 - 8) + (1+\vartheta^2) \times (4\vartheta^2 + \vartheta^4)^{1/2} \cdot \times \ln \frac{(\vartheta^4 + 4\vartheta^2 + 2) + (2+\vartheta^2)(4\vartheta^2+\vartheta^4)^{1/2}}{(\vartheta^4 + 4\vartheta^2 + 2) - (2+\vartheta^2)(4\vartheta^2+\vartheta^4)^{1/2}}\right\}, \quad (10.14a)$$

$$F_2(0) = \tfrac{2}{3} \,. \quad (10.14b)$$

Using (10.12), the distribution was calculated for $\Omega_0' = 0.2$, 0.4 and 0.6.

To estimate the error, we note that the coefficients of the F_n in (10.12) are just the Poisson distribution coefficients which sum to unity, and furthermore, if $\vartheta_s > 0$, $F_n(\vartheta_s) < F_n(0) < F_2(0) = 2/3$, so the error caused by stopping after F_2 is

$$\sum_{n=3}^{\infty} \frac{1}{n!} e^{-\Omega_0'}(\Omega_0')^n F_n(\vartheta_s) < \tfrac{2}{3}[1 - e^{-\Omega_0'}(1 + \Omega_0' + \tfrac{1}{2}\Omega_0'^2)]$$

which is 1.6% for $\Omega_0' = 0.6$, 0.53% for $\Omega_0' = 0.4$ and 0.13% for $\Omega_0' = 0.2$.

Keil *et al.* also report values of the integral of $F(m,\vartheta_s)$, namely

$$\hat{G}(m,\vartheta_s) = \int_{\vartheta_s}^\infty \vartheta d\vartheta G(m,\vartheta) \,. \quad (10.15)$$

The value of G for $\vartheta_s = 0$ is not unity because the unscattered beam is not included. For m between 1 and 4, this fact may be used to estimate m — the difference in intensity with and without the foil will give a measure of $\hat{G}(m,0)$. A more precise determination may be made if the distribution of the incident beam is known, but we shall not pursue the matter here [see Keil *et al.* (1960)].

TABLE X. Values of $G(m,\vartheta_s)$ from Keil *et al.* (1960). The notation E-2 signifies that a factor 10^{-2} is to be appended, etc.

ϑ_s	m 0.2	0.4	1	2	4	10	20
0.0	0.338	0.572	0.8808	0.8137	0.4114	0.8188E-1	0.2911E-1
0.1	0.336	0.569	0.8658	0.8020	0.4076	0.8179E-1	0.2911E-1
0.2	0.313	0.531	0.8233	0.7685	0.3969	0.8152E-1	0.2909E-1
0.4	0.253	0.431	0.6833	0.6573	0.3601	0.8049E-1	0.2901E-1
0.6	0.186	0.319	0.5229	0.5270	0.3147	0.7892E-1	0.2888E-1
0.8	0.129	0.224	0.3834	0.4096	0.2703	0.7692E-1	0.2871E-1
1.0	0.883E-1	0.155	0.2770	0.3158	0.2311	0.7459E-1	0.2849E-1
1.25	0.550E-1	0.995E-1	0.1853	0.2293	0.1900	0.7130E-1	0.2814E-1
1.5	0.348E-1	0.655E-1	0.1262	0.1688	0.1566	0.6771E-1	0.2773E-1
2.0	0.152E-1	0.288E-1	0.6276E-1	0.9574E-1	0.1073	0.6008E-1	0.2673E-1
2.5	0.741E-2	0.147E-1	0.3402E-1	0.5720E-1	0.7449E-1	0.5234E-1	0.2551E-1
3.0	0.396E-2	0.800E-2	0.1975E-1	0.3567E-1	0.5229E-1	0.4495E-1	0.2412E-1
4.0	0.139E-2	0.285E-2	0.7733E-2	0.1531E-1	0.2667E-1	0.3215E-1	0.2104E-1
6.0	0.291E-3	0.605E-3	0.1737E-2	0.3745E-2	0.7894E-2	0.1536E-1	0.1473E-1
8.0	0.948E-4	0.195E-3	0.5359E-3	0.1187E-2	0.2723E-2	0.7132E-2	0.9506E-2
10	0.381E-4	0.780E-4	0.2040E-3	0.4531E-3	0.1072E-2	0.3349E-2	0.5824E-2
15			0.3273E-4	0.7099E-4	0.1656E-3	0.5964E-3	0.1559E-2
20			0.8744E-5	0.1851E-4	0.4146E-4	0.1434E-3	0.4289E-3

Finally, these authors give the results of averaging $G(m,\vartheta_s)$ over m:

$$\overline{G}(m,\vartheta_s) = \frac{1}{m}\int_0^m G(m',\vartheta_s)dm' . \qquad (10.16)$$

Tables X and XI give some of the results of Keil, Zeitler and Zinn.

Leisegang (1952) gave preliminary results for $1 < m < 10$ using the same method as Keil *et al.*; in addition he discussed the statistical effect of irregularities in the thickness of thin films and describes experiments in which a good fit to the theory is obtained for the case $\alpha = 1.3$ when χ_α is nevertheless set equal to χ_0.

Kompaneets (1955) made a calculation similar to that quoted above for $\Omega_0' < 1$, for $\Omega_0 = 1$ and 2; the results by his rather rough method are in good agreement with those of Keil *et al.*

XI. SCATTERING AT ANGLES NEAR ZERO

A. Inelastic and Elastic Cross Sections for Single Scattering

Lenz (1954) has discussed the scattering of medium-energy electrons at very small angles (10^{-4} to 10^{-1} radian), with the inclusion of inelastic as well as elastic scattering. By medium energy is meant a range around 100 keV such that relativistic effects are not important but β is still large enough that α is small and the first Born approximation is useful.

TABLE XI. Values of the integrated distribution $\hat{G}(m,\vartheta_s)$, from Keil *et al.* (1960). Notation as in Table X.

ϑ_s	m 1	2	4	10	20
0.0	0.6321	0.8647	0.9817	1.000	0.9999
0.1	0.6728	0.8606	0.9796	0.9995	0.9998
0.2	0.6151	0.8488	0.9736	0.9983	0.9993
0.4	0.5701	0.8062	0.9509	0.9935	0.9976
0.6	0.5104	0.7475	0.9192	0.9855	0.9947
0.8	0.4478	0.6825	0.8766	0.9746	0.9907
1.0	0.3892	0.6179	0.8317	0.9609	0.9855
1.25	0.3256	0.5424	0.7729	0.9404	0.9779
1.5	0.2731	0.4749	0.7137	0.9166	0.9680
2.0	0.1956	0.3642	0.6007	0.8608	0.9441
2.5	0.1437	0.2812	0.5003	0.7977	0.9148
3.0	0.1080	0.2191	0.4146	0.7311	0.8806
4.0	0.6483E-1	0.1374	0.2840	0.5980	0.8018
6.0	0.2786E-1	0.6098E-1	0.1379	0.3752	0.6524
8.0	0.1424E-1	0.3118E-1	0.7236E-1	0.2277	0.4593
10	0.8252E-2	0.1788E-1	0.4134E-1	0.1390	0.3248
15	0.3031E-2	0.6378E-2	0.1411E-1	0.4686E-1	0.1302
20	0.1538E-2	0.3172E-2	0.6758E-2	0.2062E-1	0.5643E-1

The results are of use in electron microscopy and of interest for the light they shed on the screening calculations discussed earlier in this article.

The first step in Lenz's method is to calculate the cross-section at zero angle, without depending specifically on the Thomas–Fermi theory or Molière's (1947) or Rozental's (1935) approximations to the latter [Eqs. (6.13a) and (6.13b)]. We assume that the electron density is some known function $\rho(r)$ of the radial distance r from the nucleus, normalized so that

$$4\pi \int_0^\infty r^2 dr \rho(r) = Z . \tag{11.1}$$

Then the potential energy of a scattered electron when it is at radius r will be

$$V(r) = \frac{Ze^2}{r} - \frac{4\pi e^2}{r}\int_0^r r'^2 dr' \rho(r') - 4\pi e^2 \int_r^\infty r' dr' \rho(r') \tag{11.2}$$

which corresponds to taking the function $\omega(r/r_0)$ of (6.12) in accordance with the equation

$$Z\omega''(r/r_0) = 4\pi r r_0^2 \rho(r) . \tag{11.3}$$

If we use (11.2) in the standard Born formula for elastic scattering

$$\sigma_{el}(\chi) = \left|\frac{m}{2\pi\hbar^2}\int r^2 dr \sin\psi d\psi d\varphi \times \exp[2ik\sin(\chi/2) r\cos\psi] V(r)\right|^2 , \tag{11.4}$$

we find after some reduction the Mott (1930) result for the elastic scattering differential cross section

$$\sigma_{el}(\chi) = \frac{4(Z-f_s)^2}{a_0^2(2k\sin\chi/2)^4} \simeq \frac{4(Z-f_s)^2}{a_0^2(k\chi)^4} , \tag{11.5}$$

where f_s is the scattering factor as used in x-ray analysis:

$$f_s = 4\pi \int_0^\infty r^2 dr \rho(r) \frac{\sin[2kr\sin(\chi/2)]}{2kr\sin(\chi/2)} \tag{11.6a}$$

or in small-angle approximation,

$$_s = 4\pi \int_0^\infty r^2 dr \rho(r) \frac{\sin k\chi r}{k\chi r} . \tag{11.6b}$$

On the assumption that $\rho(r)$ falls off sufficiently rapidly with distance the integrand of (11.6b) can be expanded in powers of $k\chi$ and the integral evaluated termwise. This will be valid, for example, for the potential of (6.5), corresponding to

$$\rho(r) = (Z\mu^2/4\pi r_0^2 r) e^{-\mu r/r_0} \tag{11.7}$$

or for the potentials given by (6.13). We find

$$f_s = Z - \tfrac{1}{6}(k\chi)^2 \cdot 4\pi \int_0^\infty r^4 dr \rho(r) + O(k^4\chi^4) + \cdots . \tag{11.8}$$

Denoting the mean-square-radius integral by Θ,

$$\Theta = 4\pi \int_0^\infty r^4 dr \rho(r) \equiv Z\langle r^2\rangle_{\text{av}} , \tag{11.9}$$

we see by (11.5) that the value of $\sigma_{el}(0)$ is determined by Θ:

$$\sigma_{el}(0) = \Theta^2/9a_0^2 . \tag{11.10}$$

Let us consider values of Θ calculated by various methods. If we use the Thomas–Fermi expression (6.12), we find readily using (11.3)

$$\Theta = 6Zr_0^2 \int_0^\infty \zeta d\zeta \omega(\zeta) . \tag{11.11}$$

The integral may be evaluated numerically from knowledge of the Thomas–Fermi function $\omega(\zeta)$. This function falls off so slowly with increasing ζ that the integral must be performed with care. Lenz reports that an error of 13% is made by stopping the integration at $\zeta = 100$. (This fact alone, which brings the calculation well into the range of neighboring atoms, shows that the T–F function is unrealistic for small-angle scattering.) Several other calculations of the same function give quite different results. The various results are [using (6.6)]

$$\Theta_{\text{T.F.}} = \begin{cases} 28.1Zr_0^2 = 22.0Z^{1/3}a_0^2 & \text{(Bethe 1930)} \\ 34.5Zr_0^2 = 27.0Z^{1/3}a_0^2 & \text{(Bullard and Massey 1930)} \\ 52.5Zr_0^2 = 41.1Z^{1/3}a_0^2 & \text{(Sommerfeld 1932)} \\ 51.2Zr_0^2 = 40.1Z^{1/3}a_0^2 & \text{(Koppe 1947)} \\ 54.32Zr_0^2 = 42.5Z^{1/3}a_0^2 & \text{(Lenz 1954)} \end{cases} \tag{11.12}$$

with the last number evidently the most accurate.

The Molière approximation (6.13a) gives readily [cf. Eq. (6.45)]

$$\Theta_{\text{Mol}} = 6Zr_0^2\left[\frac{0.10}{36} + \frac{0.55}{1.44} + \frac{0.35}{0.09}\right] = 25.6Zr_0^2 = 20.1Z^{1/3}a_0^2 \tag{11.13a}$$

a result which is most sensitive to the value of the smallest exponent in (6.13a). Rozental's exponential fit (6.13b) yields

$$\Theta_{\text{Roz}} = 6Zr_0^2\left[\frac{0.164}{4.356^2} + \frac{0.581}{0.947^2} + \frac{0.255}{0.246^2}\right] = 29.2Zr_0^2 = 22.9Z^{1/3}a_0^2 . \tag{11.14}$$

Lenz (1954) points out that for substances without paramagnetism or appreciable interaction between neighboring atoms—i.e., for noble gases—a value of Θ which we denote by Θ_s can be related to the diamagnetic suscepibility.

In fact, the elementary classical theory of diamagnetism yields, for the suscepibility in Gaussian units,

$$\chi_{\mathrm{mag}} = -(Ne^2/6m_ec^2)\langle\sum r_i^2\rangle_{\mathrm{av}} \quad (11.15a)$$

where N is the number of atoms per unit volume and $\langle\sum r_i^2\rangle_{\mathrm{av}}$ is exactly the quantity Θ_s. From this result, we get in terms of atomic weight A and density d

$$\Theta_s = -1.267 \times 10^6 (A\chi_{\mathrm{mag}}/d)a_0^2 \,. \quad (11.15b)$$

For carbon, $Z = 6$, we find (graphite form, values in Hodgman, 1957)[18a]

$$\Theta_s = 28.4Zr_0^2 = 22.2Z^{1/3}a_0^2\,; \quad -170^\circ \text{ C}$$
$$= 16.6Zr_0^2 = 13.0Z^{1/3}a_0^2\,; \quad 20^\circ \text{ C}\,. \quad (11.16)$$

For beryllium, argon, and neon we obtain

$$\text{Be:} \quad \Theta_s = 5.0Zr_0^2 = 3.9Z^{1/3}a_0^2\,; \quad Z = 4,\ 20^\circ \text{ C}\,,$$
$$\text{Ar:} \quad \Theta_s = 1.11Zr_0^2 = 8.7Z^{1/3}a_0^2\,; \quad Z = 18\,,$$
$$\text{Ne:} \quad \Theta_s = 5.0Zr_0^2 = 3.9Z^{1/3}a_0^2\,; \quad Z = 10\,. \quad (11.17)$$

Gombas (1956) has given a useful summary of material on diamagnetic susceptibilities.

The most reliable results are those calculated from the Hartree or Hartree–Fock atomic distributions. For carbon, using the $3p$ ground state (Torrance, 1934) one obtains

$$\Theta_{\mathrm{H}} = 11.6Zr_0^2 = 9.10Z^{1/3}a_0^2\,; \quad Z = 6\,. \quad (11.18)$$

The formula deduced by Fleischmann (1960) for beryllium, Eq. (6.14), above, yields

$$\Theta_{\mathrm{H}} = 16.16Zr_0^2 = 12.66Z^{1/3}a_0^2\,; \quad Z = 4\,. \quad (11.19)$$

The single-scattering formula resulting from the Wentzel potential (6.5), namely, (6.11) together with (6.9a), gives

$$\Theta_{\mathrm{W}} = 6Zr_0^2/\mu^2 = (4.70/\mu^2)Z^{1/3}a_0^2\,. \quad (11.20)$$

The same result is of course, obtained for small α from the Dalitz formula (6.15). The use of $\mu = 1.80$ by Nigam, Sundaresan, and Wu (1959) yields a value $1.45\ Z^{1/3}a_0^2$, considerably below the already low values given by the diamagnetic susceptibility, and the same consequence ensues if Molière's χ_α is used in (6.63) along with (6.62), except for the smallest values of α. Even worse results ($1.05\ Z^{1/3}a_0^2$) come from the value $\mu = 2.12$, quoted by Nigam *et al.* as a result of using the analytic functions of Fock and Petrashen (1935).

[18a] Lenz (1954) has a mistakenly low value for carbon because of omission of the factor A.

It will be noticed that the chief difference between these results of Nigam *et al.* and those given in (11.13) and (11.14) arises from the terms with the smallest exponents, which represent the detail of screening omitted by Nigam *et al.*, and most relevant to the zero-angle intensity.

Although the Hartree calculations are probably the most accurate, and the unmodified Thomas–Fermi, the least, the results in condensed materials cannot be taken with much certainty because of the effects of neighboring atoms and also of crystal structure.

Lenz suggests using the Wentzel-potential form but letting the radius r_0/μ be an arbitrary value R to be determined by experiment. Thus, we set

$$\Theta = 6ZR^2\,; \quad R = r_0/\mu \quad (11.21)$$

and using (6.8)

$$\sigma_{\mathrm{el}} = \frac{4\alpha^2k^2}{(k^2\chi^2 + R^{-2})^2} \quad (11.22)$$

$$= \frac{4Z^2}{a_0^2(k^2\chi^2 + R^{-2})^2} \text{ for nonrelativistic electrons} \quad (11.22a)$$

$$= \frac{4z^2Z^2m^2}{a_0^2m_e^2(1-\beta^2)(k^2\chi^2 + R^{-2})^2} \text{ in general.} \quad (11.22b)$$

This result can be seen by (11.5) to correspond to setting

$$f_s = \frac{Z}{(k\chi R)^2 + 1} = \frac{Z}{(k^2\chi^2\Theta/6Z) + 1}\,. \quad (11.23)$$

From (11.22), we can calculate the total elastic cross section

$$\sigma_{\mathrm{el,tot}} = \frac{8\pi Z^2}{a_0^2}\int_0^\infty \frac{\chi d\chi}{(k^2\chi^2 + R^{-2})^2} = \frac{4\pi Z^2R^2}{a_0^2k^2}\,. \quad (11.24)$$

Inelastic scattering may be included in the cross section by introducing the inelastic scattering function S in the Morse [(1932); Mott and Massey (1949)] formula for the combined cross section

$$\sigma(\chi) = \frac{4}{a_0^2(k\chi)^4}\left[(Z - f_s)^2 + S\right]. \quad (11.25)$$

For inelastic scattering, Lenz refers to work of Koppe (1947) which corrected earlier work of Heisenberg (1931) and Bewilogua (1931). The latter authors used a T–F model and obtained the inelastic scattering factor S, for small $k\chi$, as proportional to $k\chi$, whereas Koppe (as corrected himself by Lenz), showed that

$$S \approx \tfrac{1}{3}\,\Theta k^2\chi^2 \quad \text{as} \quad k\chi \to 0\,. \quad (11.26)$$

Since for the smallest angles the usual inelastic scattering results (Sec. VI-H) do not give the correct limit (11.23), Lenz proposes to use the classical-physics formula of Raman (1928) and Compton (1930) which does satisfy (11.23):

$$S = Z - f_s^2/Z\,. \tag{11.27}$$

A more exact result would be to replace the second term by a sum over each electron $\sum_i f_{si}^2$, but the additional correction would not be large and Lenz has not made it. Using (11.23), we have

$$S = Z\left\{1 - \frac{1}{[(k^2\chi^2\Theta/6Z+1)]^2}\right\}. \tag{11.28}$$

As indicated just below Eq. (6.66), $k^2\chi^2$ which measures the square of the momentum transfer, should be replaced, strictly, by $k^2\chi^2 + (\epsilon/\hbar v)^2$, where ϵ is the energy loss on scattering. The chief result of such a modification will be to change the lower limit for (11.27) from $k\chi = 0$ to $k\chi_{\min} = \epsilon_1/\hbar v$ where ϵ_1 is of the order of the mean ionization energy I of the atomic species in question [Koppe (1947) takes $\epsilon_1 = I/2$]. The lower limit enters logarithmically into the total inelastic scattering cross section, but effects multiple scattering only for angles less than 10^{-4} radian.

In fact, we have $\chi_{\min} \simeq I/2pv = I/4E$, where E is the energy, assumed nonrelativistic, of the scattered particle. Taking $I = 12.5Z$ eV as a reasonable approximation for estimation purposes, we see that only for high-Z materials and energies almost too low for the first Born approximation to be valid, will we have $\chi_{\min}$ as large as 5×10^{-3}. The factor $k^2\chi^2\,\Theta/6Z = k^2\chi^2R^2$ in (11.22) or (11.23) which gets small compared to one when screening is important, is for $\chi_{\min}$, if we use (11.20) and (6.9) and $I = 12.52\,eV$

$$k^2\chi_{\min}^2R^2 = \chi_{\min}^2/\chi_0^2 = 2.9\times 10^5 Z^{4/3}/m_ec^2E\,, \tag{11.29}$$

where m_ec^2 and E are in electron volts. This number is also less than 10^{-3}, and generally much less.

The total inelastic cross section can be calculated from the term in S in (11.25). We obtain, using the smallness of $k^2\chi_{\min}R^2$,

$$\sigma_{\text{inel, tot}} = \frac{4\pi\Theta}{3a_0^2k^2}\ln\frac{6Ze^{1/2}}{k^2\chi_{\min}^2\Theta} = -\frac{8\pi ZR^2}{a_0^2k^2}\ln k^2\chi_{\min}^2R^2\,, \tag{11.30}$$

which can readily be generalized for relativistic scattering. Lenz points out that a modification of f and correspondingly of S which gives the same correct behavior for small $k\chi$, and fits the case of hydrogen more exactly, namely to use the fourth instead of the second power of the bracket in (11.28) and to replace 6 by 12, changes (11.30) by a factor 2 under the logarithm and changes the total elastic cross section by a factor 7/6. Thus, our results are not very sensitive to the exact choice of S.

The ratio of inelastic to elastic scattering at any angle χ is readily found to be from (11.23) and (11.26)

$$\frac{S}{(Z-f_s)^2} = \frac{Z+f_s}{Z(Z-f_s)} = \frac{1}{Z}\left(1 + \frac{2}{k^2\chi^2R^2}\right), \tag{11.31}$$

so that for small angles, within the screening region, the inelastic scattering may become quite a good deal larger than the elastic, while for larger angles it rapidly goes to $1/Z$ of the elastic value as discussed in section VI-F (see Fig. 27)

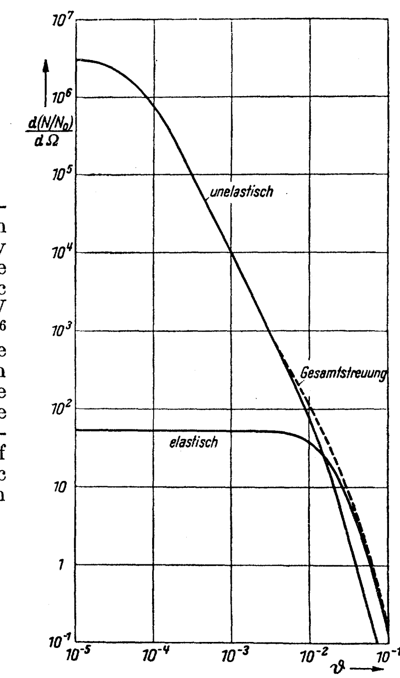


FIG. 27. The differential cross-section (here denoted by $d(N/No)/d\Omega$) for the elastic and inelastic scattering of 50 keV electrons on 10^{-6} g/cm^2 of carbon. The Hartree distribution was used for the atomic electrons. The dashed line represents the sum of elastic and inelastic cross-sections [Taken from Lenz (1954)].

The ratio of the total cross section is, from (11.24) and (11.30),

$$\frac{\sigma_{\text{inel, tot}}}{\sigma_{\text{el, tot}}} = -\frac{2}{Z}\ln k^2\chi_{\min}^2R^2\,, \tag{11.32}$$

which varies primarily with Z and only in a slow fashion with energy. Figure 28, taken from Lenz (1954) shows this ratio for 50 keV electrons as a function of Z, using $R = a_0Z^{-1/3}$ and taking values of I from the literature.

Biberman, Vtorov, Kovner, Sushkin, and Yavorskiĭ (1949) have performed an experiment with a chromium foil in an electron microscope, under conditions for which $\Omega_0 = 0.38$ and the angular range studied was from 3×10^{-3} to 2×10^{-2}. Figure 29 shows the results of their experiment for 60 keV electrons, the elastic scattering distribution using

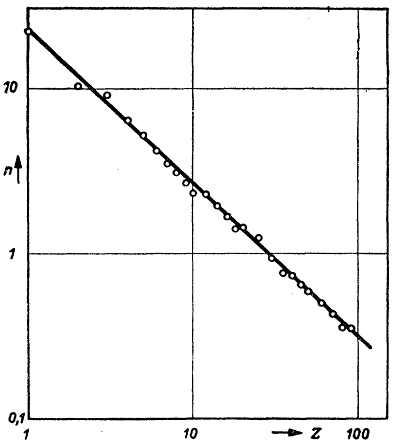


FIG. 28. The ratio n of the total inelastic scattering cross-section to the elastic for 50 keV electrons, plotted logarithmically as a function of Z, using Eq. (11.30) with values of the ionization potential I taken from the literature [Taken from Lenz (1954)].

(11.22), the total including the term with S, and the calculations of Bullard and Massey (1930). The Hartree distribution for Cr^{++} [Mooney (1939)] was integrated to give $\Theta = 20.0a_0^2 = 6.95Z^{1/3}a_0^2$. The experiments clearly show the influence of inelastic scattering. The discrepancy showing in Fig. 29 between theory and experiment may be due to interference effects, which for body-centered Cr of lattice constant 2.88 Å with 60 keV electrons should show a peak at 0.024 radians, broadened if the crystallites are quite small. The discrepancy may, of course, be due to the oversimplification of the theory. [Lenz also suggests that an experiment of Leisegang (1952), which shows a discrepancy with his theory, indicates an interference effect, this time on gold of 150 Å thickness, with 68 keV electrons].

B. Multiple Scattering

In view of the discussion above, we need to calculate $\Omega(\xi) - \Omega_0$ for the cross section

$$\sigma(\chi) = \frac{4Z}{a_0^2}\left\{\frac{ZR^4}{(k^2\chi^2R^2+1)^2} + \frac{1}{(k\chi)^4}\left[1 - \frac{1}{(k^2\chi^2R^2+1)^2}\right]\right\}$$
$$= \frac{4Z}{a_0^2}\left\{\frac{(Z+1)R^4}{(k^2\chi^2R^2+1)^2} + \frac{2R^2}{k^2\chi^2(k^2\chi^2R^2+1)^2}\right\}, \tag{11.33a}$$

or, using $\chi_\mu = 1/kR$, we have

$$2\pi\chi d\chi W(\chi,t) = 2\chi_c^2\chi d\chi \times\left[\frac{1}{(\chi^2+\chi_\mu^2)^2} + \frac{2}{Z+1}\cdot\frac{\chi_\mu^2}{\chi^2(\chi^2+\chi_\mu^2)^2}\right], \tag{11.33b}$$

where χ_c^2 is the value given in (7.4c) for electrons. Thus we seek

$$\Omega(\xi) - \Omega_0 = 2\chi_c^2\int_0^\infty \chi d\chi[J_0(\xi\chi) - 1] \times\left[\frac{1}{(\chi^2+\chi_\mu^2)^2} + \frac{2}{Z+1}\,\frac{\chi_\mu^2}{\chi^2(\chi^2+\chi_\mu^2)^2}\right]. \tag{11.34}$$

The first term, as in Eq. (8.1), gives $(\chi_c^2/\chi_\mu^2)[\xi\chi_\mu \times K_1(\xi\chi_\mu) - 1]$; the second can be evaluated in terms of the integral

$$\int_0^\infty xdx[J_0(sx) - 1]\left[\frac{1}{x^2} - \frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}\right] = [-\tfrac{1}{2}\,sK_1(s) + \tfrac{1}{2} - K_0(s) - \ln s\gamma/2] \equiv L(s)\,, \tag{11.35}$$

so that

$$\frac{4\chi_c^2\chi_\mu^2}{Z+1}\int_0^\infty \frac{\chi d\chi[J_0(\xi\chi) - 1]}{\chi^2(\chi^2+\chi_\mu^2)^2} = \frac{4(\chi_c^2/\chi_\mu^2)}{Z+1} \times[-\tfrac{1}{2}\,\xi\chi_\mu K_1(\xi\chi_\mu) + \tfrac{1}{2} - K_0(\xi\chi_\mu) - \ln(\xi\chi_\mu\gamma/2)] \equiv \frac{4\chi_c^2}{(Z+1)\chi_\mu^2}\,L(\xi\chi_\mu)\,.$$

Using $s = \xi\chi_\mu$ and $\theta_s = \theta/\chi_\mu$ as in Sec. VIII, we have

$$f_s(\theta_s) = \int_0^\infty sdsJ_0(\theta_s s) \times\exp\{\Omega_0[sK_1(s) - 1 + 4(Z+1)^{-1}L(s)]\}\,. \tag{11.36}$$

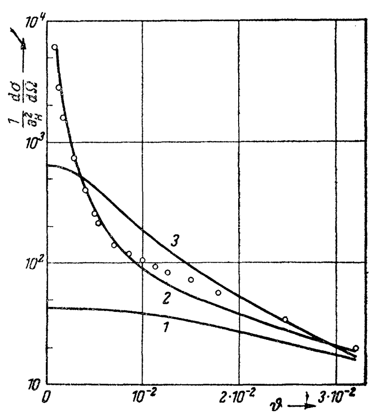


FIG. 29. Comparison of experiments of Biberman *et al.* (1949) with theory (see text). Curve 1, elastic differential cross-section; Curve 2, sum of elastic and inelastic cross-sections [Lenz (1954); Curve 3, earlier calculations of Bullard and Massey (1930) using the T–F model. Open circles, experiment. The symbol a_H signifies our a_0 [Taken from Lenz (1954)].

In this formula, we have used

$$\Omega_0 = \frac{\chi_c^2}{\chi_\mu^2} = \chi_c^2k^2R^2 = \frac{\chi_c^2k^2\Theta}{6Z} = \frac{4\pi Z^{1/3}(Z+1)N_0td(0.885)^2}{A\mu^2m_e^2v^2} \tag{11.37a}$$

which differs from Lenz's parameter p by the factor $(Z+1)/Z$:

$$\Omega_0 = p_{\text{Lenz}}(Z+1)/Z\,. \qquad (11.37\text{b})$$

Lenz has evaluated (11.36) numerically. The logarithmic term in $L(s)$ prevents rapid convergence of the integral, so he has calculated numerically the value of

$$\int_0^\infty sdsJ_0(\theta_s s)[\exp\{\Omega_0[sK_1(s) - 1 + 4(Z+1)^{-1}L(s)]\} - \exp\{\Omega_0[-(Z-1)(Z+1)^{-1} - 4(Z+1)^{-1} \times \ln s\gamma/2]\}]\,,$$

and then has established the value of the second term analytically by showing that

$$\lim_{a\to 0}\int_0^\infty dse^{-as}J_0(\theta_s s)s^{1-4\Omega_0/(Z+1)} = \tfrac{1}{2}\,\Gamma\left(1-\frac{2\Omega_0}{Z+1}\right)\left(\frac{\theta_s}{2}\right)^{-2+4\Omega_0/(Z+1)}\Big/\Gamma\left(\frac{2\Omega_0}{Z+1}\right)$$

{using properties of the hypergeometric function; cf. Erdelyi (1954) T.I.T., [8.6(7)] and H. T. F. [2.1.3(14)]}.

Figures 30–32 give Lenz's results for $f_s(\theta_s)$ for carbon ($Z = 6$), chromium ($Z = 24$) and gold ($Z = 79$). The results for gold, for $p = 16$ or $\Omega_0 = 16.2$, agree as closely as the figure can be read with the calculation of Keil *et al.* (Sec. X); for this value of Z the inelastic scattering is negligible.

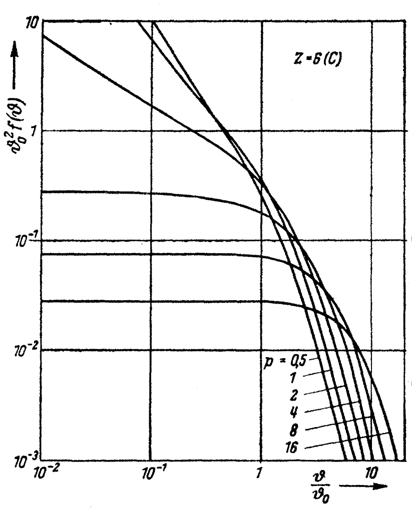


FIG. 30. Multiple scattering distribution $f(\vartheta)$ for carbon ($Z = 6$) calculated by Lenz for thicknesses corresponding to six values of his parameter p (see text). The angle ϑ_0 is χ/R [Taken from Lenz (1954)].

To show the deviation from a Gaussian distribution, Lenz gives the curves we have reproduced in Fig. 33 showing his distribution and a Gaussian one of the same maximum and same area for $p = 16$ and $Z = 6$.

Finally, Lenz takes up the case of an incident beam of finite width, either Gaussian or square, and discusses the use of measurements of $\sigma(0)$ and the width of the initial beam to obtain values for p and R, but we shall not discuss these results here.

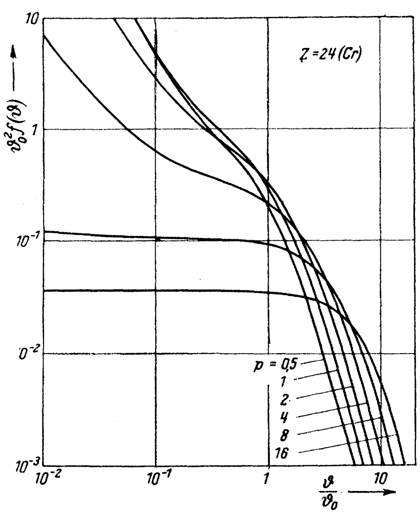


FIG. 31. Multiple scattering distribution $f(\vartheta)$ for chromium ($Z = 24$) calculated by Lenz for thickness corresponding to six values of his parameter p (see text). The angle ϑ_0 is χ/R. [Taken from Lenz (1954)].

XII. ASYMPTOTIC FORMULAS

Using the methods of Sec. IV on the calculations of Secs. VII, VIII, and IX, we can arrive at asymptotic formulas by a method independent of the Molière expansion. The relation between the formulas found by this method and by the Molière method will shed some added light on the nature of the latter expansion.

The Molière formula for $\Omega - \Omega_0$ and the various corrections derived in sec. VII, the Snyder formula using $sK_1(s) - 1$ of sec. VIII, and the Nigam, Sundaresan, and Wu formula (9.21) can, with the exception of the ξ and $\ln \xi$ terms of the last-named,[19] all be written in the general form:

$$\Omega - \Omega_0 = \Omega_0[c_1 s^2 \ln C_1 s + c_2 s^4 \ln C_2 s + c_2' s^4 \ln^2 C_2' s + c_3 s^6 \ln C_3 s + \cdots] \qquad (12.1)$$

where we use the variable s of Sec. VIII. The term with c_2' comes in only with the correction formula (7.22); that with c_3 arises if we carry the expansion of $sK_1(s) - 1$ far enough. In fact, we have for the Molière expansion, by (7.15) as well as for the Snyder expression in (8.1), and also for Eq. (9.21) by choosing s properly,

$$c_1 = \tfrac{1}{2}\,; \quad C_1 = \gamma/2e^{1/2}\,, \qquad (12.2)$$

[19] The asymptotic behavior resulting from the ξ and $\ln \xi$ terms was discussed in Sec. IX-B above.

$$\left.\begin{aligned}&s=\xi\chi_\alpha\,,\\&\text{Molière, Eq. (7.15) and Snyder, Eq. (8.1)};\\&s=\xi\chi_\alpha^{1/(1+\zeta)}(4/e)^{\zeta/2(1+\zeta)},\\&\text{Nigam } et\ al, \text{ Eq. (9.21)};\\&\Omega_0=\chi_c^2/\chi_\alpha^2,\\&\text{Molière and Snyder, Eq. (7.38)};\\&\Omega_0=(1+\zeta)\chi_c^2(e/4)^{\zeta/(1+\zeta)}\chi_\alpha^{-2/(1+\zeta)},\\&\text{Nigam } et\ al, \text{ Eq. (9.23)}\end{aligned}\right\}. \quad (12.3)$$

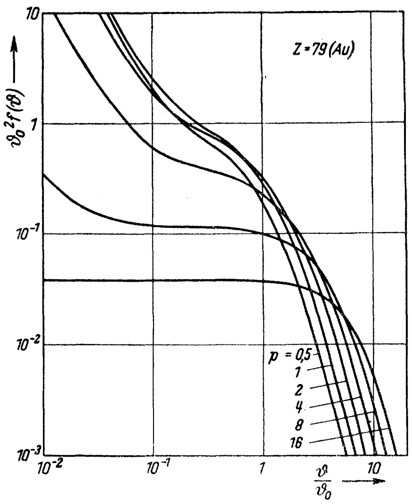


FIG. 32. Multiple scattering distribution $f(\vartheta)$ for gold ($Z = 79$) calculated by Lenz for thickness corresponding to six values of his parameter p (see text). The angle ϑ_0 is χ/R. [Taken from Lenz (1954)].

For the Snyder expression, we find from the expansion of K_1 {Watson (1952), p. 80; Erdelyi, H.T.F. (1953) [7.2(12) and (37)]}

$$\left.\begin{aligned}&c_2=1/16\,;\,C_2=\gamma/2e^{5/4}\,;\,c_2'=0\\&c_3=1/384\,;\,C_3=\gamma/2e^{5/3}\end{aligned}\right\}\text{Snyder.} \quad (12.4)$$

Let us now set $s = i\tau$, expand exp $(\Omega - \Omega_0)$ up to powers of τ^6, collect the real parts after multiplying by i as in (4.4), and integrate. In addition to formulas (4.5) and (4.6), we need the following generalizations:

$$\int_0^\infty d\tau K_0(\tau\theta)\tau^\nu \ln C\tau=\left[\ln C+\frac{\partial}{\partial\nu}\right]\int_0^\infty d\tau K_0(\tau\theta)\tau^\nu = \frac{2^{\nu-1}}{\theta^{\nu+1}}\Gamma^2\left(\frac{\nu+1}{2}\right)\left[\ln\frac{2C}{\theta}+\psi\left(\frac{\nu-1}{2}\right)\right], \quad (12.5)$$

$$\int_0^\infty d\tau K_0(\tau\theta)\tau^\nu \ln C\tau\ln C'\tau=\left[\ln C+\frac{\partial}{\partial\nu}\right]\times\left[\ln C'+\frac{\partial}{\partial\nu}\right]\int_0^\infty d\tau K_0(\tau\theta)\tau^\nu = \frac{2^{\nu-1}}{\theta^{\nu+1}}\Gamma^2\left(\frac{\nu+1}{2}\right)\left\{\left[\ln\frac{2C}{\theta}+\psi\left(\frac{\nu-1}{2}\right)\right]\times\left[\ln\frac{2C'}{\theta}+\psi\left(\frac{\nu-1}{2}\right)\right]+\tfrac{1}{2}\psi'\left(\frac{\nu-1}{2}\right)\right\}. \quad (12.6)$$

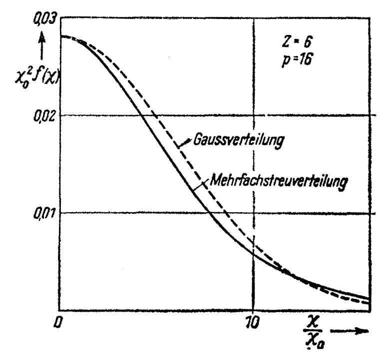
The final result is, in terms of $\theta_s = \theta\xi/s$,

$$2\pi F_s(\theta_s)\approx\Omega_0\left\{\frac{4c_1}{\theta_s^4}-\frac{64c_2}{\theta_s^6}+\frac{2304c_3}{\theta_s^8}-\frac{128c_2'}{\theta_s^6}\times[\ln(2C_2'/\theta_s)+\psi(2)]\right\}+\Omega_0^2\left\{-\frac{64c_1^2}{\theta_s^6}\times[\ln(2C_1/\theta_s)+\psi(2)]+\frac{4608c_1c_2}{\theta_s^8}\times[\tfrac{1}{2}\ln(4C_1C_2/\theta_s^2)+\psi(3)]+\frac{2304c_1c_2'}{\theta_s^8}\times\left(-\frac{\pi^2}{4}+[\ln(2C_2'/\theta_s)+\psi(3)]^2+2[\ln(2C_2'/\theta_s)+\psi(3)][\ln(2C_1/\theta_s)+\psi(3)]+\tfrac{3}{2}\psi'(3)\right)\right\}+\frac{1152\Omega_0^3c_1^3}{\theta_s^8}\left\{-\frac{\pi^2}{12}+[\ln(2C_1/\theta_s)+\psi(3)]^2+\tfrac{1}{2}\psi'(3)]\right\}+\cdots. \quad (12.7)$$

It is evident that whenever terms in θ_s^{-6} are important, (and the neglect of θ_s^{-10} is still justifiable) and when $\Omega_0 > 20$, the terms with $\Omega_0\ \theta_s^{-6}$ can be neglected compared to those with $\Omega_0^2\ \theta_s^{-6}$. Similarly, terms in $\Omega_0\ \theta_s^{-8}$ and $\Omega_0^2\ \theta_s^{-8}$ may reasonably be omitted from (12.7). It is interesting to note that if (12.3) and (12.4) are used, the first three terms are just the first three terms in the expansion of $2\Omega_0(\theta_s^2+1)^{-2}$, which is the single-scattering law in terms of θ_s. Neglect of terms beyond $2\Omega_0\ \theta_s^{-4}$ amounts to neglect of the influence of screening in the asymptotic region for that part of the entire asymptotic expansion that represents single scattering.

It is, in fact, easily seen that if we restrict the asymptotic expansion to terms of the form $\theta_s^{-2}(\Omega_0\theta_s^{-2})^n$ only the coefficients c_1 and C_1 of (12.1) will enter.

FIG. 33. Curves showing Lenz's calculation for the multiple scattering for carbon ($Z = 6$) with $p = 16$, ("Mehrfachstreuverteilung") and a Gaussian distribution of the same maximum and same area ("Gaussverteilung"). The difference between the two curves is greater for larger Z. [Taken from Lenz (1954)].



This is a justification for using only the first term of (12.1) in the Molière expansion and neglecting the various correction terms in Eqs. (7.19) to (7.22). However, for small values of Ω_0 the additional terms may become important.

If we use (12.2) for c_1 and C_1, we can write to the approximation given above

$$2\pi F_s(\theta_s) \approx \frac{2\Omega_0}{\theta_s^4} + \frac{16\Omega_0^2}{\theta_s^6}[\ln\theta_s - 1] + \frac{144\Omega_0^3}{\theta_s^8}[(\ln\theta_s - \tfrac{4}{3})^2 - 49/72], \quad (12.8)$$

where we have used $\psi(n) = -\ln\gamma + 1 + \frac{1}{2} + \cdots + 1/n$ and $\psi'(n) = \pi^2/6 - 1 - 1/2^2 - \cdots - 1/n^2$.

By use of (4.17)–(4.19), we have for the projected distribution[20]

$$f_s(\phi_s) = \frac{\Omega_0}{\phi_s^3} + \frac{6\Omega_0^2}{\phi_s^5}[\ln 2\phi_s - 19/12] + \frac{45\Omega_0^3}{\theta_s^7}[(\ln 2\phi_s - 39/20)^2 + \pi^2/12 - 5369/3600]. \quad (12.9)$$

In similar fashion, we can find asymptotic formulas for the functions $P_n(\alpha_s,\Omega_{00})$ used in Sec. VIII. We obtain

$$2\pi P_0(\alpha_s,\Omega_{00}) \approx \frac{2\Omega_{00}}{\alpha_s^4} + \frac{16\Omega_{00}^2}{\alpha_s^6}(\ln\alpha_s - 1) + \frac{144\Omega_{00}^3}{\alpha_s^8} \times [(\ln\alpha_s - \tfrac{4}{3})^2 - 49/72] + \cdots, \quad (12.10a)$$

$$2\pi P_1(\alpha_s,\Omega_{00}) \approx -\frac{8\Omega_{00}^2}{\alpha_s^6} - \frac{144\Omega_{00}^3}{\alpha_s^8}(\ln\alpha_s - \tfrac{4}{3}) - \frac{2304\Omega_{00}^4}{\alpha_s^{10}}\{[\ln\alpha_s - 19/12]^2 - 205/288\} - \cdots, \quad (12.10b)$$

$$2\pi P_2(\alpha_s,\Omega_{00}) \approx \frac{36\Omega_{00}^3}{\alpha_s^8} + \frac{1152\Omega_{00}^4}{\alpha_s^{10}} \times (\ln\alpha_s - 19/12) + \cdots, \quad (12.10c)$$

$$2\pi P_3(\alpha_s,\Omega_{00}) \approx 192\Omega_{00}^4/\alpha_s^{10} + \cdots. \quad (12.10d)$$

Now let us find the corresponding results by use of the Molière expansion. By (7.46), (9.32c), (9.32e), (A.30) and (A.32), we have, neglecting the $e^{-\vartheta^2}$ term,

$$2\pi F_{\text{red}}(\vartheta,t) \approx \frac{2}{B}\left\{\frac{1}{\vartheta^4} + \frac{4}{\vartheta^6} + \frac{18}{\vartheta^8} + \cdots\right\} - \frac{2}{B^2}\left\{\frac{8}{\vartheta^6}[\psi(2) - \ln\vartheta] + \frac{36}{\vartheta^8}[\psi(3) - \ln\vartheta] + \cdots\right\} + \cdots. \quad (12.11)$$

Using (8.8) and remembering that $F_{\text{red}}(\vartheta)\vartheta d\vartheta = F_s(\theta_s)\theta_s d\theta_s$, we have, after collecting terms:

$$2\pi F_s(\theta_s) = 2\pi F_{\text{red}}(\vartheta,t)/\Omega_0 B \approx \frac{2\Omega_0}{\theta_s^4} - \frac{16\Omega_0^2}{\theta_s^6}[\psi(2) - \ln\theta_s + \tfrac{1}{2}\ln\Omega_0 B - \tfrac{1}{2}B] - \frac{144\Omega_0^3}{\theta_s^8}\left(\frac{B}{2}\right)[\psi(3) - \ln\theta_s - \tfrac{1}{2}\ln\Omega_0 B] + \cdots. \quad (12.12)$$

If we use (7.39) for B, we obtain agreement with (12.8) for the first two terms. It is clear that only by use of $F^{(3)}(\vartheta)$ in Molière's expansion, could we obtain agreement between the θ_s^{-8} terms. Exactly the same relationship can be shown for the projected scattering.

It will be noted particularly that contributions from both $F^{(1)}(\vartheta)$ and $F^{(2)}(\vartheta)$ appear in the θ_s^{-6} term. In fact, the former term contributes the $\frac{1}{2}B$ term which is nearly canceled by $\frac{1}{2}\ln\Omega_0 B$. Thus, for large angles, we see that we need the first term of $F^{(2)}$ as soon as we need the second term of $F^{(1)}$—i.e., as soon as the single-scattering result represented by the first term of $F^{(1)}$ becomes inadequate.

The formulas of Appendix III can be used of course, to furnish the asymptotic behavior of the correction terms introduced in Sec. IX.

Bethe (1953) has given an extensive discussion of asymptotic formulas for the Molière calculation, using among other methods, one that is essentially that of our Sec. IV. He suggests that an asymptotic formula of good convergence can be obtained for R^{-1}, the reciprocal of the ratio of actual scattering to Rutherford scattering. Using (12.11), we have

$$R^{-1} \approx \frac{2/\vartheta^4}{\dfrac{2}{\vartheta^4} + \dfrac{8}{\vartheta^6} + \dfrac{36}{\vartheta^8} + \dfrac{16}{B\vartheta^6}(\ln\gamma\vartheta - \tfrac{3}{2}) + \dfrac{144}{B\vartheta^8}(\ln\gamma\theta - 11/6) + \cdots} \approx 1 - \frac{4}{\vartheta^2}\left[1 + \frac{2}{B}(\ln\gamma\vartheta - \tfrac{3}{2})\right] + \frac{2}{\vartheta^4}\left[-1 + \frac{12}{B} - \frac{4}{B}(\ln\gamma\vartheta - \tfrac{3}{2})\right] + \cdots. \quad (12.13a)$$

(Bethe does not include the term $-8(\ln\gamma\theta - \frac{3}{2})/B\vartheta^4$ and replaces $\ln\gamma - 3/2$ by its approximate value $\ln 0.4$). If we write this result in terms of θ_s we have

[20] The formula for $W(\varphi_s)$, which is $\frac{1}{2}f_s(\varphi_s)$, is given in Snyder and Scott (1949) with numerical errors. The (correct) square bracket can be written

$$[(\ln\varphi_s + 0.6159)^2 - 3.7456(\ln\varphi_s + 0.6159) + 2.8384].$$

$$R^{-1} \approx 1 - \frac{8\Omega_0}{\theta_s^2}(\ln\theta_s - 1) + \frac{2\Omega_0^2 B}{\theta_s^4}(B + 16 - 4\ln\theta_s). \tag{12.13b}$$

On the other hand, if we use (12.8), we find

$$R^{-1} \approx 1 - \frac{8\Omega_0}{\theta_s^2}(\ln\theta_s - 1) + \frac{\Omega_0^2}{\theta_s^4}(-8\ln^2\theta_s + 64\ln\theta_s - 15) + \cdots \tag{12.14a}$$

$$\simeq 1 - \frac{4}{\vartheta^2}\left[1 + \frac{2}{B}(\ln\gamma\vartheta - \tfrac{3}{2})\right] + \frac{2}{\vartheta^4}\left[-1 + \frac{12}{B} - \frac{4}{B}(\ln\gamma\vartheta - \tfrac{3}{2}) + \frac{41}{B^2} + \frac{48}{B^2}(\ln\gamma\vartheta - \tfrac{3}{2}) - \frac{8}{B^2}(\ln\gamma\vartheta - \tfrac{3}{2})^2\right] + \cdots. \tag{12.14b}$$

It is clear from the comments above that (12.14) is in general more accurate than (12.13).

Bethe has given an interesting formula for R itself, namely

$$R = \tfrac{1}{2}\vartheta_1^4 F^{(1)}(\vartheta_1) \tag{12.15a}$$

where

$$\vartheta_1 = \vartheta\left\{1 + \frac{2}{B}[\ln\gamma\vartheta - \tfrac{3}{2} + \ln(1 - 3/\vartheta^2)]\right\}^{-1/2}. \tag{12.15b}$$

The angle ϑ_1 is a new variable in place of ϑ, so chosen that the contribution of $F^{(2)}(\vartheta)$ is negligible. Its use corresponds to a certain choice of the variable a used in Eq. (7.51).

Butler (1950) has another method of calculating asymptotic expansions (see the brief description in Section XV). This method gives essentially the Snyder result (12.9), but with a choice of a certain arbitrary constant in a way that Butler considers unsatisfactory.

The other methods of calculation discussed in Sec. XV also have a bearing on asymptotic results. In particular, the calculation of Spencer and Blanchard (1954) shows how to take into account deviations at intermediate angles from the small angle approximation, and puts a limit on the extent to which the type of analysis used in this section should be pushed.

XIII. MEAN-VALUE CALCULATIONS

For many experimental purposes, some type of mean value that characterizes a scattering distribution is determined, rather than a histogram or other plot of the distribution itself. In this section, we calculate several types of mean value for small-angle multiple scattering. We shall focus attention on mean values for the Molière theory, and then specify how they may be determined when, for instance, the corrections of Nigam *et al.* (1959) must be taken into account.

By "mean values," we refer to any of the following: the νth moment or mean value of the νth power of θ or ϕ; cutoff and "shaveoff" moments for truncated distributions; the mean value of $\cos\xi_1\phi$ or $J_0(\xi_1\theta)$, where ξ_1 is a fixed number; the 1/2 or $1/e$ widths; the height of, or curvature at, the maximum; or the median value of θ or ϕ. We shall also include two measures of the "tail" of the distribution, namely the angles at which the function or its area out to ∞ has fallen to 1%.

A. Moment Calculations

It is not hard to show from (2.4b) that $\langle\theta^\nu\rangle_{\text{av}}$ and $\langle\phi^\nu\rangle_{\text{av}}$ are related by the formula

$$\langle\phi^\nu\rangle_{\text{av}} = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\pi^{1/2}\Gamma\left(\frac{\nu}{2}+1\right)}\langle\theta^\nu\rangle_{\text{av}} \tag{13.1}$$

for those values of ν for which both moments exist. Thus, it is only necessary to calculate one set for the desirable values of ν, namely 1/2 and 1. Second moments do not exist in our approximation since the long Rutherford scattering tail leads to a divergent integral. (In other words, the actually finite second moment is quite sensitive to details of the single-scattering function at large angles.) Furthermore, fluctuations in experimental estimates of the mean square are large compared to those for the first moment [for projected scattering, we refer, of course, to the "absolute-value" or "arithmetic" first moment found from $f(\phi,t)$ rather than the vanishing first moment for $F_p(\phi,t)$]. The 1/2-power moment is included because of the possibility of reducing fluctuations even more by its use.

For the spatial distribution, we can also calculate the moment for $\nu = -1$, which is useful in the "flat-chamber" distribution. By "flat-chamber" distribution we mean the distribution in one projected angle ϕ_x when the other, ϕ_y, is equal to zero within narrow limits, $\phi_y = 0 \pm \Delta\phi_y$. This function can be derived from $F(\theta)$ by setting $\theta = \phi_x = \phi$ and renormalizing. We have, in fact, for the flat-chamber distribution function $f_{\text{fc}}(\phi)$,

$$f_{\text{fc}}(\phi) = 2\pi F(\theta)/\langle\theta^{-1}\rangle_{\text{av}}, \tag{13.2}$$

so that

$$\int_0^\infty f_{\text{fc}}(\phi)d\phi = 1. \tag{13.3}$$

(Note that $\langle\theta^{-1}\rangle_{av}$ generally exists, but that $\langle\phi^{-1}\rangle_{av}$ generally does not.) We also find

$$\langle\phi^{\nu}\rangle_{av,fc} = \langle\theta^{\nu-1}\rangle_{av}/\langle\theta^{-1}\rangle_{av} . \tag{13.4}$$

Because of their widespread use in nuclear emulsion studies, we shall emphasize projected moments. Using (9.31) and (9.28), and choosing the reduced angle φ as variable [(13.1) holds, of course, when reduced angles are used on both sides], we see that evaluation of

$$\langle\varphi^{\nu}\rangle_{av} = \int_0^\infty d\varphi\varphi^\nu f_{red}(\varphi,t) \; ; \quad -1 < \nu < 2 \tag{13.5}$$

involves the calculation of the general integral

$$M_n(\nu,a) = \frac{2}{\pi}\int_0^\infty d\varphi\varphi^\nu \frac{\partial^n}{\partial a^n}\int_0^\infty d\eta \cos\varphi\eta e^{-\eta^2/4}(\eta^2/4)^{a-1/2}.$$

If we restrict ν to the range $-1 < \nu < 0$, we may interchange the order of operations and write

$$M_n(\nu,a) = \frac{2}{\pi}\frac{\partial^n}{\partial a^n}\int_0^\infty d\eta e^{-\eta^2/4}\left(\frac{\eta^2}{4}\right)^{a-1/2} \times \int_0^\infty d\varphi\varphi^\nu \cos\varphi\eta . \tag{13.6}$$

Formula [1.3(1)] of Erdelyi (T.I.T., 1954) gives

$$\int_0^\infty d\varphi\varphi^\nu \cos\varphi\eta = \frac{\pi}{2\eta^{\nu+1}\Gamma(-\nu)\cos\frac{1}{2}\nu\pi} = \frac{2^\nu\pi^{1/2}\Gamma\left(\frac{\nu+1}{2}\right)}{\eta^{\nu+1}\Gamma(-\nu/2)} . \tag{13.7}$$

We have then

$$M_0(\nu,a) = \frac{\Gamma\left(a-\frac{1}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}{\pi^{1/2}\Gamma(-\nu/2)},$$

$$M_1(\nu,a) = \frac{\Gamma\left(a-\frac{1}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)\psi\left(a-\frac{\nu}{2}-\frac{3}{2}\right)}{\pi^{1/2}\Gamma(-\nu/2)},$$

$$M_2(\nu,a) = \frac{\Gamma\left(a-\frac{1}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)\left[\psi^2\left(a-\frac{\nu}{2}-\frac{3}{2}\right)+\psi'\left(a-\frac{\nu}{2}-\frac{3}{2}\right)\right]}{\pi^{1/2}\Gamma(-\nu/2)}. \tag{13.8}$$

The restrictions on ν above-mentioned may now be relaxed, by the theory of analytic continuation, so (13.8) is valid for $-1 < \nu < 2$.

For the Molière functions, according to (9.32d) and (9.32f), we need $M_1(\nu,\frac{3}{2})$ and $M_2(\nu,\frac{5}{2})$ and find, in fact

$$\langle\varphi^\nu\rangle_{av} = \pi^{-1/2}\Gamma\left(\frac{\nu+1}{2}\right)\left\{1-\frac{\nu}{2B}\psi\left(-\frac{\nu}{2}\right) - \frac{1}{2B^2}\left(1-\frac{\nu}{2}\right)\frac{\nu}{2}\left[\psi^2\left(1-\frac{\nu}{2}\right) + \psi'\left(1-\frac{\nu}{2}\right)\right]+\cdots\right\}. \tag{13.9}$$

The value of $\langle\vartheta^\nu\rangle_{av}$ is given by the same bracket with $\Gamma(\nu/2+1)$ in front.[21] It will be noted in either case that the zeroth moments yield the normalization condition as expected—in fact, when $\nu\rightarrow 0$, only $M_0(\nu,\frac{1}{2})$ of (13.8) gives a finite, nonvanishing value.

The results for various values of ν can then be written for both distributions:

$$\langle\vartheta^0\rangle_{av} = \langle\varphi^0\rangle_{av} = 1 , \tag{13.10}$$

$$\frac{2}{\pi^{1/2}}\langle\vartheta\rangle_{av} = \pi^{1/2}\langle\varphi\rangle_{av} = \left\{1+\frac{0.9818}{B}-\frac{0.1170}{B^2}+\cdots\right\} \tag{13.11}$$

$$\frac{\langle\vartheta^{1/2}\rangle_{av}}{0.9064} = \frac{\langle\varphi^{1/2}\rangle_{av}}{0.6914} = \left\{1+\frac{0.2714}{B}-\frac{0.0774}{B^2}+\cdots\right\}, \tag{13.12}$$ [22]

$$\langle\vartheta^{-1/2}\rangle_{av} = 1.2255\left\{1-\frac{0.0569}{B}+\frac{0.1383}{B^2}+\cdots\right\}, \tag{13.13}$$

$$\langle\vartheta^{-1}\rangle_{av} = \pi^{1/2}\left\{1+\frac{0.0182}{B}+\frac{0.3693}{B^2}+\cdots\right\}. \tag{13.14}$$

The additional quantities needed in case the corrections of Nigam *et al.* are made, can easily be obtained from (13.8).

In Sec. VIII, it was noted that if $\varphi_s\Omega_0^{-1/2} = \varphi B^{1/2} = \phi/\chi_c$ is used as a variable, the distribution will be

[21] The integrals corresponding to (13.6) for the spatial-angle case may be found from $\int_0^\infty d\vartheta J_0(\eta\vartheta)\vartheta^{\nu+1} = 2^{\nu+1}\Gamma(\frac{1}{2}\nu+1)/\eta^{\nu+2}\,\Gamma(-\frac{1}{2}\nu)$ {Erdelyi, T.I.T. 1954, [8.5(7)]}.

[22] Molière (1948) seems to have a numerical error in his formula for this result.

almost independent of Ω_0. Thus we can expect that $B^{1/2}$ times the arithmetic mean value, or B times its square, will vary in a slow way with Ω_0. In fact, using (13.11)

$$\langle\varphi\rangle_{\mathrm{av}}^2 B = \frac{1}{\pi}\left\{B + 1.9636 + \frac{0.7299}{B} + \cdots\right\} \quad (13.15)$$

which, by use of (7.40), becomes

$$\langle\varphi\rangle_{\mathrm{av}}^2 B = 1.026 + 0.819 \log_{10} \Omega_0 . \quad (13.16)$$

Consequently, it is useful to plot B times the $(2/\nu)$th power of any mean value, against $\log_{10} \Omega_0$; the results give nearly straight lines over the useful range of Ω_0, whereas the relation between $\langle\varphi\rangle_{\mathrm{av}} B^{1/2}$ or any other similar product and $\log_{10} \Omega_0$ is decidely nonlinear. The linearity shown in (13.16) and in the straight lines plotted in Figs. 34 and 35 [Scott (1952); Goldberg, Snyder and Scott (1955)], which were drawn from the computed Snyder tables, may be understood as arising from the slow variation of B with Ω_0 and the relatively small $1/B$ term in (13.15). It may also be partially understood in terms of the difficult-to-calculate mean square. The mean square angle of scattering actually does exist, of course, and in accordance with the well-known theorem that the mean square for a set of independent events is proportional to the sum of the mean squares for each event, $\langle\phi^2\rangle_{\mathrm{av}}$ should be proportional to t, that is to χ_c^2, so $\langle\phi^2\rangle_{\mathrm{av}}/\chi_c^2 = \langle\varphi^2\rangle_{\mathrm{av}} B$ should be constant. Thus, it is not surprising that the other moments vary in only a slow fashion with Ω_0. In fact, if the distribution were Gaussian, none of the moments would have any variation. The smallness of the coefficient 0.819 in (13.16) and the corresponding coefficients for the other moments as given in Table XII is thus a measure of the closeness of the actual distribution to a Gaussian.

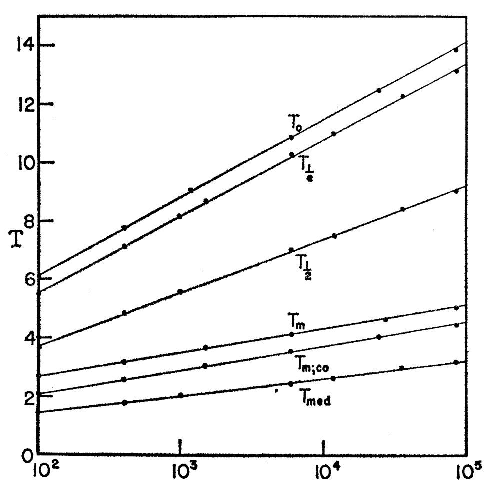


FIG. 34. Calculated values and least-square-relative-error fitted lines, for quantities related to six mean values for projected ("tangent-angle") scattering, as given in Table XIII [Scott (1952)].

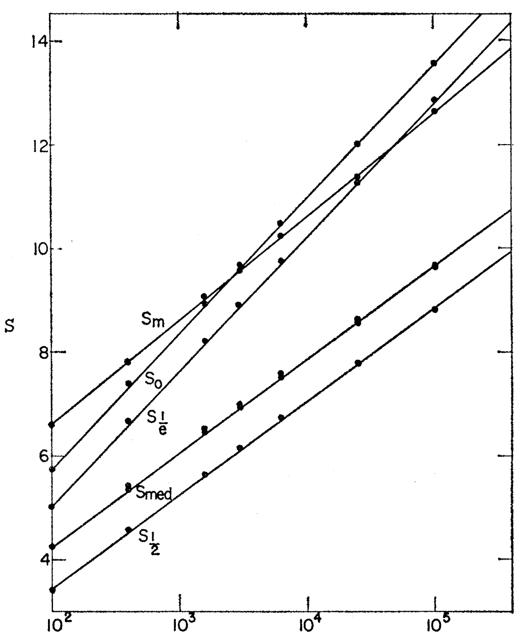


FIG. 35. Calculated values and least-square-relative-error fitted lines, for quantities related to six mean values for spatial-angle scattering, as given in Table XIII [Goldberg, Snyder, and Scott (1955)].

B. Measures of the Distribution "Tail"

Using the Snyder tables for projected scattering, it can be shown that the angle $\varphi_{1\%}$ at which the distribution function is 1% of the value at $\varphi = 0$ is

TABLE XII. Coefficients in least-squares formulas of form $a + b \log_{10} \Omega_0$ for several mean-value quantities [Scott (1952); Goldberg, Snyder, and Scott (1955)]. The symbols T and S correspond to those in Figs. 34 and 35.

		a	b
T_m	$= \langle\varphi\rangle_{\mathrm{av}}{}^2 B$	1.044	0.809
$T_{m;co}$	$= \langle\varphi\rangle_{\mathrm{av\ co}}{}^2 B^{a}$	0.418	0.818
	$\langle\varphi\rangle^2{}_{\mathrm{av\ co}} B^{a,b}$	0.998	1.221
$T_{1/2}$	$= \varphi_{1/2}{}^2 B$	0.035	1.831
$T_{1/e}$	$= \varphi_{1/e}{}^2 B$	0.253	2.636
T_{med}	$= \varphi_{\mathrm{med}}{}^2 B$	0.222	0.596
T_0	$= \varphi_0{}^2 B$	0.806	2.656
S_m	$= \langle\vartheta\rangle_{\mathrm{av}}{}^2 B$	2.601	2.002
$S_{1/2}$	$= \vartheta_{1/2}{}^2 B$	−0.146	1.799
$S_{1/e}$	$= \vartheta_{1/e}{}^2 B$	−0.170	2.601
S_{med}	$= \vartheta_{\mathrm{med}}{}^2 B$	0.675	1.801
S_0	$= \vartheta_0{}^2 B$	0.529	2.614

[a] Calculated for cutoff ratio $k_1 = 4.0$ (See Table XIII)
[b] Not included in Fig. 34.

given to 2% accuracy for Ω_0 from 600 to 84 000 by either of the rules

$$\varphi_1\% = 6.14B^{-1/2}\Omega_0^{0.035}, \quad (13.17a)$$

$$\varphi_1^2\% B = 30.5 + 10.4 \log_{10} \Omega_0 . \quad (13.17b)$$

A formula which gives the angle $\varphi_1\%_{\text{tail}}$ for which the area under the tail is 0.01, and is good to a percent or so for Ω_0 from 100 to 100 000, is

$$\varphi_1\%_{\text{tail}} B^{1/2} = 7.25 + 0.79 \log_{10} \Omega_0 ,$$

$$\int_{\varphi_1\%_{\text{tail}}}^{\infty} f(\varphi)d\varphi = 0.01 . \quad (13.18)$$

Note that this formula uses the angle times $B^{1/2}$ instead of the square of this quantity.

C. Cutoff and Shaveoff Moments

Cutoff moments are defined by truncating the projected distribution at some angle ϕ_1, and renormalizing. We write

$$\langle\phi^\nu\rangle_{\text{av,co.}} = \int_0^{\phi_1} \phi^\nu d\phi f(\phi) \Big/ \int_0^{\phi_1} d\phi f(\phi) , \quad (13.19)$$

where the upper limit ϕ_1 is chosen to be some multiple of $\langle\phi\rangle_{\text{av co}}$, say 3.5 or 4.0 times the latter.

So-called "shaveoff" moments are constructed from the distribution found by counting all angles greater than ϕ_1 as equal to it. Renormalization is not needed. We have

$$\langle\phi^\nu\rangle_{\text{av s.o.}} = \int_0^{\phi_1} \phi^\nu d\phi f(\phi) + \phi_1^\nu \int_{\phi_1}^{\infty} d\phi f(\phi) . \quad (13.20)$$

Since both the cutoff and shaveoff distributions possess mean squares, calculations of dispersion are meaningful. Calculations were carried out by numerical integration of the Molière functions for smaller values of ϕ and analytic use of the asymptotic expressions for large ϕ. They were made for ϕ_1 from 1.6 to 3.0 and B = 6,9,12, and 15, and the results found as functions of $k_1 = \phi_1/\langle\phi\rangle_{\text{av}}$. Straight-line plots against $\log_{10} \Omega_0$ were easily found, good to 1/2% or better; the coefficients of visually-fitted lines are given in Table XIII along with some values of the dispersion. It will be noticed that the shaveoff moments have the expected larger values of dispersion. Calculations involving the corrections of Nigam *et al.* have not been carried out.

D. Transform Mean Values

The mean value of cos $\xi_1\phi$ for the projected distribution and of $J_0(\xi_1\theta)$ for the spatial-angle one, where ξ_1, is an arbitrarily-chosen constant, is readily calculable, for it is just the transform itself evaluated for $\xi = \xi_1$. By (2.8), (2.17), (2.18), and (2.37), we have

$$\langle\cos \xi_1\phi\rangle_{\text{av}} = \int_0^{\infty} d\phi f(\phi,t) \cos \xi_1\phi = \tilde{f}(\xi_1)$$
$$= \tilde{F}(\xi_1) = 2\pi \int_0^{\infty} \theta d\theta F(\theta,t) J_0(\xi_1\theta)$$
$$= \langle J_0(\xi_1\theta)\rangle_{\text{av}} = \exp [\Omega(\xi_1) - \Omega_0] . \quad (13.21)$$

Such mean values take more computation from experimental data than simple averages, and the choice of ξ_1 to get the optimal dispersion depends on the resulting determination of χ_0, so that iteration is necessary. However, this type of mean depends largely on the values of $f(\phi)$ and $F(\theta)$ for small angles and hence should be less subject to fluctuation arising from the rarer large-angle events. In fact, the second moment (mean of cosine-squared, etc.) can readily be found and the dispersion calculated. Lipkin, Rosendorff and Yekutieli (1955) and Rosendorff and Eisenberg (1958) have utilized this method of obtaining information from multiply scattered tracks, and have shown how to choose ξ_1 so as to give almost the smallest possible dispersion. However, they indicate that the precision so obtained is only a small amount greater than that obtained by use of the simpler arithmetic mean value.

TABLE XIII. Coefficients in visually-fitted formulas of form $a + b \log_{10} \Omega_0$ for cutoff and shaveoff mean values in projected scattering, for several values of the ratio $k_1 = \varphi_1/\langle\varphi\rangle_{\text{av co}}$ or $\varphi_1/\langle\varphi\rangle_{\text{av so}}$, and values of the relative dispersion $[\langle\varphi^2\rangle_{\text{av co}} - \langle\varphi\rangle^2_{\text{av co}}]/\langle\varphi\rangle^2_{\text{av co}}$ and $[\langle\varphi^2\rangle_{\text{av so}} - \langle\varphi\rangle^2_{\text{av so}}]/\langle\varphi\rangle^2_{\text{av so}}$. Discrepancies with Table XII are produced by use of different fitting methods.

k_1	$\langle\varphi\rangle^2_{\text{av co}}$ a	b	$\langle\varphi^2\rangle_{\text{av co}}$ a	b	Dispersion $B = 6$	$B = 15$
3.1	0.191	0.765	0.375	1.147	0.55	0.52
3.5	0.363	0.781	0.609	1.214		
4.0	0.456	0.808	0.883	1.250	0.64	0.59
4.5	0.551	0.811	1.083	1.264		
5.0	0.602	0.816	1.293	1.264	0.71	0.62
k_1	$\langle\varphi\rangle^2_{\text{av so}}$ a	b	$\langle\varphi^2\rangle_{\text{av so}}$ a	b	Dispersion $B = 6$	$B = 15$
3.0	0.593	0.813	1.170	1.231	0.64	0.57
3.5	0.673	0.813	1.402	1.257		
4.0	0.743	0.813	1.607	1.262	0.75	0.64
4.5	0.803	0.813	1.797	1.262	0.79	0.66

E. Other Measures

The approximately Gaussian behavior of the Molière distributions can be used helpfully in evaluating the remaining types of mean-value measures. The 1/2 and $1/e$ widths can be accurately determined by interpolating log $f_{\text{red}}(\varphi)$ against φ^2 and log $F_{\text{red}}(\vartheta)$

against ϑ^2. The median values can be found by using unnormalized Gaussians which agree with the 1/2 or $1/e$ widths and with the zero-angle values, or by using a combination of direct integration and interpolation.

The height of the maximum or zero-angle-value quantity can be taken directly from the Snyder tables. It also can be calculated by using the general mean-value formula (13.9). We have, in fact,

$$f_{\text{red}}(0) = \frac{2}{\pi}\int_0^\infty d\eta e^{-\eta^2/4} \times\left[1+\frac{\eta^2}{4B}\ln\frac{\eta^2}{4}+\frac{1}{2}\left(\frac{\eta^2}{4B}\ln\frac{\eta^2}{4}\right)^2+\cdots\right], \tag{13.22}$$

so that the calculation of $\langle\varphi^\nu\rangle_{\text{av}}$ can be used if we remove the factor $\pi/2\Gamma(-\nu)\cos\frac{1}{2}\nu\pi$ and then set $\nu=-1$. We find

$$f_{\text{red}}(0) = \frac{2}{\pi^{1/2}}\left\{1+\frac{1}{2B}\psi\left(\frac{1}{2}\right)+\frac{3}{8B^2}\left[\psi^2\left(\frac{3}{2}\right)+\psi'\left(\frac{3}{2}\right)\right]+\cdots\right\} = \frac{2}{\pi^{1/2}}\left\{1+\frac{0.0182}{B}+\frac{0.3693}{B^2}+\cdots\right\} \tag{13.23}$$

and by using the same calculation (since $J_0(0)=1$) with $\nu=-2$,

$$2\pi F_{\text{red}}(0) = 2\left\{1+\frac{\psi(1)}{B}+\frac{\psi^2(2)+\psi'(2)}{B^2}+\cdots\right\} = 2\left\{1+\frac{0.4228}{B}+\frac{1.2465}{B^2}+\cdots\right\}. \tag{13.24}$$

A normalized Gaussian with the same $F_{\text{red}}(0)$ would have a $1/e$ width given by

$$\vartheta_0 = [\pi F_{\text{red}}(0)]^{-1/2}. \tag{13.25}$$

For the projected distribution, the relation is

$$\varphi_0 = 2/\pi^{1/2} f_{\text{red}}(0). \tag{13.26}$$

We express the curvature at the maximum in terms of the Gaussian width of a function with the same relative curvature. If we set

$$f_{\text{red}}(\varphi) \approx c\exp[-\varphi^2/\varphi^2_{\text{curv}}], \tag{13.27}$$

and assume a strict equality in the neighborhood of $\varphi=0$, we find

$$\varphi^2_{\text{curv}} = -2f_{\text{red}}(0)/f''_{\text{red}}(0) \tag{13.28}$$

and for spatial-angle scattering

$$\theta^2_{\text{curv}} = -2F_{\text{red}}(0)/F''_{\text{red}}(0). \tag{13.29}$$

The values of $f''_{\text{red}}(0)$ and $F''_{\text{red}}(0)$ can be found by the same technique as just used for $f_{\text{red}}(0)$ and $F_{\text{red}}(0)$; this time we use $\nu=-3$ and -4. We find

$$-f''_{\text{red}}(0) = \frac{4}{\pi^{1/2}}\left\{1+\frac{3}{2B}\psi\left(\frac{3}{2}\right)+\frac{15}{8B^2}\left[\psi^2\left(\frac{5}{2}\right)+\psi'\left(\frac{5}{2}\right)\right]+\cdots\right\} = \frac{4}{\pi^{1/2}}\left\{1+\frac{1.0547}{B}+\frac{2.9012}{B^2}+\cdots\right\} \tag{13.30}$$

$$-2\pi F''_{\text{red}}(0) = 4\left\{1+\frac{2}{B}\psi(2)+\frac{3}{B^2}[\psi^2(3)+\psi'(3)]+\cdots\right\} = 4\left\{1+\frac{1.8456}{B}+\frac{5.5848}{B^2}+\cdots\right\}. \tag{13.31}$$

We finally obtain from (13.23), (13.28), and (13.30),

$$\varphi^2_{\text{curv}} = 1-\frac{1.0365}{B}-\frac{1.4195}{B^2} \tag{13.32}$$

and from (13.24), (13.29), and (13.31),

$$\vartheta^2_{\text{curv}} = 1-\frac{1.4226}{B}-\frac{0.9321}{B^2}. \tag{13.33}$$

Extension of the results for height-of-maximum and curvature-at-maximum to cover the new functions introduced by Nigam *et al.* is straightforward. To find corrections to the Molière values of the 1/2 and $1/e$ widths and the median, tables of the derivatives and integrals of all the functions have been prepared.[23]

F. Formulas

The behavior of squares of mean values as functions of $\log_{10}\Omega_0$ referred to in part A of this section also holds for the miscellaneous group of mean values. Table XII gives the coefficients for a set of quantities labeled T (for "tangent," as opposed to "chord") and S (for "spatial"). Figures 34 and 35 show the straight lines and calculated points to which they were fitted. The table also includes coefficients for the cutoff mean and cutoff mean square for projected scattering with $k_1=4.0$.

The logarithmic increase of all mean values, along with the absence of a mean square, is a sign of the

[23] Available with the tables of D_n functions from the American Documentation Institute (cf. footnote 17).

deviation of the distributions from Gaussian shape, as indicated above. It is interesting to compare ratios of the various mean values to each other with the corresponding ratios in the Gaussian case. If a Gaussian distribution for positive φ only is written in the form (13.25), with $c = 2/\varphi_{\text{curv}}\,\Pi^{1/2}$, then the various mean values are given by

$$\begin{aligned}
\varphi_0 = \varphi_{1/e} &= \varphi_{\text{curv}}\,, \\
\varphi_{1/2} &= 0.8325\varphi_{\text{curv}}\,, \\
\varphi_{\text{med}} &= 0.4769\varphi_{\text{curv}}\,, \\
\langle\varphi\rangle_{\text{av}} &= 0.5642\varphi_{\text{curv}}\,, \\
\langle\varphi\rangle_{\text{av,co}} &= 0.5614\varphi_{\text{curv}} \quad \text{for } k_1 = 4.0\,, \\
\langle\varphi\rangle_{\text{av,so}} &= 0.5640\varphi_{\text{curv}} \quad \text{for } k_1 = 4.0\,. \qquad (13.34)
\end{aligned}$$

Table XIV shows the agreement among the coefficients of $\log_{10}\Omega_0$ in expressions for $\varphi_{\text{curv}}{}^2B$ calculated from the various T's and S's. It is evident that except for mean values, which involve the tail end of the distribution more than do the other quantities, the effective Gaussian widths agree within 1% for large Ω_0, even though they steadily increase. For small Ω_0, however, they do not agree well at all. So we see that Gaussian approximations to the multiple scattering distributions will be more accurate the larger is Ω_0.

TABLE XIV. Coefficients for formulas of form $a + b\log_{10}\Omega_0$ for $\varphi_{\text{curv}}{}^2B$, calculated by use of (13.34) from the various T's.

Source	a	b
$T_{1/e}$	0.253	2.636
T_0	0.806	2.656
$T_{1/2}$	0.051	2.642
T_{med}	0.976	2.621
T_m	3.280	2.542
$T_{m;\text{co}}$	1.326	2.595

In passing, we should note that the mean-square angle calculated in the usual form of Gaussian approximation may be simply related to the mean-value formulas we have given. From Rossi (1952), p. 67 we find

$$\langle\vartheta^2\rangle_{\text{av}}B = 4\ln(183Z^{-1/3}) \qquad (13.35)$$

where the factor 183 depends on the particular method selected of making a cutoff related to finite nuclear size. From Birkhoff (1958), p. 116 we have

$$\langle\vartheta^2\rangle_{\text{av}}B = 2\ln(1.13\chi_c/\chi_0) \simeq \ln\Omega_0\,, \qquad (13.36)$$

where the cutoff has been arbitrarily taken at χ_c.

The various mean values can be evaluated for any of the track characteristics discussed in Secs. III-C and VII-c by using the variable $X^{(j)}/(\chi_c^2C_1{}^{(j)}B)^{1/2}$ in place of φ. Since the value of B is not greatly different for the different quantities, the mean values for each $X^{(j)}$ will be essentially proportional to $C_1{}^{(j)1/2}$. The correlation between adjacent $\hat{\phi}$'s, α's, and $\hat{\alpha}$'s becomes evident when we note that (using angular brackets to refer to any mean value)

$$\begin{aligned}
\langle\hat{\phi}_j + \hat{\phi}_{j+1}\rangle^2 : 2\langle\hat{\phi}_j\rangle^2 : \langle\hat{\phi}_j - \hat{\phi}_{j+1}\rangle^2 &\simeq \frac{61}{35} : \frac{52}{35} : \frac{43}{35} \\
\langle\alpha_j + \alpha_{j+1}\rangle^2 : 2\langle\alpha_j\rangle^2 : \langle\alpha_j - \alpha_{j+1}\rangle^2 &\simeq \frac{5}{3} : \frac{4}{3} : \frac{3}{3} \\
\langle\hat{\alpha}_j + \hat{\alpha}_{j+1}\rangle^2 : 2\langle\hat{\alpha}_j\rangle^2 : \langle\hat{\alpha}_j - \hat{\alpha}_{j+1}\rangle^2 &\simeq \frac{46}{30} : \frac{33}{30} : \frac{20}{30}\,.
\end{aligned} \qquad (13.37)$$

If these quantities were uncorrelated, the three successive squares would be equal; indeed, the ϕ's are uncorrelated and their C_1's have this property.

More exact relations between various mean values are derivable from the formulas of Table XII relating the T's to Ω_0. For instance, for the arithmetic mean, we have, correcting Ω_0 by (7.58),

$$T = \langle X^{(j)}\rangle_{\text{av}}^2/\chi_c^2C_1^{(j)} = 1.044 + 0.809(\log_{10}\Omega_0 + \Delta\log_{10}\Omega_0)\,. \qquad (13.38)$$

The ratios of the arithmetic means for the second, third and fourth differences are important for the use of these differences in the study of tracks in nuclear emulsions. We have from Table III for the squares of the ratios of the mean third and fourth differences to the mean second difference

$$(\overline{\Delta^3}/\overline{\Delta^2})^2 = \frac{3}{2}\,\frac{1.279 + 0.809\log_{10}\Omega_0}{1.135 + 0.809\log_{10}\Omega_0} \qquad (13.39)$$

$$(\overline{\Delta^4}/\overline{\Delta^2})^2 = 4\,\frac{1.258 + 0.809\log_{10}\Omega_0}{1.135 + 0.809\log_{10}\Omega_0}\,. \qquad (13.40)$$

The multipliers of the factors 3/2 and 4 in these two expressions are close to unity, but slightly larger, by amounts depending on Ω_0. For $\Omega_0 = 10^2$, for instance, the two fractions are 1.052 and 1.045, respectively, whereas for $\Omega_0 = 10^5$, the values are 1.028 and 1.024. These numbers may be taken as indications of the extent to which the Gaussian approximation does not apply to the correlated events.

XIV. CALCULATIONS OF POLARIZATION EFFECTS

According to Mühlschlegel and Koppe (1958), the application of the first Born approximation to the calculation of the functions $f(\theta)$ and $g(\theta)$ in the matrix $A(\theta,\beta)$ of Eq. (5.33) yields the result, valid for small angles, that $|f(\theta)|^2$ is just the screened Rutherford cross section (6.10) [using (7.7)]:

$$|f(\theta)|^2 = (\chi_c^2/\pi N)q(\theta)/\theta^4 \tag{14.1}$$

and that $g(\theta)$ is given by

$$g(\theta) = -\tfrac{1}{2}[1-(1-\beta^2)^{1/2}]f(\theta) . \tag{14.2}$$

As stated in Sec. V, $g(\theta)$ vanishes as $\beta \to 0$ (non-relativistic limit). The function $f(\theta)$ itself is just the first Born amplitude, given for instance by Eq. (11.4). The results of Dalitz (1951) on the second Born approximation can be used to obtain better formulas for f and g, but this has not been done.

Since according to (14.2), f^*g is real, $D(\theta)$ as given in (5.36) vanishes—Mott polarization disappears at small angles. We have then from (5.36)

$$\begin{aligned} J(\theta) &= |f(\theta)|^2 + \theta^2|g(\theta)|^2 \\ &= \{1+\tfrac{1}{4}\theta^2[1-(1-\beta^2)^{1/2}]\}(\chi_c^2/\pi N)q(\theta)/\theta^4 , \end{aligned} \tag{14.3a}$$

$$D(\theta) = 0 , \tag{14.3b}$$

$$E(\theta) = [1-(1-\beta^2)^{1/2}](\chi_c^2/\pi N)q(\theta)/\theta^4 , \tag{14.3c}$$

$$G(\theta) = \tfrac{1}{2}[1-(1-\beta^2)^{1/2}]^2(\chi_c^2/\pi N)q(\theta)/\theta^4 . \tag{14.3d}$$

We can now find the five transforms given in (5.46). After multiplying by t, we have

$$t\tilde{\jmath}(\xi) = \Omega(\xi) + \tfrac{1}{2}t\tilde{g}(\xi) , \tag{14.4a}$$

$$t\tilde{d}(\xi) = 0 , \tag{14.4b}$$

$$t\tilde{e}(\xi) = -[1-(1-\beta^2)^{1/2}]d\Omega/d\xi , \tag{14.4c}$$

$$t\tilde{g}(\xi) = -\tfrac{1}{2}[1-(1-\beta^2)^{1/2}]^2[d^2\Omega/d\xi^2 + \xi^{-1}d\Omega/d\xi] , \tag{14.4d}$$

$$t\tilde{h}(\xi) = -\tfrac{1}{4}[1-(1-\beta^2)^{1/2}]^2\xi^{-1}d\Omega/d\xi , \tag{14.4e}$$

where we have used certain properties of Bessel functions:

$$\begin{aligned} &J_0'(x) = -J_1(x) ; \\ &J_0''(x) + x^{-1}J_0'(x) + J_0(x) = 0 , \\ &J_0(x) + J_2(x) = 2x^{-1}J_1(x) = -2x^{-1}J_0'(x) . \end{aligned}$$

It might seem that in small-angle approximation the $\theta^2|g(\theta)|^2$ term in (14.3a) could be omitted, and it is true that this can be done in the lowest nonvanishing approximation for polarization effects. However, as shown by Mühlschlegel and Koppe, failure to include this term leads in the next lowest approximation to an apparent increase in the degree of polarization on scattering, which is nonphysical and wrong.

In fact it can be shown by straightforward algebra that the four functions in (5.36) obey the identity

$$(J-\sin^2\theta G)^2 + \sin^2\theta(D^2+E^2) = J^2 . \tag{14.5}$$

From (5.38b) we find, after lengthy calculation

$$\mathbf{1}_0\cdot\mathbf{P}' = (\mathbf{1}_0\cdot\mathbf{P} + \sin\theta D/J)/(1+\mathbf{1}_0\cdot\mathbf{P}\sin\theta D/J) \tag{14.6}$$

and with (14.3),

$$\begin{aligned} (1-\mathbf{P}'\cdot\mathbf{P}') &= (1-\sin^2\theta D^2/J^2)(1-\mathbf{P}\cdot\mathbf{P}) \\ &\quad\times(1+\mathbf{1}_0\cdot\mathbf{P}\sin\theta D/J)^{-2} . \end{aligned} \tag{14.7}$$

When $D = 0$, we have

$$\mathbf{1}_0\cdot\mathbf{P}' = \mathbf{1}_0\cdot\mathbf{P} \quad , \quad P'^2 = P^2 . \tag{14.8}$$

so that the degree of polarization P' is the same as P, and its component in the direction $\mathbf{1}_0$ is unaltered; $\mathbf{P}$ rotates around $\mathbf{1}_0$ during a scattering. The identities (14.5), (14.6) and (14.7) would not be correct if the $\sin\theta$ or θ were omitted from (14.3a).

The angle θ_p between $\mathbf{P}'$ and $\mathbf{P}$ may be found in the case $D = 0$, by calculating $\mathbf{P}\cdot\mathbf{P}' = P'P\cos\theta_p$ from (5.38b). We find

$$\sin(\theta_p/2) = (G/2J)^{1/2}\sin\theta\sin\alpha , \tag{14.9a}$$

where α is the angle between $\mathbf{P}$ and $\mathbf{1}_0$:

$$\mathbf{1}_0\cdot\mathbf{P} = P\cos\alpha . \tag{14.10}$$

Using (14.1) and (14.2) we find in small-angle Born approximation

$$\theta_p = [1-(1-\beta^2)^{1/2}]\theta\sin\alpha \tag{14.9b}$$

for the rotation of the polarization vector in a single scattering.

The term $\tfrac{1}{2}t\,\tilde{g}(\xi)$ in (14.4a) leads to the awkward consequence that $\tilde{\jmath}_0$ is infinite [if we use the Molière formula (7.15) for $\Omega(\xi)$ in (14.4d)]. This difficulty, however, only affects the asymptotic results for large angles. It arises because polarization effects are proportionately more affected by the few scatterings at large angles than is the angular distribution itself. The failure of $\Omega''(0)$ to be finite is as we have seen equivalent to the absence of a finite mean square for $F(\theta)$, and represents an inadequacy of the small-angle approximation. Mühlschlegel and Koppe assume that errors arising from this source will only affect results for large θ, and that an expansion of the integrand that gives correct results when the integrand is large is justified even if it is incorrect for ξ very near zero.

We write then

$$t(\tilde{\jmath}(\xi)-\tilde{\jmath}_0) = \Omega(\xi) - \Omega_0 + \tfrac{1}{2}t\tilde{g}(\xi) . \tag{14.11}$$

To calculate the polarization after scattering for an initially longitudinally-polarized beam of electrons, we substitute (14.4) into (5.57)–(5.60) for the case $P_{ox} = P_{oy} = 0$, and $P_{oz} = P_o$. The distributions F and Π_z then become independent of the angle β. We also take β equal to zero, so $\Pi_y = 0$—the y-axis becomes the axis of deflection (otherwise we will have Π_x proportional to $\cos\beta$ and Π_y to $\sin\beta$ with the same coefficient.)

In the formulas for F, Π_x, and Π_z, we make the Molière change of variable [cf. (7.35) and (7.43)] which gives in place of (14.4c), (14.4d), and (14.4e) and (14.9) (using primes for derivatives with respect to η)

$$t\tilde{e}(\eta) = -[1-(1-\beta^2)^{1/2}]\chi_c B^{1/2}\Omega'(\eta) = -s^{1/2}\Omega'(\eta)\,, \tag{14.12a}$$

$$t\tilde{g}(\eta) = -\tfrac{1}{2}[1-(1-\beta^2)^{1/2}]^2\chi_c^2 B[\Omega''(\eta)+\eta^{-1}\Omega'(\eta)]$$
$$= -\tfrac{1}{2}s[\Omega''(\eta)+\eta^{-1}\Omega'(\eta)]\,, \tag{14.12b}$$

$$t\tilde{h}(\eta) = -\tfrac{1}{4}[1-(1-\beta^2)^{1/2}]^2\chi_c^2 B\eta^{-1}\Omega'(\eta)$$
$$= -(s/4\eta)\Omega'(\eta) \tag{14.12c}$$

$$t[\tilde{j}(\eta)-\tilde{j}_0] = \Omega(\eta)-\Omega_0+\tfrac{1}{2}t\tilde{g}(\eta)\,, \tag{14.12d}$$

where we have used s for the small parameter that measures the polarization effects:

$$s = [1-(1-\beta^2)^{1/2}]^2\chi_c^2 B\,. \tag{14.13}$$

The ratio $s/\chi_c^2 B$ ranges from 0.02 at $\beta = 5$ to 0.3 at $\beta = .9$ and 1.0 at $\beta = 1.0$; $\chi_c^2 B$ is, of course, small in the small-angle approximation.

We also have, from (5.61),

$$t\tilde{\mu}(\eta) = t\tilde{h}(\eta) = -(s/4\eta)\Omega'(\eta)\,, \tag{14.14a}$$

$$t\tilde{\nu}(\eta) \simeq |\tilde{e}(\eta)|[1+\tilde{h}^2(\eta)/2\tilde{e}^2(\eta)]$$
$$\simeq [s^{1/2}+s^{3/2}/32\eta^2+\cdots]\,. \tag{14.14b}$$

We then find, expanding the hyperbolic functions,

$$2\pi F_{\text{red}}(\vartheta,t) = \int_0^\infty \eta d\eta \exp[\Omega(\eta)-\Omega_0+\tfrac{1}{2}t\tilde{g}(\eta)]J_0(\vartheta\eta), \tag{14.15a}$$

$$2\pi\Pi_{x\,\text{red}}(\vartheta,0,t) = \int_0^\infty \eta d\eta \exp[\Omega(\eta)-\Omega_0-\tfrac{1}{2}t\tilde{g}(\eta) + t\tilde{h}(\eta)]\cdot[t\tilde{e}(\eta)+\tfrac{1}{6}t^3\tilde{\nu}^2(\eta)\tilde{e}(\eta) +\cdots]J_1(\vartheta\eta)\,, \tag{14.15b}$$

$$2\pi\Pi_{z\,\text{red}}(\vartheta,t) = \int_0^\infty \eta d\eta \exp[\Omega-\Omega_0-\tfrac{1}{2}t\tilde{g}(\eta)+t\tilde{h}(\eta)] \times[1-t\tilde{h}(\eta)+\tfrac{1}{2}t^2\tilde{e}^2(\eta) +\tfrac{1}{2}t^2\tilde{h}^2(\eta)+\tfrac{1}{24}t^4\tilde{e}^4(\eta) -\tfrac{1}{6}t^3\tilde{e}^2(\eta)\tilde{h}(\eta)]J_0(\vartheta\eta)\,. \tag{14.15c}$$

The method of Mühlschlegel and Koppe is to expand these expressions in powers of s [according to (14.10)] and then to reduce the integrals by partial integration. However, these authors proceeded first without the term $\frac{1}{2}t\,g$ in (14.4a) and (14.11) and then showed that the results to order s^2 were patently incorrect.

Unfortunately, when they introduced the term $\frac{1}{2}t\,\tilde{g}$ to make a correction, they did it in a way that involved a mathematical error. They did not expand this term in the formula (14.15a) for F, but chose to express Π_x and Π_z in relation to F. This method of incomplete expansion is not wrong in itself, but as mentioned below, it involved an integration by parts for which inaccuracies in the expression for $\frac{1}{2}t\,\tilde{g}$ played a major role rather than being suppressed as in the rest of their calculation.

A consistent treatment that avoids this difficulty involves treating the $\frac{1}{2}t\,\tilde{g}$ term right along with $t(\tilde{h}-\tilde{g})'$ which is, of course, of the same order of magnitude.

When this is done, and (14.12) is used, we find

$$2\pi F_{\text{red}}(\vartheta,t) = \int_0^\infty \eta d\eta e^{\Omega-\Omega_0}[1-\tfrac{1}{4}s(\Omega''+\Omega'/\eta) + \tfrac{1}{32}s^2(\Omega''+\Omega'/\eta)^2]J_0(\vartheta\eta)\,, \tag{14.16a}$$

$$2\pi\Pi_{x\,\text{red}}(\vartheta,0,t) = P_0\int_0^\infty \eta d\eta e^{\Omega-\Omega_0}[-s^{1/2}\Omega' - s^{3/2} \times(\tfrac{1}{4}\Omega'\Omega''+\tfrac{1}{6}\Omega'^3)]J_1(\vartheta\eta)\,, \tag{14.16b}$$

$$2\pi\Pi_{z\,\text{red}}(\vartheta,t) = P_0\int_0^\infty \eta d\eta e^{\Omega-\Omega_0}\{1+\tfrac{1}{4}s(2\Omega'^2+\Omega'' +\Omega'/\eta)+\tfrac{1}{24}s^2[\tfrac{3}{4}(\Omega''+\Omega'/\eta)^2 +\Omega'^4+\Omega'^3/\eta+3\Omega'^2\Omega'']\}J_0(\vartheta\eta)\,. \tag{14.16c}$$

These formulas can be reduced by several steps of partial integration, if we assume that Ω' and $\Omega''\eta$ approach 0 as $\eta\to 0$, and that $e^{\Omega-\Omega_o}\to 0$ as $\eta\to\infty$. This last is not strictly correct, as discussed in Sec. II, but the contribution of the term containing $e^{-\Omega_0}$ that must be added to give a transform that vanishes properly at infinity is negligible for any thickness of material in which multiple-scattering polarization effects are observable. The vanishing at $\eta = 0$ of Ω' and $\Omega''\eta$ holds for Molière's form for Ω, and also for the form of Nigam *et al.* if (9.15b) and (9.17b) are taken into account.

We find then

$$2\pi F_{\mathrm{red}}(\vartheta,t) = \int_0^\infty \eta d\eta e^{\Omega-\Omega_0} J_0(\vartheta\eta)[1 - \tfrac{1}{4} s(\Omega'' + \Omega'/\eta) + (s^2/32)(\Omega''^2 + 2\Omega'\Omega''/\eta + \Omega'^2/\eta^2) + \cdots], \quad (14.17\mathrm{a})$$

$$2\pi \Pi_{x\,\mathrm{red}}(\vartheta,0,t) = P_0 s^{1/2}\vartheta \int_0^\infty \eta d\eta e^{\Omega-\Omega_0} J_0(\vartheta\eta) + P_0 s^{3/2} \times \int_0^\infty d\eta e^{\Omega-\Omega_0}[(\tfrac{1}{6}\vartheta^2\Omega' - \tfrac{1}{12}\Omega''') \times \eta J_1(\vartheta\eta) - (\tfrac{1}{4}\vartheta\eta\Omega'' + \tfrac{1}{6}\vartheta\Omega') \times J_0(\vartheta\eta)], \quad (14.17\mathrm{b})$$

$$2\pi \Pi_{z\,\mathrm{red}}(\vartheta,t) = P_0 \int_0^\infty \eta d\eta e^{\Omega-\Omega_0} J_0(\vartheta\eta) + \tfrac{1}{4} P_0 s \times \int_0^\infty d\eta e^{\Omega-\Omega_0}[2\vartheta\eta\Omega' J_1(\vartheta\eta) - (\Omega' + \eta\Omega'')J_0(\vartheta\eta)] + \tfrac{1}{24} P_0 s^2 \times \int_0^\infty d\eta e^{\Omega-\Omega_0}[(\tfrac{3}{4}\Omega''^2\eta + \tfrac{3}{2}\Omega'\Omega'' + 3\Omega'^2/4\eta + 3\Omega''\vartheta^2\eta + \Omega'\vartheta^2)J_0(\vartheta\eta) + (-\Omega'\vartheta^3\eta + 2\Omega'''\eta\vartheta)J_1(\vartheta\eta)]. \quad (14.17\mathrm{c})$$

For evaluation, we use the Molière form for Ω and properties of its derivatives:

$$\Omega(\eta) = -\frac{\eta^2}{4} + \frac{\eta^2}{4B}\ln\frac{\eta^2}{4}$$
$$\eta\Omega''(\eta) - \Omega'(\eta) = \eta/B,$$
$$\eta\Omega'''(\eta) = 1/B. \quad (14.18)$$

We also introduce the following abbreviations

$$\rho(\vartheta) = 2\pi F_{\mathrm{red}}(\vartheta)|_{s=0} = \int_0^\infty \eta d\eta e^{\Omega-\Omega_0} J_0(\vartheta\eta), \quad (14.19)$$

$$\tau(\vartheta) = \int_0^\vartheta \vartheta'\rho(\vartheta')d\vartheta' = \vartheta\int_0^\infty d\eta e^{\Omega-\Omega_0} J_1(\vartheta\eta), \quad (14.20)$$

$$\sigma(\vartheta) = \int_0^\infty d\eta e^{\Omega-\Omega_0}\Omega'(\eta)\Omega''(\eta)J_0(\vartheta\eta). \quad (14.21\mathrm{a})$$

This last function can be expressed as follows

$$\sigma(\vartheta) = \sigma(0) - \int_0^\vartheta d\vartheta' \int_0^\infty d\eta e^{\Omega-\Omega_0}\Omega'\Omega''\eta J_1(\vartheta'\eta) = \sigma(0) + \int_0^\vartheta d\vartheta' \int_0^\infty \eta d\eta e^{\Omega-\Omega_0}[\Omega''' J_1(\vartheta\eta) + \vartheta\Omega'' J_0(\vartheta\eta)]$$

and with (14.15), (14.16) and (14.17),

$$\sigma(\vartheta) = \sigma(0) + \frac{1}{B}\int_0^\vartheta d\theta' \frac{\tau(\vartheta')}{\vartheta'} + \frac{\tau(\vartheta)}{B} - \tfrac{1}{2}\vartheta^2 + \int_0^\vartheta \vartheta'\tau(\vartheta')d\vartheta' \quad (14.21\mathrm{b})$$

$$\sigma(0) = \int_0^\infty d\eta e^{\Omega-\Omega_0}\Omega'(\eta)\Omega''(\eta). \quad (14.21\mathrm{c})$$

The function $\rho(\vartheta)$ was given in Sec. VII, Eq. (7.46) and values may be taken from Table VII [see Eq. (9.32)]. The integrals involved in (14.20) and (14.21b) may be evaluated by use of suitable formulas in Appendix III in terms of the D_n functions of Sec. IX. Finally, $\sigma(0)$ can be evaluated by expansion in inverse powers of B and use of tables (see footnote 17) for values of the D_n functions for $\vartheta = 0$. We find

$$\sigma(0) = \tfrac{1}{2}D_0(1,1,0) + (1/B)[\tfrac{1}{2}D_1(2,1,0) - D_1(1,1,0) - 2D_0(1,1,0)] + (1/B^2)[\tfrac{1}{4}D_2(3,1,0) - D_2(2,1,0) + \tfrac{1}{2}D_2(1,1,0) - 2D_1(2,1,0) + 2D_1(1,1,0) + \tfrac{3}{2}D_1(1,1,0)] + \cdots$$
$$= 0.5000 - 1.2114/B + 0.2886/B^2 + \cdots. \quad (14.21\mathrm{d})$$

Table XV gives values of $\tau(\vartheta)$. The function $\sigma(\vartheta)$ has not been computed. The values given for τ by Mühlschlegel and Koppe appear to be grossly in error.

TABLE XV. Coefficients for $\tau(\vartheta) = \tau^{(0)}(\vartheta) + \tau^{(1)}(\vartheta)/B + \tau^{(2)}(\vartheta)/B^2$.

ϑ	$\tau^{(0)}(\vartheta)$	$\tau^{(1)}(\vartheta)$	$\tau^{(2)}(\vartheta)$
0	0	0	0
0.2	0.0392	0.0555	0.0455
0.4	0.1479	0.0464	0.1373
0.6	0.3023	0.0581	0.1839
0.8	0.4727	+0.0218	0.1322
1.0	0.6321	−0.0651	+0.0126
1.2	0.7631	−0.1783	−0.0936
1.4	0.8591	−0.2825	−0.1235
1.6	0.9227	−0.3501	−0.0733
1.8	0.9608	−0.3719	+0.0129
2.0	0.9817	−0.3550	0.0844
2.2	0.9921	−0.3148	0.1147
2.4	0.9968	−0.2666	0.1062
2.6	0.9988	−0.2205	0.0766
2.8	0.9996	−0.1815	0.0436
3.0	1.0000	−0.1504	0.0174
3.2	1.0000	−0.1264	+0.0007
3.4	1.0000	−0.1078	−0.0081
3.6	1.0000	−0.0933	−0.0117
3.8	1.0000	−0.0817	−0.0125
4.0	1.0000	−0.0723	−0.0120
5.0	1.0000	−0.0437	−0.0069
6.0	1.0000	−0.0294	−0.0039
7.0	1.0000	−0.0213	−0.0023
8.0	1.0000	−0.0161	−0.0015

The final evaluation of F and $\mathbf{\Pi}$ then reads

$$2\pi F_{\mathrm{red}}(\vartheta,t) = \rho(\vartheta) - s\left[\frac{\rho(\vartheta)}{4B} + \tfrac{1}{2}\tau(\vartheta) - \tfrac{1}{2}\right] + s^2\left[\frac{\rho(\vartheta)}{32B^2} + \tfrac{1}{8}\sigma(\vartheta)\right] + \cdots \quad (14.22\mathrm{a})$$

$$2\pi\Pi_{x\ \mathrm{red}}(\vartheta,0,t) = P_0 s^{1/2}\vartheta\rho(\vartheta) + P_0 s^{3/2}\left[-\frac{\vartheta^3\rho(\vartheta)}{6} - \frac{\tau(\vartheta)}{12B\vartheta} - \frac{5\vartheta\tau(\vartheta)}{12} + \frac{5}{12}\vartheta - \frac{\vartheta\rho(\vartheta)}{4B}\right] + \cdots$$
$$= P_0 s^{1/2}\vartheta\cdot 2\pi F_{\mathrm{red}}(\vartheta,t) + P_0 s^{3/2} \times\left[-\frac{\vartheta^3\rho(\vartheta)}{6} + \frac{\vartheta\tau(\vartheta)}{12} - \frac{\vartheta}{12} - \frac{\tau(\vartheta)}{12B\vartheta}\right], \qquad (14.22\mathrm{b})$$

$$2\pi\Pi_{z\ \mathrm{red}}(\vartheta,t) = P_0\rho(\vartheta) - P_0 s\left[\tfrac{1}{2}\vartheta^2\rho(\vartheta) + \frac{\rho(\vartheta)}{4B} + \tfrac{1}{2}\tau(\vartheta) - \tfrac{1}{2}\right] + P_0 s^2\left[\tfrac{1}{8}\sigma(\vartheta) + \frac{\vartheta^2\rho(\vartheta)}{8B} + \frac{\rho(\vartheta)}{32B^2} + \frac{\vartheta^2\tau(\vartheta)}{6} - \tfrac{1}{6}\vartheta^2 + \frac{\vartheta^4\rho(\vartheta)}{24} + \frac{\tau(\vartheta)}{12B}\right] + \cdots$$
$$= P_0\cdot 2\pi F_{\mathrm{red}}(\vartheta,t) - \tfrac{1}{2}P_0 s\vartheta^2\cdot 2\pi F_{\mathrm{red}}(\vartheta,t) + P_0 s^2\left[-\frac{\vartheta^2\tau(\vartheta)}{12} + \frac{\vartheta^2}{12} + \frac{\vartheta^4\rho(\vartheta)}{24} + \frac{\tau(\vartheta)}{12B}\right]. \qquad (14.22\mathrm{c})$$

These results agree with those of Mühlschlegel and Koppe with the exception that their formula (51) for Π_z includes an extra term $-P_0 s^2/4B$. This term arises from an integration by parts in which the infinite value of Ω'/η at $\eta = 0$ is not properly canceled by suitable power of η in functions multiplying it, as mentioned earlier.

The degree of polarization P may be found from (14.22) and is given by

$$P^2 = P_x^2 + P_z^2 = (\Pi_x^2 + \Pi_z^2)/F^2 = P_0^2 \qquad (14.23)$$

to order s^2; the coefficients of both s and s^2 vanish as expected from (14.8) (in contradistinction to Mühlschlegel and Koppe who find a depolarization).

The angle θ_p through which $\mathbf{P}$ has turned may be found by writing, to order $s^{3/2}$,

$$P_x = \Pi_x/F = P_0 s^{1/2}\vartheta\left\{1 + s\left[-\frac{\vartheta^2}{6} + \frac{\tau(\vartheta)}{12\rho(\vartheta)} - \frac{1}{12\rho(\vartheta)} - \frac{\tau(\vartheta)}{12B\rho(\vartheta)\vartheta^2}\right]\right\} = P_0\sin\theta_p\,, \qquad (14.24)$$

so that

$$\theta_p \simeq s^{1/2}\vartheta[1 - sa(\vartheta)] \qquad (14.25\mathrm{a})$$

with $a(\vartheta)$ the same as given by Mühlschlegel and Koppe:

$$a(\vartheta) = \frac{1}{12}\left[\frac{1-\tau(\vartheta)}{\rho(\vartheta)} + \frac{\tau(\vartheta)}{B\vartheta^2\rho(\vartheta)}\right]. \qquad (14.25\mathrm{b})$$

The term $-\vartheta^2/6$ in 14.24 disappears if the second term in the expansion of $\sin^{-1}(P_x/P_o)$ is taken into account.

The results given here may be used to calculate the transverse component of polarization [Sundaresan (1960)] and the self-depolarization of a beta source [Mühlschlegel (1959); Mühlschlegel and Koppe (1958), Sec. VII). The numbers given by these authors will need correction by use of Table XV.

An extension of the theory to large scattering angles is given by Toptygin (1959).

XV. SURVEY OF OTHER CALCULATIONS

In this section, we give a brief survey of other methods of deriving the distribution functions for small-angle multiple scattering, including methods used for including the effects due to finite nuclear size. Detailed accounts of these methods are beyond the scope of this article.

A simple method for calculating the scattering for larger angles (asymptotic series) that is applicable to various forms for the "tail" of the single-scattering distribution is given by Butler (1950). He divides the integral in (2.42) into two parts at a transition angle θ_1, effectively considering $W(\chi,t)$ to be made of two separate functions. The first part, for angles less than θ_1, is treated by the Gaussian approximation method given in Sec. II-E, and the second part is treated as a small perturbation in which the value of F derived from the first part alone is to be inserted. By a suitable choice of θ_1, the magnitude of the perturbation is kept small. The results agree closely with those of Molière or those of Snyder, depending on the choice of θ_1.

Monte Carlo calculations can be used successfully to generate an approximation to the Molière distribution if the single-scattering formula (6.63) is appropriately sampled for a large number of simulated particle trajectories [Humphrey (1962)]. This method is useful if multiple scattering and other particle events are to be treated simultaneously.

Another application of Monte Carlo methods is to the lateral and angular spread of a beam of particles over a wide range, sampling from the Molière distribution for short path lengths. Sidei, Higasimura, and Kinosita (1957) have demonstrated the utility of this method for study of the penetration of thick layers, backscattering, etc. Extensive use of Monte Carlo methods is reported by Berger (1962).

Spencer (1952, 1953) has given a general method for evaluating inverse transforms of the type met in multiple scattering theory, and Spencer and Blanchard (1954) applied it to a calculation of the Molière result with the inclusion of relativistic effects at angles up to 20° or so.

Spencer's method is to fit the function $\exp[\Omega(\xi) - \Omega_0]$ with a combination of functions which have known or easily calculable inverse transforms, and which also embody all available information about the behavior of the given function or of the final distribution function. The functions used for the fitting contain adjustable parameters, and suitable procedures are developed for evaluating them so as to fit $e^{\Omega-\Omega_0}$ adequately over the important range of ξ.

The types of functions considered by Spencer are Gaussian, Gaussian plus a contribution using the Bessel function K_1 (to fit the "tail" of the distribution in θ), Gaussian times ξK_1, and a series of Bessel functions K_m. The fitting procedure involved choosing a set of n parameters to fit n pieces of information about the function $e^{\Omega-\Omega_0}$, such as values of the function, values of some combination of derivatives, or moments.

The results given by Spencer and Blanchard (1954) are consistent with the Molière results as modified by the conjecture of Bethe (1953)—namely, that the multiple-scattering "tail" should be multiplied by the ratio of the correct single-scattering cross-section to the Rutherford result—and show that other modifications of the transform could be handled in this way.

Cooper and Rainwater (1955) gave two methods for evaluating multiple scattering on extended nuclei, one of which is of interest as a general method for any distribution. It follows the method of Butler in dividing the single-scattering law at some angle θ_1 into two parts, treating the small-angle part as producing a Gaussian distribution (θ_1 is chosen to be equal to approximately $\chi_c B^{1/2}$ of Sec. VII). The large-angle part, in agreement with our theory of transforms in Sec. II, should be "folded" according to Eq. (2.19) with the Gaussian after its inverse transform is calculated.

To calculate the distribution arising from angles larger than θ_1, Rainwater and Cooper first calculate for a fraction, say 1/8, of the final thickness of material. In terms of transforms, if $\omega_1(\xi)$ is the exponent of the transform of the large-angle part of $W(\chi,t)$, they calculate the inverse of $\exp[\omega_1(\xi)/8] \simeq 1 + \omega_1(\xi)/8 + \omega_1^2(\xi)/128$, using only two terms of this expansion (a delta function plus 1/8 the single-scattering distribution). A folding of this result on itself yields the distribution corresponding to $1 + \omega_1(\xi)/4 + \omega_1^2(\xi)/64$; the first two terms being easily calculated, the third "error" term can be found.

Two more self-foldings yield the inverse of

$$[1 + \omega_1(\xi)/8]^8 = 1 + \omega_1(\xi) + 56\omega_1^2(\xi)/128 + \cdots \tag{15.1}$$

in place of the actually sought inverse of

$$\exp[\omega_1(\xi)] = 1 + \omega_1(\xi) + 64\omega_1^2(\xi)/128 + \cdots, \tag{15.2}$$

so that the error of $\omega_1^2/16$ can be added in from the previous calculation.

The final folding with the Gaussian part was accomplished with a simplifying graphical device; the reader is referred to the original paper for details.

The nuclear size effect may be calculated, as implied in Sec. VI-I, by including a suitable nuclear form factor $\mathfrak{F}_N(\chi)$ so that (7.8) for the point-nucleus case is replaced by

$$\Omega(\xi) - \Omega_0 = 2\chi_c^2 \int_0^\infty \frac{d\chi}{\chi^3}\, \bar{q}(\chi)\mathfrak{F}_N(\chi)[J_0(\xi\chi) - 1], \tag{15.3}$$

which we may write as

$$\Omega(\xi) - \Omega_0 = (\Omega - \Omega_0)_{\text{point nucleus}} + (\Omega - \Omega_0)_{\text{corr}}, \tag{15.4a}$$

$$(\Omega - \Omega_0)_{\text{corr}} = 2\chi_c^2 \int_0^\infty \frac{d\chi}{\chi^3}\, \bar{q}(\chi)[\mathfrak{F}_N(\chi) - 1] \times [J_0(\xi\chi) - 1]. \tag{15.4b}$$

Since $\mathfrak{F}_N$ differs from 1 only for angles approximately 10^3 times as large as those for which screening is important, we may set $\bar{q}(\chi) = 1$ in the last equation (overlooking the modifications introduced by Nigam *et al.* (1959), as shown in Figs. 19 and 20).

Cooper and Rainwater (1955) use the Molière change of variables, (7.35) and (7.43) and treat $(\Omega - \Omega_0)_{\text{corr}}$ as a term in $1/B$ to be expanded along with $(\eta^2/4B)\ln(\eta^2/4)$. They used a form factor for projected scattering which was fitted numerically to nuclear scattering data then available, and evaluated the final integrals by Weddle's rule. They also used an alternative derivation which follows the method of Sec. VII a step further, including $\mathfrak{F}_N$ before making the expansion in powers of $1/B$.

In view of the developments since 1955 both in knowledge of nuclear form factors and of μ-meson scattering, we shall not discuss the details of these authors' results.

Ter-Mikayelian (1959) has also made a calculation of multiple scattering including nuclear size effects. He chose for $\mathfrak{F}_N$ the function $[1 + k^2r_N^2\theta^2]^{-4}$ with r_N a nuclear "radius." This function was chosen for analytical convenience although it can only be considered as physically meaningful for electrons of momentum of the order of 100 MeV/c (it corresponds to an exponential distribution of nuclear charged density which is known to be wrong). His calculation then proceeded by complete evaluation of the exponent in terms of the Bessel functions K_0 and K_1 of argument ξ/kr_N, similar to that discussed in Eqs. (11.34) and (11.35), and evaluated the final distribution numerically.

As a matter of fact, the results for many types of nuclear form factors can be evaluated by expressing the resulting $(\Omega - \Omega_0)_{\mathrm{corr}}$ in terms of Molière and Gaussian transforms, in the manner of Spencer discussed above, so that the final distributions can be expressed in terms of the D_n functions of Sec. IX, but we shall not go into details here [Scott (1955)].

Some mention should be made of the sharp cutoff calculations of Olbert (1952). Olbert set $\mathfrak{F}_N(\theta)$ equal to zero for $\theta > 1/kr_N$ and evaluated $\Omega - \Omega_0$ for this case. Cutting off the "tail" of the single-scattering curve amounts to eliminating the $\ln \xi$ behavior in the exponent, although this elimination occurs only for very small ξ in most cases. This means that the final distribution is essentially Gaussian in behavior. The results do not apply to the case of scattering by extended nuclei, but have some meaning if a cutoff in single scattering is introduced observationally—e.g., by visually rejecting bubble chamber or emulsion tracks that show single scatterings bigger than some easily observable angle, such as 0.1 radian. The chief difficulty with such calculations is that they cannot be done in reduced angular units, such as $\chi_c B^{1/2}$, χ_α or $1/kr_N$, because the cutoff angle is fixed in units of degrees or radians.

We have restricted our considerations in this review to the multiple scattering of fast charged particles at small angles. Extension to larger angles requires a shift from the use of Fourier–Bessel transforms to Legendre polynomial expansions. Goudsmit and Saunderson (1940 a,b) as mentioned in Sec. I have given a theory valid for all angles, with however the two serious restrictions that (a) the independent variable t is the actual path length traversed by the particles, rather than their depth of penetration in a given direction, and (b) the scattering medium must be infinite in all directions. Lewis (1950) has generalized this theory and derived the results of Molière in the limit of small angles. Wang and Guth (1951) have given a comprehensive survey of methods of tackling the above-mentioned difficulties [see also Mertens (1953, 1954)]. A specific calculation of Butler's type for an angle of 60° was made by Teichmann (1951). Some progress in handling the boundary problem for a thin layer has been made by Breitenberger (1959); Molière (1958, 1959) has improved the treatment of the transport equation. The path-length problem has been handled by Yang (1951) in Gaussian approximation [see also Scott (1949b)]. Much work on the general problem of electron straggling and penetration has been carried out by Spencer and coworkers [Spencer (1955); Spencer and Coyne (to be published)], but they have not dealt specifically with multiple scattering in thin foils at large angles.

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APPENDIX I

We give here formulas for the expansion of certain indefinite Bessel-function integrals used in Sec. IV, valid for small values of the argument. Using the relations

$$dJ_0(x)/dx = -J_1(x)\ ; \quad d[xJ_1(x)]/dx = xJ_0(x)\ , \tag{A1}$$

we can readily verify the following:

$$\int_x^\infty dtJ_0(t)/t^2 = J_0(x)/x - \int_x^\infty dtJ_1(t)/t\ , \tag{A2}$$

$$\int_x^\infty dtJ_0(t)/t^3 = J_0(x)/2x^2 - J_1(x)/4x - \int_x^\infty dtJ_0(t)/4t\ , \tag{A3}$$

$$\int_x^\infty dtJ_0(t)/t^4 = J_0(x)/3x^3 - J_1(x)/9x^2 - \int_x^\infty dtJ_0(t)/9t^2\ , \tag{A4}$$

$$\int_x^\infty dtJ_0(t)/t^5 = J_0(x)/4x^4 - J_1(x)/16x^3 - \int_x^\infty dtJ_0(t)/16t^3\ . \tag{A5}$$

Using a form of Bessel's integral for $J_0(t)$,

$$J_0(t) = \frac{2}{\pi}\int_0^{\pi/2} \cos\,(t\cos\phi)d\phi\ , \tag{A6}$$

we can write

$$\int_x^\infty dtJ_0(t)/t = \frac{2}{\pi}\int_0^{\pi/2} d\phi \int_x^\infty dt \cos\,(t\cos\phi)dt/t = \frac{2}{\pi}\int_0^{\pi/2} d\phi \int_{x\cos\phi}^\infty du \cos u/u = \frac{2}{\pi}\int_0^{\pi/2} d\phi \times [-\ln\,(\gamma x\cos\phi) + (x\cos\phi)^2/4 - (x\cos\phi)^4/96 + \cdots]\ ,$$

using the expansion of the cosine integral for small argument (Jahnke–Emde, 1943, p. 3). The number γ is $1.781\cdots$; $\ln\gamma = 0.5772\cdots$ is Euler's constant. The ϕ integrals are given in ordinary tables, and we find

$$\int_x^\infty dt J_0(t)/t = -\ln(\gamma x/2) + \frac{x^2}{8} - \frac{x^4}{256}\cdots. \quad (A7)$$

For the J_1 integral, we use the formula

$$J_1(t) = \frac{2}{\pi}\int_0^{\pi/2} d\phi \cos\phi \sin(t\cos\phi) \quad (A8)$$

and find

$$\int_x^\infty dt J_1(t)/t = \frac{2}{\pi}\int_0^{\pi/2} d\phi\cos\phi \int_{x\cos\phi}^\infty \frac{du \sin u}{u}.$$

We can readily show that

$$\int_y^\infty \frac{du\sin u}{u} = \int_0^\infty \frac{du \sin u}{u} - \int_0^y \frac{du\sin u}{u}$$
$$= \frac{\pi}{2} - y + \frac{y^3}{18} - \frac{y^5}{600}\cdots, \quad (A9)$$

so that we finally get

$$\int_x^\infty dt \frac{J_1(t)}{t} = 1 - \frac{x}{2} + \frac{x^3}{48} - \frac{x^5}{1920}\cdots. \quad (A10)$$

Combining the results of (A.2–5), (A.7) and (A.10) along with the expansions of $J_0(x)$ and $J_1(x)$, we find the formulas we seek

$$\int_{\chi_1}^\infty d\chi[J_0(\chi\xi) - 1]/\chi^2 = -\xi + \tfrac{1}{4}\chi_1\xi^2 - \frac{\chi_1^3\xi^4}{192}\cdots, \quad (A11)$$

$$\int_{\chi_1}^\infty d\chi[J_0(\chi\xi) - 1]/\chi^3 = \frac{\xi^2}{4}\left(\ln\frac{\gamma\chi_1\xi}{2} - 1\right) - \frac{\chi_1^2\xi^4}{128} + \frac{\chi_1^4\xi^6}{9216} + \cdots, \quad (A12)$$

$$\int_{\chi_1}^\infty d\chi \frac{[J_0(\chi\xi) - 1]}{\chi^4} = -\frac{\xi^2}{4\chi_1} + \frac{\xi^3}{9} - \frac{\chi_1\xi^4}{64} + \frac{\chi_1^3\xi^6}{6912}\cdots, \quad (A13)$$

$$\int_{\chi_1}^\infty d\chi \frac{[J_0(\chi\xi) - 1]}{\chi^5} = -\frac{\xi^2}{\chi_1^2} - \frac{\xi^4}{64}\left(\ln\frac{\gamma\chi_1\xi}{2} - \frac{3}{2}\right) + \frac{\xi^6\chi_1^2}{4608}\cdots, \quad (A14)$$

The various properties of Bessel functions used here are readily available [Jahnke–Emde (1943); Watson (1952); Erdelyi H.T.F. (1954)].

Another set of integrals that are needed [for calculating correction terms—see Eqs. (7.16) ff.] involve $\ln x$. By means of complex integration [using the method of Watson (1952), Sec. 13.6] we can establish the following

$$\int_x^\infty dt J_0(t)\frac{\ln t}{t} \simeq \tfrac{1}{2}\ln^2(\gamma/2) - \tfrac{1}{2}\ln^2 x + \tfrac{1}{8}x^2 \times (\ln x - \tfrac{1}{2}) - (x^4/256)(\ln x - \tfrac{1}{4})\cdots \quad (A15)$$

and

$$\int_x^\infty dt J_1(t)\frac{\ln t}{t} \simeq -2\ln 2 - \frac{x}{2}(\ln x - 1) + \frac{x^3}{48} \times (\ln x - \tfrac{1}{3}) - \frac{x^5}{1920}(\ln x - \tfrac{1}{5})\cdots. \quad (A16)$$

We can then by a similar process to the one just used arrive at the following desired results:

$$\int_{\chi_1}^\infty \frac{d\chi}{\chi^4}[J_0(\chi\xi) - 1]\ln\chi = -\frac{\xi^2}{4\chi_1}(\ln\chi_1 + 1) + \xi^3\left(\frac{5}{27} - \frac{1}{9}\ln 4\xi\right) - \frac{\xi^4\chi_1}{64}(\ln\chi_1 - 1) + \frac{\xi^6\chi_1^3}{6912} \times (\ln\chi_1 - \tfrac{1}{3}) + \cdots, \quad (A17)$$

$$\int_{\chi_1}^\infty \frac{d\chi}{\chi^5}[J_0(\chi\xi) - 1]\ln\chi = -\frac{\xi^2}{8\chi_1^2}(\ln\chi_1 + \tfrac{1}{2}) + \frac{\xi^4}{128}\left[\ln^2\left(\frac{\xi\gamma}{2}\right) - \ln^2\chi_1 - 3\ln\left(\frac{\xi\gamma}{2}\right) + \frac{23}{8}\right] + \frac{\xi^6\chi_1^2}{4608}(\ln\chi_1 - \tfrac{1}{2}) + \cdots. \quad (A18)$$

APPENDIX II

We wish to prove two useful Bessel-function theorems.

If the azimuthal behavior of a function whose transform is sought is proportional to the component of a vector in an arbitrary direction, i.e., to $\cos(\beta - \beta_1)$ where β_1 is constant, then the transform is proportional to $2\pi i J_1(\xi\theta)$ times $\cos(\alpha - \beta_1)$—i.e., a vector at angle β transforms to $2\pi i J_1(\xi\theta)$ times a vector at angle α.

We have to calculate

$$\int_0^{2\pi} d\beta e^{i\xi\theta\cos(\beta-\alpha)}\cos(\beta - \beta_1)$$
$$= \int_0^{2\pi} d\beta e^{i\xi\theta\cos\beta}\cos(\beta + \alpha - \beta_1)$$
$$= \int_0^{2\pi} d\beta e^{i\xi\theta\cos\beta}[\cos\beta\cos(\alpha - \beta_1) - \sin\beta\sin(\alpha - \beta_1)].$$

The integral over $\sin\beta$ vanishes by symmetry. The $\cos\beta$ integral is by (2.15) $\pi i J_1(\xi\theta) + \pi i^{-1}J_{-1}(\xi\theta) = 2\pi i J_1(\xi\theta)$, so we have

$$\int_0^{2\pi} d\beta e^{i\xi\theta\cos(\beta-\alpha)} \cos(\beta-\beta_1) = 2\pi i J_1(\xi\theta)\cos(\alpha-\beta_1)\,. \quad \text{(A19)}$$

If now the integrand contains the products of two components in any two directions, we have a more complicated result, namely,

$$\int_0^{2\pi} d\beta e^{i\xi\theta\cos(\beta-\alpha)} \cos(\beta-\beta_1)\cos(\beta-\beta_2) = \pi[J_0(\xi\theta) - J_2(\xi\theta)]\cos(\alpha-\beta_1)\cos(\alpha-\beta_2) + \pi[J_0(\xi\theta)+J_2(\xi\theta)]\cos(\alpha-\pi/2-\beta_1) \times \cos(\alpha-\pi/2-\beta_2)\,. \quad \text{(A20)}$$

The two components yield a term with the corresponding components of a vector at angle α, and a term for a vector at $\alpha - \pi/2$ (or $\alpha + \pi/2$). The steps corresponding to those given for the first theorem involve $\cos(\beta+\alpha-\beta_1)\cos(\beta+\alpha-\beta_2)$ which is readily transformed to

$$\tfrac{1}{2}(1+\cos 2\beta)\cos(\alpha-\beta_1)\cos(\alpha-\beta_2) - \tfrac{1}{2}\sin 2\beta\sin(2\alpha-\beta_1-\beta_2) + \tfrac{1}{2}(1-\cos 2\beta) \times \sin(\alpha-\beta_1)\sin(\alpha-\beta_2)\,,$$

where again the $\sin 2\beta$ term yields zero, and with (2.15) we get (A.16).

If $\beta_1 = \beta_2$, we have the simpler result

$$\int_0^{2\pi} d\beta e^{i\xi\theta\cos(\beta-\alpha)}\cos^2(\beta-\beta_1) = \pi[J_0(\xi\theta) - J_2(\xi\theta)\cos 2(\beta-\beta_1)]\,. \quad \text{(A21)}$$

APPENDIX III

We give in this Appendix some relevant properties of the functions $D_n(\alpha,\beta,z)$ defined in (9.29). The ordinary series expansion for ${}_1F_1$ yields

$$D_0(\alpha,\beta,-\vartheta^2) = \Gamma(\beta)\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)(-\vartheta^2)^k}{k!\Gamma(\beta+k)}\,, \quad \text{(A22)}$$

$$D_1(\alpha,\beta,-\vartheta^2) = \Gamma(\beta)\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)(-\vartheta^2)^k}{k!\Gamma(\beta+k)} \times \psi(\alpha-1+k)\,, \quad \text{(A23)}$$

$$D_2(\alpha,\beta,-\vartheta^2) = \Gamma(\beta)\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)(-\vartheta^2)^k}{k!\Gamma(\beta+k)} \times [\psi^2(\alpha-1+k)+\psi'(\alpha-1+k)]\,. \quad \text{(A24)}$$

The derivatives and integrals of these functions are also useful. We have after a little manipulation

$$\frac{\partial D_n(\alpha,\beta,-\vartheta^2)}{\partial\vartheta} = \frac{-2\vartheta}{\beta}D_n(\alpha+1,\beta+1,-\vartheta^2) \quad \text{(A25)}$$

$$\int_0^{\vartheta}\vartheta' d\vartheta' D_n(\alpha,1,-\vartheta'^2) = \frac{\vartheta^2}{2}D_n(\alpha,2,-\vartheta^2) \quad \text{(A26a)}$$

$$\int_0^{\varphi} d\varphi' D_n(\alpha,\tfrac{1}{2},-\varphi'^2) = \varphi D_n(\alpha,\tfrac{3}{2},-\varphi^2)\,. \quad \text{(A26b)}$$

As indicated by (9.30), (9.31), (A.23), and (A.24) we wish to consider the values of $n = 0, 1, 2$; $\beta = \frac{1}{2}, 1, 3/2, 2$, and numerous values of α up to 7/2.

A second set of series is derivable from Kummer's transformation for ${}_1F_1(\alpha,\beta,z)$:

$${}_1F_1(\alpha,\beta,z) = e^z{}_1F_1(\beta-\alpha,\beta,-z) \quad \text{(A27)}$$

and with the use of properties of the gamma function, explicit formulas for all the functions may be written, each involving a factor $e^{-\vartheta^2}$.

Asymptotic formulas are derivable from the expression for ${}_1F_1(\alpha,\beta,z)$ when z is large and negative [Erdelyi, H.T.F. (1954), Vol. I, p. 278]:

$${}_1F_1(\alpha,\beta,z) = -\frac{\Gamma(\beta)\sin\pi(\alpha-\beta)}{\pi\Gamma(\alpha)}\sum_{k=0}^{n} \times \frac{\Gamma(\alpha+k)\Gamma(\alpha-\beta+1+k)}{k!(-z)^{k+\alpha}} + R_{n+1}\,, \quad \text{(A28)}$$

where the remainder R_{n+1} is of the order of the series term with $k = n+1$ plus a term involving e^z which is negligibly small for the useful range of this formula.

The terms of this series all vanish when $\alpha - \beta$ is an integer m, corresponding to the fact that $D_0(\beta + m,\beta,z)$ is expressible as e^z times a polynomial, for which there exists no asymptotic expansion. Nonvanishing results are obtained when $\alpha - \beta = m + \frac{1}{2}$ and also for all α and β after differentiating with respect to α. Consequently, we have

$$D_0(\beta+m+\tfrac{1}{2},\beta,-\vartheta^2) \approx \frac{(-1)^{m+1}\Gamma(\beta)}{\pi\vartheta^{2\beta+2m+1}}\sum_{k=0}^{n} \times \frac{\Gamma(\beta+m+\frac{1}{2}+k)\Gamma(m+\frac{3}{2}+k)}{k!\vartheta^{2k}}\,, \quad \text{(A29)}$$

$$D_1(\beta+m,\beta,-\vartheta^2) \approx \frac{(-1)^{m+1}\Gamma(\beta)}{\vartheta^{2\beta+2m}}\sum_{k=0}^{n} \times \frac{\Gamma(\beta+m+k)(m+k)!}{k!\vartheta^{2k}}\,, \quad \text{(A30)}$$

$$D_1(\beta+m+\tfrac{1}{2},\beta,-\vartheta^2) \approx \frac{(-1)^{m+1}\Gamma(\beta)}{\pi\vartheta^{2\beta+2m+1}}\sum_{k=0}^{n} \times \frac{\Gamma(\beta+m+\frac{1}{2}+k)\Gamma(m+\frac{3}{2}+k)}{k!\vartheta^{2k}} \times \{\psi(\beta+m+k-\tfrac{1}{2})+\psi(m+k+\tfrac{1}{2})-\ln\vartheta^2\}\,, \quad \text{(A31)}$$

$$D_2(\beta+m,\beta,-\vartheta^2) \approx \frac{2(-1)^{m+1}\Gamma(\beta)}{\vartheta^{2\beta+2m}}\sum_{k=0}^{n} \times \frac{\Gamma(\beta+m+k)(m+k)!}{k!\vartheta^{2k}}\{\psi(\beta+m-1+k) + \psi(m+k)-\ln\vartheta^2\}\,, \quad \text{(A32)}$$

$$D_2(\beta + m + \tfrac{1}{2},\beta,-\vartheta^2) \approx \frac{(-1)^{m+1}\Gamma(\beta)}{\pi\vartheta^{2\beta+2m+1}} \sum_{k=0}^{n} \times \frac{\Gamma(\beta + m + \frac{1}{2} + k)\Gamma(m + \frac{3}{2} + k)}{k!\vartheta^{2k}} \times \{-\pi^2 + [\psi(\beta + m - \tfrac{1}{2} + k) + \psi(m + \tfrac{1}{2} + k) - \ln \vartheta^2]^2 + \psi'(\beta + m - \tfrac{1}{2} + k) + \psi'(m + \tfrac{1}{2} + k)\} . \quad \text{(A33)}$$

A number of recursion formulas for D_0 may be derived from (A.22) [see also Jahnke–Emde (1943), p. 275 and Jahnke–Emde–Lösch (1960), p. 276] and differentiated to give results for D_1 and D_2. Among the more useful we give the following:

$$D_0(\alpha + 1,\beta,z) = \alpha D_0(\alpha,\beta,z) + (z/\beta) \times D_0(\alpha + 1,\beta + 1,z) , \quad \text{(A34)}$$

$$D_0(\alpha + 1,\beta,z) = (\alpha + z)D_0(\alpha,\beta,z) + (z/\beta)(\alpha - \beta) \times D_0(\alpha,\beta + 1,z) , \quad \text{(A35)}$$

$$D_0(\alpha + 1,\beta,z) = (2\alpha - \beta + z)D_0(\alpha,\beta,z) - (\alpha - \beta)(\alpha - 1)D_0(\alpha - 1,\beta,z) , \quad \text{(A36)}$$

$$D_0(\alpha + 1,\beta + 1,z) = \beta D_0(\alpha,\beta,z) + (\alpha - \beta) \times D_0(\alpha,\beta + 1,z) , \quad \text{(A37)}$$

$$D_1(\alpha + 1,\beta,z) = D_0(\alpha,\beta,z) + \alpha D_1(\alpha,\beta,z) + (z/\beta) \times D_1(\alpha + 1,\beta + 1,z) , \quad \text{(A38)}$$

$$D_1(\alpha + 1,\beta + 1,z) = D_0(\alpha,\beta + 1,z) + (\alpha - \beta) \times D_1(\alpha,\beta + 1,z) + \beta D_1(\alpha,\beta,z) , \quad \text{(A39)}$$

$$D_2(\alpha + 1,\beta,z) = 2D_1(\alpha,\beta,z) + \alpha D_2(\alpha,\beta,z) + (z/\beta) \times D_2(\alpha + 1,\beta + 1,z) , \quad \text{(A40)}$$

$$D_2(\alpha + 1,\beta + 1,z) = 2D_1(\alpha,\beta + 1,z) + (\alpha - \beta) \times D_2(\alpha,\beta + 1,z) + \beta D_2(\alpha,\beta,z) . \quad \text{(A41)}$$

These functions have been evaluated for the values of α and β that are indicated in (9.30) and (9.31), as well as for the values of the parameters that will give the integrals and derivatives in accordance with (A.25) and (A.26), using an IBM 7090 computer.[24] The program was based on the series derived from (A.27), and on the asymptotic expressions (A.29)–(A.33). A sampling of the results is given in Tables VII and VIII and graphs of the functions are shown in Figs. 21–26.[25] The results agree with those of Bethe (1953) and Molière (1948). It will be noted that the integrals from 0 to ∞ of the functions required in (9.30) and (9.31) may be obtained by using (A.26) and the appropriate asymptotic formula from (A.29) to (A.33). Equations (A.22)–(A.24) with (A.25) show that all derivatives vanish at $\vartheta = 0$.

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[25] The complete tables are on file at the American Documentation Institute (cf. footnote 17).

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