we obtained an explicit time dependence which we could handle.[37]

ACKNOWLEDGMENTS

I am very grateful to my advisor Professor David S. Saxon, to Professor A. V. Balakrishnan, Professor George E. Uhlenbeck, Professor Mark Kac, Professor Norbert Wiener, and the Physics Department staff and would like to express deep appreciation for their time and interest to them and to several others, particularly, Robert C. Heimiller, Dr. Donald F. DuBois, Dr. Dominic G. B. Edelen, and Dr. A. V. Gafarian.

Also, I would like to thank Dr. E. U. Condon for his editing and the management of Hughes Aircraft Company for substantial financial support and typing.

[37] Some of the examples of the dissertation and further applications will be discussed in forthcoming papers and in *Mathematics in Science and Engineering*, edited by Richard Bellman [Academic Press Inc., New York (to be published, 1963)].

REVIEWS OF MODERN PHYSICS VOLUME 35, NUMBER 1 JANUARY 1963

The Theory of Waves in Stratified Fluids Including the Effects of Gravity and Rotation*†

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CONTENTS

1. INTRODUCTION

THIS article outlines the linear theory of waves in stratified, compressible fluids in a gravity field, at rest with respect to inertial or rotating coordinates, and neglecting viscosity and heat conduction. From a mathematical standpoint, this is one of the simpler branches of hydrodynamics, since it involves the solution of second-order differential equations which are linear. Despite its formal simplicity, the field has not been truly preempted. Eckart's recent and interesting book[1] on the subject is clear testimony to this.

There are four principal effects in the assumed model: compressibility, stratification, gravity, and rotation. Considering individually the cases for which these features are either present or absent, one could construct many different models, the simplest of which, obtained by ignoring all but one or two of these effects, are well known and understood. Thus, one may think of sound waves, surface gravity waves, tidal oscillations, and the effect of Coriolis forces upon these. On the other hand, the effects of density stratification and gravity upon very-low-frequency sound in the atmosphere and the behavior of internal gravity waves in the oceans and atmosphere have certainly not been as systematically studied, although Eckart's work[1,2] has done much to remedy this situation. The formal simplicity of the equations may lead one to feel that the answers to all these problems, even if not actually known in detail, can at least be deduced from available solutions. This point of view, although justifiable perhaps from a purely mathematical standpoint, is of little actual help to the physicist involved with these questions. More often than not the propagation of the various possible modes of motion is dispersive and anisotropic, depending upon several physical parameters, and a surprising variety of possibilities lurks beneath the deceptively simple appearance of the equations.

* Columbia University, Hudson Laboratories Contribution No. 157.
† This work was supported by the Office of Naval Research.
[1] Carl Eckart, *Hydrodynamics of Oceans and Atmospheres* (Pergamon Press, New York, 1960).

[2] C. Eckart and H. G. Ferris, Rev. Mod. Phys. **28**, 48 (1956).

The geophysicist, oceanographer, or meteorologist needs more than the statement that a mathematical or numerical solution is feasible. He is required to make idealizations and approximations, and for this he needs a precise understanding of both the nature and the order of magnitude of the phenomena involved. Otherwise he will make mistakes, and, according to Eckart,[2] mistakes of this type are not infrequent in the oceanographic and meteorological literatures.

One is led to wonder why no systematic treatment of the field had appeared until 1960. Perhaps this was because the wave motions predicted by theory for a fluid at rest may often be masked in practice by other natural motions due to currents, winds, turbulence, etc., so that people were prompted to investigate the more complex cases only, by-passing the simpler but nonetheless fundamental problems for a fluid at rest.

Another possible reason is the dearth of unambiguous experimental data. Take the case of deep ocean measurements of internal waves. It appears very likely that such waves have in fact been observed (see Cox,[3] Haurwitz *et al.*,[4] Pochapsky,[5] and LaFond[6]), but the results are ambiguous in the sense that it is conceivable that these measurements do not correspond to waves at all. All the data have been taken at a single point, and as a result the true wave nature of the observed motion has not been established, e.g., Haurwitz *et al.*,[4] and Pochapsky.[5] Measurements at several fixed points will be required to establish the travel-time and diagnostic dispersion curves essential for showing that these are waves of a given type. This raises some practical problems in the case of deep water. In shallow seas the experimental obstacles are less severe: LaFond,[7] Ufford,[8] Lee,[9] and others have obtained unambiguous results, and traveling internal wave systems have been positively identified. In the atmosphere the "lee waves," forming to leeward of a mountain range over which a steady wind is blowing,[10,11] have been shown to be internal waves (which can be viewed as excited by a moving source, i.e., a wave wake in which the dominant wavelengths are those whose phase velocity matches the relative velocity between the source and the receiver, as in Lamb's fishline problem[12]). It has also been suggested that certain types of banded cloud structures are explainable along these lines.[13]

It has seemed worthwhile to recast the subject matter so as to provide a point of view different from the usual ones resulting from linearization of the equations of fluid mechanics. Thus, the formulation of the equations of motion is greatly simplified by using a variational approach (Sec. 2). It will be shown that this leads to equations for the *displacements* from equilibrium (Sec. 3), which are directly measurable and intuitively visualizable quantities. These equations are identical to those one obtains from Biot's theory for waves in prestressed elastic media[14] (by making the rigidity vanish).

However, this article is more than a reformulation of problems discussed elsewhere. It is also an effort to shed further light on the basic mechanisms of wave motion in stratified fluids, in the course of which we shall examine a series of simple, limiting cases obtained by considering the principal physical effects either singly or in partial groupings. This brings out some interesting phenomena and prepares one for coping with more realistic and complete models. This outline is self-contained and does not not presuppose any prior acquaintance with the subject matter.

2. ENERGY DENSITIES, LAGRANGIAN

Consider a vertically stratified, compressible fluid at rest in a gravity field. We consider a stationary Cartesian coordinate system, x, y, z, with z the vertical axis (pointing up) and $\mathbf{g}$ the gravity field (pointing down). Let ξ, η, ζ be the x, y, z components of displacement of a fluid element from its equilibrium position, and $\rho(z)$ the equilibrium density of the fluid (Fig. 1).

The *kinetic energy* density is

$$T = \tfrac{1}{2}\,\rho(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)\,, \qquad (2.1)$$

[3] C. Cox, Meeting of the International Union of Geodesy and Geophysics, Helsinki, Finland, August, 1960 (unpublished).

[4] B. Haurwitz, H. Stommel, and W. H. Munk, in *The Atmosphere and the Sea in Motion*, Rossby Memorial Volume, edited by B. Bolin (Rockefeller Institute Press, New York, 1959).

[5] T. E. Pochapsky, Deep-Sea Research **8**, 269 (1961).

[6] E. C. LaFond, J. Geophys. Research **67**, 3573 (1962).

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[11] L. Prandtl, *Essentials of Fluid Dynamics* (Hafner Publishing Company, Inc., New York, 1952).

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[13] E. E. Gossard and W. H. Munk, J. Meteorol. **11**, 259 (1954).

[14] M. A. Biot, J. Appl. Phys. **11**, 522 (1940). Also a general variational theory by Biot, incorporating a number of new results has appeared as a U. S. Air Force O.S.R. report (September 1962), under the title "Generalized Theory of Internal Gravity Waves" (to be published as a paper).

where the dots represent the differentiation $\partial/\partial t$ with respect to time.

The *potential energy* density is a sum of several terms. First of all, the fluid being compressible, there is an *elastic energy*, which is

$$V_1 = \tfrac{1}{2}\lambda\epsilon^2 , \tag{2.2}$$

where

$$\epsilon = \left(\frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z}\right) \tag{2.3}$$

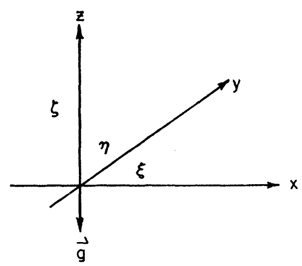

FIG. 1. Coordinate system used. g is the constant gravity field.

is the incremental volume change or strain, and

$$\lambda = \rho c^2 \tag{2.4}$$

is the bulk modulus, c being the velocity of sound.

There is also a *gravitational energy*. Thus, if in a stratified fluid an element is given a vertical displacement, it is moved into surroundings of different density. There will then be a buoyancy or Archimedes force acting on it to bring it back to its proper equilibrium level. This Archimedes force is, simply,

$$F = -g\Delta\rho , \tag{2.5}$$

where

$$\Delta\rho = \rho_i - \rho_0 \tag{2.6}$$

is the difference between the density inside the fluid element ρ_i and the density outside ρ_0.

For a vertical displacement ζ, in an *incompressible fluid*,

$$\Delta\rho = -\zeta(d\rho/dz) \tag{2.7}$$

and

$$F = g\zeta(d\rho/dz) . \tag{2.8}$$

The corresponding potential is, therefore,

$$V_2 = -\tfrac{1}{2}g\zeta^2(d\rho/dz) . \tag{2.9}$$

If the fluid is *compressible* there is an additional term, representing a correction to Eq. (2.9). Suppose that through some external agency the fluid element is compressed. Equation (2.5) is still valid, and the potential energy associated with a displacement ζ is

$$V_3 = \zeta g\Delta\rho , \tag{2.10}$$

where now the form of $\Delta\rho$ depends upon the agency setting up the compression. If this compression is simply that due to the displacement field in the fluid,

$$\Delta\rho = -\rho\epsilon \tag{2.11}$$

and

$$V_3 = -\rho g\zeta\epsilon . \tag{2.12}$$

Thus, the *Lagrange density* is

$$L = T - (V_1 + V_2 + V_3) \tag{2.13}$$

or

$$L = \tfrac{1}{2}\rho(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2) - \tfrac{1}{2}\lambda\epsilon^2 + \tfrac{1}{2}g\zeta^2(d\rho/dz) + \rho g\zeta\epsilon , \tag{2.14}$$

where ϵ is defined by Eq. (2.3).

From Eq. (2.14) we can now easily deduce the equations of motion by applying Euler's variational equations. Before we do so, let us resume the reasoning used in the derivation of the gravitational energies. Consider a single element of fluid displaced vertically from its equilibrium position, independently of its neighbors, e.g., a sphere of unit volume which we may conveniently visualize as surrounded by a perfectly flaccid, thin membrane.[1] When we release it, its motion will obey the equation

$$\rho\ddot{\zeta} = -g\Delta\rho , \tag{2.15}$$

where $\Delta\rho$ is compounded of two terms:

$$\Delta\rho = \Delta\rho_1 + \Delta\rho_2 . \tag{2.16}$$

The first term ignores compressibility and is given by Eq. (2.7):

$$\Delta\rho_1 = -\zeta(d\rho/dz) . \tag{2.17}$$

The other term, $\Delta\rho_2$, is the change in density due to compressibility, which, in this case, is produced by the change in hydrostatic head P:

$$\Delta P = \zeta(dP/dz) . \tag{2.18}$$

But we know that for adiabatic expansion

$$\Delta\rho = (1/c^2)\Delta P . \tag{2.19}$$

The fundamental equation of hydrostatics being

$$dP/dz = -\rho g \tag{2.20}$$

we have

$$\Delta\rho_2 = -(g/c^2)\rho\zeta , \tag{2.21}$$

and Eq. (2.15) gives

$$\ddot{\zeta} - g[(d/dz)\ln\rho + g/c^2]\zeta = 0\,. \qquad (2.22)$$

Thus, when the term in brackets is negative, the element of fluid goes into simple harmonic motion such that the square of its angular frequency is

$$N^2 = -g[(d/dz)\ln\rho + g/c^2]\,. \qquad (2.23)$$

The manner in which we have chosen our coordinate system implies that $(d/dz)\ln\rho < 0$. Thus, for a sufficiently pronounced density variation

$$|(d/dz)\ln\rho| > g/c^2 \qquad (2.24)$$

the situation is *stable*, and the medium is characterized for each value of z by a natural, or resonant, frequency of oscillation known as the *Väisälä frequency*, defined by Eq. (2.23).

Note once more the two terms in Eq. (2.23) play opposing roles. One is the effect of density stratification, and the other the effect of adiabatic expansion. In general, terms of the type g/c^2 will represent buoyancy effects due to the compressibility of a fluid in a gravity field.

3. EQUATIONS OF MOTION

Given a Lagrange density L that is a function of the generalized coordinates q and of their partial derivatives q_x, q_y, q_z, q_t, the principle of least action leads to the Euler-Lagrange equations:

$$\frac{\partial}{\partial t}\cdot\frac{\partial L}{\partial q_t} + \frac{\partial}{\partial x}\cdot\frac{\partial L}{\partial q_x} + \frac{\partial}{\partial q}\cdot\frac{\partial L}{\partial q_y} + \frac{\partial}{\partial z}\cdot\frac{\partial L}{\partial q_z} - \frac{\partial L}{\partial q} = 0\,. \qquad (3.1)$$

Applying this to Eq. (2.14) and the coordinates ξ, η, ζ we have

$$\begin{aligned} \rho\ddot{\xi} - \frac{\partial}{\partial x}\lambda\epsilon + \rho g\frac{\partial\zeta}{\partial x} &= 0\,, \\ \rho\ddot{\eta} - \frac{\partial}{\partial y}\lambda\epsilon + \rho g\frac{\partial\zeta}{\partial y} &= 0\,, \\ \rho\ddot{\zeta} - \frac{\partial}{\partial z}\lambda\epsilon - \rho g\left(\frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y}\right) &= 0\,. \end{aligned} \qquad (3.2)$$

These are the general equations of motion of a stratified compressible fluid in a constant gravity field, in an inertial Cartesian frame x, y, z, expressing the fact that the forces of inertia are balanced by those of elasticity (or constraint) and those of buoyancy. If one wishes to consider a rotating system of coordinates, one must add to the r.h.s. the appropriate components of the Coriolis and centrifugal forces. The latter can be considered as incorporated in the gravity field, at least in localized areas on the surface of a planet.[15] In this approximation, we must add to the r.h.s. of the ξ, η, ζ equations the x, y, z components of

$$\mathbf{F} = \rho\mathbf{V}\Lambda 2\boldsymbol{\omega}_p\,, \qquad (3.3)$$

where $\mathbf{V}$ is the particle velocity vector and $\boldsymbol{\omega}_p$ the angular velocity of the planet. To conform with standard notation we define the Coriolis vector $\boldsymbol{\Omega}$ as

$$\boldsymbol{\Omega} = 2\boldsymbol{\omega}_p\,, \qquad (3.4)$$

and the components of Eq. (3.3) are given by expanding:

$$\mathbf{F} = \rho\begin{vmatrix} \mathbf{1}_x & \mathbf{1}_y & \mathbf{1}_z \\ \dot{\xi} & \dot{\eta} & \dot{\zeta} \\ \Omega_x & \Omega_y & \Omega_z \end{vmatrix}\,, \qquad (3.5)$$

where $\mathbf{1}_x$, $\mathbf{1}_y$, $\mathbf{1}_z$ are the x, y, z unit vectors and Ω_x, Ω_y, Ω_z the x, y, z components of $\boldsymbol{\Omega}$.

Equations (3.2) are a limiting form of *Biot's equations* for a prestressed elastic solid,[14] obtained by making the rigidity vanish and assuming that the state of prestress is that of a hydrostatic pressure field due to the fluid's weight.

We see that the derivation of these equations from first principles is very direct and clear. They can also be derived by applying the usual perturbation techniques to the nonlinear Euler equations, but this derviation is laborious and requires the use of thermodynamic concepts. These equations are completely equivalent to Eckart's.[1,16] They have the advantage of describing displacements from an equilibrium state, i.e., quantities which are both easily visualized and directly observable. They also bring out clearly the two essential physical parameters: the density ρ and the bulk modulus λ.

Note that the *pressures* p associated with the displacements in Eqs. (3.2) can be deduced by referring to Euler's equations, e.g., in the absence of Coriolis forces,

$$\partial p/\partial x = -\rho\ddot{\xi}\,. \qquad (3.6)$$

4. SEPARATION OF THE EQUATIONS OF MOTION

The equations of motion are separable for a vertically stratified fluid, i.e., when ρ, λ are functions of z only. Separation is accomplished in the usual way by assuming ξ, η, ζ proportional to $e^{i(\alpha x + \beta y - \omega t)}$.

[15] The resultant of the centrifugal and gravitational fields is then taken as defining both the direction of the vertical and the constant magnitude of g. This ceases to be a valid approximation for small scale problems with rapid rates of rotation in which the centrifugal forces may drastically affect the equilibrium geometry.

[16] A proof of this statement is given in the Appendix.

When this assumption is applied to the equations of motion (3.2), with the Coriolis force terms [Eq. (3.5)] on the r.h.s., after elimination of ξ, η, one obtains a second-order differential equation for ζ of the form:

$$\zeta_{zz} + f(z)\zeta_z + r(z)\zeta = 0 . \qquad (4.1)$$

The standard transformation

$$\zeta = h \exp\left[-\tfrac{1}{2}\int^z f dz\right] \qquad (4.2)$$

gives

$$h_{zz} + \gamma^2 h = 0 \qquad (4.3)$$

with

$$\gamma^2 = r - \tfrac{1}{4}f^2 - \tfrac{1}{2}f_z . \qquad (4.4)$$

Equation (4.3) is the usual separated form of the wave equation. The function h has an oscillatory type solution when $\gamma^2 > 0$ and γ may be interpreted as the vertical component of the wave number, α and β being the horizontal components. When $\gamma^2 < 0$, its behavior is exponential or hyperbolic.

There are no difficulties in obtaining Eq. (4.1) for the general case when Coriolis forces are present. However, there is not much point in doing this except for special orientations of the Coriolis vector, because, in application to actual problems, the concept of Coriolis forces in a Cartesian system of axes is useful merely for qualitative, locally valid approximations. We shall be interested only in the *type* of effect produced by Coriolis forces, and for this it is sufficient to examine the two limiting cases of either a purely vertical or a purely horizontal Coriolis vector $\mathbf{\Omega}$.

First we make a sample derivation of Eqs. (4.1) and (4.3) in the absence of Coriolis effects, to illustrate the mathematical steps in their simplest form. When $\Omega = 0$ it is possible to consider the case of plane waves propagating in the x and z directions without loss of generality, and the equations of motion (3.2) become

$$\rho\ddot{\xi} - (\partial/\partial x)\lambda\epsilon + \rho g\zeta_x = 0 ,$$
$$\rho\ddot{\zeta} - (\partial/\partial z)\lambda\epsilon - \rho g\xi_x = 0 , \qquad (4.5)$$

with

$$\epsilon = \xi_x + \zeta_z = i\alpha\xi + \zeta_z . \qquad (4.6)$$

Using the assumption of proportionality to $e^{i(\alpha x - \omega t)}$, Eqs. (4.5) become

$$-\rho\omega^2\xi + \lambda\alpha^2\xi - \lambda i\alpha\zeta_z + \rho g i\alpha\zeta = 0 \quad (4.7a)$$

$$\rho\omega^2\zeta + \lambda_z(i\alpha\xi + \zeta_z) + \lambda(i\alpha\xi_z + \zeta_{zz}) + \rho g i\alpha\xi = 0 . \qquad (4.7b)$$

Since $\lambda = \rho c^2$ (4.7a) gives

$$\xi = i\alpha \frac{c^2\zeta_z - g\zeta}{\alpha^2 c^2 - \omega^2} . \qquad (4.8)$$

Substituting in Eq. (4.7b), we get Eq. (4.1) with

$$f(z) = (d/dz) \ln (\rho/b^2) \qquad (4.9)$$

$$r(z) = b^2 - \frac{\alpha^2}{\omega^2} g \frac{d}{dz} \ln\left(\frac{\rho}{b^2}\right) - \frac{\alpha^2}{\omega^2}\frac{g^2}{c^2} , \qquad (4.10)$$

where b represents the "pure" acoustical wave number in the absence of gravity and density effects:

$$b^2 = \omega^2/c^2 - \alpha^2 . \qquad (4.11)$$

It is clear from Eq. (4.10) that the gravity buoyancy forces dominate at very low frequencies, but become negligible at high frequencies. Note that f, r depend upon both the density and sound velocity gradients, but the latter only plays a role in practice for "pure" acoustical waves in the high frequency range (see Sec. 9). Thus, we assume for present purposes

$$c = \text{const} . \qquad (4.12)$$

Then

$$f = (d/dz) \ln \rho , \qquad (4.13)$$

$$r = \omega^2/c^2 - \alpha^2 + \alpha^2(N^2/\omega^2) . \qquad (4.14)$$

The transformation of Eq. (4.2) is

$$\zeta = h\rho^{-1/2} \qquad (4.15)$$

and the equation in h,

$$h_{zz} + \gamma^2 h = 0 , \qquad (4.3)$$

with

$$\gamma^2 = \frac{\omega^2}{c^2} - \alpha^2 + \frac{\alpha^2}{\omega^2}N^2 - \tfrac{1}{4}\left(\frac{d}{dz}\ln\rho\right)^2 - \tfrac{1}{2}\frac{d^2}{dz^2}\ln\rho . \qquad (4.16)$$

This is the equation describing the propagation of waves in a density stratified compressible fluid in a gravity field, with constant sound velocity c.

If ρ is an exponential function of z,

$$\rho = \rho_0 e^{-2\nu z} , \qquad (4.17)$$

N^2 is a constant and $(d^2/dz^2) \ln \rho = 0$. Equation (4.16) defines then a *constant coefficient:*

$$\gamma^2 = \frac{\omega^2}{c^2} - \alpha^2 + \frac{\alpha^2}{\omega^2}N^2 - \nu^2 . \qquad (4.18)$$

An exponential distribution of density Eq. (4.17) is realized in the isothermal model of the atmosphere for which

$$2\nu = \frac{g}{c^2}\frac{c_p}{c_V} = 1.4\frac{g}{c^2}, \tag{4.19}$$

where c_p/c_V is the ratio of specific heats.

Unfortunately, the assumption of constant coefficients is seldom realized in nature. Typical $N(z)$ graphs show a series of maxima and minima in the atmosphere (shown in Fig. 12 below) and usually at least one pronounced maximum near the surface for oceans or lakes.[1] However, we shall have occasion to dwell at length on this idealized model, since it will illustrate some of the basic phenomena in their simplest form.

When Coriolis forces are included in the picture, there arises a distinction between the x,y components, and the full three-dimensional equations (3.2) must be used, with the appropriate Coriolis forces on the r.h.s.

If $\mathbf{\Omega}$ is purely *vertical*,

$$\mathbf{\Omega} = \mathbf{1}_z\Omega_V, \tag{4.20}$$

and the Coriolis force components are, by Eq. (3.5), $\rho\dot{\eta}\Omega_V$, $-\rho\dot{\xi}\Omega_V$, and 0 along the x,y,z axes. It is then a simple matter, assuming as before proportionality to $e^{i(\alpha x + \beta y - \omega t)}$, to obtain from the first two equations (3.2)

$$\xi = (g\zeta - c^2\zeta_z)\frac{i\alpha - \beta(\Omega_V/\omega)}{\omega^2 - \Omega_V^2 - c^2(\alpha^2+\beta^2)} \tag{4.21}$$

$$\eta = (g\zeta - c^2\zeta_z)\frac{i\beta + \alpha(\Omega_V/\omega)}{\omega^2 - \Omega_V^2 - c^2(\alpha^2+\beta^2)}. \tag{4.22}$$

Substituting Eqs. (4.21) and (4.22) into the third equation (3.2) and writing

$$k^2 = \alpha^2 + \beta^2, \tag{4.23}$$

one finds, using Eq. (4.12),

$$f(z) = (d/dz)\ln\rho \tag{4.24}$$

$$r(z) = \frac{\omega^2}{c^2} - k^2\frac{\omega^2}{\omega^2-\Omega_V^2} + k^2\frac{N^2}{\omega^2-\Omega_V^2}, \tag{4.25}$$

and in Eq. (4.3) we put

$$\gamma^2 = \frac{\omega^2}{c^2} - k^2\frac{\omega^2}{\omega^2-\Omega_V^2}\left(1-\frac{N^2}{\omega^2}\right) - \tfrac{1}{4}\left(\frac{d}{dz}\ln\rho\right)^2 - \tfrac{1}{2}\frac{d^2}{dz^2}\ln\rho. \tag{4.26}$$

This describes approximately the behavior of waves in the polar regions of the planet.

On the other hand, if $\mathbf{\Omega}$ is horizontal, we may choose the y axis parallel to $\mathbf{\Omega}$:

$$\mathbf{\Omega} = \mathbf{1}_y\Omega_H. \tag{4.27}$$

The Coriolis force x,y,z components to be introduced on the r.h.s. of Eqs. (3.2) are, respectively, $-\rho\dot{\zeta}\Omega_H$,0, and $\rho\dot{\xi}\Omega_H$. It is easily verified that,

$$\xi = \frac{i}{\omega^2 - k^2c^2}\left[\zeta\left(\alpha g + \beta^2c^2\frac{\Omega_H}{\omega} - \omega\Omega_H\right) - \alpha c^2\zeta_z\right], \tag{4.28}$$

$$\eta = \frac{i}{\omega^2 - k^2c^2}\left[\zeta\left(\beta g - \alpha\beta c^2\frac{\Omega_H}{\omega}\right) - \beta c^2\zeta_z\right], \tag{4.29}$$

and

$$\gamma^2 = \frac{\omega^2}{c^2} - k^2 + \frac{N^2}{\omega^2}\left(k^2 - \frac{\alpha\omega\Omega_H}{g}\right) + \frac{\Omega_H}{\omega}\frac{1}{c^2}\left(\alpha g + \beta^2c^2\frac{\Omega_H}{\omega} - \omega\Omega_H\right) - \frac{1}{4}\left(\frac{d}{dz}\ln\rho\right)^2 - \frac{1}{2}\frac{d^2}{dz^2}\ln\rho. \tag{4.30}$$

This is an approximate model for the behavior of waves in the equatorial regions.

5. GENERAL PROPERTIES OF THE EQUATIONS OF MOTION

In the previous section we showed how the study of waves in a general stratified fluid reduces to the equation

$$h_{zz} + \gamma^2h = 0, \tag{5.1}$$

and we derived expressions for γ^2 valid for a broad variety of conditions. Also we obtained explicit expressions for the fluid displacements.

Several comments of a quite general nature should be made at this point.

The first remarks concern the implications of the form of Eq. (4.16), and Eqs. (4.26) and (4.30), for γ^2. When $\gamma^2 > 0$, the solutions of Eq. (5.1) can be analyzed into plane progressive waves, γ being the vertical component of the wave number. Equations (4.16), (4.26), and (4.30) provide an explicit relationship between it, the two horizontal components α, β, and the frequency ω of the form:

$$\gamma = F(\alpha,\beta,\omega). \tag{5.2}$$

Now, since α, β, γ are the three components of a vector in a Cartesian system, it is clear that, if in Eq. (5.2) they do not occur in the simple combination $\alpha^2+\beta^2+\gamma^2 = K^2 = \omega^2/V^2$, the properties of the propagation are going to depend upon direction, i.e., there is *anisotropy*. Also, if Eq. (5.2) is not of the form $K^2V^2 = \omega^2$, the phase velocity ω/K is not a constant, i.e., it is frequency dependent and there is *dispersion*. This dispersion is *structural* in origin, i.e., it is a characteristic property of the fluid stratification and is quite independent of any boundary conditions: plane waves in an indefinitely extended medium will be dispersive. This is to be contrasted with geometric dispersion, which is brought about by interference effects due to the reflection at boundaries. A simple example of geometric dispersion is provided by perfect electromagnetic and acoustic waveguides. On the other hand, a mechanism of dispersion which is not geometric in origin is illustrated by the "anomalous" dispersion of light caused by the existence of atomic or ionic resonators in the medium. It seems that *structural and intrinsic dispersions* are always connected with such *internal resonant frequencies*. Thus, in the problems to be discussed here there are three such resonances: one is the *Väisälä* frequency (see Sec. 2), another is a distributed mass and spring effect (acoustical resonance) (see Sec. 6), and a third is gyroscopic resonance due to precessional motions in a rotating fluid (see end of Sec. 6).

The *anisotropy* is, of course, due to the fact that gravity and/or rotation establish a preferred orientation in the system. In anisotropic systems it is often convenient to consider Eq. (5.2) as the equation of a surface in α, β, γ space, with ω as a parameter. This is known as the *propagation surface*,[1] which, in the cases we shall be concerned with, can be put in the form

$$\pm k^2/a^2 \pm \gamma^2/b^2 = 1$$
$$k^2 = \alpha^2 + \beta^2 , \qquad (5.3)$$

defining an ellipsoid (two plus signs) or hyperboloid of revolution about z of half-axes a,b. The radius vector drawn from the center of the quadric intersects the latter at a point defining the magnitude of the wave number, i.e., the index of refraction, corresponding to this direction. Thus, this quadric is the analog of the ellipsoid of indices used in crystal optics.

There can be two kinds of hyperboloid:

$$k^2/a^2 - \gamma^2/b^2 = 1 \qquad (5.4)$$

is a hyperbolid of one sheet, whereas

$$-k^2/a^2 + \gamma^2/b^2 = 1 \qquad (5.5)$$

has two sheets. In both cases

$$\theta_a = \tan^{-1}(a/b) \qquad (5.6)$$

is the half-angle of the asymptotic cone, measured from the vertical. In the case of Eq. (5.5) this means that only angles $\theta > \theta_a$ are permissible, whereas for Eq. (5.6) one must have $\theta < \theta_a$ (Fig. 2).

Our third remark concerns curl or *vorticity*. If $\mathbf{d}$ is the vector displacement (ξ,η,ζ), it is seen from Eqs. (4.8), etc., that, in general,

$$\text{curl } \mathbf{d} \neq 0 . \qquad (5.7)$$

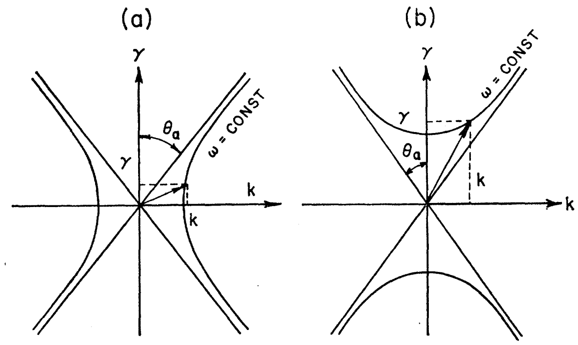

FIG. 2. Hyperboloid propagation surfaces.

For the case of no rotation [Eq. (4.8)] we have explicitly

$$\text{curl } \mathbf{d} = \mathbf{1}_y(\xi_z - \zeta_x) = \frac{i\alpha}{\alpha^2c^2 - \omega^2} \times [c^2\zeta_{zz} - g\zeta_z + (\omega^2 - \alpha^2c^2)\zeta] . \qquad (5.8)$$

For the special cases $g = 0$, $\gamma^2 = \omega^2/c^2 - \alpha^2$ (*acoustic waves* in fluid of constant density) or $\zeta = e^{\alpha z}$, $\omega^2 = \alpha g$ (*surface gravity waves*), however, the bracket vanishes and

$$\text{curl } \mathbf{d} = 0 . \qquad (5.9)$$

6. INFINITE MEDIA: SPECIFIC EXAMPLES OF BODY WAVES

We can now illustrate the general features discussed in Sec. 5 by means of specific examples, first reminding the reader that γ^2, as given by Eq. (4.16), (4.26), or (4.30), may contain four separate effects: those of compressibility, density stratification, gravity, and Coriolis forces.

The compressibility enters through terms of the form ω^2/c^2 and g^2/c^2 [second term in N^2, Eq. (2.23)]. An incompressible model corresponds to $c = \infty$.

The stratification comes into N^2 [first term in Eq. (2.23)] and into the last terms of γ^2, which are of the form

$$-\frac{1}{4}\left(\frac{d}{dz}\ln\rho\right)^2 - \frac{1}{2}\frac{d^2}{dz^2}\ln\rho .$$

In a homogeneous model $(d/dz)\ln\rho = 0$.

Gravity enters in the definition of N^2 and, when the Coriolis vector is not vertical, through additional terms.

The terms affected by rotation are obvious in Eqs. (4.26) and (4.30). Note that for vertical Ω they correspond to corrections of the order of Ω^2/ω^2.

Depending upon which of these effects are considered, one may construct a great number of distinct models, corresponding to different types of wave behavior, which may be entirely different in some frequency ranges or almost identical in others. A complete study of all these combinations would be too long and rather unnecessary: it is enough to examine the important cases.

(*a*) *Incompressible homogeneous fluid without rotation.* In an incompressible homogeneous fluid without rotation, $\gamma^2 = -\alpha^2$, and no real propagating wave systems exist in the absence of boundaries. The effect of gravity enters in this case through the boundary conditions only. Thus, surface gravity waves and waves at the interface between two incompressible media of different densities are pure boundary waves and will be taken up in a later section.

(*b*) *Compressible homogeneous fluid without gravity or rotation.* We need only point out that for a compressible homogeneous fluid without gravity or rotation Eq. (4.16) reduces to

$$\gamma^2 = \omega^2/c^2 - \alpha^2 \tag{6.1}$$

and thus h,ζ obey the acoustical wave equation.[17]

(*c*) *Density stratified compressible fluid.* For a density stratified compressible fluid, without gravity or rotation, our equations give results equivalent to Bergmann's.[18] In particular, if

$$\rho = \rho_0 e^{-2\nu z}, \tag{6.2}$$

we get from Eq. (4.16)

$$\gamma^2 = \omega^2/c^2 - \alpha^2 - \nu^2 . \tag{6.3}$$

We see that the propagation surface is a sphere and that propagation is therefore *isotropic*. We define the wave number in the direction of propagation:

$$K = [\alpha^2 + \gamma^2]^{1/2} . \tag{6.4}$$

Thus,

$$\omega = c[K^2 + \nu^2]^{1/2}$$
$$K = [\omega^2/c^2 - \nu^2]^{1/2} . \tag{6.5}$$

The characteristic curve in the ω,K plane is a hyperbola (Fig. 3). There is a cutoff frequency at

$$\omega_0 = \nu c . \tag{6.6}$$

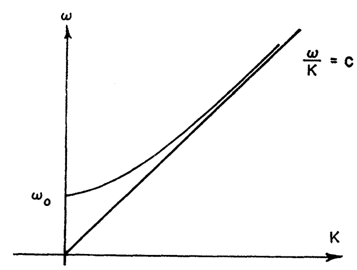

FIG. 3. Structural dispersion of acoustic waves due to density.

The group and phase velocities U, V are, by Eq. (6.5),

$$U = d\omega/dK = c(1 - \omega_0^2/\omega^2)^{1/2} , \tag{6.7}$$

$$V = \omega/K = c(1 - \omega_0^2/\omega^2)^{-1/2} , \tag{6.8}$$

$$UV = c^2 . \tag{6.9}$$

At $\omega = \omega_0$, $U = 0$, $c = \infty$, and the cutoff ω_0 is a *resonant frequency* for propagating waves. This resonance is characteristic of the fluid body considered as a distributed mass and spring system of variable constants. This is the second type of resonance mentioned in Sec. 5 as being a source of structural dispersion in stratified media.

The *particle motion* can easily be shown, with the aid of Eqs. (4.8) and (6.3), to possess a transverse component and to have a nonvanishing vorticity or curl, $\xi_z - \zeta_x$, as long as $\omega_0 \neq 0$, i.e., if the density is not constant. Only in media of constant density are sound waves purely longitudinal and irrotational.

It may be of interest to point out some orders of magnitude. For the so-called *isothermal atmosphere* Eq. (6.2) holds and

$$2\nu = \frac{g}{R\theta} = \frac{g}{c^2}\,\frac{c_p}{c_V} = 1.4\,\frac{g}{c^2} . \tag{6.10}$$

When c_p/c_V is the ratio of specific heats, $R = 2.9 \times 10^6$ erg/g deg. At $\theta = 273°K$, $c \sim 3.3 \times 10^4$ cm sec^{-1} and $2\nu \sim 1.2 \times 10^{-6}$ cm^{-1}, corresponding to periods $T_0 = 2\pi/\omega_0 \sim 5$ min. It is seen from Eqs. (6.7) and (6.8) that departures of 5% or more from the propagation velocity c will occur for $\omega_0{}^2/\omega^2 \geq 10^{-1}$, i.e., $T \geq 1.7$ min.

[17] In the case of variable c, constant ρ, the displacement field is irrotational, and it is well known that one may define a displacement potential ϕ which obeys the usual wave equation

$$\phi_{zz} + \gamma^2\phi = 0 ,$$

where γ^2 is given by Eq. (6.1), with c a function of z. Differentiating this equation with respect to z and using $\zeta = \phi_z$, we obtain Eq. (4.1) with f,g defined as in Eqs. (4.10), (4.11), and $(d/dz)\ln\rho = 0$. Thus, the *displacement* ζ does not obey the wave equation in this case, although the potential ϕ and the acoustical pressure $\rho\,\partial^2\phi/\partial t^2$ do.

[18] P. Bergmann, J. Acoust. Soc. Am. **17**, 329 (1946).

In a typical *ocean* Eq. (6.2) does not hold. Laws of variation of N^2 [Eq. (2.23)] are discussed in Eckart's book[1] with reference to the real ocean. However, if we assume, for estimating orders of magnitude, that N^2 is constant and has the value $\overline{N}^2$, we may apply Eq. (6.2). A reasonable figure, corresponding to a 2% variation of ρ in 4km, is $2\nu \sim 5 \times 10^{-8}$ cm^{-1}. Since $c \sim 1.5 \times 10^5$ cm sec^{-1} this gives $\omega_0 \sim 3 \times 10^{-3}$ and $T_0 = 2\pi/\omega_0 \sim 35$ min. Effects exceeding 5% of the velocity of propagation of sound waves will obtain for periods in excess of 10 min. This is hardly likely to be an observable effect.

(*d*) *Stratified incompressible fluid with gravity and no rotation: internal waves.* In this case, supposing once more that Eq. (6.2) holds, Eq. (4.16) is

$$\gamma^2 = \alpha^2(N^2/\omega^2 - 1) - \nu^2 \tag{6.11}$$

and

$$N^2 = 2\nu g\,. \tag{6.12}$$

The propagation surface is of the type (5.4) (hyperboloid of one sheet) with

$$a^2 = \frac{\nu^2}{N^2/\omega^2 - 1}\,, \qquad b^2 = \nu^2\,. \tag{6.13}$$

The asymptotic cone is defined by its half-angle (Fig. 2a):

$$\theta_\alpha = \tan^{-1}(N^2/\omega^2 - 1)^{-1/2}\,. \tag{6.14}$$

Only the region $\theta > \theta_a$ corresponds to real waves. Propagation is therefore both intrinsically dispersive and anisotropic. When ν is sufficiently small, and α, γ sufficiently large, the propagation surface is indistinguishable from the asymptotic cone, and for any given frequency there is essentially but one direction of propagation.[19]

In general we see that one must have $\omega \leq N$: The resonant *Väisälä* frequency is thus a high-frequency cutoff. One may define the horizontal and vertical components of the group and phase velocities as $\partial\omega/\partial\alpha$, $\partial\omega/\partial\gamma$ and ω/α, ω/γ, respectively.

From Eq. (6.11) we deduce

$$\alpha = \left[\frac{\gamma^2 + \nu^2}{N^2/\omega^2 - 1}\right]^{1/2} \tag{6.15}$$

and

$$\omega = \frac{\alpha N}{(\alpha^2 + \gamma^2 + \nu^2)^{1/2}}\,. \tag{6.16}$$

[19] This is the meaning of a rather recondite result displayed by L. Landau and E. Lifshitz in their book *Fluid Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959), Sec. 14.

Thus, for instance,

$$U_x = \frac{\partial\omega}{\partial\alpha} = \frac{\omega^3}{N^2}\left(\frac{N^2}{\omega^2} - 1\right)^{3/2}(\gamma^2 + \nu^2)^{-1/2} \tag{6.17}$$

$$V_x = \omega/\alpha = \omega(N^2/\omega^2 - 1)^{1/2}(\gamma^2 + \nu^2)^{-1/2}\,. \tag{6.18}$$

Thus, the largest group and phase velocities occur near $\omega = 0$ and for $\gamma = 0$. As $\omega \to 0$

$$V_x \to U_x \to N/(\gamma^2 + \nu^2)^{1/2}\,. \tag{6.19}$$

The characteristic curves in the ω,α plane are shown in Fig. 4, where γ is used as a parameter.

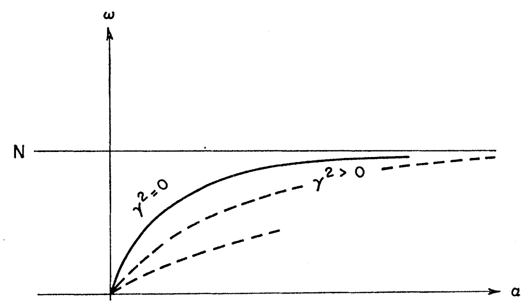

FIG. 4. Dispersion of plane internal waves in infinite incompressible medium.

The *particle motion* is easily obtained as follows by Eqs. (4.8), (4.15), and (6.2):

$$\xi = (i/\alpha)\zeta_z = (i/\alpha)(\nu + i\gamma)\zeta \tag{6.20}$$

for upgoing waves. Therefore, if

$$\zeta = \cos\omega t$$
$$\xi = (\nu/\alpha)\sin\omega t - (\gamma/\alpha)\cos\omega t\,, \tag{6.21}$$

eliminating time, we obtain the equation for the particle orbit

$$\xi^2 + \zeta^2\left(\frac{N^2}{\omega^2} - 1\right) + 2\frac{\gamma}{\alpha}\xi\zeta = \frac{\nu^2}{\alpha^2}\,. \tag{6.22}$$

For $\omega < N$ this is the equation of an ellipse.

For $\omega \to N$, $\alpha \to \infty$, $\xi = 0$, and the motion is purely vertical.

For $\gamma = 0$, (horizontal propagation) we have an ellipse of vertical semi-axis $\nu^2\alpha^{-2}(N^2/\omega^2 - 1)^{-1}$ and horizontal semi-axis $\nu^2\alpha^{-2}$.

For $\omega \sim N$ the motion is essentially transverse.

For $N/2^{1/2} < \omega < N$ the motion is elliptical with vertical axis longer than horizontal.

At $\omega = N2^{-1/2}$ the orbit is circular, and for $0 < \omega < N2^{-1/2}$ it is a horizontally elongated ellipse. Finally for $\omega \to 0$ the motion becomes purely horizontal and longitudinal.

(e) *Compressible stratified fluid with gravity, no*

Coriolis effects. We have, with the assumption (6.2)

$$\gamma^2 = \omega^2/c^2 - \alpha^2 + \alpha^2(N^2/\omega^2) - \nu^2 . \quad (6.23)$$

We may expect to find here two branches: an acoustical branch of the type discussed under (c), somewhat modified by gravity; and the internal waves discussed under (d), modified by compressibility.

Insofar as internal waves are concerned, we saw that, in the incompressible case, N is a high-frequency cutoff. N is typically very small. Since c is large, the first term ω^2/c^2 in Eq. (6.23) will therefore be a relatively unimportant term, of the nature of a correction. One may expect that the chief effect of compressibility is to change the numerical value of N^2 from that given in Eq. (6.12) to

$$N^2 = 2\nu g - g^2/c^2 . \quad (6.24)$$

On the other hand, the acoustic modes have a low-frequency cutoff characteristic of another type of resonance, which we called ω_0 [Eq. (6.6)]. It is clear from Eq. (6.23) that $\gamma^2 = 0$, $\alpha^2 = 0$ corresponds here also to $\omega = \omega_0$.

One may rewrite Eq. (6.23) in the form

$$\omega^4/c^2 - (\alpha^2 + \gamma^2 + \nu^2)\omega^2 + \alpha^2 N^2 = 0 . \quad (6.25)$$

This is a quadratic in ω^2 with two roots corresponding to the acoustic and internal wave branches.

Let

$$\omega_a^2 = (\alpha^2 + \gamma^2 + \nu^2)c^2 \quad (6.26)$$

$$\omega_i^2 = \alpha^2 N^2(\alpha^2 + \gamma^2 + \nu^2)^{-1} . \quad (6.27)$$

We recognize these as the characteristic equations, (6.3), for pure acoustic waves in the presence cf a density gradient, and Eq. (6.11) for internal waves in an incompressible medium of constant N^2 [except that here N^2 has the meaning of Eq. (6.24)]. Equation (6.25) is then

$$\omega^4/\omega_a^2 - \omega^2 + \omega_i^2 = 0 . \quad (6.28)$$

Its roots are

$$\omega^2 = \tfrac{1}{2}\,\omega_a^2[1 \pm (1 - 4(\omega_i^2/\omega_a^2))^{1/2}] . \quad (6.29)$$

Assuming $\omega_i/\omega_a \ll 1$ and expanding the square root we have for one root [+ sign in Eq. (6.29)]:

$$\omega_A^2 = \omega_a^2(1 - \omega_i^2/\omega_a^2 + \cdots) \quad (6.30)$$

and for the other

$$\omega_I^2 = \omega_i^2(1 + \omega_i^2/\omega_a^2 + \cdots) . \quad (6.31)$$

We see that corrections to the internal wave equation (6.23) due to compressibility, and to the acoustical waves [Eq. (6.22)] due to gravity are of the order of

$$\epsilon = \omega_i^2/\omega_a^2 \quad (6.32)$$

for given α, γ, i.e.,

$$\epsilon = \frac{N^2}{c^2}\,\frac{\alpha^2}{(\alpha^2 + \gamma^2 + \nu^2)^2} . \quad (6.33)$$

We may estimate, for given γ, the maximum size of this correction term from the fact that

$$\partial\epsilon/\partial\alpha = 0$$

at

$$\alpha = (\gamma^2 + \nu^2)^{1/2} . \quad (6.34)$$

Since ϵ decreases monotonically with increasing γ, the absolute maximum of ϵ occurs at $\gamma = 0$, i.e.,

$$\alpha = \nu$$

$$\epsilon_{\max} = N^2/4\nu^2c^2 . \quad (6.35)$$

For the isothermal model of Sec. 6(c), letting $\sigma = c_p/c_V$, $2\nu = g\sigma/c^2$ and $\epsilon_{\max} = (\sigma - 1)/\gamma \sim 2 \times 10^{-1}$ ($\lambda \sim 100$km), whereas for our deep ocean sample $\epsilon_{\max} \sim 1\times10^{-1}$ for $\lambda \sim 2500$km.

In the ω, α plane one distinguishes, therefore, three regions separated by the curves $\gamma^2 = 0$, i.e., by $\omega_I = f(\alpha)$, $\omega_A = g(\alpha)$ with $\gamma = 0$. These are the dispersion curves for horizontally traveling internal gravity and acoustic plane waves (Fig. 5). In the region between these curves γ is imaginary, and there are no propagating body waves in an infinite medium.

The propagation surfaces and particle orbits are similar to those of Secs. 6c and d. For the acoustical branch the propagation surface is an ellipsoid of semi-axes in the ratio $1 - N^2/\omega^2$, i.e., essentially a sphere for $\omega \gg \omega_0 > N$.

(f) Effects of rotation: gyroscopic (inertial) waves in homogeneous incompressible fluid. In this case, Eq. (4.26) gives, for a vertical axis of rotation,

$$\gamma^2 = k^2[\omega^2/(\Omega^2 - \omega^2)] . \quad (6.36)$$

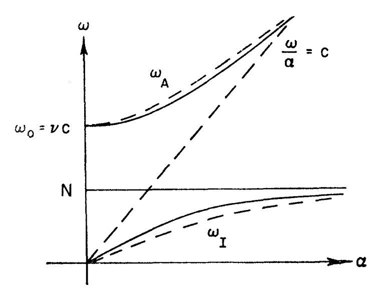

FIG. 5. Characteristic curves $\gamma = 0$ for compressible stratified fluid in gravity field. The dashed curves correspond to the $\omega_a(\alpha)$ and $\omega_i(\alpha)$ curves for which one neglects gravity (ω_a) or compressibility (ω_i).

It thus appears that real propagating wave systems can exist for $\omega < \Omega$.

Since we assume $c = \infty$, Eqs. (4.21) and (4.22) give

$$\xi = (1/k^2)\zeta_z(i\alpha - \beta[\Omega/\omega]) \quad (6.37)$$

$$\eta = (1/k^2)\zeta_z(i\beta + \alpha[\Omega/\omega]) . \quad (6.38)$$

Since the propagation direction in the x,y plane is immaterial we may suppose it to coincide with the x axis, i.e., $\beta = 0$, $\alpha = k$. Thus,

$$\xi = (i/k)\zeta_z = -(\gamma/k)\zeta \quad (6.39)$$

$$\eta = (1/k)(\Omega/\omega)\zeta_z = i(\gamma/k)(\Omega/\omega)\zeta \quad (6.40)$$

for waves traveling up, in the z positive direction. Since, by definition,

$$\zeta/\xi = -k/\gamma = -\tan\theta\,, \quad (6.41)$$

we see that, in the x,z plane, motion is along a straight line of slope $-\theta$ (see Fig. 6). Along this line, the

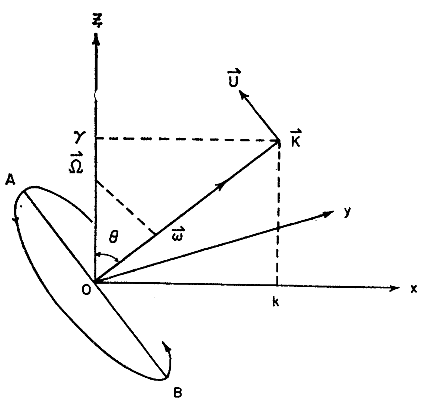

FIG. 6. Kinematics of gyroscopic waves.

motion is simple harmonic. Let a be the component of displacement along this line:

$$a = R\cos\omega t = (\xi^2 + \zeta^2)^{1/2}\cos\omega t = (|\zeta|/\sin\theta)\cos\omega t \quad (6.42)$$

and, by Eq. (6.40),

$$\eta = (\Omega/\omega)(\gamma/k)|\zeta|\sin\omega t\,. \quad (6.43)$$

But now Eq. (6.36) is also

$$\omega^2 = (\gamma^2/K^2)\Omega^2\,, \quad (6.44)$$

where

$$K^2 = k^2 + \gamma^2\,, \quad (6.45)$$

K being the wave number absolute magnitude. Therefore (see Fig. 6), Eq. (6.44) is simply

$$\omega^2 = \Omega^2\cos^2\theta\,. \quad (6.46)$$

Substituting in Eq. (6.43) immediately gives

$$\eta = R\sin\omega t\,. \quad (6.47)$$

From this result and Eq. (6.42) it is clear that the fluid particle describes a circular orbit in a plane normal to the vector K, with angular velocity ω. The trace of this plane on the x,z plane is the line AB in Fig. 6. The characteristic equation (6.46) or (6.36) can then be interpreted as the gyroscopic equation for a fluid particle constrained to move with the angular velocity ω along a circular path: Ω is the angular velocity of precession and θ the inclination of the circle with respect to the horizontal. These waves, therefore, deserve to be called *gyroscopic waves.* Equation (6.36) or (6.46) implies that for given ω there is only one permissible direction of propagation. The *phase velocity* in the direction of K is

$$V = (\Omega\cos\theta)/K = \Lambda(\Omega/2\pi)\cos\theta\,, \quad (6.48)$$

Λ being the wavelength in this direction.

The corresponding *group velocity* is

$$U = d\omega/dK = V - \Lambda(dV/d\Lambda) = 0 \quad (6.49)$$

and must therefore be perpendicular to $\mathbf{K}$. Its components are, by Eq. (6.36),

$$U_x = \partial\omega/\partial k = -(k\gamma/K^3)\Omega\,, \quad (6.50)$$

$$U_z = \partial\omega/\partial\gamma = (k^2/K^3)\Omega \quad (6.51)$$

and the magnitude of U is thus

$$U = (U_x^2 + U_z^2)^{1/2} = \Lambda(\Omega/2\pi)\sin\theta\,. \quad (6.52)$$

It is, however, probable that these waves will be difficult to detect in the earth's oceans and atmosphere, because for short wavelengths (of the order of a few kilometers), the velocities are small (of the order of a few centimeters per second) compared to background motions, and for longer wavelengths the present treatment must be modified to include sphericity of the earth.

One should also note that this treatment applies to the case $N = 0$. By Eq. (4.26) it can be seen that for $N > \Omega$ waves of this type will not exist as free body waves. They will exist, however, for $N < \Omega$, in modified form.

If the fluid is homogeneous but compressible, Eq. (4.26) reveals the existence of two branches: a dispersive acoustic mode and modified gyroscopic waves. The acoustic mode is discussed by Eckart[1] in the limit of horizontal incidence ($\theta = \pi/2$), when it obeys the dispersion law:

$$\omega^2 = \Omega^2 + K^2c^2\,. \quad (6.53)$$

7. BOUNDARY CONDITIONS

At a *rigid wall,* the normal component of displacement vanishes. Thus, if the wall is horizontal

$$\zeta = 0\,. \quad (7.1)$$

At a *free surface* the boundary condition can be expressed in a number of equivalent ways. Since we have formulated our equations in terms of particle displacements, we shall derive this boundary condition in terms of ζ, ζ_z. We use the statement that if

$$F(x,y,z,t) = 0 \tag{7.2}$$

is the equation of the surface, then one must always have

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\dot{\xi} + \frac{\partial F}{\partial y}\dot{\eta} + \frac{\partial F}{\partial z}\dot{\zeta} + \frac{\partial F}{\partial t} = 0 \,, \tag{7.3}$$

where the dots indicate differentiation with respect to time. This is equivalent to the statement that a particle at the surface stays on it.[12] Now, if $\zeta(x,y,z,t)$ is the vertical displacement of the fluid,

$$F = z - \zeta \,, \tag{7.4}$$

and Eq. (7.3) gives for a simple harmonic wave system

$$i\alpha\xi + i\beta\eta + \zeta_z = 0 \,. \tag{7.5}$$

This is the general form of the free surface boundary condition, which can be put into the desired form involving only ζ and ζ_z by use of Eqs. (4.8) or (4.21–22), or (4.28–29), depending upon circumstances.

In the absence of rotation Eq. (4.8) gives

$$\zeta_z/\zeta = \alpha^2 g/\omega^2 \,. \tag{7.6}$$

For a vertical axis of rotation

$$\frac{\zeta_z}{\zeta} = \frac{k^2 g}{\omega^2 - \Omega_V^2} \,. \tag{7.7}$$

Note that Eq. (7.5) is equivalent to the usual criterion for vanishing of the incremental pressure for a compressible fluid:

$$p = \lambda\epsilon \,. \tag{7.8}$$

At a *boundary between two compressible fluids* of different densities ρ_1, ρ_2 we require continuity of ζ and p, i.e.,

$$\zeta_1 = \zeta_2 \tag{7.9}$$

$$\lambda_1(i\alpha\xi_1 + i\beta\eta_1 + \zeta_{1_z}) = \lambda_2(i\alpha\xi_2 + i\beta\eta_2 + \zeta_{2_z}) \,. \tag{7.10}$$

In the absence of rotation and for $c_1 \neq c_2$, Eq. (7.10) is simply

$$\rho_1(\alpha^2 g\zeta_1 - \omega^2\zeta_{1z}) = \rho_2(\alpha^2 g\zeta_2 - \omega^2\zeta_{2z})\,\frac{c_2^2}{c_1^2}\,\frac{\alpha^2 c_1^2 - \omega^2}{\alpha^2 c_2^2 - \omega^2} \,. \tag{7.11}$$

Or, in view of Eq. (7.9), for $c_1 = c_2$

$$\alpha^2 g(\rho_1 - \rho_2)\zeta_1 = \omega^2(\rho_1\zeta_{1z} - \rho_2\zeta_{2z}) \,. \tag{7.12}$$

Since c has canceled out, this is also the correct form of the condition at the interface between two *incompressible* fluids of different densities. When there is *no gravity field* Eq. (7.12) gives

$$\rho_1\zeta_{1z} = \rho_2\zeta_{2z} \,. \tag{7.13}$$

If there is *no density discontinuity* one has, with or without a gravity field,

$$\zeta_{1z} = \zeta_{2z} \,. \tag{7.14}$$

8. REFLECTION AND REFRACTION OF PLANE WAVES. BODY WAVE MODES

(*a*) *General comments*. The well-known procedure for deriving reflection and transmission coefficients for plane waves in media with *constant coefficients* is to assume an incident wave $e^{i(\alpha x + \gamma_1 z - \omega t)}$ in medium (1) and reflected and transmitted waves $Re^{i(\alpha x - \gamma_1 z - \omega t)}$, $Te^{i(\alpha x + \gamma_2 z - \omega t)}$ in media (1) and (2), respectively. In the problems considered here there are *two* boundary conditions at an interface between two media, giving two inhomogeneous equations which one may solve for R, T. An interesting point arises in connection with the possibility of change in type of wave motion upon reflection or transmission. Thus, it was shown in Sec. 6 that, in a fluid of constant sound velocity and exponential density variation (constant *Väisälä* frequency), in the absence of rotation, two types of body waves could exist: the acoustic and the internal gravity wave modes. One is led to ask under what conditions an incident wave of one type can produce reflected or transmitted waves of another.

First of all, it is clear that this *cannot* occur upon *reflection:* α, ω are invariant and in order that transformations of this kind be possible, the characteristic curves $\omega(\alpha)$ for the two types of modes must be able to intersect. This is not possible since the straight line $\omega/\alpha = c$ separates the acoustic and internal wave regions (see Fig. 5). From the mathematical standpoint this is due, of course, to the fact that the equations of motion are second order and that only two boundary conditions are available at an interface. It is only when the equations of motion are of higher order, as in elastic solids, that transformations of this type are possible (P to S conversions).

But, upon *refraction*, it is theoretically conceivable that, for instance an acoustic wave incident from (1) on the interface may give an internal gravity wave in (2) upon transmission. For this to occur it is sufficient that the $\omega(\alpha)$ acoustic curves for medium

(1) intersect the corresponding internal wave curves for medium (2). This will be possible if the speed of sound c_1 is appreciably less than c_2 (for media with constant c, N). A necessary (but not sufficient) condition for this is that the slope of the internal wave curve for (2) near $\omega = 0$ exceed c_1. This initial slope is given by Eq. (6.19), so that for $\gamma_2 = 0$ one has

$$N_2/\nu_2 > c_1 . \tag{8.1}$$

Another necessary condition is, of course, that

$$N_2 > \nu_1 c_1 ; \tag{8.2}$$

otherwise there can be no overlap in the permissible frequencies. However, conditions (8.1)–(8.2) are not sufficient of themselves since they do not guarantee intersection of the ω_{A1} and ω_{I2} curves.

For models with *nonconstant coefficients* one may visualize a series of characteristic $\omega(\alpha)$ graphs, each characteristic of a given depth. If the $\omega(\alpha)$ curves for one type of mode and for some depth z_1 can intersect those for another mode type and depth z_2, then there may be transformation of type, through a sort of "high-velocity" barrier in which $\gamma^2 < 0$.

(*b*) *Specific forms of the reflection coefficient.* We have noted that the effect of gravity on the acoustic branch is not, as a rule, very important. Thus in cases involving reflection of acoustic waves at a boundary of discontinuity of the sound velocity, one obtains essentially the usual Rayleigh reflection coefficient with a correction term. This term is only important at very low frequencies and involves corrections of the order of a few percent. We will not derive this coefficient in its most general form: This would unnecessarily lengthen our presentation and, besides, the reader can easily do so for himself.

We shall examine first the case for which there are discontinuities in density and density gradient, in the presence of a gravity field. This brings out an interesting phenomenon. We consider two half-spaces in contact at $z = 0$. In the overlying half-space $z > 0$

$$\rho_1 = \rho_{10} e^{-2\nu_1 z} \tag{8.3}$$

and in the underlying region $z < 0$

$$\rho_2 = \rho_{20} e^{-2\nu_2 z} , \tag{8.4}$$

with

$$c_1 = c_2 = \text{const} . \tag{8.5}$$

We write, as in Eq. (4.15),

$$\zeta = \zeta_1 = \rho_1^{-1/2} h_1 ,\ z > 0 \tag{8.6}$$

$$\zeta = \zeta_2 = \rho_2^{-1/2} h_2 ,\ z < 0 , \tag{8.7}$$

with

$$h_1 = e^{i(\alpha x - \gamma_1 z - \omega t)} + R e^{i(\alpha x + \gamma_1 z - \omega t)} \tag{8.8}$$

$$h_2 = T e^{i(\alpha x - \gamma_2 z - \omega t)} . \tag{8.9}$$

The boundary conditions Eqs. (7.9) and (7.12) applied at $z = 0$ give

$$\rho_{10}^{-1/2}(1 + R) = \rho_{20}^{-1/2} T \tag{8.10}$$

$$(\alpha^2 g/\omega^2)(\rho_{10} - \rho_{20})(1 + R)\rho_{10}^{-1/2} = \rho_{10}^{1/2} \times [\nu_1(1 + R) + i\gamma_1(R - 1)] - \rho_{20}^{1/2}(\nu_2 - i\gamma_2)T . \tag{8.11}$$

Writing

$$\rho_{20}/\rho_{10} = a \tag{8.12}$$

and solving Eqs. (8.10) and (8.11) for R give

$$R = \frac{i\gamma_1 + (\alpha^2 g/\omega^2)(1 - a) + a\nu_2 - \nu_1 - ia\gamma_2}{i\gamma_1 - (\alpha^2 g/\omega^2)(1 - a) - a\nu_2 + \nu_1 + ia\gamma_2} . \tag{8.13}$$

If γ_2 is imaginary and γ_1 real,

$$\gamma_2 = i\gamma_2' . \tag{8.14}$$

One has

$$R = e^{-2i\chi} , \tag{8.15}$$

with

$$\chi = \tan^{-1}(1/\gamma_1)[(\alpha^2 g/\omega^2)(1 - a) + a\nu_2 - \nu_1 + a\gamma_2'] . \tag{8.16}$$

This corresponds to total reflection of plane body waves. If $\nu_1 < \nu_2$, $N_1 < N_2$, only acoustic waves incident from above are totally reflected off the surface of the underlying half-space (2) [see Eq. (6.23) or Eqs. (8.18) and (8.19)], for frequencies and wave numbers falling below the characteristic acoustic branch of medium (2). Conversely, if $\nu_1 > \nu_2$, $N_1 > N_2$ only internal gravity waves may be similarly subject to total reflection.

One may visualize a layer of thickness h, density (8.3), inside an infinite space of density (8.4). Then total reflection may occur at both upper and lower boundaries, and energy may be trapped within the layer by total reflection: We thus have a *waveguide* (for internal gravity waves if $N_1 > N_2$, and for acoustic waves if $\nu_1 < \nu_2$).

Consider, for instance, the especially simple case in which, neglecting gravity, there is a jump in density gradient but none in density ($a = 1$, $g = 0$). Let $\nu_2 > \nu_1$. Then, as shown in Sec. 6, the acoustic waves are characterized by a dispersion which is described in each half space by a hyperbola (as in Fig.

3). This means that, in the α,ω plane, real waves correspond, in each case, to points which are, for given ω, to the left of the characteristic hyperbolas shown in Fig. 7(f). If k_1 is the wave number along a ray,

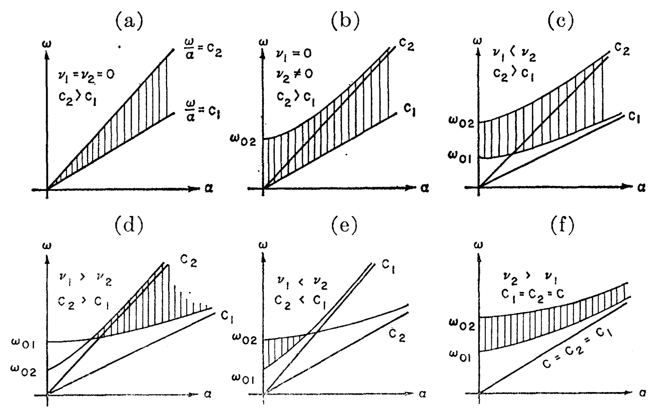

FIG. 7. Shaded region corresponds to total reflection of plane acoustic waves traveling in medium (1) and incident upon the surface of medium (2).

then, for given ω, the angle of incidence in medium (1) is

$$\theta_1 = \sin^{-1}(\alpha/k_1) . \quad (8.17)$$

Then, any point in the shaded region in Fig. 7 corresponds to traveling waves in medium (1), $z > 0$, totally reflected at $z = 0$.

An even simpler example of total reflection of acoustic waves by density gradients is given by a homogeneous half-space $z > 0(\nu_1 = 0, \omega_{01} = 0)$. In this case the whole region below the curve

$$\omega = [\alpha^2 c^2 + \omega_{02}^2]^{1/2} \quad (8.18)$$

corresponds to total reflection, with a change in phase:

$$\chi = \tan^{-1}\left[\frac{\nu_2 + (\nu_2^2 - k^2\cos^2\theta_1)^{1/2}}{k\cos\theta_1}\right]. \quad (8.19)$$

In terms of wavelength λ measured along the ray, points below the curve Eq. (8.18) correspond to

$$\lambda_1 > (2\pi/\nu_2)\cos\theta_1 . \quad (8.20)$$

This is the condition for total reflection. At normal incidence ($\theta_1 = 0$) wavelengths longer than $2\pi/\nu_2$ are totally reflected. On the other hand, at grazing angles ($\theta_1 = \pi/2$), all wavelengths are subject to total reflection if $\nu_2 \neq 0$.

It is not difficult to obtain the corresponding formulae for the case when there is also a jump in the sound velocity. Suppose once more that there is no density jump ($a = 1$) and that gravity is neglected. Then Eq. (7.11) reads, at $z = 0$,

$$\zeta_{1z} = \zeta_{2z}(b_1^2/b_2^2) , \quad (8.21)$$

where

$$b_1^2 = \omega^2/c_1^2 - \alpha^2$$
$$b_2^2 = \omega^2/c_2^2 - \alpha^2 \quad (8.22)$$

are the squares of the usual acoustical z wave numbers in the absence of a density gradient. This implies that in Eq. (8.11) we simply multiply T by b_1^2/b_2^2, i.e.,

$$R = \frac{i\gamma_1 - \nu_1 + (\nu_2 - i\gamma_2)b_1^2/b_2^2}{i\gamma_1 + \nu_1 - (\nu_2 - i\gamma_2)b_1^2/b_2^2}. \quad (8.23)$$

In the case $\nu_1 = \nu_2 = 0$, $b_1 = \gamma_1$, $b_2 = \gamma_2$ one obtains the usual Rayleigh coefficient (for displacements) in the absence of density contrast:

$$R_{\text{Rayleigh}} = \frac{b_2 - b_1}{b_2 + b_1}. \quad (8.24)$$

The corresponding region of total reflection is bounded by the two straight lines of slopes c_1, c_2 in Fig. 7(a). On the other hand, if $\nu_1, \nu_2 \neq 0$, and $\nu_1 < \nu_2$, the zone of total reflection is as shown in Fig. 7(c). If $\nu_1 = 0$, $\nu_2 \neq 0$ we have the situation illustrated by Fig. 7(b).

Similar diagrams, illustrating the conditions required for the total reflection of *internal gravity waves*, can be drawn. Thus, under the assumption Eq. (8.5), for $\nu_1 > \nu_2$ we have $N_1 > N_2$, and internal gravity waves incident in medium (1) upon the boundary $z = 0$ will be totally reflected for the range of frequencies and wave numbers (or wavelengths and angles of incidence) corresponding to the shaded region of Fig. 8. Here we may note that if $\nu_1 = \nu_2$

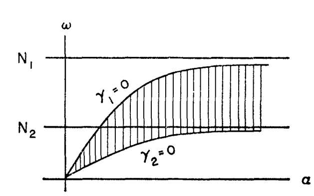

FIG. 8. Total reflection of plane internal gravity waves.

internal waves will suffer reflection at the boundary $z = 0$ if Eq. (8.5) does not hold, i.e.,

$$c_1 > c_2 . \quad (8.25)$$

Then, by Eq. (6.24),

$$N_1 > N_2 \quad (8.26)$$

and total reflection of internal gravity waves may occur as in Fig. 8, even though $-(d/dz) \ln \rho = 2\nu_1 = 2\nu_2 = \text{const}$.

Anticipating the discussion of Sec. 10, it is clear that if we have a layer (1) of finite thickness lying between two half-spaces such that the conditions of total reflection are fulfilled at both boundaries, energy will be trapped and the layer may act as a waveguide for either acoustic or internal gravity waves. For example, the previous discussion leads us to the remarkable result that a homogeneous layer, or a layer of low-density gradient, imbedded in a space of higher density gradient may act as an acoustical waveguide even in the absence of any variation of the sound velocity.[20]

To conclude this section we derive the change in phase upon total reflection at a free surface. In this case we take medium (1) to be vacuum and consider an up-traveling wave with its reflection:

$$h = e^{i(\alpha x+\gamma z-\omega t)} + Re^{i(\alpha x-\gamma z-\omega t)}, \qquad (8.27)$$

with the boundary condition Eq. (7.6) (in the absence of rotation) giving

$$\nu_1(1 + R) + i\gamma(1 - R) = (\alpha^2 g/\omega^2)(1 + R) \qquad (8.28)$$

and

$$R = e^{2i\chi}, \qquad (8.29)$$

with

$$\chi = \tan^{-1}(1/\gamma)(\alpha^2 g/\omega^2 - \nu) . \qquad (8.30)$$

It is interesting to note here that pure acoustic waves in the absence of any density gradient ($\nu = 0$) suffer a change in phase upon total reflection due to surface gravity effects:

$$\chi = \tan^{-1}(1/\gamma)(\alpha^2 g/\omega^2) . \qquad (8.31)$$

At grazing angles of incidence $\gamma \to 0$, $\theta \to \pi/2$ and this change in phase is not trivially small, as is usually assumed in the acoustical literature: the "free" surface acts, for grazing incidence, as a rigid wall, since $\chi \to \pi/2$, $R \to -1$ (for displacements).

(*c*) *Body wave modes.* The modes of a stratified fluid are the standing waves obtained by superposing waves of equal amplitude, traveling in opposite directions. In the case of total reflection, at some boundary $z = \text{const}$, it is sufficient to consider the superposition of an incident wave train with its reflection.

For example, consider a compressible half-space $z < 0$, with a free surface at $z = 0$. If it is density stratified according to our usual exponential law, the modes are, in terms of the displacements,

$$\zeta = B[e^{i\gamma z} + e^{i(2\chi-\gamma z)}]e^{\nu z} = Ae^{\nu z} \cos(\gamma z - \chi) , \qquad (8.32)$$

γ being defined by Eq. (6.23) and χ by Eq. (8.30). These are either acoustic or internal gravity wave modes, depending upon the range of ω, α values being considered. In the case of an incompressible fluid, γ is defined by Eq. (6.11), and Eq. (8.32) represents pure internal gravity modes.

9. BOUNDARY WAVES

These are waves whose energy is concentrated at a boundary of discontinuity of some parameter and correspond to $\gamma^2 < 0$ on both sides of the boundary, i.e., to an exponential variation of amplitude. This variation must be such as to give vanishing energy density at infinity. Their existence is quite independent of that of body waves in the medium. Typical examples of "boundary" (or "surface") waves are gravity waves at the surface of an incompressible homogeneous fluid, Rayleigh waves at the free surface of an elastic solid, interface waves at the boundary between an elastic solid and a liquid (Stoneley waves), etc. In these well-known examples it is easy to see the special nature of these waves. Thus, there are no body waves of any kind in an incompressible homogeneous fluid: Surface waves are an entirely distinct mode of propagation existing by virtue of the possibility of two methods of energy storage at the interface (kinetic and gravitational potential energies). One may also think of Stoneley waves at a solid-liquid interface: It has been pointed out by Biot[21] that these may exist at the boundary between a massless solid ($\rho = 0$) and an incompressible fluid, i.e., when neither of the two media in contact can propagate waves of any kind by itself. Here energy is transferred back and forth across the interface, being purely kinetic on one side (incompressible fluid) and purely elastic on the other (massless solid). This independence from body waves is an essential feature of true surface waves, as distinguished from other types which may have only some of the characteristics of boundary waves. Thus, Love waves, which are simply body waves (horizontal shear) trapped in a low velocity layer, are often improperly classified as surface waves in the seismological literature. Likewise it would be inconsistent to describe Lamb waves as boundary waves,[1] since they are but

[20] I. Tolstoy, Columbia University, Hudson Laboratories Contribution No. 146. Presented at the Fourth International Congress on Acoustics, Copenhagen, Denmark, August 21–28, 1962 (to be published).

[21] M. A. Biot, Bull. Seismol. Soc. Am. **42**, 81 (1952).

body acoustic waves traveling parallel to the density stratification.

Stratified compressible fluids in a gravity field can exhibit a number of types of discontinuity supporting *bona fide* boundary waves. We confine ourselves here to a quick description of some of the simpler cases.

If we first consider two half-spaces (as in Sec. 8), Eq. (7.9) implies that solutions for the displacement are

$$\zeta_1 = e^{\nu_1 z} e^{-\gamma'_1 z} e^{i(\alpha x - \omega t)}, \; z > 0 \tag{9.1}$$

$$\zeta_2 = e^{\nu_2 z} e^{\gamma'_2 z} e^{i(\alpha x - \omega t)}, \; z < 0 \,. \tag{9.2}$$

These solutions are always consistent with the requirement of vanishing energy density for $z \to \pm \infty$.

The general equation describing the possible solutions of this type is Eq. (7.11), i.e.,

$$\rho_1 \left[\frac{\alpha^2 g}{\omega^2} - (\nu_1 - \gamma'_1) \right] = \rho_2 \left[\frac{\alpha^2 g}{\omega^2} - (\nu_2 + \gamma'_2) \right] \frac{b_1^2}{b_2^2} \,, \tag{9.3}$$

where b_1, b_2 are defined by Eqs. (8.22). A complete analysis of the roots of this equation would be cumbersome at this point. We first note that if $c_1 = c_2 = c$, it reduces to

$$\rho_1 \left[\frac{\alpha^2 g}{\omega^2} - (\nu_1 - \gamma'_1) \right] = \rho_2 \left[\frac{\alpha^2 g}{\omega^2} - (\nu_2 + \gamma'_2) \right]. \tag{9.4}$$

This equation contains a variety of known solutions as limiting cases.

Thus, the characteristic equation for *surface gravity waves* over an *incompressible homogeneous half-space* is obtained by taking

$$\rho_1 = 0 \,, \tag{9.5}$$

$$c = \infty \,, \tag{9.6}$$

$$\nu_2 = 0 \,. \tag{9.7}$$

Then, by Eqs. (6.23) and (8.14)

$$\gamma'_2 = \alpha \,, \tag{9.8}$$

and Eq. (9.4) gives

$$\alpha g = \omega^2 \,, \tag{9.9}$$

i.e., the well-known dispersion equation for surface gravity waves in "deep water." If the half-space is *density stratified* according to the law Eq. (8.4), and still *incompressible*, we drop condition Eq. (9.7) and obtain

$$\gamma'_2 = [\alpha^2 - \alpha^2(2\nu_2 g/\omega^2) + \nu_2^2]^{1/2} \,, \tag{9.10}$$

and Eq. (9.4) is

$$\alpha^2 g/\omega^2 = \nu_2 + \gamma'_2 \,. \tag{9.11}$$

But then Eq. (9.9) still gives a root of Eq. (9.11), since Eq. (9.10) becomes

$$\gamma'_2 = (\alpha^2 - 2\nu_2\alpha + \nu_2^2)^{1/2} = \pm(\alpha - \nu_2) \,. \tag{9.12}$$

In practice, $\alpha \gg \nu_2$ and we keep the plus sign. Then we have by Eq. (9.11)

$$\alpha^2 g/\omega^2 = \alpha \,, \tag{9.13}$$

which is verified by virtue of Eq. (9.9), which we had assumed to hold. If the half-space is *density stratified* and *compressible*, the characteristic equation (9.4) still has the form (9.11), but with

$$\gamma'_2 = \left[\alpha^2 - \frac{\omega^2}{c^2} - \alpha^2 \left(\frac{2\nu_2 g}{\omega^2} - \frac{g^2}{\omega^2 c^2} \right) + \nu_2^2 \right]^{1/2} . \tag{9.14}$$

Equation (9.11) has two roots in this case. It is easily verified once more that Eq. (9.9) is still valid for $\alpha > \nu_2$. Thus, the surface gravity wave solutions are essentially unaffected by compressibility or by density stratification.

The second solution corresponds to

$$\omega = \alpha c \tag{9.15}$$

for it is easily verified by substitution into Eq. (9.14) that one then has

$$\gamma'_2 = g/c^2 - \nu_2 \,. \tag{9.16}$$

Substitution of Eqs. (9.15) and (9.16) into Eq. (9.11) shows that Eq. (9.15) is indeed a solution. Equation (9.16) implies

$$\zeta_2 = e^{(g/c_2)z} \,, \; z < 0 \,. \tag{9.17}$$

This represents a *horizontally traveling sound wave* with a small vertical component arising from buoyancy effects in the gravity field. Note that for the infinite half-space, the condition of vanishing energy density at $z = -\infty$, i.e., of vanishing

$$\rho\zeta^2 = e^{2\gamma'_2 z} \,, \tag{9.18}$$

requires that $\gamma'_2 > 0$, and $g/c^2 > \nu_2$, a condition which may or may not be verified in practice (for instance, it is verified for the case of the isothermal model of the atmosphere). However, in actual problems one is always dealing with media of finite extent and may thus expect a solution of this type to exist. It is *not*, strictly speaking, a surface wave but an ordinary, nondispersive, acoustic wave traveling in a horizontal direction and slightly modified by the density stratification and by buoyancy effects. It appears as a possible root of Eq. (9.11) because of its (weak) exponential behavior as a function of z. It is essentially similar to the *Lamb wave*, which is usually introduced for a density stratified half-space overlying a rigid wall. In this case one assumes $\zeta = 0$ everywhere, and condition Eq. (7.1) is obeyed automatically. Then, by Eq. (4.7b), ξ is determined by a first-order differential equation:

$$(d/dz)\lambda\xi = -(g/c^2)\lambda\xi \tag{9.19}$$

or

$$\lambda\xi = Ce^{-(g/c^2)z}, \; z > 0\,, \tag{9.20}$$

i.e.,

$$\xi = \xi_0 e^{(2\nu_1 - g/c^2)z}, \; z > 0\,. \tag{9.21}$$

Now, stability of the medium requires that $2\nu_1 > g/c^2$ (Sec. 2), and the amplitude of ξ *increases* exponentially as $z \rightarrow +\infty$. But the energy density is proportional to

$$\rho\xi^2 = \xi_0^2 e^{2(\nu_1 - g/c^2)z} \tag{9.22}$$

and will tend to zero for $g/c^2 > \nu$, a condition fulfilled by the isothermal model. However, it should be kept in mind that in nature one usually has models of finite extent, so that these considerations of convergence are not always binding.

Two remarks may be made in connection with the Lamb waves [Eq. (9.21)] or Lamb-type waves [Eq. (9.17)].

First of all, they are not true boundary waves, but modified body waves (acoustic).

Second, in the case of a free surface, we see that the characteristic curves Eqs. (9.9) and (9.15) intersect in the ω,α plane at the point

$$\alpha = g/c^2\,, \tag{9.23}$$

and one is led to ask why there is no interaction between the surface wave branch [Eq. (9.9)] and the acoustic (Lamb) wave [Eq. (9.15)]. The answer is simply that, by virtue of Eq. (5.8), the surface gravity wave is irrotational, whereas the Lamb wave is not: In the absence of viscosity there can be no transfer of energy between the two.

For the case of two *incompressible, homogeneous half-spaces* of different densities, $\nu_1 = \nu_2 = 0$, $\gamma_1' = \gamma_2' = \alpha$ and Eq. (9.4) gives the well-known equation[12]

$$\omega^2 = \alpha g \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\,. \tag{9.24}$$

When $\rho_1 > \rho_2$, these are *bona fide* boundary waves in the sense used here. They should be distinguished from internal waves in a stratified fluid, which are body waves. Of course, a stepwise variation of ρ is merely an extreme case of stratification, and these interface waves may be considered as a limiting case of internal waves [in a layer of *Väisälä* frequency $g(\Delta\rho/\rho)\delta(z)$]. However, this is altogether too formal an analogy for physical purposes, and one should distinguish between these two cases. One may refer to the waves of Eq. (9.24) as *interface gravity waves*, to separate them from the internal gravity body waves.

This completes our review of the simplest types of solution of Eq. (9.4) and of the corresponding boundary waves. A more thorough investigation of Eqs. (9.3) and (9.4), dealing with the effects of compressibility stratification, jumps in sound velocity, etc., remains to be done and would undoubtedly be of interest. This would, however, lengthen our discussion beyond its intended bounds.

10. LAYERED WAVEGUIDES

If, in a stratified fluid, wave energy can be trapped or channeled within a zone of finite thickness h, one has a *waveguide*. This occurs when the conditions for total reflection of plane waves are met at two plane parallel boundaries.

The classic example of this is the perfect acoustic waveguide in which sound waves are trapped between two rigid walls or between a rigid wall and a free surface. Now it is well known that under these conditions propagation is dispersive, the dispersion being due to interference between up- and downgoing sound waves: Energy at a given frequency can only propagate at certain angles of incidence without losses due to destructive interference. The condition of constructive interference, giving the dispersion law, has then the same form as the diffraction grating equation.[22,23] This is often known as "geometrical dispersion" to distinguish it from other types of dispersion not due to interferences between reflections. Thus, the dispersion of plane acoustic waves in an infinite, density stratified, fluid [Eq. (6.5)], or of plane internal gravity waves [Eq. (6.11)], is due to a quite different process. This type of dispersion is best referred to as "structural dispersion" since it is determined entirely by the assumed stratification and the resulting natural resonant frequencies of the medium.

If a density stratified fluid is bounded by two perfect reflectors, we may expect the observed dispersion to be due to a superposition of these effects: There will be dispersive effects due to interference between

[22] L. M. Brekhovskikh, *Waves in Layered Media* (Academic Press Inc., New York, 1960).
[23] Ivan Tolstoy, J. Acoust. Soc. Am. **27**, 274 (1955).

downgoing and upgoing plane waves, but these plane waves are also dispersive of themselves, even in the absence of boundaries.

As a *first example*, illustrating the combined effects of geometrical and structural dispersion, consider a layer of constant sound velocity c, density stratified according to the law (8.3), with a free surface at $z = 0$ and a rigid bottom at $z = -h$. This model provides an average, qualitative picture of the types of propagation to be expected in the ocean and will illustrate the gross properties of the various modes. On the basis of the preceding sections we may expect these to be of two types: *sinusoidal modes*, with $\gamma^2 > 0$ [Eqs. (4.16) and (6.23)], corresponding to standing, plane, body waves (acoustic and internal gravity modes), and *exponential modes*, with $\gamma^2 < 0$, which will be essentially the boundary waves of Sec. 9, modified here by the condition of finite depth.

In Sec. 8 we saw that a system of *sinusoidal modes* corresponding to total reflection at a free surface $z = 0$ has the form:

$$\zeta = Ae^{\nu z} \cos (\gamma z - \chi) . \qquad (10.1)$$

In the present case we must also satisfy the boundary condition $\zeta = 0$ at $z = -h$, i.e.,

$$\gamma h + \chi = (m + \tfrac{1}{2})\pi , \quad m = 0,1,2 \cdots \qquad (10.2)$$

where χ is given by Eq. (8.30). This has the form of the usual geometric dispersion law for waveguides. The fact that propagation is also structurally dispersive, i.e., that the individual interfering plane waves are by themselves dispersive, is reflected in the definition of γ by Eq. (6.23). It is seen that, in the absence of gravity and stratification ($g = 0$, $\nu = 0$), γ is simply the usual acoustical z wave number b and also that $\chi = 0$. Equation (10.2) then becomes the familiar dispersion equation for the homogeneous acoustical waveguide with one rigid wall and a free surface.

Exponential modes corresponding to $\gamma = i\gamma'$, $\chi = i\chi'$ are more conveniently treated by changing χ to ϕ in Eq. (8.30):

$$\phi = \chi + \pi/2 = -\tan^{-1} \frac{\gamma}{\alpha^2 g/\omega^2 - \nu} . \qquad (10.3)$$

Thus,

$$\zeta = A \sin (\gamma z - \phi) \qquad (10.4)$$

and Eq. (10.2) becomes

$$\gamma h + \phi = m\pi , \quad m = 0,1,2 \cdots , \qquad (10.5)$$

for exponential modes γ, ϕ become imaginary and only $m = 0$ is possible. Thus,

$$\gamma = i\gamma'$$
$$\phi = i\phi' = -i \tanh^{-1} \frac{\gamma'}{\alpha^2 g/\omega^2 - \nu} \qquad (10.6)$$

and

$$\gamma' h + \phi' = 0 \qquad (10.7)$$

or

$$\tanh \gamma' h = \frac{\gamma'}{\alpha^2 g/\omega^2 - \nu} . \qquad (10.8)$$

For the limiting case $h \to \infty$

$$\gamma' = \alpha^2 g/\omega^2 - \nu , \qquad (10.9)$$

and we return to Eq. (9.11), with the two roots (9.9) and (9.15) corresponding to the surface gravity wave and the Lamb-type wave. The Lamb wave does not actually exist in the case of our finite depth model, since the condition $\alpha c = \omega$ would imply, by virtue of Eq. (4.8), either that $\zeta = 0$ everywhere or $\zeta = e^{g/c^2 z}$. In the former choice it is not possible to satisfy the free surface condition unless $\xi = 0$, and in the latter choice the rigid boundary condition at $z = -h$ cannot be obeyed. The Lamb wave, with $\zeta = 0$, would exist in the case of two rigid boundaries: Thus, in these perfect waveguide problems, it plays the role of the so-called zeroth acoustical mode which, as is well known, exists only for the case of two rigid walls. We may expect Eq. (10.8) to have only one physically significant root, corresponding to gravity waves on the surface of a layer of finite depth, slightly modified by the density stratification. Note that if $h \to 0$, Eq. (10.8) becomes

$$\alpha^2 g/\omega^2 = \nu + 1/h . \qquad (10.10)$$

For the ocean

$$\nu \ll 1/h \qquad (10.11)$$

and

$$\omega^2/\alpha^2 = V^2 = gh , \qquad (10.12)$$

which is the usual long wavelength approximation for gravity waves.

The properties of the *sinusoidal modes* can also be explored by means of approximations.

First of all, we have already noted that if gravity and density gradients are neglected, Eq. (10.2) is the usual acoustic waveguide equation. As seen in Section 6, g and ν can be taken equal to zero for $\omega \gg \omega_0 = \nu c$. But the lowest acoustical mode ($m = 1$) low-frequency cutoff ω_c is, for our model,

$$\omega_c = (\pi/2)c/h . \qquad (10.13)$$

In view of Eq. (10.11), this implies $\omega_c \gg \omega_0$. Thus, the usual *acoustical modes* of the ocean are, to a very high order of accuracy, unaffected by either gravity or density gradients.

The other solutions of Eq. (10.2) correspond to *trapped internal gravity waves*, with

$$\omega \leqslant N . \quad (10.14)$$

In Sec. 6 we showed that if compressibility is neglected one does not incur errors of more than 10%. Thus, the wave number γ is then quite satisfactorily represented by

$$\gamma = [(\alpha^2/\omega^2)N^2 - \alpha^2 - \nu^2]^{1/2} . \quad (10.15)$$

Assuming

$$\alpha \gg \nu \quad (10.16)$$

limits us to wavelengths in the x direction:

$$\lambda_x = 2\pi/\alpha \ll 2\pi/\nu . \quad (10.17)$$

In terms of the average orders of magnitude assumed in Sec. 6 for an ocean, this implies

$$\lambda_x \ll 2.5 \times 10^3 \text{ km} . \quad (10.18)$$

Then, a good approximation is

$$\gamma = \alpha[N^2/\omega^2 - 1]^{1/2} , \quad (10.19)$$

and Eq. (10.2) becomes

$$\gamma h = \alpha h[N^2/\omega^2 - 1]^{1/2} = (m + \tfrac{1}{2})\pi - \chi . \quad (10.20)$$

This means that γ is of the order of magnitude of π/h for moderate values of m. Now, for short wavelengths the phase velocity of body internal waves is very small, i.e.,

$$\text{as} \quad \omega \to N , \quad \lambda \to 0 , \quad \omega/\alpha \to 0 , \quad (10.21)$$

and from Eq. (8.30) it is seen that

$$\chi \to \pi/2 , \quad (10.22)$$

so that the characteristic equation becomes

$$\alpha h[N^2/\omega^2 - 1]^{1/2} = m\pi . \quad (10.23)$$

This approximation is quite good even for wavelengths of the order of hundreds of kilometers. Equation (10.23) gives the family of dispersion curves shown in Fig. 9.

The shape of the *particle orbit* depends upon the depth, mode number, and frequency. The particle displacements are, approximately,

$$\zeta = A \sin \gamma z \sin (\alpha x - \omega t)$$
$$\xi = A(\gamma/\alpha) \cos \gamma z \cos (\alpha x - \omega t) . \quad (10.24)$$

Nodes for ζ are antinodes for ξ and vice versa. The ratio of maximum ξ,ζ amplitudes is

$$|\xi|/|\zeta| = \gamma/\alpha = (N^2/\omega^2 - 1)^{1/2} \quad (10.25)$$

by Eqs. (10.23) and (10.19). Thus, for $N \sim \omega$, the motion is mostly vertical (except at the ζ nodes), and

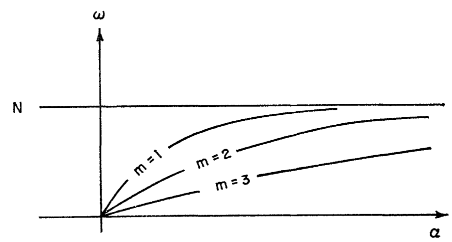

FIG. 9. Dispersion of normal modes of perfect internal gravity wave waveguide.

for $\omega \ll 2^{-1/2} N$ the horizontal amplitude is much greater. At tidal frequencies $\omega \ll N$, and

$$|\xi|/|\zeta| \backsim N/\omega . \quad (10.26)$$

For these very low frequencies, Eq. (10.23) simplifies to

$$\alpha h = m\pi(\omega/N) , \quad (10.27)$$

and the phase and group velocity for $\omega \ll N$ are of the order of

$$U \backsim V \backsim Nh/m\pi . \quad (10.28)$$

Note that the wavelength is

$$\lambda = 2Nh/m\omega . \quad (10.29)$$

For our averaged ocean model $N \sim 2.8 \times 10^{-3}$, $h \sim 5$ km. For a semidiurnal period, $\omega \sim 1.4 \times 10^{-5}$, and $\lambda \sim 2000$ km, $U \sim V \sim 5$m/sec for $m = 1$. [These are but rough estimates, since, by Eq. (10.17), our approximations become inaccurate for such long wavelengths.]

It is of interest to compute the *pressure* perturbations connected with an internal wave system. Using Eqs. (3.6) and (10.24) we have

$$p \backsim A\rho(\omega^2/\alpha^2)\gamma \cos \gamma z \sin (\alpha x - \omega t) . \quad (10.30)$$

For $\lambda \ll h$ and low modes, the maximum pressure amplitude $|p|$ is related to the maximum displacement $|\zeta|$ as follows:

$$|p| \backsim (1/4\pi)\rho N^2(\lambda^2/h)m|\zeta| , \quad (10.31)$$

whereas for $\lambda \gg h$,

$$|p| \backsim (1/4\pi)\rho N^2(4h/m)|\zeta| . \quad (10.32)$$

In our idealized "average" deep ocean model, $N^2 \sim 8 \times 10^{-6}$, h $\sim$ 5 km, and for $\lambda = 1$ km, $m = 1$, $|p| = 1.5 \times 10^{-2}\ |\zeta|$ dyn/cm² (ζ in cm), whereas for $\lambda \gg h$, $|p| \sim 1.3\ |\zeta|$ dyn/cm². In shallow seas the *Väisälä* period may be much shorter, e.g., $\sim$ 2 min, $N^2 \sim 2.5 \times 10^{-3}$, and for a depth of, say, 10 *m*, $\lambda \gg 10\ m$, $m = 1$, $|p| \sim |\zeta|$ dyn/cm².

In the present approximation, the free surface is a node for the vertical displacement ζ, due to the assumption Eq. (10.2). Dropping this assumption one finds a small vertical displacement of the free surface connected with any internal wave system. A brief calculation shows, as one would expect, that the maximum pressure amplitude $|p|$ is simply the change in hydrostatic head due to the surface displacement ζ_0, i.e.,

$$|p| = \rho g|\zeta_0| \, . \tag{10.33}$$

It is therefore easy, using Eqs. (10.24), (10.30), and (10.33), to estimate the vertical displacement of the surface. It is greatest for wavelengths $\lambda \gg h$ and is then about 10^{-3} of the maximum displacements observed at depth.

The effect of a Coriolis field can be included (Eckart[1]). Its influence, for a vertical $\mathbf{\Omega}$ vector, depends upon whether N is larger or smaller than Ω. If $N > \Omega$ the effect of rotation is to cut off the internal wave modes at $\omega = \Omega$, $\alpha = 0$ (the curves in Fig. 9 then acquire a horizontal tangent at this point). If $N < \Omega$, the internal wave modes are replaced by gyroscopic waves of the type examined in Sec. 6 with a low-frequency cutoff at $\omega = N$.

A *second example* of waveguide propagation is given by acoustic waves in a medium of constant sound velocity c, but with a zone of minimum density

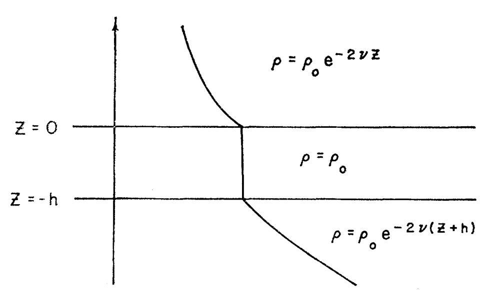

FIG. 10. A simple waveguide for acoustic waves with c = const.

gradient. The simplest case of this sort of trapping occurs for a homogeneous layer of thickness h imbedded in an infinite medium which is density stratified according to the law Eq. (8.3) (Fig. 10).

In this case, acoustic waves will be trapped in the layer of constant density, as explained in Sec. 8. This trapping is due to total reflection with a change in phase [Eqs. (8.15)–(8.19)]. Letting the subscript 2 correspond to the two half-spaces and 1 to the homogeneous layer, we have

$$\chi = \chi_{\downarrow} = \tan^{-1}(1/\gamma_1)(\nu + \gamma_2') \tag{10.34}$$

at the lower boundary and

$$\chi = \chi_{\uparrow} = \tan^{-1}(1/\gamma_1)(-\nu + \gamma_2') \tag{10.35}$$

at the upper.

The dispersion of these trapped waves is purely geometrical and is described by the interference (grating) condition:

$$\gamma_1 h = \chi_{\uparrow} + \chi_{\downarrow} \tag{10.36}$$

or

$$\tan \gamma_1 h = \gamma_2'/\gamma_1 \, , \tag{10.37}$$

since

$$\gamma_2' = [\nu^2 - \gamma_1^2]^{1/2} \, ; \tag{10.38}$$

the condition of total reflection corresponds to $0 \leq \gamma_1 \leq \nu$. At $\gamma_1 = \nu$ the r.h.s. of Eq. (10.37) is zero, and the l.h.s. is tan νh. At $\gamma_1 = 0$, the r.h.s. is ∞ and the l.h.s. is zero. There is, therefore, at least one root $0 < \gamma_1 < \nu$. If $(n+1)\pi > \nu h > n\pi$, there will be $n + 1$ roots, or modes of propagation. If $\gamma_{1m} = x_m$ is a root, then the dispersion of the corresponding mode is given by

$$\omega_m^2 = c^2(\alpha^2 + x_m^2) \, , \tag{10.39}$$

i.e., the characteristic curve is a hyperbola in the ω,α plane, with vertical axis, intersecting the ω axis at the cutoff frequency

$$\omega_{0m} = cx_m \, . \tag{10.40}$$

The group and phase velocities U, V obey the well-known law

$$UV = c^2 \, .$$

Equation (10.37) may also be obtained in the customary but more laborious fashion by considering the elementary solutions for ζ in the half-spaces and in the layer and making them satisfy the conditions of continuity for ζ and ζ_z. The amplitudes vary sinusoidally with depth in the layer and exponentially in the half-spaces. Note that in the latter they are

$$\zeta = Ae^{\nu z}e^{\gamma'_2 z} \, , \quad z < -h \tag{10.41}$$

$$\zeta = Be^{\nu z}e^{-\gamma'_2 z} \, , \quad z > 0 \, , \tag{10.42}$$

but $\gamma_2' < \nu$, and $|\zeta| \to \infty$ as $z \to +\infty$. Similarly, it may be verified that the pressure $\rho c^2 \epsilon \to \infty$ as z

$\rightarrow -\infty$. But the energy densities $\rightarrow 0$ for $z \rightarrow \pm\infty$, and we are in fact dealing with trapped energy in a waveguide. For example, the kinetic energy is proportional to $\rho\zeta^2$, i.e.,

$$E \backsim e^{-2\gamma'_z z}, z > 0$$
$$\backsim e^{2\gamma'_z z}, z < 0 . \quad (10.43)$$

Figure 11 shows some typical dispersion curves. Note

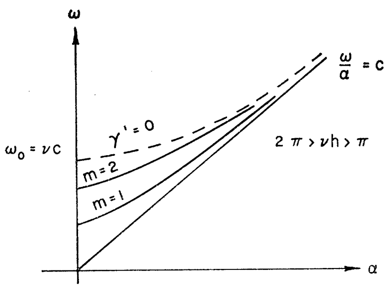

FIG. 11. Dispersion curves for guided modes in model of Fig. 10.

that under suitable conditions we may have acoustic waves trapped in a layer of higher sound velocity, providing the contrast in density gradients is sufficient [corresponding to total reflection of the type shown in Fig. 7(e)].

It is clear from Eqs. (10.43) and (10.37) that the concentration of energy is, at the most, of the order of magnitude of the concentration of specific volume in the medium. Thus, in the case of weak gradients in shallow bodies of fluid as in oceans, these effects are negligible. But in planetary atmospheres they may be important.

11. WAVEGUIDES WITH CONTINUOUSLY VARYING PARAMETERS

When the parameters ρ, λ do not vary in the simple ways considered so far, the quantitative description of wave propagation becomes more difficult. N^2 and c may be arbitrary functions of z, and the coefficient γ^2 in Eq. (4.3) is no longer a constant. The properties of such equations are well known and require no elaboration. However, only a small number of types of $\gamma^2(z)$ variation can actually be solved in terms of well-known and adequately tabulated functions. In most cases of practical interest one must use numerical procedures or approximations.

In order to visualize the behavior of the various solutions, it is often convenient to consider a continuously stratified medium as a superposition of a great many very thin layers with constant coefficients. It then becomes clear that for a given z one may consider a locally valid ω,α characteristic diagram, such as that of Fig. 5, based upon the local values of N, c. In this manner one may distinguish easily the acoustic and internal gravity wave branches: There are local values for the *Väisälä* frequency N and the acoustic resonance ω_0:

$$\omega_0^2 = c^2[\tfrac{1}{4}((d/dz)\ln\rho)^2 + \tfrac{1}{2}(d^2/dz^2)\ln\rho] . \quad (11.1)$$

It becomes clear that the energy of internal gravity waves of frequency ω can be captured by total internal reflection (i.e., turning of the rays) in a zone of maximum $N(z)$ such that $\omega < N_{\max}$. Acoustic waves, on the other hand, may be trapped by two different mechanisms of total reflection. One such mechanism has been discussed in Sec. 10 and can be translated into the present context by saying that a zone of minimum $\omega_0(z)$ may act as a waveguide independently of what $c(z)$ does, e.g., it may be constant. The other mechanism is classic and needs no discussion here: it is simply the trapping of acoustic energy in zones of minimum $c(z)$, independently of the density stratification, e.g., ω_0 may be zero. (This is typified by the SOFAR channel in the oceans.) In nature all three of these effects may coexist, although one or the other may turn out to be negligible upon closer inspection, depending upon the frequency range and model considered.

The Earth's Oceans

Eckart[1] has made a thorough study of an ocean model with a maximum N at some finite depth. He assumes that the sound velocity is constant, which is legitimate for low frequencies: compared to N, the usual "acoustical frequencies" subject to SOFAR propagation are so high that the effects of density stratification and gravity are entirely nigligible. He also assumes $N = \omega_0$. As a result of this, one may expect the surface and bottom layers of the ocean, where $N = \omega_0$ is small, to act as acoustical waveguides. This effect shows up in Eckart's solutions but, as pointed out in Sec. 10, it is of somewhat academic interest since the energy concentration will be extremely weak. This fact can be used to justify his assumption $N = \omega_0$: The acoustic trapping is so small, that it does not really matter what assumptions are made for $\omega_0(z)$. Eckart discusses at length the properties of the internal gravity wave waveguide due to N maximum and shows that reasonable results can be obtained from WKB approximations.

The Earth's Atmosphere

Recent data on the earth's atmosphere[24] suggest

[24] U. S. Air Force Geophysics Research Directorate, *Handbook of Geophysics* (The Macmillan Company, New York, 1960).

$N(z)$, $\omega_0(z)$, and $c(z)$ curves of the type shown in Fig. 12. Clearly there is here a variety of waveguide conditions. There may be two distinct channels for internal gravity waves [maxima of $N(z)$ in the stratosphere and at the top of the mesosphere], two acoustic waveguides of the ordinary type [corresponding to

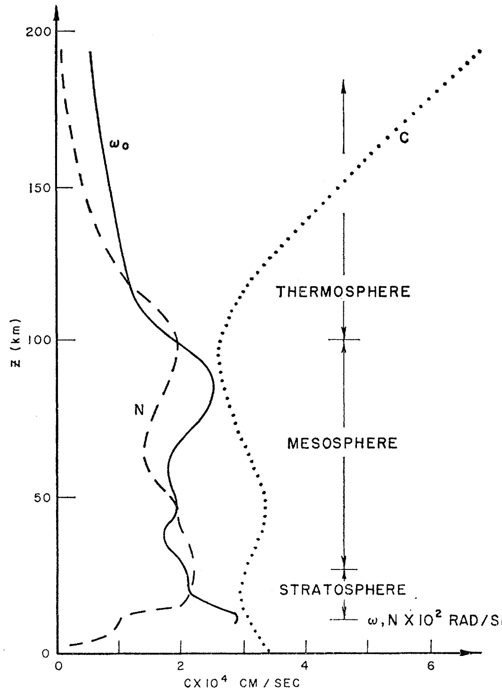

FIG. 12. Structure of the earth's atmosphere.

$c(z)$ minima in the stratosphere and upper mesosphere] and possibly several more acoustic waveguides due to minima of $\omega_0(z)$. Of the last, the most important is associated with the thermosphere itself, in the 100–200 km height region. Thus, it would require a very elaborate analysis to account for observed properties of low frequency atmospheric waves. It is probable, for instance, that Scorer's[25] and Pekeris'[26] studies fail to provide an adequate description for the *acoustic* waves generated by the great Siberian meteor because of insufficient data concerning the upper atmosphere. It seems likely that a good part of the difficulty was due to ignoring the ordinary $c(z)$, upper mesospheric waveguide and the "lid" effect of the thermosphere. But quantitative agreement will also certainly require the inclusion of correct $\omega_0(z)$ laws since the observed periods are in the 1- to 2-min range.

In addition, there should be late, very-low-frequency arrivals. Part of these would be due to internal gravity waves. However, these are probably not very efficiently excited by explosions, since, as we have seen (Sec. 6), the effect of compressibility on the waves is minor (except in determining the value of N). Other low-frequency arrivals would correspond to the acoustic branches—here the sharp increase of $c(z)$ in the thermosphere and the decrease of $\omega_0(z)$ with height create an interesting situation.

In order to illustrate the *type* of effect one may expect, we consider the model shown in Fig. 13,

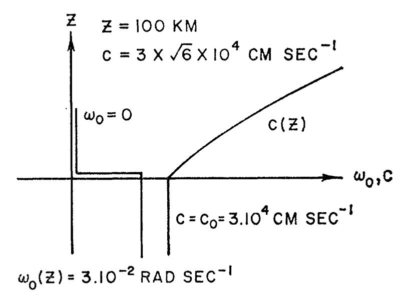

FIG. 13. An idealized model having acoustical properties qualitatively similar to those of the earth's thermosphere.

letting

$$
\begin{aligned}
c_0 &= 3 \times 10^4 \text{ cm/sec} \\
c_h &= 3 \times 6^{1/2} \times 10^4 \text{ cm/sec} \\
\omega_0 &= 0\,, \quad z > 0 \\
\omega_0 &= 3.10^{-2}, z < 0 \\
h &= 10^2 \text{ km} = 10^7 \text{ cm}\,.
\end{aligned}
\qquad (11.2)
$$

We assume, for mathematical purposes,

$$c(z) = (pz + q)^{-1/2} \qquad (11.3)$$

where, in accordance with (11.2),

$$
\begin{aligned}
p &= -9.26 \times 10^{-17} \\
q &= 1/c_0^2 = 1.111 \times 10^{-9}\,.
\end{aligned}
\qquad (11.4)
$$

We write

$$S = \int_0^{z_T} b(z) \quad dz = -\frac{2}{3\omega^2 p} b^3\,, \qquad (11.5)$$

z_T being the turning point $b(z_T) = 0$ and b is defined by Eq. (4.11). Assume[1,22] that as a wave suffers total internal reflection near a turning point it changes its phase by $\pi/4$. The waveguide interference condition, or Bohr-Sommerfeld eigenvalue equation, is

$$S + \pi/4 = \chi + m\pi \qquad (11.6)$$

[25] R. S. Scorer, Proc. Roy. Soc. (London), **A201**, 137–157 (1950).

[26] C. L. Pekeris, Proc. Roy. Soc. (London) **A171**, 434 (1939).

where χ is given by Eq. (10.34) with $\gamma_1 = b_0$:

$$\chi = \tan^{-1}(\nu + \gamma')/b_0 \tag{11.7}$$

$$\gamma' = [\nu^2 - b_0^2]^{1/2}. \tag{11.8}$$

The numerical solution of Eq. (11.6) gives the characteristic $\omega(\alpha)$ curves shown in Fig. 14.

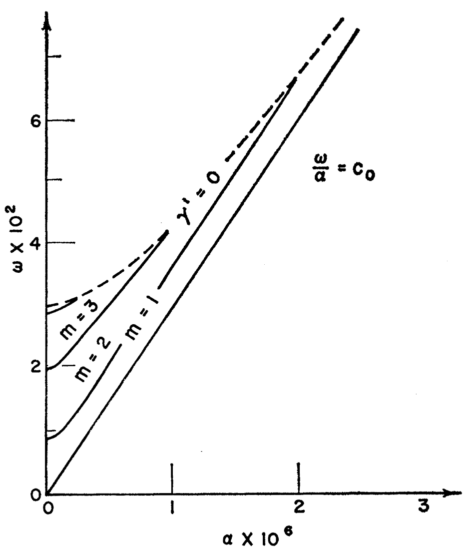

FIG. 14. Dispersion curves for guided modes (solid curves) of model in Fig. 13.

A somewhat more realistic solution would require a rigid boundary at $z = -h$, to simulate the earth's surface. But since ζ decreases exponentially as $z \to -\infty$, the effect of this boundary upon the branches shown in Fig. 14 is very slight. However, they would now be continued into the region $\gamma^2 > 0$, since total reflection is assured at $z = -h$.

The energy concentration in the waveguide of Fig. 13 is quite strong. In the upper half-space it is the usual $c(z)$ effect, with pressures falling off like $b^{-1/2}\, e^{-S}$ as $z \to \infty$. In the lower half-space, the displacement falls off as in Eq. (10.41) as $z \to -\infty$. The energy density behaves like $e^{2\gamma' z}$ [Eq. (10.43)]. The pressure, on the other hand, increases like $e^{-(\nu - \gamma')z}$. Thus, at the earth's surface the pressure perturbations may be quite large. Of course, the question of validity of these equations at high altitudes may be raised, the mean free path of the molecules being of the order of meters. However, at these very low frequencies the acoustic waves have very long wavelengths, and the displacements ζ may be orders of magnitude longer than the mean free path without violating the conditions of linearity. As long as these conditions are met, it appears justifiable to use the simple equations of this article.

12. CONCLUSIONS

We have not ventured deeply into studies of more special and realistic models of planetary atmospheres and oceans.[27–31] Such studies are based essentially upon familiar properties of the wave equation with variable coefficients, upon well-known techniques of solution, and do not shed any further light upon the basic properties of acoustic, internal, or gyroscopic type waves. Our purpose here has been only to review these basic properties in as simple and clear a manner as possible. Thus, we have tried to bring into focus the nature of the dispersion of acoustic and internal waves in density stratified media, the central role played by the acoustic resonance ω_0 and the *Väisälä* frequency N, the importance of total reflection phenomena, the twofold character (structural and geometrical) of dispersion in density stratified acoustic and internal gravity wave ducts, the presence of gyroscopic waves in rotating fluids, the fundamental difference between surface and body waves, etc.

We have found it convenient to use equations for the displacements from equilibrium to describe the propagation of small amplitude waves. We believe that their use is to be recommended since displacements are perhaps the most easily visualized physical quantities and, also, because they are so simple to derive. The reasoning used in arriving at Eqs. (3.2) is easily generalized to include finite rigidity. Indeed, our equations are but a special case of Biot's equations for waves in prestressed solids. Other formulations for the equations of motion are usually employed in the literature, derived by perturbation methods applied to the Eulerian equations of fluid motion. Eckart[1,29] has made use of a formulation giving a slight gain in mathematical conciseness which must, however, be paid for by using derived variables difficult to visualize [such as the entropy perturbation divided by $(\rho c)^{1/2}$].

We have omitted discussion of possible methods of excitation of the various modes of propagation. High altitude explosions are an obvious, man-made source of acoustic and internal gravity waves in the atmos-

[27] J.E. Fjeldstad, Geofys. Publikasjoner Norske Videnskaps-Akad. Oslo **10**, 3 (1933).
[28] P. Groen, Koninkl. Ned. Meteor. Inst. Bilt, B, **2**, No. 2 (1948).
[29] Carl Eckart, Phys. Fluids **4**, 791 (1961).
[30] F. Press and D. Harkrider, J. Geophys. Res. **67**, 3889 (1962).
[31] Yu. L. Gazaryan, Soviet Phys.—Acoust. **7**, 17 (1961).

phere at the low frequencies for which the effects discussed here become of interest. A realistic treatment of this problem remains to be done. A natural mechanism which has been definitely observed and treated theoretically is that of flow past an obstacle, e.g., in the formation of lee waves in the atmosphere[10,11] to leeward of a mountain range. This effect probably occurs also near seamounts in the ocean[6] and is analogous to all problems of wave production by moving sources, in the sense that the major phenomenon is the appearance of those particular wavelengths for which the phase velocity matches the relative velocity of medium and obstacle (Lamb's fishline problem, bow waves, Čerenkov radiation, etc.). Adequate methods exist for calculating amplitudes and details of the wavefield under these conditions.

Although the whole subject of generation of long period acoustic and internal waves is of great interest, a systematic and meaningful treatment is not feasible at present. The chief reason for this is that, in most cases, there is still no agreement as to how the wave trains observed in nature are in fact created. Thus, the reasons for the almost universal occurrence of what appear to be internal waves of tidal frequencies are not understood.[32,33] Furthermore, the shorter period internal waves may occur in conjunction with shear flow, and it is often not clear whether one is dealing with the type of motion discussed in this article or with waves of the type existing in stable shear flows. Thus, although the basic properties of waves in stratified media are understood theoretically, the problem of their identification and observation in nature is still far from resolved.

APPENDIX

To prove the equivalence of our equations of motion to the first-order (perturbation) form of Euler's equations of fluid dynamics we proceed as follows:

One needs the thermodynamic condition connecting the perturbations of pressure p, specific volume v, and entropy η:

$$\Delta p = -X\Delta v + Y\Delta\eta \,. \tag{A1}$$

For the stratified equilibrium state p_0, $v_0 = 1/\rho_0$, η_0, this gives

$$\frac{dp_0}{dz} = X_0 \frac{1}{\rho_0^2}\frac{d\rho_0}{dz} + Y_0 \frac{d\eta_0}{dz}\,. \tag{A2}$$

For small perturbations p_1, ρ_1, η_1 from this equilibrium state Eq. (A1) gives

$$p_1 = X_0(\rho_1/\rho_0^2) + Y_0\eta_1 \,, \tag{A3}$$

where[1]

$$X_0 = \rho_0^2 c_0^2 \tag{A4}$$

and, in the absence of net heat accession (no conduction, no heat sources),

$$\eta_1 = -\zeta(d\eta_0/dz)\,, \tag{A5}$$

ζ being the vertical displacement from equilibrium.

Now, the linearized equations of small motion, obtained by perturbation methods from Euler's equations are, in two dimensions,[1]

$$\rho_0\frac{\partial u}{\partial t} = -\frac{\partial p_1}{\partial x} \tag{A6a}$$

$$\rho_0\frac{\partial w}{\partial t} + g\rho_1 = -\frac{\partial p_1}{\partial z} \tag{A6b}$$

$$\frac{\partial \rho_1}{\partial t} + w\frac{d\rho_0}{dz} + \rho_0\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = 0\,, \tag{A6c}$$

where u, w are the velocities $\dot{\xi}$, $\dot{\zeta}$ in the x, z directions.

Equations (A3)–(A5) give

$$\frac{\partial p_1}{\partial t} = c^2\frac{\partial \rho_1}{\partial t} - Y_0\frac{d\eta_0}{dz}w\,. \tag{A7}$$

Substituting this into the equation of continuity (A6c), and using Eq. (A2),

$$\frac{\partial p_1}{\partial t} = -w\frac{dp_0}{dz} - c^2\rho_0\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right). \tag{A8}$$

But in the equilibrium state

$$dp_0/dz = -\rho_0 g \tag{A9}$$

and thus

$$\partial p_1/\partial t = w\rho_0 g - \rho_0 c^2(\partial u/\partial x + \partial w/\partial z)\,. \tag{A10}$$

In conjunction with (A6a) and (A6b) this gives

$$\rho_0\frac{\partial^2 u}{\partial t_2} = -\rho_0 g\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\left[\rho_0 c^2\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)\right]$$

$$\rho_0\frac{\partial^2 w}{\partial t^2} = \rho_0 g\frac{\partial u}{\partial x} + \frac{\partial}{\partial z}\left[\rho_0 c^2\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)\right]. \tag{A11}$$

But these are simply Eqs. (4.5) differentiated with respect to time. Since we are dealing with perturbations about an equilibrium state, Eqs. (4.5) do not admit of time independent solutions, and Eqs. (A11) and (4.5) are completely equivalent.

[32] Albert Defant, J. Marine Research **9**, 111 (1950).
[33] Maurice Rattray, Jr., Tellus **12**, 54 (1960).