The Measuring Process in Quantum Theory

J. M. BURGERS
Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland

1. INTRODUCTION

IT has been proved by von Neumann that the outcome of a measuring process is independent of the way in which one describes the effect of the measuring apparatus, either by applying the formalism for observations considered as an irreversible perturbation of the wave function of the system, or by introducing a Schrödinger equation valid for the observed system plus the measuring apparatus.[1] The subject has been pursued and was presented in a transparent form by Everett, in what he called the *relative state* formulation of quantum mechanics.[2] The essence of what was brought forward by Everett, however, was already contained in a paper by Groenewold.[3] Since interest in von Neumann's deductions is still alive,[4] it may be worthwhile to give a unified presentation of the ideas of the three authors mentioned which may bring out more clearly a few of the assumptions contained in the argument.

We consider a collection of systems which during a certain interval of time come into interaction with other systems of such nature that the latter can produce a record of certain features connected with the interaction. For convenience we call the first type of systems *particles* and assume that they are all of the same nature. The other type of systems are called measuring apparatus. It is supposed that the records produced by them remain more or less intact during the future history, so that the records can be read and interpreted by a human observer at some later time.

The following history is analyzed. A collection of particles is first subjected to a preliminary observation (0) that throws each particle into one of a series of eigenstates ϕ_l of the operator connected with the observation. It is then supposed that we are able to select those particles which have been thrown into one definite eigenstate ϕ_k, whereas all other particles are rejected. Thus the preliminary observation serves to prepare a collection of particles all having the same initial state. Let this be accomplished at the instant t_0 and write $(\phi_k)_0$ for the eigenfunction at this instant.

The particles are then left to themselves during the interval $t_0 < t < t_1$. During the next interval $t_1 < t < t_2$ they are subjected to the interaction with the measuring apparatus. After this, during the interval $t_2 < t < t_3$, they are left to themselves again. In order to obtain information concerning the state of the particles at the instant t_3, we subject them to a second measuring process. We follow two ways to calculate the expected outcome of this second measuring process; first we consider the interaction that takes place in the interval $t_1 < t < t_2$ as a perturbation; second, we analyze this interaction with the aid of the formulas for a continuous process. Our purpose is to find out whether the two descriptions give the same result for the expected outcome of the second measurement, and which conditions must be satisfied in order that the same result is obtained.

The two periods in which the particles are left to themselves have been introduced for completeness, in order to show that what happens during these periods has no influence upon the result. They could be left out for simplification.

2. FIRST DESCRIPTION

At t_0 the wave function for the particles has the form $(\phi_k)_0$. During the interval $t_0 < t < t_1$ the state of the particles may change due to processes taking place either within the particles or to interaction with certain fields of force, not forming part of the apparatus. This may bring about a unitary transformation S_I, which can be derived from the Hamiltonian governing the change, and at the instant t_1 the particles will be in a state $(\phi_k)_1$ given by

$$(\phi_k)_1 = S_I(\phi_k)_0 \,. \tag{1}$$

During the interval $t_1 < t < t_2$ the particles are subjected to a measuring process, which measures the observable A, with eigenvalues α_i. The apparatus records the eigenvalues obtained and the record will show which percentages of the particles have been thrown into each of the eigenstates $(\psi_i)_2$ correspond-

[1] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Verlag Julius Springer, Berlin 1932), Chap. VI, pp. 222–237. [English translation by R. T. Beyer, under the title *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, New Jersey, 1955), Chap. VI, pp. 417–445 ff.

[2] H. Everett, III, Rev. Mod. Phys. **29**, 454 (1957).

[3] H. J. Groenewold, Koninkl. Ned. Akad. Wetenschappen, Proc. **B55**, 219 (1952).

[4] See, for instance, A. Komar, Phys. Rev. **126**, 365 (1962).

ing to the eigenvalues. The subscript 2 has been added to indicate that these eigenstates are obtained at the instant t_2.

To find the percentages we need the projections of the eigenstate $(\phi_k)_1$ upon the new eigenstates $(\psi_i)_2$, as given by

$$a_{ki} = \int d\mathbf{x}(\psi_i)_2^*(\phi_k)_1 \,, \tag{2}$$

the asterisk denoting the conjugate value, while the integration is extended over all values of the coordinates $\mathbf{x}$ entering into the description of the particles. For convenience we suppose that a maximum observation is carried out; the eigenvalues α_i will stand each for a set of values. We have

$$(\phi_k)_1 = \sum\nolimits_i a_{ki}(\psi_i)_2 \,. \tag{3}$$

The statistical frequences or percentages are given by

$$a_{ki}a_{ki}^* \,, \quad \text{with} \quad \sum\nolimits_i a_{ki}a_{ki}^* = 1 \,. \tag{4}$$

The collection of particles, which formerly could be considered as a homogeneous collection in which all particles belonged to the same eigenstate ϕ_k, has now become a set of sub-collections, each characterized by an eigenstate ψ_i. We suppose that during the interval $t_2 < t < t_3$ the particles are left to themselves. Their wave functions change and we write

$$(\psi_i)_3 = S_{\mathrm{II}}(\psi_i)_2 \,, \tag{5}$$

S_{II} being an operator depending upon a Hamiltonian valid for the particles in this period, which operator may act differently upon each of the eigenstates.

As mentioned in the introduction, the particles are subjected to a second measuring process at the instant t_3, since this is the only means for obtaining information concerning their states. It is convenient to assume that the second observation measures an observable B not commuting with A. The new observation then throws the particles into eigenstates χ_j, corresponding to this observable, with eigenvalues β_j.

The statistical frequency of the eigenvalue β_j, for the subcollection of particles in the state $(\psi_i)_3$, is determined by the projection of $(\psi_i)_3$ upon χ_j, as given by

$$b_{ij} = \int \mathbf{dx}\; \chi_j^*(\psi_i)_3 \,. \tag{6}$$

The statistical frequency is then given by

$$b_{ij}b_{ij}^* \,, \quad \text{with} \quad \sum\nolimits_j b_{ij}b_{ij}^* = 1 \,.$$

When we add together all subcollections and ask for the total statistical frequencies of the results β_j recorded in the second measurement, we obtain

$$P(\beta_j) = \sum\nolimits_i a_{ki}a_{ki}^* b_{ij}b_{ij}^* \,. \tag{7}$$

3. SECOND DESCRIPTION

The interaction of the particles with the apparatus during the interval $t_1 < t < t_2$ will not be considered as an observation, but as a part of the development in time of the combined system formed by a particle together with the apparatus. We need a wave function Φ for apparatus and particle combined, while it is necessary also to consider the wave function Ψ for the apparatus alone, in the periods when there is no interaction with the particles. The wave functions for the apparatus depend upon a set of coordinates $\boldsymbol{\xi}$ describing the apparatus, which coordinates are different from the $\mathbf{x}$, describing the particles.

Until the instant t_1, particles and apparatus have been independent. At this instant the wave function for particles and apparatus combined will be the product of $(\phi_k)_1$ and Ψ_1:

$$\Phi(\mathbf{x},\boldsymbol{\xi})_1 = (\phi_k)_1\Psi_1 \,. \tag{8}$$

During the interval $t_1 < t < t_2$ the function Φ is subjected to a Schrödinger equation with a Hamiltonian H_A, resulting in a certain transformation of Φ, which we write as

$$\Phi_2 = T_A\Phi_1 = T_A\{(\phi_k)_1\Psi_1\} \,. \tag{9}$$

We must next follow the history of the combined system during the interval $t_2 < t < t_3$, which elapses before the second measurement is made. For simplicity we assume that the apparatus does not change during this interval, so that the coordinates $\boldsymbol{\xi}$ retain the values they had at t_2. Hence, there is only the transformation expressed by S_{II}, acting upon the particle functions, and at the instant t_3 we have

$$\Phi_3 = S_{II}T_A\{(\varphi_k)_1\Psi_1\} = S_{II}T_A[\sum\nolimits_i a_{ki}(\psi_i)_2\Psi_1] \,, \tag{10}$$

if we make use of (3).

We must now specify the nature of the Hamiltonian H_A and the transformation T_A. Those terms in H_A that operate on the coordinates $\mathbf{x}$ of the particles must have the effect considered before: They produce the eigenstates $(\psi_i)_2$ and record eigenvalues α_i. When a particle was already in an eigenstate $(\psi_i)_2$, it will remain in that state, according to the principle holding for immediate repetition of the same observation.[5] This makes it necessary that the coordinates $\mathbf{x}$

[5] Compare reference 2, p. 458 (first column); and reference 3, p. 220, in connection with Eq. (2.07).

occur in H_A only in the form of the operator A corresponding to the observable which is measured. For convenience, we assume that the operator T_A can be represented as a power series with respect to A, of the form

$$T_A = \sum_n \Gamma_n A^N . \tag{11}$$

where the functions Γ_n depend exclusively upon the coordinates ξ of the apparatus. Since

$$A^n(\psi_i)_2 = \alpha_i^n(\psi_i)_2 , \tag{12}$$

we then find

$$\Phi_3 = S_{II} \sum_n \Gamma_n \sum_i a_{ki}\alpha_i^n(\psi_i)_2\Psi_1 .$$

This separates the variables ξ from the $\mathbf{x}$. We write[6]

$$(\Psi_i)_2 = \sum_n \Gamma_n\alpha_i^n\Psi_1 , \tag{13}$$

and at the same time make use of (5). This leads to the result

$$\Phi_3 = \sum_i a_{ki}(\psi_i)_3(\Psi_i)_2 . \tag{14}$$

When we now refer to (6) we can transform this into

$$\Phi_3 = \sum_{ij} a_{ki}b_{ij}\chi_j(\Psi_i)_2 = \sum_j \{\sum_i a_{ki}b_{ij}(\Psi_i)_2\}\chi_j . \tag{15}$$

The form of the last expression suggests that the coefficients of the χ_j determine the statistical frequencies with which the eigenvalues β_j are obtained in the second measuring process, carried out at the instant t_3 to obtain information about the state of the particles. These coefficients, however, contain the coordinates ξ of the apparatus. When we do not wish to make use of the record of this apparatus, but consider it as a feature of the external field to which the particles have been subjected, we must get rid of these coordinates by integrating over them. Thus we expect that the statistical frequency of a result β_j in the second measuring process is given by the integral

$$\int d\xi \{\sum_i a_{ki}b_{ij}(\Psi_i)_2\}\{\sum_l a_{kl}^*b_{lj}^*(\Psi_l^*)_2\} . \tag{16}$$

We now introduce the important condition that, when the apparatus which measures the observable A is a good one, its indications must be such that the various possible results are clearly distinguished. This requires that the various states $(\Psi_i)_2$ be orthonormal to each other, so that

$$\int d\xi(\Psi_i)_2(\Psi_l^*)_2 = \delta_{il} . \tag{17}$$

This condition is mentioned without further comment by Groenewold.[7] When it holds, formula (16) reduces to

$$\sum_i a_{ki}a_{ki}^*b_{ij}b_{ij}^* , \tag{18}$$

which is identical with (7).

Von Neumann made use of an example, which is also used by Everett,[8] in which it is supposed that, for a simple case, the operator H_A has the form

$$H_A = (h/2\pi i)A(\partial/\partial\xi) .$$

When during the interval $t_1 < t < t_2$ its exponential acts upon one of the terms

$$a_{ki}(\psi_i)_2\Psi_1$$

occurring at the end of (10), we obtain

$$a_{ki}(\psi_i)_2\Psi(\xi - \alpha_i t_A) ,$$

where $t_A = t_2 - t_1$. For sharp measuring properties von Neumann requires that $\Psi(\xi)$ has the nature of a Dirac function

$$\Psi(\xi) = \delta(\xi) .$$

The required orthonormality for different α_i then follows immediately.

4. ADDITIONAL REMARKS

The identity of the results (18) and (7) is obtained as a consequence of the integration with respect to $d\xi$ applied in (16), combined with the orthonormality condition (17). We can express this somewhat differently as follows.

The outcome of the second measuring process depends upon the projection of the wave function Φ_3, as obtained in (14), upon the eigenfunctions connected with the second observation. The eigenfunctions χ_j, however, refer only to the coordinates $\mathbf{x}$ of the particle. Let us therefore extend the second measuring apparatus with an imaginary part refering to the coordinates ξ and independent of the $\mathbf{x}$. This additional part should project the apparatus functions $(\Psi_i)_2$ upon a set of eigenfunctions ϑ_m, corresponding to some feature of the apparatus that can be observed. We then obtain projection coefficients

$$c_{jm} = \int d\mathbf{x} \int d\xi\ \chi_j^*\vartheta_m^*\Phi_3$$
$$= \int d\mathbf{x} \int d\xi \sum_i a_{ki}\chi_j^*(\psi_i)_3\vartheta_m^*(\Psi_i)_2 = \sum_i a_{ki}b_{ij}f_{im} , \tag{19}$$

[6] Equation (13) is similar to Groenewold's Eq. (2.09) (see reference 3).

[7] Reference 3, p. 220, Eq. (2.08).

[8] Reference 1, German edition, p. 236; translation p. 443; Everett quotes this example on p. 456, second column, Eq. (4) (see reference 2).

where

$$f_{im} = \int d\xi \, \vartheta_m^*(\Psi_i)_2 , \quad \text{with} \quad \sum_m f_{im} f_{im}^* = 1 . \quad (20)$$

This enables us to write

$$\Phi_3 = \sum_{ijm} a_{ki} b_{ij} f_{im} \chi_j \vartheta_m . \quad (21)$$

The statistical frequency of the result β_j, admitting all possible eigenstates ϑ_m for the apparatus, is then given by

$$\sum_{im} a_{ki} a_{ki}^* b_{ij} b_{ij}^* f_{im} f_{im}^* = \sum_i a_{ki} a_{ki}^* b_{ij} b_{ij}^* , \quad (22)$$

as before. The assumption that the measuring apparatus is a good one, apparently is characterized already fully by (11), and (17) must be seen as a necessary conclusion.

Von Neumann's proof has again been discussed in a recent paper by Komar.[4] Komar's interest is directed to the question whether von Neumann's proof definitely shows that the use of hidden variables never can account for the dispersion in the results of experiments, or whether certain features of the proof need reformulation; in this connection an alternative proof is presented. Since Komar's notation is greatly different from the one used in the present paper, it may be useful to mention that his Eq. (8) corresponds to our (8), and his Eq. (9) is our (9). Further, his Eq. (10) has some analogy with our (21), if we consider his n and k as corresponding to our i and m, respectively. Our j has no analogy in Komar's notation, as he does not explicitly introduce what we called the *second measurement*. Although a detailed comparison is difficult, it may be of interest to make the following observations.

We write Eq. (21) back in terms of the $(\psi_i)_3$ instead of the χ_j [making use of Eq. (6)], so that no explicit reference is made to the second measurement. At the same time we express $(\psi_i)_3$ in terms of $(\psi_i)_2$ with the aid of (5). This gives

$$\Phi_3 = S_{\text{II}} \sum_{im} a_{ki} f_{im} (\psi_i)_2 \vartheta_m .$$

When the coefficients a_{ki} and f_{im} are expressed through the integrals (2) and (20), respectively, this becomes

$$\Phi_3 = S_{II} \sum_{im} \psi_i \vartheta_m \iint d\mathbf{x} d\xi \, \psi_i^* \vartheta_m^* \phi_k \Psi_i \quad (23)$$

(for convenience we omit the subscripts 2). It is this equation, rather than (21), which can be compared with Komar's Eq. (10). The double integral

$$\iint d\mathbf{x} d\xi \, \psi_i^* \vartheta_m^* \phi_k \Psi_i \quad (24)$$

implicitly refers to some assumed initial state of the measuring apparatus [entering into the Ψ_i through their relation to Ψ_1, as indicated in Eq. (13)]; this initial state is specified by Komar by a label j.

If we now look at Komar's postulates (a), (b), and (c),[9] we see that postulate (a) corresponds to our introduction of the function Ψ_1 in (8). Postulate (c) is also a fundamental point in our reasoning, leading to our Eqs. (11) and (12).

Komar's postulate (b), requiring that the measuring apparatus should throw the particle into an eigenstate which is a unique function of ϕ_k and Ψ_1, has not been introduced into our description. It would assert that, for a given initial state j of the apparatus and a given initial state ϕ_k (in our notation) of the particle, there should be a unique i (Komar's n), for which the double integral (24) will be different from zero, at least for some m (Komar's k). This, however, is in contradiction with the assumption that all our functions belong to a linear vector space, since according to this assumption we can always write

$$\phi_k = \sum_h \alpha_h \psi_h .$$

The double integral then reduces to

$$\alpha_i \int d\xi \, \vartheta_m^* \Psi_i ,$$

and for any choice of i there will always be values of m for which this integral is not zero. Hence we cannot say that for any i there is only a unique k, or inversely a unique i for any k. We have therefore accepted that the interaction with the apparatus always leads to the appearance of many eigenstates, with statistical frequencies $b_{ij} \, b_{ij}^*$.

5. PHILOSOPHICAL EXCURSION

Komar concluded his article with some remarks of a philosophical nature, which go beyond the realm of pure physical theory. He mentions that there is a possibility of being led to the consideration of teleological forces in nature. Since this is a point of far reaching consequences, which is also open to various interpretations, it may perhaps be permitted to state that my preference is to accept the point of view developed in Whitehead's philosophical works,[10] in

[9] Reference 4, p. 367, second column.

[10] A. N. Whitehead, *Science and the Modern World* (The Macmillan Company, New York, 1925); *Process and Reality* (Cambridge University Press, London, 1929); *Adventures of Ideas* (Cambridge University Press, London, 1933). (Paperback editions have appeared of *Science and the Modern World* and *Adventures of Ideas*.) In the presentation of the ideas as given in the text, I have introduced a certain formalism not found in Whitehead's books, in order to bring out in shaper light what I consider to be the main assumptions.

which the idea of a stepwise functioning of the universe is introduced as a basic assumption. Whitehead has applied this postulate in order to present a new view concerning the notion of causality. In this view it is assumed that each subsequent step, while arising from the results of past steps, is not fully determined by them: It is supposed that the way in which the new step accepts the experience of the past is affected by an emphasis on certain forms of relationship, selected in view of future possibilities. The influences from the past provide the basis for the ultimate appearance of such relations as are studied in physics and its related sciences; the assumption of a selective or discriminating activity is a step beyond physics.

The reason for the introduction of this metaphysical principle is twofold. In the first place, physics itself is no more than the formulation of chains of relationship or rules of succession between events, which always necessitate empirical specifications of the situation or the objects to which one or other of these rules can be applied. For instance, there are electric and magnetic fields, which have different influences on the motion of charged particles, but there is no explanation of what a charge is, nor what leads to the distinction between electric and magnetic (or other) fields. To make a full description of the universe, a metaphysical concept would be needed which should provide a ground for these distinctions. Without going into any detail, Whitehead assumes that there exists an abstract principle of discrimination, a conceptual activity (which cannot be defined in further terms), from which the distinctions can follow. This brings into the picture the acceptance of something related to mind, which in this system must be a fundamental feature of the universe, and not something accidental, or something merely additional to the physical relations. With this point of view we reach the second reason for Whitehead's introduction of the postulate, viz., to obtain a picture of the universe which is wide enough to include both the physical phenomena and the phenomena of life. I share Whitehead's conviction that the present philosophy of physics is too narrow for a discussion of life without submitting it to far-reaching distortion, and with Whitehead, I prefer to believe that initiative and creativity should rank as fundamental features in a philosophical picture.

To make use of the possibilities for explanation thus obtained, one can start by making a construction in more or less abstract terms, which construction must obtain its meaning through the relations introduced between these terms, as well as through a comparison with the relations that we find around us, that is, on the one hand the results of physics and its related sciences; and on the other hand, the subjective phenomena which we experience within ourselves, in the belief that they derive from general effects functioning in all forms of life. When this system of relations is condensed to its most important items, we arrive at the following statements:

The ultimate reality of the universe is a multiple and never ending complex of steps, for convenience to be called *processes*, developing out of each other.

Each process is a mode of functioning, which arises out of the experience of existing facts, and which, by its completion, passes into a new fact. Thus, facts are the results of processes, and facts are material for experience out of which new processes arise.

Each process is an instance of conceptual activity, which means, it is the experience of facts with an emphasis on certain forms of relatedness recognized in the experience.

There are infinitely many forms of relatedness, which embrace, e.g., forms of physical relationship, quantitative relations, esthetic relations, ethic relations, the latter two to be considered not as accidental byproducts of the former ones, but deriving from fundamental aspects of the universe (to be known, of course, only from what we can deduce from experience, with all its uncertainties). Each fact is experienced together with forms of relatedness, and forms of relatedness can be known through the part they play in conceptual activity.

The recognition of definite forms of relatedness in experience, in preference to other possible forms, is the essential feature of conceptual activity. Thus in each instance of conceptual activity certain forms of relatedness are emphasized or preferred, while others are rejected or driven to a background of unimportance.

Through their entrance in conceptual activity, the preferred forms of relatedness represent potentialities for connections between future facts. Hence some recognition of future possibilities is an essential feature of conceptual activity. This is the teleological element accepted in the picture, which is the basis for the recognition of values. It is not to be construed as directed to definite "final" ends: its "vision," to use this term, is limited only to a near future.

New facts lead to new instances of conceptual activity. Thus conceptual activity is rhythmic (is "quantized"). It will never end so long as it is impossible to give full expression to all forms of relatedness in the outcome of a single process.

Each instance of conceptual activity (each process) has a certain freedom in giving emphasis to particular forms of relatedness. This assumption is the expression of a belief in *creativity* as a fundamental feature of the universe.

The freedom of selection, however, is limited by the assumption that there is also a certain persistence of the emphasis and of the selection which occurred in the immediately preceding instance of conceptual activity. This persistence is the germ of *tradition* in emphasis. It makes possible the transmission of certain forms of relatedness through complexes of processes, and it leads to the appearance of societies of processes. Societies of processes bring increased strength of persistence in the selected forms of relatedness through mutual reinforcement, and thus permit great intensity of these forms.

Societies in this way can lead to the establishment of definite lines of tradition, in which forms of relatedness are transmitted from step to step with little or no alteration. This is particularly the case in societies in which conceptual activity has become dormant. This means that tradition has grown so strong that the awareness of possibilities for the selection of new forms of relatedness has dwindled to practically nothing. Consequently there is no awareness of anything which would mean freedom, and traditions develop into more and more rigid laws in these societies. These traditions are the basis of the physical laws—it is here that *physics* can come into the scheme, as the analysis of persistent traditions in the relations between processes. The picture thus does not give an absolute character to the physical laws (nor to what we call *physical constants*); as traditions, although extending over immense periods, they are still subjected to the possibility of changes leading to new traditions.

The societies which are the carriers of physical laws, constitute *matter*; the term is used here so as to include both matter in the ordinary sense and fields, and in general anything the comportment of which is governed by definite laws.

We know that the physical laws are of a statistical character; this indicates that in each transmission of forms of relatedness by mere tradition certain features are lost. As there is practically no conceptual activity in the societies considered here, which could provide the possibility of making decisions with regard to the lost forms of relatedness, there is randomness with respect to them in the outcome of the processes occurring under these circumstances. In other words, for certain forms of relationship all possible results have the same weight, if a method is introduced for the distinction between different results, such as is specified in quantum theory, which itself should be an outcome of the relations between processes.

There are, however, also other complexes of processes, for which we must assume that conceptual activity remains effective, so that, at least on certain occasions, it can exert a more or less decisive influence directed towards certain preferred forms of relatedness. We can say that conceptual activity then gives evidence of a certain purposiveness. The assumption of this possibility leads to a description of *life*, in which the essential aspect of life is not sought in the phenomena of self-reproduction, but in the possibilities of choosing particular, and sometimes new forms of relatedness. A consequence of this is that systems of this nature to a certain extent can combat the loss of definiteness which appears in societies with dormant conceptual activity.

The development of the picture, of which this sketch gives no more than the starting points, towards the physical as well as towards the biological side, needs a discussion of various further relations and additional assumptions.[11]

---

[11] I hope to present elsewhere an account of those aspects which I consider to be most important for establishing a connection between these philosophical ideas and the conceptions of physics and biology, although I am aware that there are many problems which require extensive investigation, even if the main lines of the picture should be found to be attractive.