dict a region of deformation where the proton number goes from 50 to 82 and the neutron number from 126 to 184 (i.e., in the highly neutron-excess rare earths). However, at present the possibility of the experimental observation of deformation in these nuclei seems remote.

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The Two-Nucleon Interaction

B. P. NIGAM
Department of Physics, University of Buffalo, Buffalo, New York

1. INTRODUCTION

In recent years a considerable amount of progress has been achieved in the study of the two-nucleon interaction. The experimental investigation of polarization and the triple scattering parameters of *p-p* scattering at high energies has made it possible to look into several features of the two-nucleon interaction which did not exhibit themselves in data on cross section at lower energies. The first set of triple scattering experiments was performed by Chamberlain, Segrè, Tripp, Wiegand, and Ypsilantis[1] who measured five scattering parameters; cross section σ, polarization P, depolarization D, rotation parameters R, and A for *p-p* scattering at 310 MeV. Similar experiments have since been performed at 150 MeV,[2,3] 210 MeV[4] and at other energies. The theoretical investigation of the experimental data has followed two main procedures, viz., (i) phase shift analysis of the experimental data, and (ii) phenomenological potential models. In the first approach one obtains several sets of phase-shift solutions which fit the experimental data and then the problem is that of discriminating among the various sets of solutions on the basis of other experimental evidence. Stapp, Ypsilantis, and Metropolis[5] have carried out the phase shift analysis of the experiment of Chamberlain *et al.*, obtaining several phase-shift solutions out of which finally only four were acceptable. A later modified analysis of the *p-p* scattering data at 310 MeV was carried out by Cziffra, MacGregor, Moravcsik, and Stapp[6] in which the partial waves G and higher were calculated from the one-pion-exchange pole in the scattering amplitude. This procedure left only two distinguishable solutions. In the second approach followed by Signell and Marshak,[7] Gammel and Thaler,[8] Otsuki,[9] Watari,[10] and Tamagaki,[11] one starts with a phenomenological potential and calculates a set of phase shifts which finally enable one to calculate the experimental scattering parameters. Both Signell and Marshak (SM)[7] and Gammel and Thaler (GT)[8] found it necessary to add to the central and tensor potentials a strong spin-

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orbit short-range potential in order to explain the polarization and the triple scattering data at 150 and 310 MeV. The introduction of the spin-orbit potential was earlier criticized by the Japanese group[9–11,12] but now its presence is quite well established.[13] The more recent phenomenological potentials are due to Bryan,[14] Saylor, Bryan, and Marshak,[15] and Hamada[16] and these differ from the SM and GT potentials in explicitly recognizing that the two-nucleon interaction in the asysmptotic region is given by the one-pion-exchange contribution, which can be determined unambiguously.

The attempts to obtain the spin-orbit interaction of the requisite strength from meson theory have not been too successful. The two-pion-exchange interaction predicts a spin-orbit term,[17–20] but the strength is too small. Besides, the various authors differ in the results because of the ambiguities involved in the calculations and the treatment of the effects of nucleon recoil. However, the approach of dispersion theory[21] (Mandelstam representation[22]) is expected to be more promising. In this the covariant S matrix for the scattering process is obtained from the general requirements of field theory, viz., analyticity and unitarity, and the mass spectrum of the strongly interacting particles.

The recent review articles on the nucleon–nucleon system have been written by Phillips,[23] Gammel and Thaler,[24] Marshak,[25] MacGregor, Moravcsik, and Stapp,[26] Moravcsik and Noyes.[27] The present article is shorter in size and does not go into specific details of the problems involved. It is mainly intended to provide a broad background of the various approaches used in the study of nucleon–nucleon interaction.

2. PHASE SHIFT ANALYSIS

The wave function which describes the scattering of a two nucleon system can asymptotically be written as follows[5,28]:

$$\psi \simeq e^{i\mathbf{k}\cdot\mathbf{r}}\chi_s^{m_s} + \frac{e^{ikr}}{r}\sum_{m'_s} M^{m_s m'_s}(\theta,\varphi)\chi_s^{m'_s}\,, \qquad (1)$$

where $\chi_s^{m_s}$ is the singlet ($s = 0, m_s = 0$) and the triplet ($s = 1, m_s = 0, \pm 1$) spin states of the two nucleon system and $M^{m_s m'_s}(\theta,\varphi)$ is the matrix, in spin space, describing the scattering of an incident spin state $\chi_s^{m'_s}$ into the final spin state $\chi_s^{m_s}$; the total spin s being conserved for interactions conserving parity. $\mathbf{k}$, (θ,φ) are the momentum and the scattering angles in the center-of-mass system. The asymptotic expressions ($r \to \infty$) for the incoming and the scattered waves, in the $lsm_lm_s(=\lambda)$ representation, are given by

$$\psi^{inc}_{s,m'_s}(\theta',\varphi',r) \simeq -\frac{1}{r}\sum_{l',m'_l} e^{-i(kr-\pi l'/2)} g(\lambda')\, Y_{l'}^{m_l'}(\theta',\varphi')\chi_s^{m'_s} \qquad (2)$$

$$\psi^{sc}_{s,m_s}(\theta,\varphi,r) \simeq \frac{1}{r}\sum_{l,m_l} e^{i(kr-\pi l/2)} f(\lambda)\, Y_l^{m_l}(\theta,\varphi)\chi_s^{m_s} = \frac{e^{ikr}}{r}\sum_{m'_s} M^{m_s m'_s}(\theta,\varphi)\chi_s^{m'_s}\,, \qquad (3)$$

where $Y_l^{m_l}(\theta,\varphi)$ is the spherical harmonic for angular momentum l with z component m_l, s the spin with the z component m_s, and the amplitude g as given by a plane-wave expansion of a wave proceeding along the direction $\mathbf{k}/|\mathbf{k}| = \hat{k}$ is

$$g = (4\pi/2i_k)e^{il\pi/2}Y_{l'}^{m_l'*}(\hat{k})\,.$$

The amplitudes f and g are connected by the R matrix ($R = S - I$) by the relation

$$f(\lambda) = \textstyle\sum_{\lambda'} \langle\lambda|R|\lambda'\rangle g(\lambda')\,. \qquad (4)$$

Substituting Eq. (4) in Eq. (3), we obtain

$$\textstyle\sum_{m'_s} \chi_s^{m^*_s} M^{m_s m'_s}(\theta,\varphi)\chi_s^{m'_s} = \sum Y_l^{m_l}(\theta,\varphi) \times M(lsm_lm_s;sm'_s) \qquad (5)$$

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where

$$M(lsm_lm_s;sm_s') = \frac{4\pi}{2ik}e^{-il\pi/2} \times \sum_{l'm_{l'}} \langle\lambda|R|\lambda'\rangle e^{i\pi l'/2} Y_{l'}^{m_{l'}*}(\hat{k}') . \quad (6)$$

We can choose the z axis along the direction of the incident beam so that $m_{l'} = 0$, and $\hat{k}' = (\theta = 0,\varphi = 0)$. Since the total angular momentum j, its z component m_j, and the spin S are conserved, the R matrix is diagonal in these quantum numbers and it is therefore convenient to express the R matrix in the $lsjm_j$ representation using the relation

$$|lsm_lm_s\rangle = \sum_{j,m_j} \langle lsm_lm_s|jm_j\rangle|lsjm_j\rangle \quad (7)$$

where $\langle lsm_lm_s|jm_j\rangle$ are the usual Clebsch-Gordon coefficients. The matrix elements of R in the $lsjm_j$ representation are then directly related to the phase shifts through the S matrix. Denoting the matrix elements of the R matrix by α_l for scattering in the spin singlet state, α_{lj} and α^j for the diagonal and the nondiagonal (in l) scattering in the spin triplet state, i.e.,

$$\langle l0lm_j|R|l0lm_j\rangle = \alpha_l$$
$$\langle l1jm_j|R|l1jm_j\rangle = \alpha_{l_j}$$
$$\langle j \pm 1,1,j,m_j|R|j \mp 1,1,j,m_j\rangle = \alpha^j \quad (8)$$

where the α's can be expressed in terms of the phase shifts δ's and the coupling parameters ϵ_j, which couple the states of angular momenta $j-1$ and $j+1$ for a given j, as follows

$$\alpha_l = e^{2i\delta_l} - 1$$
$$\alpha_{l_j} = e^{2i\delta_l} - 1 , \quad \text{for} \quad l = j$$
$$\alpha_{j\pm1,j} = \cos^2 \epsilon_j e^{2i\delta_{j\pm1,j}} + \sin^2 \epsilon_j e^{2i\delta_{j1\mp,j}} - 1$$
$$\alpha^j = \tfrac{1}{2} \sin 2\epsilon_j (e^{2i\delta_{j\mp1,j}} - e^{2i\delta_{j\pm1,j}}) \quad (8a)$$

If we want to include the effects of Coulomb interaction for p-p scattering in a nonrelativistic manner, then in the asymptotic expressions for ψ^{inc} and ψ^{sc} we should replace kr by $kr - \eta \ln 2kr$ where $\eta = e^2/\hbar v$, v being the velocity in the laboratory system. Further $R = S - 1 \equiv (S - S_c) + (S_c - 1)$, where S_c is the Coulomb scattering matrix. In the part $S - S_c$, S_c is expressed in terms of Coulomb phase shifts

$$\varphi_l = \sum_{n=1}^{l} \arctan \frac{\eta}{n}$$

and thereby the unity on the r.h.s. of the equations for α's, Eq. (8a), is replaced by $e^{2i\varphi_l}$. The part $S_c - 1$ gives rise to the Coulomb scattering amplitude

$$f_c(\theta) = -[\eta/k(1 - \cos\theta)] \times \{\exp[-i\eta \ln \tfrac{1}{2}(1 - \cos\theta)]\} . \quad (8b)$$

Substituting Eqs. (8) and (7) in Eq. (5), one obtains

$$M^{m_sm'_s}(\theta,\varphi) = f_c^{m_sm'_s}(\theta) + \textstyle\sum_j [\{a_{j-1j}^{m_sm'_s}(\theta,\varphi)\alpha_{j-1j} + a_{jj}^{m_sm'_s}\alpha_{jj} + a_{j+1j}^{m_sm'_s}\alpha_{j+1j}\} + a^{jm_sm'_s}\alpha^j] + \textstyle\sum_l a_l^{m_sm'_s}\alpha_l , \quad (9)$$

where $f_c^{ss}(\theta) = f_c(\theta) + f_c(\pi - \theta)$, $f_c^{11} = f_c^{00} = f_c(\theta) - f_c(\pi - \theta)$, all others equal to zero, and because of Pauli's exclusion principle the first summation over j goes over values corresponding to odd (even) l and the second summation over l is over even (odd) values for the nucleons in the isotopic spin state $T = 1(0)$. The coefficients $a^{m_sm'_s}$ are given in Table III of Stapp *et al.*[5] From Eqs. (8a) and (9), it is easy to obtain the following expression for the M matrix in terms of phase shifts[13]

$$M^{m_sm'_s}(\theta,\varphi) = f_c^{m_sm'_s}(\theta) + \textstyle\sum_j \{ \sum_{mn} a_{mn}^{m_sm'_s}(\theta,\varphi) \times \langle m'n,\varphi_m| + f(\epsilon_j, a_{j-1j}^{m_sm'_s}, a_{j+1j}^{m_sm'_s}, a^{jm_sm'_s}/2) \times \langle j-1j,j+1j|\} , \quad (9a)$$

where mn take values $jj, j-1j, j+1j$, and j and $m' = m$ except when $m = j-1$, $m' = j+1$, and

$$\langle A,B| = 2ie^{i(\delta_A+\delta_B)} \sin(\delta_A - \delta_B)$$
$$f(\epsilon_j,a,b,c) = a\cos^2\epsilon_j + b\sin^2\epsilon_j + c\sin 2\epsilon_j . \quad (9b)$$

3. S-MATRIX APPROACH

The scattering matrix for nucleon–nucleon scattering can also be written from the general considerations of charge symmetry and invariance with respect to rotation, parity and time reversal. If $\mathbf{k}_i$ is the initial momentum in the center-of-mass-system, $\mathbf{k}_f$ the final momentum and $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ spins of nucleons 1 and 2, then the scattering matrix for each isotopic spin state $T = 1,0$ is given by[28]

$$M(\theta) = A + C(\sigma_{1n} + \sigma_{2n}) + B\sigma_{1n}\sigma_{2n} + \tfrac{1}{2}G(\sigma_{1K}\sigma_{2K} + \sigma_{1P}\sigma_{2P}) + \tfrac{1}{2}H(\sigma_{1K}\sigma_{2K} - \sigma_{1P}\sigma_{2P}) \quad (10)$$

where

$$\sigma_{1n} = \boldsymbol{\sigma}_1\cdot\mathbf{n}, \text{ etc.}, \quad \mathbf{n} = \frac{\mathbf{k}_i \times \mathbf{k}_f}{|\mathbf{k}_i \times \mathbf{k}_f|}, \quad \mathbf{P} = \frac{\mathbf{k}_i + \mathbf{k}_f}{|\mathbf{k}_i + \mathbf{k}_f|},$$
$$\mathbf{K} = \frac{\mathbf{k}_f - \mathbf{k}_i}{|\mathbf{k}_f - \mathbf{k}_i|}, \quad (11)$$

and the coefficients A,C,B,G, and H are complex and are functions of energy and angle. We can express the coefficients of the M matrix in terms of its elements

$M^{m_s m'_s}$ in spin space of the two nucleons by multiplying Eq. (10) on the left by $\chi_s^{m_s *}$ and by $\chi_s^{m'_s}$ on the right. It is then easy to obtain from Eq. (10) the following relations,

$$\begin{aligned} A &= \tfrac{1}{4}\,(2M^{11} + M^{00} + M^{ss}) \\ C &= \tfrac{1}{4}\,i\sqrt{2}(M^{10} - M^{01}) \\ B &= \tfrac{1}{4}\,(-2M^{1-1} + M^{00} - M^{ss}) \\ G &= \tfrac{1}{2}\,(M^{11} + M^{1-1} - M^{ss}) \\ H &= (1/2\cos\theta)(M^{00} + M^{1-1} - M^{11}) \\ &= -\tfrac{1}{2}\,(\sqrt{2}/\sin\theta)(M_{10} + M_{01}) \end{aligned} \tag{12}$$

with

$$M^{11} = M^{-1-1}\,, \quad M^{-11} = M^{1-1}\,,$$
$$M^{01} = -M^{0-1}\,, \quad M^{10} = -M^{-10}\,. \tag{13}$$

The M matrix for p-p, n-n, and n-p scattering can be expressed in terms of the M matrices M_1 and M_0 for the total isotopic spin states $T = 1$ and $T = 0$, respectively, by transforming from the $\tau_1\tau_2\tau_{1Z}\tau_{2Z}$ representation to the $\tau_1\tau_2 TT_Z$ representation, making use of Eq. (7) and the fact of rotational invariance of M matrix in isospin space. We then obtain (neglecting Coulomb corrections)

$$\begin{aligned} \langle pp|M|pp\rangle &= \langle nn|M|nn\rangle = M_1 \\ \langle np|M|np\rangle &= \tfrac{1}{2}\,(M_1 + M_0) \\ \langle np|M|pn\rangle &= \tfrac{1}{2}\,(M_1 - M_0)\,. \end{aligned} \tag{14}$$

We can rewrite Eq. (10) in the form

$$M(\theta) = bS + c(\sigma_{1n} + \sigma_{2n}) + \tfrac{1}{2}\,g(\sigma_{1K}\sigma_{2K} + \sigma_{1P}\sigma_{2P})T + \tfrac{1}{2}\,h(\sigma_{1K}\sigma_{2K} - \sigma_{1P}\sigma_{2P})T + n\sigma_{1n}\sigma_{2n}T\,, \tag{15}$$

where

$$S = \tfrac{1}{4}\,(1 - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\,, \quad T = \tfrac{1}{4}\,(B + \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\,,$$
$$n = A + B\,, \quad g = 2A + G\,,$$
$$b = A - B - G\,, \quad c = C\,, \quad h = H\,,$$

and use has been made of the relations

$$\sigma_{1n}\sigma_{2n}T = \sigma_{1n}\sigma_{2n} + S \quad \text{and}$$
$$\sigma_{1P}\sigma_{2P} + \sigma_{1K}\sigma_{2K} + \sigma_{1n}\sigma_{2n} = \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\,.$$

It is now easy to see[29] from the requirement of antisymmetry of the two nucleon system with respect to interchange of space (implying $\mathbf{k}_f \to -\mathbf{k}_f$), spin, and isotopic spin coordinates, that the coefficients $b_1(\theta), h_1(\theta), c_1(\theta), g_0(\theta), n_0(\theta)$ remain unchanged for $\theta \to \pi - \theta$ while $b_0(\theta), h_0(\theta), c_0(\theta), g_1(\theta), n_1(\theta)$ change sign, where the suffixes 1 and 0 refer to the value of the isotopic spin T. Thus it is necessary to know the coefficients only over the range 0 to $\pi/2$.

It has been further shown by Puzikov, Ryndin, and Smorodinski[30] that the unitarity of the S matrix leads to five integral equations of restraints on the coefficients of the M matrix. The unitarity of the S matrix implies

$$SS^+ = 1 = (R+1)(R^+ + 1)\,,$$
$$\text{i.e., } \langle\Omega|R + R^+|\Omega'\rangle = -\textstyle\sum_{\Omega''}\langle\Omega|R|\Omega''\rangle\langle\Omega''|R^+|\Omega'\rangle\,, \tag{16}$$

where $\Omega \equiv (\theta,\varphi) \equiv \hat{k}$. From Eqs. (2)–(4) it is easy to obtain

$$M(\mathbf{k},\mathbf{k}') = (4\pi/2ik)\langle\Omega|R|\Omega'\rangle\,, \tag{17}$$

so that Eq. (16) becomes

$$-i(M - M^+) = \operatorname{Im} M(\mathbf{k},\mathbf{k}') = \frac{k}{4\pi}\int d\Omega'' M(\mathbf{k},\mathbf{k}'')M^+(\mathbf{k}'',\mathbf{k}')\,. \tag{18}$$

From Eqs. (10) and (18), the following integral equations can be obtained

$$\begin{aligned} 4\pi \operatorname{Im} A(\theta) &= \frac{k}{4}\int d\Omega''\, \mathrm{Tr}\,[M(\mathbf{k},\mathbf{k}'')M^+(\mathbf{k}'',\mathbf{k}')] \\ 4\pi \operatorname{Im} B(\theta) &= \frac{k}{4}\int d\Omega''\, \mathrm{Tr}\,[M(\mathbf{k},\mathbf{k}'')M^+(\mathbf{k}'',\mathbf{k}')\sigma_{1n}\sigma_{2n}] \\ 4\pi \operatorname{Re} C(\theta) &= \frac{ik}{8}\int d\Omega'' \\ &\quad \times \mathrm{Tr}\,[M(\mathbf{k},\mathbf{k}'')M^+(\mathbf{k}'',\mathbf{k}')(\sigma_{1n} + \sigma_{2n})] \\ 4\pi \operatorname{Im}\,[G(\theta) + H(\theta)] &= \frac{k}{2}\int d\Omega'' \\ &\quad \times \mathrm{Tr}\,[M(\mathbf{k},\mathbf{k}'')M^+(\mathbf{k}'',\mathbf{k}')\sigma_{1K}\sigma_{2K}] \\ 4\pi \operatorname{Im}\,[G(\theta) - H(\theta)] &= \frac{k}{2}\int d\Omega'' \\ &\quad \times \mathrm{Tr}\,[M(\mathbf{k},\mathbf{k}'')M^+(\mathbf{k}'',\mathbf{k}')\sigma_{1P}\sigma_{2P}] \end{aligned} \tag{19}$$

It therefore follows from Eq. (19) that, as a consequence of the unitarity of the S matrix, the ten complex (or twenty real) coefficients A_T, B_T, C_T, G_T, and H_T $(T = 0,1)$ are not completely independent. For each isotopic spin state T, there are five integral equations of constraints the solutions of which determine five parts (real or imaginary) of the five complex coefficients provided the remaining five parts are known over the angular range 0 to $\pi/2$ at the energy under consideration. Thus the M matrix

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for p-p (n-p) scattering at a given angle and energy requires a knowledge of 10 (20) real coefficients (phase factor being determined too) at this angle and energy or alternatively 5 (10) real coefficients over the angular range 0 to $\pi/2$ at this energy. It may be remarked, that the number of elements of the matrix $M^{m_s m'_s}$ ($m_s, m'_s = 0, \pm 1$, and s) is also 10 (20) for p-p (n-p) scattering. It has been pointed out by Golovin, Dzhelepov, Nadezhdin, and Satarov[31] that the performance of each pair of experiments to determine the same characteristics of the p-p system for $0 \leqslant \theta \leqslant \pi/2$ and the n-p system for the angular range 0 to π provides information about three real functions which describe scattering. Therefore, in order to determine all the ten complex coefficients (except for the common phase factor) it is necessary to carry out six pairs of identical experiments on p-p and n-p scattering, giving information regarding eighteen real functions, and one more p-p or n-p scattering experiment.

It has been pointed out by Schumacher and Bethe[32] that because of the bilinear form of the expressions of scattering parameters ambiguities arise in the construction of the nucleon-nucleon scattering matrix from data at one angle and energy. They have shown that these ambiguities can be eliminated by a knowledge of the polarization transfer tensor K_{ik} which measures the polarization of the recoil (scattered) nucleon in the scattering of a polarized (unpolarized) beam from an unpolarized (polarized) target. Their method does not make use of the condition of unitarity and is therefore also applicable at energies at which inelastic processes occur.

4. EXPERIMENTAL SCATTERING PARAMETERS

The scattering parameters describing nucleon–nucleon scattering are referred to as single, double, and triple scattering parameters. The cross section for the unpolarized beam is the single scattering parameter while the polarization P of the beam once scattered by the target is the double scattering parameter. The triple scattering parameters, depolarization D, rotation parameters R,A,R', and A', describe how the second scatterer changes the direction or magnitude or both of the polarization —the first scatterer acts as a polarizer and the final scatterer as an analyzer. These coefficients can be defined by expressing the final-state polarization $\langle \boldsymbol{\sigma}_f \rangle$ in terms of the initial-state polarization $\langle \boldsymbol{\sigma}_i \rangle$. If $\mathbf{k}_i$ is the initial momentum of the nucleon and $\mathbf{k}_f$ its final momentum, then it is convenient to express $\langle \boldsymbol{\sigma}_i \rangle$ in terms of the three mutually perpendicular unit vectors $\hat{k}_i, \hat{k}_i \times \hat{k}_f$, and $\hat{k}_i \times (\hat{k}_i \times \hat{k}_f)$ and similarly express $\langle \boldsymbol{\sigma}_f \rangle$ in terms of the vectors $\hat{k}_f, \hat{k}_i \times \hat{k}_f$ and $\hat{k}_f \times (\hat{k}_i \times \hat{k}_f)$. Assuming that $\langle \boldsymbol{\sigma}_f \rangle$ depends linearly on $\langle \boldsymbol{\sigma}_i \rangle$, and observing that $\boldsymbol{\sigma}$ is a pseudovector, provided parity is conserved, we can express $\langle \boldsymbol{\sigma}_f \rangle$ in terms of the components of $\langle \boldsymbol{\sigma}_i \rangle$ along the three mutually perpendicular directions $\hat{k}_i, \hat{k}_i \times \hat{k}_f$, and $\hat{k}_i \times (\hat{k}_i \times \hat{k}_f)$ as follows[28]

$$\begin{aligned} I_f \langle \boldsymbol{\sigma}_f \rangle = {} & I_i [\{P + D\langle \boldsymbol{\sigma}_i \rangle \cdot (\hat{k}_i \times \hat{k}_f)\} (\hat{k}_i \times \hat{k}_f) \\ & + \{A\langle \boldsymbol{\sigma}_i \rangle \cdot \hat{k}_i + R\langle \boldsymbol{\sigma}_i \rangle \cdot \hat{k}_i \times (\hat{k}_i \times \hat{k}_f)\} \hat{k}_f \times (\hat{k}_i \times \hat{k}_f) \\ & + \{A'\langle \boldsymbol{\sigma}_i \rangle \cdot \hat{k}_i + R'\langle \boldsymbol{\sigma}_i \rangle \cdot \hat{k}_i \times (\hat{k}_i \times \hat{k}_f)\} \hat{k}_f] \,, \end{aligned} \tag{20}$$

where I_f is the cross section for all cases, I_i is the cross section for an unpolarized beam, and only those terms occur on the right-hand side which transform like a pseudovector. From Eq. (20), it is now easy to understand the significance of each of the scattering parameters P,D, A, A', R, and R' in the transformation of the initial polarization $\langle \boldsymbol{\sigma}_i \rangle$ to the final polarization $\langle \boldsymbol{\sigma}_f \rangle$. These are illustrated in Fig. 1.

In order to obtain expressions for the scattering parameters, we quote a result, due to Wolfenstein,[28] which relates the final-state spin operator to the initial-state spin operator by means of the M matrix which transforms from the spin space of the initial channel to the spin space of the final channel. If the incident and the target particles have spins s and s_t, respectively, and S^μ denotes the base matrices, where μ runs from 1 to $(2s+1)^2(2s_t+1)^2$, then

$$I_0 \langle \bar{S}^\mu \rangle_f = \frac{1}{(2s+1)(2s_t+1)} \sum_\nu \langle \bar{S}^\nu \rangle_i \operatorname{Tr}(M S^\nu M^+ S^\mu) \,, \tag{21}$$

where I_0 is the differential scattering cross section. Comparing the various coefficients of Eqs. (20) and (21), we obtain

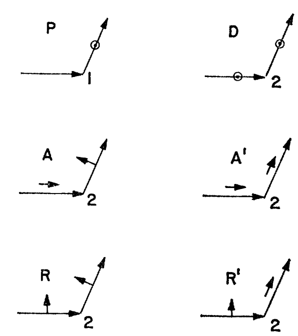

FIG. 1. Diagrammatic representation of the scattering parameters. 1 and 2 denote the first and second scattering. The horizontal line describes the beam before the first or the second scattering and the inclined after scattering. The arrows indicate the direction of the polarization vector, the circle ⊙ indicating polarization direction normal to the page.

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[31] B. Golovin, V. Dzhelepov, V. Nadezhdin, and V. I. Satarov, Soviet Phys.—JETP 9, 302 (1959).

[32] C. R. Schumacher and H. A. Bethe, Phys. Rev. 121, 1543 (1961).

$$I_0P = \tfrac{1}{4}\,\mathrm{Tr}\,[MM^+(\boldsymbol{\sigma},\mathbf{k}_i\times\mathbf{k}_f)]$$
$$I_0D = \tfrac{1}{4}\,\mathrm{Tr}\,[M(\boldsymbol{\sigma},\mathbf{k}_i\times\mathbf{k}_f)M^+(\boldsymbol{\sigma},\mathbf{k}_i\times\mathbf{k}_f)]$$
$$I_0A = \tfrac{1}{4}\,\mathrm{Tr}\,[M(\boldsymbol{\sigma},\mathbf{k}_i)M^+(\boldsymbol{\sigma},\mathbf{k}_f\times(\mathbf{k}_i\times\mathbf{k}_f))]$$
$$I_0A' = \tfrac{1}{4}\,\mathrm{Tr}\,[M(\boldsymbol{\sigma},\mathbf{k}_i)M^+(\boldsymbol{\sigma},\mathbf{k}_f)]$$
$$I_0R = \tfrac{1}{4}\,\mathrm{Tr}\,[M(\boldsymbol{\sigma},\mathbf{k}_i\times(\mathbf{k}_i\times\mathbf{k}_f))M^+ \times(\boldsymbol{\sigma},\mathbf{k}_f\times(\mathbf{k}_i\times\mathbf{k}_f))]$$
$$I_0R' = \tfrac{1}{4}\,\mathrm{Tr}\,[M(\boldsymbol{\sigma},\mathbf{k}_i\times(\mathbf{k}_i\times\mathbf{k}_f))M^+(\boldsymbol{\sigma},\mathbf{k}_f)]\,. \quad (22)$$

Using $\alpha\alpha,\alpha\beta,\beta\alpha$, and $\beta\beta$ as the basis spin states for the two nucleons, the matrices $\boldsymbol{\sigma}$ can be explicitly written down. Further expressing the elements of the M matrix in this representation, viz., $\langle s_1s_2m_1m_2|M|s_1s_2m'_1m'_2\rangle$ in terms of elements $\langle s_1s_2sm_s|M|s_1s_2sm'_s\rangle = M_{m_sm'_s}$ in the triplet-singlet representation, by means of Eq. (7), one obtains the following expressions for the scattering parameters:

$$I_0P = (\sqrt{2}/4)\,\mathrm{Re}\,\{i(M_{10}-M_{01})(M_{11}-M_{1-1}+M_{00})^*\}\,,$$
$$I_0D = \tfrac{1}{2}\,\mathrm{Re}\,\{(M_{11}+M_{1-1})M_{ss}^* + (M_{11}-M_{1-1})M_{00}^*\} - \mathrm{Re}\,M_{10}M_{01}^*\,,$$
$$\frac{I_0R}{\cos\theta/2} = \tfrac{1}{2}\,\mathrm{Re}\left\{\left[M_{00}+(\cos\theta-1)\frac{\sqrt{2}M_{10}}{\sin\theta}\right](M_{11}+M_{1-1})^* + \left[M_{00}+\cos\theta\frac{\sqrt{2}M_{10}}{\sin\theta}\right]M_{ss}^*\right\},$$
$$\frac{I_0A}{\sin\theta/2} = -\tfrac{1}{2}\,\mathrm{Re}\left\{\left[M_{00}+\cos\theta\frac{\sqrt{2}M_{10}}{\sin\theta}-\frac{\sqrt{2}M_{01}}{\sin\theta}\right](M_{11}+M_{1-1})^* + \left[M_{00}+(\cos\theta+1)\frac{\sqrt{2}M_{10}}{\sin\theta}\right]M_{ss}^*\right\}$$
$$\frac{I_0R'}{\sin\theta/2} = \tfrac{1}{2}\,\mathrm{Re}\left\{\left[M_{00}+(\cos\theta+1)\frac{\sqrt{2}M_{10}}{\sin\theta}\right](M_{11}+M_{1-1})^* + \left[M_{00}+\cos\theta\frac{\sqrt{2}M_{10}}{\sin\theta}-\frac{\sqrt{2}M_{01}}{\sin\theta}\right]M_{ss}^*\right\},$$
$$\frac{I_0A'}{\cos\theta/2} = \tfrac{1}{2}\,\mathrm{Re}\left\{\left[M_{00}+\cos\theta\frac{\sqrt{2}M_{10}}{\sin\theta}+\frac{\sqrt{2}M_{10}}{\sin\theta}\right](M_{11}+M_{1-1})^* + \left[M_{00}+(\cos\theta-1)\frac{\sqrt{2}M_{10}}{\sin\theta}\right]M_{ss}^*\right\},$$
$$I_0 = \tfrac{1}{2}|M_{11}|^2 + \tfrac{1}{4}|M_{00}|^2 + \tfrac{1}{4}|M_{ss}|^2 + \tfrac{1}{2}|M_{10}|^2 + \tfrac{1}{2}|M_{01}|^2 + \tfrac{1}{2}|M_{1-1}|^2\,, \quad (23)$$

where the expressions for $M_{m_sm'_s}(\theta,\varphi)$ in terms of the phase shifts are given by Eq. (9).

The other experimental parameters are the spin correlation coefficients C_{nn}, C_{KP}, C_{PP}, and C_{KK}. In these experiments an unpolarized nucleon beam is scattered by nucleons and components of the polarization of the scattered nucleon and the recoil nucleon along the directions indicated by the indices, are measured in coincidence. Thus $C_{nn} = \langle\sigma_{1n}\sigma_{2n}\rangle$, etc., the following expressions for these coefficients can be obtained.

$$\tfrac{1}{2}\,I_0C_{nn} = \mathrm{Re}\,A^*B + |C|^2 - \tfrac{1}{4}|G|^2 + \tfrac{1}{4}|H|^2\,,$$
$$I_0C_{KP} = -2\,\mathrm{Im}\,C^*H\,,$$
$$\tfrac{1}{2}\,I_0(C_{PP}+C_{KK}) = \mathrm{Re}\,G^*(A-B)\,,$$
$$\tfrac{1}{2}\,I_0(C_{PP}-C_{KK}) = -\mathrm{Re}\,H^*(A+B)\,. \quad (24)$$

5. PHASE-SHIFT ANALYSIS OF EXPERIMENTAL PARAMETERS

A complete set of experiments on proton–proton scattering at 310 MeV was carried out by Chamberlain, Segrè, Tripp, Wiegand, and Ypsilantis.[1] The experiment consisted in the measurement of five scattering parameters I_0, P, D, R, and A. As Puzikov, Ryndin and Smorodinsky[30] have shown, because of the identity of particles it is necessary to make measurements only in the angular range 0 to $\pi/2$ and if the unitarity of the S matrix is used, only five scattering parameters need be measured at a given energy. It has been further pointed out by Marshak[25] that the S matrix will be uniquely determined only if at least one of the experiments involves scatterings which are not in the same plane. The phase-shift analysis of the 310 MeV proton–proton scattering experiment of Chamberlain *et al.*[1] was carried out by Stapp, Ypsilantis, and Metropolis[5] and led to 8 possible sets of phase-shift solutions because of inaccuracies of the experiments. Three of the solutions (solutions number 5, 7, and 8) were excluded because they did not fit the production of pions in the process $p+p\to d+\pi^+$. The solution number 6 is possible to discard because of the measurement[33] of C_{KP} at $\pi/2$ at 382 MeV, thus leaving four sets of solutions. A modified phase-shift analysis of the p-p scattering data of the experiment at 310 MeV has been carried out by Moravcsik *et al.*,[6] which makes it almost possible to obtain a unique phase-shift solution. Their procedure is based

[33] A. Ashmore, A. N. Diddens, and G. B. Huxtable, Proc. Phys. Soc. (London) **73**, 957 (1959).

on the philosophy of the Japanese group, led by Taketani, that the nucleon–nucleon interaction at large distances is adequately described by the one-pion-exchange potential. They therefore evaluate the partial waves G and higher from the one-pion-exchange pole in the scattering amplitude and calculate the remaining lower partial waves from the experimental data. Moravcsik *et al.*[6] arrive at the conclusion that the modified solutions 1 and 2 converge, respectively, to solutions 3 and 4, so that one is left essentially with only two solutions.

Experiments similar to those of Chamberlain *et al.*[1] have been carried out at Harvard[2] and Harwell[3] at an energy of 150 MeV. The group at Rochester[4] has measured proton–proton scattering data–cross section σ, polarization P, and the triple scattering parameters A, R, and D at 210 MeV. MacGregor and Moravcsik[34] have applied their modified phase-shift analysis to the Rochester data,[4] and obtained four sets of phase-shift solutions. Of these two have been excluded primarily because they do not correspond to one of the two acceptable phase-shift solutions at 310 MeV.

6. PHENOMENOLOGICAL TWO-NUCLEON POTENTIAL

The scattering matrix approach, though very general, has its limitations. In order to determine the wave functions of the two-nucleon system, its bound states, and off-the-energy-shell matrix elements, it is necessary to know the Hamiltonian for the system. If the two-nucleon potential can be specified, then it is possible to determine the phase shifts uniquely, and hence, all the scattering parameters. An unambigous derivation of the complete potential from meson theory would solve this problem, but as yet this does not seem possible; only the second-order one-pion-exchange potential is unambigously known. We shall therefore confine ourselves to the phenomenological and the semiphenomenological potentials based on concepts of meson theory.

Wigner[35] proposed the first general form of the phenomenological nucleon–nucleon potential under the restrictions that (i) the potential depends on the spins $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, the relative separation $\mathbf{r}$, and the relative momentum $\mathbf{p}$ of the two nucleons so that the center-of-mass motion is separable and the total momentum is conserved, (ii) the potential has rotational invariance so that the total angular momentum is conserved, and (iii) the potential depends, at the most, linearly on the relative momentum $\mathbf{p}$. Okubo and Marshak[36] have given the most general velocity-dependent proton–proton potential by dropping restriction (iii). They determine the potential under the restrictions of (i) translational invariance, (ii) rotational invariance, (iii) Galilean invariance, (iv) space reflection invariance, (v) time reversal invariance, (vi) charge independence, (vii) permutation symmetry, and (viii) Hermiticity and arrive at the following expression:

$$\begin{aligned} V = V_0 &+ (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)V_1 + S_{12}V_2 + (\mathbf{L}\cdot\mathbf{S})V_3 \\ &+ \tfrac{1}{2}[(\boldsymbol{\sigma}_1\cdot\mathbf{L})(\boldsymbol{\sigma}_2\cdot\mathbf{L}) + (\boldsymbol{\sigma}_2\cdot\mathbf{L})(\boldsymbol{\sigma}_1\cdot L)]V_4 \\ &+ (\boldsymbol{\sigma}_1\cdot\mathbf{p})(\boldsymbol{\sigma}_2\cdot\mathbf{p})V_5 + \text{H.C.} \end{aligned} \tag{25}$$

where $S_{12} = 3(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})/r^2 - (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$ and the functions $V_i = V_i(r^2,p^2,L^2)$. In the case of the n-p system, the potential is the sum of the isospin $T = 1$ and $T = 0$ potentials each one of which is of the form of Eq. (25). For elastic scattering (on the energy shell), it can be shown, that the V_5 term can be dropped out and $V_i = V_i(\mathbf{r}^2,L^2)$.

Equation (25) describes the form of the most general potential. Since the phenomenological potentials have been obtained by fitting the experimental data over a certain energy range, the general practice has been to require determination of as few parameters in the potential as possible unless it is inevitable to introduce complexities by including more terms in the potential in order to fit the experiments. A very thorough effort was made by Gammel, Thaler, and Christian[37] to fit the nucleon–nucleon scattering data at 170 and 310 MeV by central and tensor potentials of the Yukawa type outside the hard core; the radius of the hard core assumed to be independent of the parity. Their analysis led them to the following results:

(i) The calculated n-p polarizations agree with the experimental n-p polarization even at the highest energies so that the potential describes the triplet even-parity interaction correctly.

(ii) There is qualitative discrepancy between the calculated and the experimental values of the high-energy p-p polarization. The polarization predicted by the potential being opposite in sign to the experimentally observed.

(iii) For a singlet range ${}^1r_0 = 0{\cdot}4 \times 10^{-13}$ cm, the ${}^1S_0$,${}^1D_2$, and ${}^1G_4$ phase shifts are of the type of Stapp's

[34] M. H. MacGregor and M. J. Moravcsik, Phys. Rev. Letters **4**, 524 (1960).
[35] L. Eisenbud and F. Wigner, Proc. Natl. Acad. Sci. U. S. **27**, 281 (1941).
[36] S. Okubo and R. E. Marshak, Ann. Phys. (N. Y.) **4**, 166 (1958).
[37] J. L. Gammel, R. Christian, and R. M. Thaler, Phys. Rev. **105**, 311 (1957).

solutions,[5] 1 and 3, so that the potential describes the singlet even-parity interaction correctly.

The essential main conclusion of the analysis of Gammel, Thaler, and Christian[37] was that the triplet odd-parity interaction is not capable of being described by central and tensor potentials (and hard cores) alone. Gammel and Thaler[8] noting the fact that central and tensor potentials though successful in many respects fails to predict the p-p polarization data at 170 and 310 MeV correctly, observed that Stapp's[5] phase shift analysis of the 310 MeV p-p scattering indicates that the triplet p-wave phase shifts ${}^3P_{0,1,2}$ are split in a manner inconsistent with the tensor force. They therefore looked for a potential which will fit the low-energy scattering experiments and the phase shift solution 1 of Stapp *et al.*[5] In order to achieve this, it was found that the potential should have the qualitative features that (i) the tensor force is long range and attractive in the ${}^3P_0$ state, and (ii) that a strong short-range spin-orbit force which is repulsive in the ${}^3P_0$ state is necessary. Thus the general form of the potential taken was

$$\begin{aligned} V &= +\infty\ , \quad \text{for} \quad r < r_0\ , \\ &= V_c(r) + S_{12}V_T(r) + \mathbf{L}\cdot\mathbf{S}V_{LS}(r)\ , \quad \text{for} \quad r > r_0, \\ \text{where} \quad V(r) &= Ve^{-\mu r}/_{\mu r}\ . \end{aligned} \tag{26}$$

Assuming charge independence of nuclear forces, a spin-orbit term in the triplet even-parity potential of the same short range (but lesser depth) as the triplet odd-parity spin-orbit term was also introduced. In the following we list the parameters of their triplet-odd and singlet-even potentials (the p-p system).

$$\begin{aligned} &{}^3r_0^- = 0.4125 \times 10^{-13}\ \text{cm}\ , && {}^1r_0^+ = 0.4 \times 10^{-13}\ \text{cm}\ , \\ &{}^3V_c^- = 0\ , && \\ &{}^3V_T^- = -22\ \text{MeV}\ , && {}^3\mu_T^- = 0.8 \times 10^{13}\ \text{cm}^{-1}\ , \\ &{}^3V_{LS}^- = 7317.5\ \text{MeV}\ , && {}^3\mu_{LS}^- = 3.7 \times 10^{13}\ \text{cm}^{-1}\ , \\ &{}^1V_c^+ = 425.5\ \text{MeV}\ , && {}^1\mu_c^+ = 1.45 \times 10^{13}\ \text{cm}^{-1}\ . \end{aligned} \tag{27}$$

The p-p scattering data was well fitted by the above potential, Eqs. (26) and (27), except that the differential cross section was slightly low in the forward direction at 90 and 156 MeV.

An alternative potential which also included a spin-orbit interaction besides the central and the tensor interactions was independently proposed by Signell and Marshak[7] at about the same time as Gammel and Thaler[8] had proposed their potential. It was noted by Signell and Marshak[7] that though several meson-theoretic two-nucleon potentials give a reasonable fit of the data at low energies, yet all of these, viz., Levy[38] and Gartenhaus[39] potentials, fail to fit the 100 and 150 MeV p-p scattering data. The phase shift analysis by Ohnuma and Feldman[40] of the experimental cross-sections at 150 MeV favored the inclusion of a spin-orbit potential. Signell and Marshak SM[7] therefore decided to add a spin-orbit term to the Gartenhaus[39] potential. The Gartenhaus potential was derived by applying Chew and Low's Static nucleon extended source (cutoff) p-wave pion $PS(PV)$ interaction Hamiltonian to the two-nucleon problem in the second- and the fourth-order non-relativistic perturbation theory, omitting the so-called ladder corrections, using coupling constant (renormalized) $f_0^2 = 0.089$ and a cutoff $\omega_m = 6\mu$, where μ is the pion mass. Gartenhaus potential was chosen by Signell and Marshak[7] because it appeared to have the most plausible meson-theoretic basis and because it fitted the low-energy data very well. The following is the form of the SM potential[7]

$$\begin{aligned} V &= V_G + \mathbf{L}\cdot\mathbf{S}\,\frac{V_0}{x_c}\,\frac{d}{dx}\left(\frac{e^{-x}}{x}\right)\Bigg|_{r=r_c} && \text{for } r \leqslant r_c \\ V &= V_G + \mathbf{L}\cdot\mathbf{S}\,\frac{V_0}{x}\,\frac{d}{dx}\left(\frac{e^{-x}}{x}\right) && \text{for } r > r_c\ , \end{aligned} \tag{28}$$

where V_G is the Gartenhaus potential[39] which has central plus tensor parts, and $x = r/r_0, x_c = r_c/r_0, r_c = 1/M = 0{\cdot}21 \times 10^{-13}$ cm, $r_0 = 1{\cdot}07 \times 10^{-13}$ cm, $V_0 = +30$ MeV, the sign of V_0 being chosen to that needed for the shell model. This potential was found to be in very good agreement with experimental scattering data up to 150 MeV. Later recognizing the fact that Klein[17] and Greene[18] found evidence of the short range spin-orbit potential in the fourth-order $PS(PS)$ field theory provided nucleon recoil is taken into account, Signell, Marshak, and Zinn[41] modified the spin-orbit part of the SM potential so that the spin-orbit potential has a range of $1/2\mu$ corresponding to exchange of two mesons by the nucleons. This is called the SM1 potential[41] given by

$$V_{LS} = \frac{V_0}{x}\,\frac{d}{dx}\left(\frac{e^{-2x}}{x}\right)\ , \tag{29}$$

where $x = \mu r, V_0 = 21$ MeV, and the Gartenhaus triplet-odd potential is modified to include an infinitely repulsive core out to $x_\circ = 0.37$ in order to avoid the bound ${}^3P_2$ state. The SM1 potential improved the fit of the data up to 150 MeV.

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[38] M. M. Lèvy, Phys. Rev. **88**, 725 (1952).
[39] S. Gartenhaus, Phys. Rev. **100**, 900 (1955).
[40] S. Ohnuma and D. Feldman, Phys. Rev. **102**, 1641 (1956).
[41] P. S. Signell, R. Zinn, and R. E. Marshak, Phys. Rev. Letters **1**, 416 (1958).

A separate effort to solve the two-nucleon problem was in progress in Japan. The Japanese group led by Taketani[42] attempted to solve the two-nucleon problem with much stronger faith in the meson-theoretic calculations. Their approach was discussed in details by Iwadare, Otsuki, Tamagaki, and Watari.[12] One of their essential steps consisted in the division of the inter-nucleon distance into three regions.

(*i*) *Region I*. This is the outer region $x \gtrsim 1.5$ (in units of $\hbar/\mu c$) in which the potential due to one-pion exchange is dominant and corrections to it can be neglected. The asymptotic form of the one-pion-exchange potential does not depend on the detailed form of the coup ing or on the approximations, so that the potential in this region can be specified unambigously.

(*ii*) *Region II*. This is the intermediate region $0.7 < x < 1.5$, where the two-pion-exchange potential is important and starts dominating over the one-pion-exchange potential. The nucleon recoil effects contribute appreciably to the two-pion-exchange potential and since it depends very much on the coupling (p wave or other), the high-energy pion field cutoff procedure, and the shape of the source function, the many derivations of it are not free from ambiguities.

(*iii*) *Region III*. This is the inner region $x < 0.7$, where many-pion-exchange, heavy-meson exchange, and other effects render this region beyond the scope of theoretical investigation and therefore can be treated only phenomenologically by means of hard cores or by specifying the value of the logarithmic derivative of the wave function determined by fitting the experimental parameters.

In addition to the above division of the inter-nucleon distance, wide use was made of the "impact parameter" considerations. By an extensive calculation Matsumoto and Watari[43] have shown that the partial wave with angular momentum L is hardly affected by the nuclear potential inside about $b/2$ where the impact parameter $b = [L(L+1)]^{1/2}\hbar/p$, p being the momentum of the nucleon in the laboratory system. The Japanese group hoped to explain the two-nucleon scattering experiments by means of hard cores and central and tensor potentials arising out of one-and two-pion-exchange processes, and strongly criticized the use of spin-orbit interaction by Gammel and Thaler[8] at 310 MeV and by Signell and Marshak[7] at 150 MeV as unwarranted. Otsuki,[9] Watari,[10] Tamagaki[11] had reasonable success in fitting the cross section and polarization at 90 and 150 MeV by purely central and tensor potentials. However, it was pointed out by Nigam[13] that if the Harvard data on depolarization parameter D at 150 MeV (in contrast to the Harwell data) is correct, viz., that D goes over to positive values at 75° in the c.m. system, it is very unlikely that pure central and tensor potentials can achieve this result in spite of the fact that polarization can be fitted. It was found by Nigam that D is a very sensitive function of the $^3P_0$ phase shift and a positive depolarization requires a small or even negative $^3P_0$ phase shift, a result which can be obtained by introducing spin-orbit interactions as done by Gammel and Thaler[8] and Signell and Marshak.[7] With purely central and tensor forces the depolarization at 150 MeV was predicted to be too negative as was also found out later by Otsuki *et al.*[12] It was also suggested by Nigam that to fit the Harvard data on depolarization, the spin-orbit interaction should be strengthened compared to its value in the SM1 potential.[41] A considerably improved fit to the high-energy p-p scattering data from 40 to 310 MeV range was obtained by Bryan[14] by choosing the static potentials (central, tensor, and spin-orbit) of the general form

$$V = \sum_{n=2}^{5} A_n x^{-n} e^{-2x} + V_2(\text{OPEP}) , \qquad (30)$$

together with infinite repulsive cores for the central

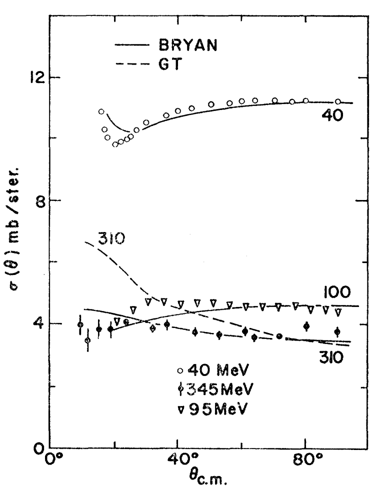

FIG. 2. Comparison of 40, 100, and 310 MeV p-p differential cross section predictions with experiment.

potentials, where x is the distance in units of $\hbar/\mu c$, $V_2(OPEP)$ stands for the one-pion-exchange potential, and A_n are constants to be fitted from the scattering data. The curves obtained by Bryan[14] are illustrated in Figs. 2 to 5.

[42] M. Taketani, S. Nakamura, and M. Sasaki, Progr. Theoret. Phys. (Kyoto) **6**, 581 (1951).

[43] M. Matsumoto and W. Watari, Progr. Theoret. Phys. (Kyoto) **12**, 503 (1954).

An approach similar to that of Bryan[14] was also carried out successfully by Hamada.[16] Since the linear spin-orbit potential vanishes in the singlet states, a quadratic spin-orbit potential was added to fit the singlet even parity states. The various potentials included terms corresponding to ranges of one-, two-, and three-pion Compton wavelengths. A two-

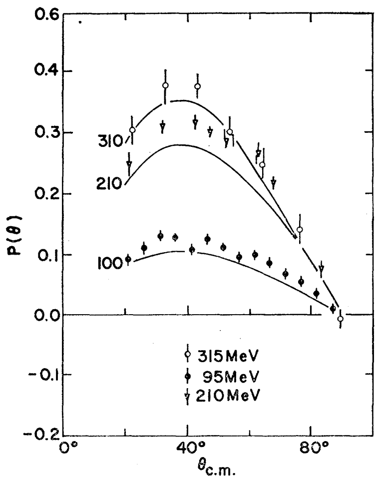

FIG. 3. Comparison of 100, 210, and 310 MeV p-p polarization predictions with experiment.

nucleon potential for the isotopic spin state $T = 0$ was also determined which when added to the $T = 1$ potential was found to reproduce the n-p experimental data below 300 MeV. The triplet even parity potential was not strictly energy independent, the

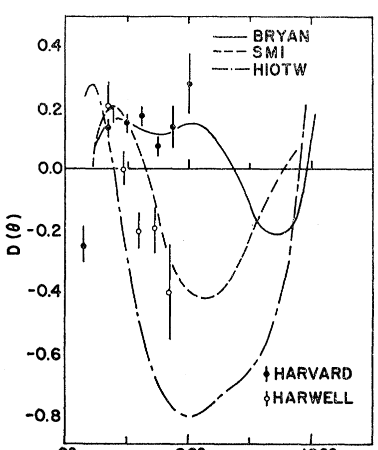

FIG. 4. Comparison of 150 MeV p-p $D(\theta)$ predictions with experiments.

energy dependence being rather small and mainly confined to the core region. The $T = 0$ quadratic spin-orbit potential is stronger than required in the $T = 1$ state. Since the first triplet-even state affected by the linear spin-orbit potential is the D-state while the first triplet-odd state affected is the P state, the linear spin-orbit potential is not as important in the $T = 0$ state as in $T = 1$ state. On the other hand, the quadratic spin-orbit potential plays an important role in the $T = 0$ state.

The boundary condition model which was first suggested by Breit and Bouricius[44] and used to describe S-wave scattering was extended to higher waves by Feshbach and Lomon[45] who however predicted phase shifts of the type of solution 6 of Stapp *et al.*[5] which solution was found to be in disagreement with the measurement of C_{KP} at 380 MeV. Saylor, Bryan, and Marshak,[15] noting that the Bryan potential[14] model for proton–proton scattering defines in the triplet states a region of great strength between $0.6/\mu$ and the radius of the infinite core, $0.38/\mu$, and outside the potential is weak, developed a boundary condition model with potential tails outside with considerable success. Both the Taketani–

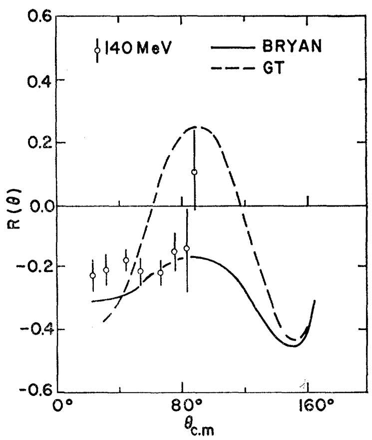

FIG. 5. Comparison of 150 MeV p-p $R(\theta)$ predictions with experiments.

Machide–Ohnuma[46] potential and the one-pion-exchange potential in the outside region with suitable energy-independent boundary conditions for each partial waves (at $r_0 = 0{\cdot}53/\mu$ for singlet states and at $r_0 = 0{\cdot}56/\mu$ for triplet states) were found to fit the proton–proton scattering data from 40 MeV to 310 MeV very well predicting a phase shift set consistent with solution 1 of MacGregor, Moravcsik, and Stapp.[6] The predicted cross section was found to be little low in the forward direction.

A study of the two-nucleon phenomelogic phase shifts with energy has been extensively pursued by

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[44] G. Breit and W. G. Bouricius Phys. Rev. **79**, 1029 (1949), A. M. Saperstein and L. Durand, *ibid.* **104**, 1102 (1956).

[45] H. Feshbach and E. L. Lomon, Phys. Rev. **102**, 891 (1956).

[46] M. Taketani, S. Machida, and S. Ohnuma, Progr. Theoret. Phys. (Kyoto) **6**, 638 (1951).

Breit, Hull, Pyatt, Fischer, Lassila, and Degges at Yale[47] by expressing the phase shifts as some function of energy-containing parameters that can be varied so as to obtain a fit of the data at several energies. They have fitted the data over the whole energy range up to 345 MeV. Starting with one of the phase-shift solutions, say, the Signell–Marshak[7] (or Gammel–Thaler[8]) corrections were suitably introduced into the preliminary phase-parameters so that the mean weighted sum of the squares of deviations from experimental values was minimized. The quality of the fits to angular distribution curves for the scattering parameters was also used in deciding on the correction functions to be tried. A similarity of results was observed whether one started with SM or the GT potentials. This was interpreted as an indication of existence of a region in phase-parameter space favored by experiment. Also, by adding the one-pion-exchange phase shifts for the higher orbital angular momenta to the phase-shift expression in the low angular momentum states, the quality of the fit was found to improve.

7. MESON THEORETIC POTENTIAL

The theoretical derivation of the two-nucleon potential from meson theory involves the calculation of the interaction between the two nucleons arising from the exchange of one pion (second-order calculation), two pions (fourth-order calculation); and higher number of pions between the two nucleons. The calculations have been carried out with both the pseudoscalar (pseudoscalar) [$PS(ps)$] and the pseudoscalar (pseudovector) [$PS(pv)$] interactions between the meson field and the nucleon. The second order one-pion-exchange potential in the static limit (lowest order terms in μ/M) is given by

$$V^{(1)}(x) = \tfrac{1}{3}\,(g^2/4\pi)\mu c^2(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2) \times \{(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) + S_{12}(1+3/x+3/x^2)\}e^{-x}/x\,,$$

and all authors agree on the form of this part of the interaction. Also, the correctness of the one-pion-exchange potential is now well established in view of the detailed analysis of the nucleon-nucleon scattering data. The fourth-order contribution to the static two-nucleon potential has, among various authors, probably best been derived by Gartenhaus[39] who used the nonrelativistic p-wave extended source (cutoff) model of Chew and Low and carried out a nonrelativistic perturbation calculation. Gartenhaus was successful in obtaining a good fit of the low energy nucleon–nucleon scattering data with his potential. However, Gartenhaus' method is incapable of yielding a spin-orbit interaction since the interaction Hamiltonian used is completely static. Klein[17] and Greene[18] have found that short-range spin-orbit potential originated in the fourth-order field theory provided nucleon recoil is taken into account. The static limit consists in taking the lowest order in μ/M and the so-called "adiabatic approximation" involves the limit $\mathbf{p}/M \to 0$, where $\mathbf{p}$ is the nucleon momentum. Since both μ/M and $\mathbf{p}/M$ are not very small the nonstatic and the nonadiabatic corrections which will take into account the nucleon recoil may be important at high energies. Several authors[19,20] have calculated the two-pion-exchange contribution to the two-nucleon potential taking into account the nucleon recoil. In general, all of them have reported a spin-orbit ($\mathbf{L}\cdot\mathbf{S}$) interaction term, though there is general lack of agreement on its sign and magnitude. The most favorable $\mathbf{L}\cdot\mathbf{S}$ interaction obtained meson-theoretically is due to Tzoar, Raphael, and Klein[20] who adopted the procedure used by Klein and McCormick[48] to construct the potential for pion–nucleon scattering. The $\mathbf{L}\cdot\mathbf{S}$ term of Tzoar *et al.*[20] is quite similar to the one introduced phenomenologically by Signell and Marshak.[7] Taketani and Machida,[49] Hoshizaki and Machida,[50] Otsuki, Tamagaki, and Watari[51] have recently carried out a detailed investigation of the two-nucleon potential with full recoil.

It has been pointed out by Charap and Fubini[52] and by Gupta[53] that the calculation of the nonstatic corrections is not an unambiguous procedure. For instance, for the static potential, the μ/M limit must be taken at the beginning of the calculations, as otherwise it leads to ambiguities. Further, the adiabatic limit $\mathbf{p}/M \to 0$ is unambiguous only if $\mathbf{p}$ is the initial and the final nucleon momentum. Also the nonadiabatic corrections cannot be separated unambiguously from the higher order adiabatic corrections. All these factors make the derivation of the two-pion-exchange contribution to the two-nucleon potential highly dependent on the approximations used in the calculation.

It is worth mentioning regarding the three-pion-

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[47] G. Breit, *Proceedings of the Conference on Nuclear Forces and the Few-Nucleon Problem* (Pergamon Press, New York, 1959), p. 23; G. Breit, M. H. Hull, K. E. Lassila, and K. D. Pyatt, Jr., Phys. Rev. **120**, 2227 (1960).

[48] A. Klein and B. H. McCormick, Phys. Rev. **104**, 1747 (1956).

[49] M. Taketani and S. Machida, Progr. Theoret. Phys. (Kyoto) **24**, 1317 (1960).

[50] N. Hoshizaki and S. Machida, Progr. Theoret. Phys. (Kyoto) **24**, 1325 (1960).

[51] S. Otsuki, R. Tamagaki, and W. Watari, Progr. Theoret. Phys. (Kyoto) (to be published).

[52] J. M. Charap and S. P. Fubini, Nuovo Cimento **14**, 540 (1959).

[53] S. N. Gupta, Phys. Rev. **117**, 1146 (1960).

exchange potential that not only will it be a prohibitive job to calculate it but also that with the present status of meson theory it is quite unnecessary to do so. The range of the three-pion-exchange potential is $\sim 0.47 \times 10^{-13}$ cm which region is very much masked by the replusive core introduced phenomenologically and yet understood in terms of meson theory.

8. MANDELSTAM REPRESENTATION

In recent years, wide use has been made of the dispersion theory to compute the scattering matrices for various processes through analytic properties of the matrices and the requirement of unitarity (probability conservation). If the transition matrices are boundary values of analytic functions of energy variables, then, if they are known over a finite region, they are fully determined over the entire region of analyticity. The principle of causality, viz., light signals cannot travel with a velocity larger than the velocity of light, leads to a dispersion relation for the real part of the scattering amplitude in terms of an integral over its imaginary part (total cross section). Karplus and Ruderman[21] and Goldberger *et al.*[21] obtained dispersion relations in quantum field theory using causality. There are two important concepts[54] developed in the theory of dispersion relations: (i) The scattering amplitudes which are analytic functions of the energy variables have singularities which can be determined from the knowledge of the mass spectrum and the quantum numbers of the strongly interacting particles, and (ii) near the singularities, the behavior of the amplitudes is entirely determined by the singularities; the singularities nearest the region of investigation playing a more important role than farther ones.

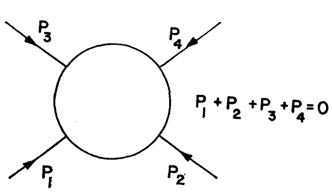

FIG. 6. Feynman diagram describing two-particle scattering.

Major progress has been achieved in the theory of dispersion relation through the work of Mandelstam.[22] Mandelstam has written down double dispersion relations for two-particle scattering amplitudes (Fig. 6). The Feynman diagram in the figure is capable of describing scattering in the three channels (i) $P_1 + P_2 \to (-P_3) + (-P_4)$, (ii) $P_1 + \overline{P}_3 \to (-\overline{P}_2) + (-P_4)$, and (iii) $P_2 + \overline{P}_3 \to (-\overline{P}_1) + (-P_4)$, where the P's are the 4-momenta satisfying the conservation relation $P_1 + P_2 + P_3 + P_4 = 0$, and the bars on P signify the antiparticle with the corresponding 4-momentum. The energy variables for each of the channels are $s = (P_1 + P_2)^2$, $t = (P_1 + P_3)^2$, and $\bar{t} = (P_2 + P_3)^2$, respectively, so that $s + t + \bar{t} = m_1^2 + m_2^2 + m_3^2 + m_4^2$. For each channel, the other variables correspond to momentum transfer. The following features are worth noting:

(a) The two-particle scattering amplitudes are functions of only two relativistic invariants, the energy invariant and the momentum transfer invariant.

(b) If the initial two-particle state and the final two-particle state can both assume the same quantum numbers as a single particle of mass m, then the nearest singularity to the physical region is a pole at the invariant variable corresponding to the square of the total four-momentum equal to m^2.

(c) Substitution rule: All three channels are described by a single analytic function, viz., the scattering amplitude for channel (i) is the boundary value of the analytic function as the energy variable s approaches the positive real axis in the physical energy range; one of the momentum transfer variables being held fixed at a physical value.

(d) Crossing symmetry: If two or more identical particles are involved in the scattering, exchange of two identical particles does not change the value of the amplitude (except, perhaps, by a sign).

(e) The imaginary part of the two-particle scattering amplitude in each channel is given by the unitary condition in terms of the two-particle amplitudes. Thus for channel (i) we have

$$\text{Im}\,(P_1,P_2|(-P_3),(-P_4)) = \int (P_1,P_2|nn) \times (nn|(-P_3),(-P_4))d\tau_n\,,$$

i.e., $Im(P_1,P_2|(-P_3),(-P_4))$ is expressed by inserting a complete set of "intermediate" physical states $|nn)$. Thus in the Mandelstam representation the two-particle elastic scattering amplitude can be expressed as a double spectral dispersion relation in energy variables for the three channels with branch cuts starting at the lowest mass value corresponding to the quantum numbers in that channel. Mandelstam representation therefore leads to an integral equation for the transition amplitude for a given

[54] G. F. Chew, Ann. Rev. Nucl. Sci. 9, 29 (1959).

channel. Since the integral equation involves the scattering amplitudes for the other channels (crossed channels) also, we finally have a set of coupled integral equations to solve.

We will now discuss the features involved in solving the two-nucleon scattering problem using Mandelstam representation. One thing to note in dispersion theory is that one always uses physical masses and coupling constants so that there are no renormalizations in the theory. Now since the amplitudes for the three channels are coupled, the nearest single-particle intermediate state is a one-pion state arising in the $(N\bar{N}|N\bar{N})$ (bar for antinucleon) channel, giving rise to a pole term rather than a branch cut. (In the *n-p* channel the pole contribution is due to the deuteron state.) The next singularity, again in the $(N\bar{N}|N\bar{N})$ channel, enters through the amplitude $(N\bar{N}|\pi\pi)$ when using the unitarity condition a two-pion intermediate state is introduced in this process. The $(N\bar{N}|\pi\pi)$ amplitude is however required in the highly unphysical region $t > 4\mu^2$ (physical region being $t > 4M^2$). If we now analyze the $(N\bar{N}|\pi\pi)$ amplitude in the Mandelstam representation the crossed channel involves the $(N\pi|N\pi)$ amplitude which has as its nearest singularity the famous pole in the pion–nucleon scattering. Further since

$$\mathrm{Im}\,(N\bar{N}|\pi\pi) = \int (N\bar{N}|\pi_n\pi_n)(\pi_n\pi_n|\pi\pi)d\tau_n\,,$$

where now $(N\bar{N}|\pi\pi)$ would be a crossed channel needed in the evaluation of the $(N\pi|N\pi)$ amplitude, we find that we also must take into account the $\pi - \pi$ scattering. We thus arrive at the conclusion that in order to solve the nucleon–nucleon scattering problem, it is necessary to know firstly the $(\pi\pi|\pi\pi)$ scattering amplitude required for the evaluation of the $(\pi N|\pi N)$ amplitude. We can then obtain the $(N\bar{N}|\pi\pi)$ amplitude, and hence determine the $(NN|NN)$ amplitude from the coupled integral equations of the Mandelstam representation.

It has been noted from a study of the electromagnetic structure of the proton and the neutron that the $\pi - \pi$ system should have a resonance in the isotopic spin state $I = 1$ and P state. Frazer and Fulco[55] have carried out this calculation. Frautschi and Walecka[56] obtained the pion–nucleon amplitude from Frazer and Fulco's work on the $\pi - \pi$ system. Though the calculation of $\pi - \pi$ and the $\pi - N$ scattering amplitudes is not completely satisfactory, Ball and Wong[56] have used it to estimate the $(N\bar{N}|\pi\pi)$ amplitude. Cini, Fubini, and Stanghellini[57] have applied the Mandelstam representation to nucleon–nucleon scattering and have been able to obtain the cross section at 90°. Several[58] calculations which take into account the two-pion interaction in nucleon–nucleon scattering have been carried out and some are in progress.

9. CONCLUSION

It seems fair to say that in recent years the understanding of the two-nucleon interaction has improved considerably. The phenomenological study of nucleon–nucleon scattering and its comparison with the experimental data is definitely in favor of the existence of a strong spin-orbit interaction. There is also evidence that the two-body spin-orbit interaction is capable of explaining spin-orbit interaction in complex nuclei as desired by the shell theory. Meson theoretic verification of the spin-orbit interaction has not yet been satisfactorily accomplished because an unambiguous treatment of the two-pion-exchange interaction without neglecting nucleon recoil contributions is still lacking. However, the validity of the one-pion-exchange potential in the outer region has been well demonstrated through the work of Iwadare *et al.*,[12,59] the modified phase-shift analysis by Moravcsik *et al.*,[6] Breit *et al.*,[47,60] and Saylor, Bryan, and Marshak.[15] The application of dispersion relation theory to nucleon–nucleon scattering has offered us with the possibility of carrying out an unambiguous calculations in terms of one-pion and two-pion-exchange contributions. The two-pion-exchange calculations are highly involved and first require the solution of the pion–pion and the pion–nucleon scattering amplitude problems. It is however hoped that a satisfactory solution of the nucleon–nucleon scattering problem within the framework of dispersion theory will be available in the near future.

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