Systematics of Deformations of Atomic Nuclei*

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INTRODUCTION

DEFORMATIONS of atomic nuclei have been calculated by many authors. These calculations have both included and excluded pairing. The initial calculations carried out by Mottelson and Nilsson[1] indicated immediately that one could obtain an estimate of the total nuclear energy as a function of the deformation by summing the one-particle energies of Nilsson.[2] The deformation then corresponds to the minimum in the energy so summed. In this simple calculation, a definite orbital assignment is made for each nucleon and the total energy is computed as a function of deformation for each configuration. Recently, it has been shown that the kind of scalloped curves obtained from these calculations are considerably smoothed out if one adds the pairing correlation to the nuclear model. This is not surprising, since the pairing model, in effect, smears out the Fermi surface, making possible the partial population of many levels, with the corresponding smearing out of the potential-energy curves. This is shown schematically in Fig. 1.

In general, however, calculations of deformation with or without pairing have been made only for those nuclei for which experimental deformations are available. In our work we calculate deformations generally over large regions of the nuclear periodic table. Our purposes are at least twofold. In the first place, we search for nuclear regions where deformations might be theoretically expected, but not yet experimentally observed. In the second place, even in those regions where nuclear deformations are expected, it is interesting to see what the general contour of deformations is over a much broader range of nuclei than experimentally observed. Therefore, it seems worthwhile to calculate deformations systematically, not only as a guide to present and future experiments, but also to see how deformation varies in nuclei much more widely differing than those experimentally studied.

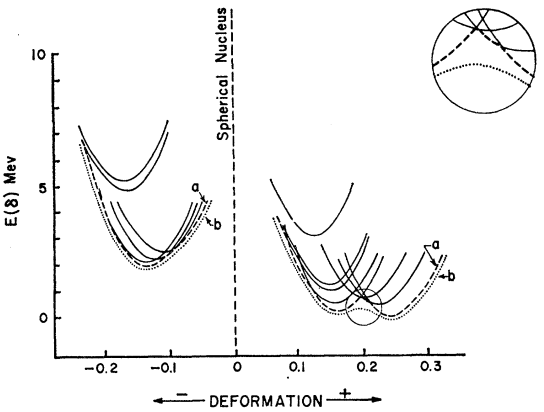

FIG. 1. A schematic representation of the determination of deformation and energy of deformation, showing the summation of Nilsson particle levels. The dashed curve marked a is the envelope of the Nilsson levels, whereas the dotted curve marked b shows the smoothing which results when a pairing effect is added to the calculation. The inset shows the particularly complex region in better detail.

Figure 2 is a schematic presentation of the nuclear periodic table. The odd banana-shaped curve represents very approximately the general area in which nuclei have been previously studied. Horizontal and vertical lines represent proton and neutron closed shells, respectively. Previously observed regions of deformation are indicated by crosshatching. Those

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[1] B. R. Mottelson and S. G. Nilsson, Phys. Rev. 99, 1615 (1955).
[2] S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. 29, No. 16 (1955).

regions are $A \approx 19$ to 28, 150 to 190, 222 up, and probably $A \approx 8$. They are numbered 2, 4, 5, and 6 in Fig. 2. It would be worthwhile to investigate further the limits of these regions of deformation and

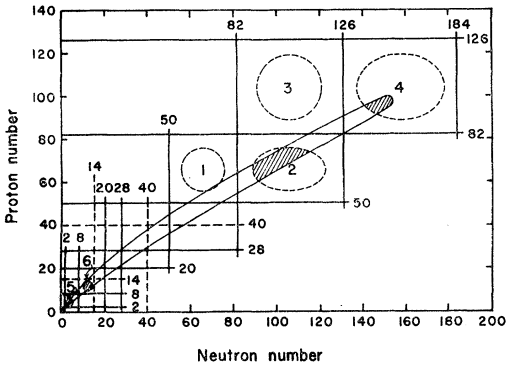

FIG. 2. A schematic representation of the nuclear periodic table showing the closed shells of neutrons (vertical solid lines) and protons (horizontal solid lines). The dashed vertical and horizontal lines represent semi-closed shells, which have an effect on regions of deformation. The thin banana-shaped curve approximately encloses nuclei that have been experimentally studied. Regions where nuclei have been experimentally observed to be deformed are indicated with cross-hatching. Additional regions where it is reasonable to expect to find deformed nuclei are labeled 1 and 3.

to ask the question why there are not other regions of deformation. For example, it is particularly strange that the region of deformation from $A \approx 19$ to 28 does not extend considerably further toward the double closed shell Ca^{40}. It is also surprising that in the region $A \approx 8$ deformation does not extend further. Finally, in the region around Sr and Zr, one would expect to find a fairly extended region of deformation. It seems probable that all these abnormalities may be explained on the basis of semi-closed shells of 6, 14, and 40 neutrons or protons. If spin-orbit coupling in nuclei were somewhat stronger, both neutron and proton numbers 14 and 6 would be closed shells. On the other hand, if spin-orbit coupling were somewhat weaker, 40 neutrons or protons would be a closed shell. Consequently, in spite of the fact that these numbers of neutrons and protons do not represent completely legitimate closed shells, the additional rigidity expected from these configurations keeps nuclei in these regions from becoming deformed. In Fig. 2, therefore, these numbers of neutrons and protons are indicated by dotted vertical and horizontal lines. In the case of neutron and proton numbers 6, the complexity of the diagram does not allow the semi-closed shell dotted lines to be drawn. Figure 2 does, however, point up clearly two additional regions of deformation which have not been experimentally or theoretically observed. They are indicated as regions 1 and 3 in Fig. 2. It is our purpose in these calculations to make detailed studies of the contour of the deformation and the deformation energy in regions 1, 2, 3, and 4 (Fig. 2).

In addition to suggesting new regions of deformation like regions 1 and 3 (Fig. 2), we have another purpose for these calculations. Consider region 2 (Fig. 2). This is enlarged in Fig. 3. Since calculations of the deformation until now have largely been attempts to reproduce experimental deformations, the regular rare-earth region, shown in Fig. 3, has been studied experimentally and theoretically. In both cases, the deformation rises very rapidly at the beginning of the region of deformation marked A, quickly reaches the region of maximum deformation marked B, and falls off gradually over a considerable region of deformation in the area marked C. Thus, nuclei have been studied in which there are both proton particles and neutron particles (region A), and proton holes and neutron holes (region C_3). On the other hand, we have neither experimentally nor theoretically studied those nuclei represented by the regions D and E in Fig. 3. These regions represent proton holes and neutron particles D, and proton particles and neutron holes E. Therefore, theoretical

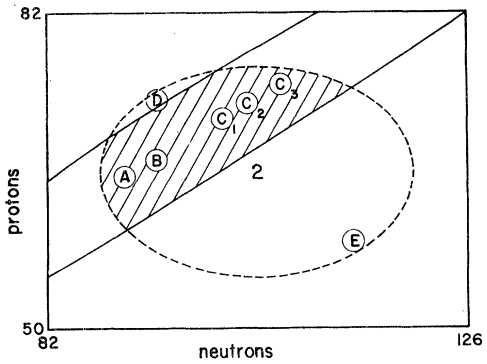

FIG. 3. An enlargement of region 2 from Fig. 2. This figure points out that in general the experimentally studied regions of deformation represent, even in the most advantageous experimental situations, only a small fraction of the total region available for deformation. Of particular interest are the regions marked D and E in this figure, where nuclei representing proton holes and neutron particles, and proton particles and neutron holes, respectively, are to be expected. Calculations of the deformations and energies of deformations for these types of nuclei should be especially worthwhile.

calculations that indicate whether or not regions D and E are expected to be similar to the experimentally observed regions should be of considerable value.

II. DETAILS OF CALCULATION

If one neglects the pairing interaction, the calculation of equilibrium deformations is most elementary. Given the single-particle eigenvalues of the deformed self-consistent field, one chooses for each value of the deformation parameter the configuration that minimizes the sum of the single-particle energies. The equilibrium deformation corresponds to that value of the deformation for which the sum is an absolute minimum. In our calculation, the eigenvalues of the Nilsson-model Hamiltonian[2] were chosen as approximations to the self-consistent energies. Beginning with the lowest level, the successive orbitals for each value of ϵ are filled with two nucleons apiece corresponding to the double degeneracy of states with opposite projections of angular momentum on the z axis. The summations were performed at intervals of $\epsilon = 0.01$, for various combinations of neutron and proton numbers, and the minimum value of ϵ in the range covered was found with a maximum error of ± 0.01. The range covered was $0 \leqslant \epsilon \leqslant 0.39$ in the rare earths, and $0 \leqslant \epsilon \leqslant 0.30$ in the actinides. All computations were performed on the IBM 650 computer.

In summary, we minimize the sum

$$W = \sum_i 2\mathcal{E}_i(\epsilon) , \tag{1}$$

where $\mathcal{E}_i$ are single-particle eigenvalues of the deformed field,

$$\hat{H}\Psi_i = \mathcal{E}_i\Psi_i \tag{2}$$

with

$$\hat{H} = \tfrac{1}{2} m[\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2] - \kappa\hbar\omega_0^0(2\vec{l}\cdot\vec{s} + \mu l^2) ,$$

$$\omega_x = \omega_y = \omega_0(\epsilon)(1 + \tfrac{1}{3}\epsilon) ,$$

$$\omega z = \omega_0(\epsilon)(1 - \tfrac{2}{3}\epsilon) ,$$

$$\omega_0(\epsilon) = \omega_0^0[1 - \tfrac{1}{3}\epsilon^2 - \tfrac{2}{27}\epsilon^3]^{-1/3} \approx \omega_0^0(1 + \tfrac{1}{9}\epsilon^2) ,$$

and

$$\hbar\omega_0^0 \approx 41/A^{1/3}\ \text{MeV}. \tag{3}$$

The use of the deformation parameter ϵ, instead of δ, approximately takes into account neglected couplings of the quadrupole-moment operator between different major shells, as explained by Nilsson.[2] The relationship between ϵ and δ is

$$\epsilon = \delta + \tfrac{1}{6}\delta^2 + O(\delta^3) .$$

It is essential to include terms higher than second order in the expansion of $W_0(\epsilon)$ because an unreasonably large deformation may otherwise result. On the other hand, if the original parametrization in terms of δ is used (neglect of intershell couplings), unreasonably small deformations are obtained.

It has been previously emphasized that a calculation of the total nuclear energy as a function of deformation requires that one subtract one-half of the potential energy from Eq. (1) to avoid counting twice the two-body interactions that give rise to the self-consistent field.[2,3] That is,

$$E = \sum_i 2\mathcal{E}_i - \sum_i \langle v_i \rangle \tag{4}$$

Moszkowski,[4] among others, has argued that such a correction is necessary only for calculation of total binding energies, but not for calculating energy differences for Hartree fields. Recent theoretical discussions of equilibrium distortions taking pairing effects into account also seem to indicate that such a correction is unnecessary.[5,6] At any rate, Inglis and Lee have shown that such a correction is quite small.

Bes and Szymanski,[7] and Szymanski,[8] applying the Bardeen–Cooper–Schrieffer theory, have calculated equilibrium deformations for experimentally studied rare earths and actinides; they minimize

$$W = \sum_\nu 2\mathcal{E}_\nu(\epsilon) V_\nu^2(\epsilon) - \Delta^2(\epsilon)/G , \tag{5}$$

where V_ν^2 is the probability that the single-particle state with energy $\mathcal{E}_\nu$ is occupied, corresponding to the diffuseness of the Fermi surface, Δ is one-half of the energy gap, and G is the "strength parameter" of the pairing force.

In the limit of vanishing superfluidity, that is, in the case of a sharp Fermi surface, Eq. (5) reduces to Eq. (1).

Their calculations also indicate that the pairing force does not significantly alter large values of the deformation, unless the potential-energy curve is very shallow.[7,8] On the other hand, the pairing force is important in determining the boundaries of a region of deformation, for it is just the pairing force which makes the spherical shape stable even with a few nucleons outside the closed shells. In the absence of the pairing force, a single nucleon outside the closed shells may cause polarization.

In this work we neglected pairing to avoid the necessity of solving the nonlinear superconductivity equations for the energy gap and for the Fermi level as a function of deformation. Because of the large number of nuclei studied, inclusion of the superfluidity would have made even a machine calculation

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[3] B. R. Mottelson and S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 1, No. 8 (1959).
[4] S. A. Moszkowski, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, Germany, 1957), Vol. 39.
[5] S. T. Belyaev, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. 31, No. 1 (1959).
[6] M. Baranger, Phys. Rev. 122, 992 (1961).
[7] D. Bès and Z. Szymanski, Nucl. Phys. 28, 42 (1961).
[8] Z. Szymanski, Nucl. Phys. 28, 63 (1961).

excessively long. Even so, it is possible to roughly estimate the boundaries of a region of deformation by use of the magnitude of the deformation and the difference in energy between spherical and deformed shapes. It has been estimated that the pairing stabilizes the spherical shape by 1 or 2 MeV relative to a strongly deformed shape.[9] If the energetic economy of a large deformation exceeds this amount, the spherical shape will be unstable.

Another factor of importance in calculating equilibrium distortions is the contribution to the energy due to Coulombic repulsion between protons. It is possible that the differences between neutron and proton levels in the Nilsson scheme reflect the effect of the Coulomb force (as has been suggested elsewhere[8]). The differences are due primarily to a different choice of the coefficient μ of the l^2 term in the Nilsson Hamiltonian, the choice being empirically motivated. Of course, there is no *a priori* reason why this term should have anything to do with the Coulomb force. It is included only to simulate some of the effects of a square well. However, Lemmer has shown that such a term could arise from a nonlocal interaction.[10] At any rate, since the term in the Hamiltonian is spherically symmetrical, it is unlikely that it gives the right deformation dependence of the Coulomb energy.[11]

A crude, but convenient, way to account for the Coulomb energy is to add to the sum of Eq. (1) the electrostatic energy of a uniformly charged spheroid:

$$E_c = \tfrac{3}{5}\,[(Ze)^2/R_0](1 - \tfrac{4}{45}\epsilon^2\cdots)\,. \qquad (6)$$

Bes and Szymanski[7] use a trapezoidal charge distribution, but the difference is not significant. Possibly, a better method would be to include the contribution of the Coulomb interaction in the deformed field of the protons. This could be done by assuming a larger coupling constant for the coupling of a proton to the average quadrupole field of the protons than for the coupling to the field of the neutrons.

In our calculation, it was found that inclusion of Coulomb effects without pairing leads to anomolously large deformations. This is because the restoring force of the independent-particle potential is in many cases too weak to overcome the Coulomb repulsion. In the work of Szymanski[8] both pairing and Coulomb effects are included, with reasonable results indicating that the pairing largely counteracts the disruptive influence of the electrostatic repulsion. At the beginning of the rare-earth region, where the potential-energy curves are shallow, the deformations calculated by Bes and Szymanski, which include both effects, are about 15% larger than our results, which neglect both. Actually, our results are slightly closer to experimental values, indicating that our method of including electrostatic forces may be somewhat inaccurate. In the actinide region, where the Coulomb forces are even more important, Szymanski's deformations are generally 10 to 20% greater than ours and closer to the experimental values.

In performing the calculations, some modifications of the Nilsson scheme are necessary. Mottelson and Nilsson[3] point out that slightly different choices of the parameter μ from those made in the original paper of Nilsson[2] are needed to account exhaustively for the observed spectra of odd-mass deformed nuclei. A change in μ results chiefly in a translation of the subshells of an affected major shell along the energy coordinate of the Nilsson-level scheme. The modifications recommended by Mottelson and Nilsson have been made in our work. In addition to the observed levels occuring between successive magic numbers, there are sharply sloping levels from lower filled shells and from upper unoccupied shells which penetrate into the regions of interest. Insofar as these sharply sloping levels are not observed, it seems necessary to shift them to prevent interference with the proper sequence. That is, the values of μ should be readjusted to increase the gap between the closed and open shells.

The shift of the upper levels does not seem at all improper, since many of them would not appear at all if a more realistic finite well were used instead of an infinite oscillator well. Moreover, it was found that unless such shifts are made, the calculated deformations become anomalously large at the boundaries of a region, instead of getting smaller. Consequently, the modifications made in the Nilsson scheme are biased against very large deformations. This should weigh against any suspicion that the values predicted in the unstudied regions are artifacts of our adjustments.

Table I gives the minimum shifts that are necessary to produce reasonable deformations in the experimentally studied regions. The same levels were applied to the neutron-deficient regions. One exception is the scheme for neutron levels $50 \leqslant N \leqslant 82$ used in the neutron-deficient rare earths. Although there is no empirical basis for modifying these, it was found that unless the shifts are introduced, we obtain values of $\epsilon > 0.40$ in most cases, which seems ex-

[9] B. L. Birbrair, Soviet Phys.—JETP 6, 951 (1958).
[10] R. H. Lemmer, Phys. Rev. 117, 1555 (1960).
[11] We are indebted to Dr. Sven Gösta Nilsson for clarifying this point.

tremely implausible. Finally, it should be emphasized that it is only the relative shifts which are important for the calculation.

There still remains one more question essential to the validity of the calculations performed here and by Bes and Szymanski.[7] The question is whether the

TABLE I. Parameters used in defining the Nilsson-level spectrum used in the calculations.

Region		Shifts
	Protons	
$50 \leq Z \leq 82$	$N = 5$ when $h_{11/2}$	$-0.20\hbar\omega_0^0$
	$N = 5$ except $h_{11/2}$	$+0.15\hbar\omega_0^0$
	$N = 6$	$+0.15\hbar\omega_0^0$
$82 \leq Z \leq 126$	$N = 4$ except $g_{9/2}$	$-0.25\hbar\omega_0^0$
	$N = 6$ when $i_{13/2}$	$-0.35\hbar\omega_0^0$
	Neutrons	
$50 \leq N \leq 82$	$N = 4$ when $g_{9/2}$	$-0.30\hbar\omega_0^0$
	$N = 5$ except $h_{11/2}$	∞
	$N = 6$	∞
$82 \leq N \leq 126$	$N = 4$ except $g_{9/2}$	$-0.37\hbar\omega_0^0$
	$N = 6$ except $i_{13/2}$	$+0.15\hbar\omega_0^0$
	$N = 5$ when $p_{3/2}$	$+0.04\hbar\omega_0^0$
$126 \leq N \leq 182$	$N = 5$ except $h_{11/2}$	$-0.25\hbar\omega_0^0$
	$N = 7$ when $j_{15/2}$	$-0.06\hbar\omega_0^0$

model potential used fulfills the condition of self-consistency; namely, that the anisotropy of the potential, characterized by a deformation parameter which measures the eccentricity of the equipotential surfaces, must equal the anisotropy of the density distribution of nucleons moving in the potential. It has been shown by Moszkowski[12] that for a harmonic oscillator this condition is exactly fulfilled at equilibrium if the equipotential volumes are independent of deformation. The condition also holds in case residual interactions are included if these are independent of deformation. This general result has been verified by the calculations by Bes and Szymanski.[7]

III. RESULTS

The calculated results are divided into the four regions indicated as 1 to 4 in Fig. 2. The classification is as follows:

Designation in Fig. 2	Neutron number	Proton number	Description of region
Region 1	50 to 82	50 to 82	neutron-deficient rare-earth elements
Region 2	82 to 126	50 to 82	rare-earth elements
Region 3	82 to 126	82 to 126	neutron-deficient heavy elements
Region 4	126 to 184	82 to 126	heavy elements

[12] S. A. Moszkowski, Phys. Rev. **103**, 1328 (1956).

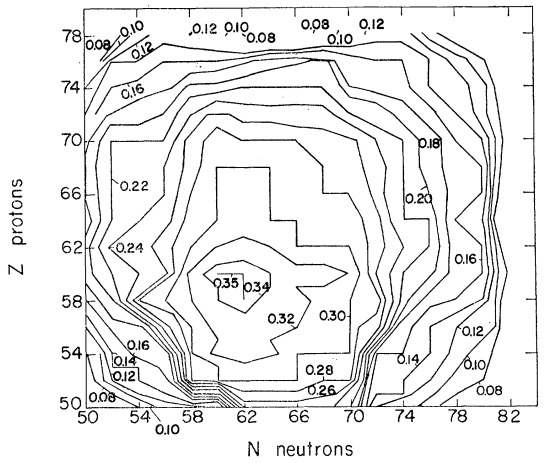

FIG. 4. A contour map of the deformation vs neutron and proton numbers in the region where the protons and neutrons both go from 50 to 82. The numbers on the contour lines are values for the deformation, $\epsilon \approx 0.95\beta$.

In some cases, the calculations have not been carried out for the entire region. This is especially true if the nuclei are not easy to synthesize or not important in a definition of the region of deformation. For each region, we have plotted contour maps for deformation and for energy of deformation (the energy difference between the deformed nucleus and spherical nucleus) against neutron and proton numbers for even-even nuclei. In all, there are eight figures, Figs. 4 through 11. We present the data as contour maps rather than tables to emphasize trends and at the same time to de-emphasize exact values for deformation and energy of deformation because

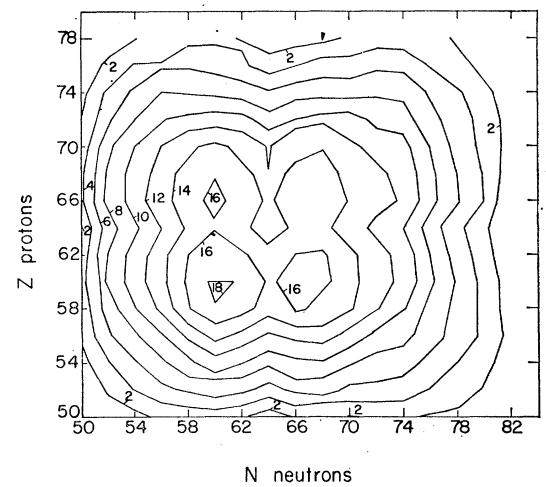

FIG. 5. A contour map of the energy of deformation vs neutron and proton number in the region where the protons and neutrons both go from 50 to 82. The numbers on the contour lines are the values for the energy differences between spherical and deformed nuclei in MeV.

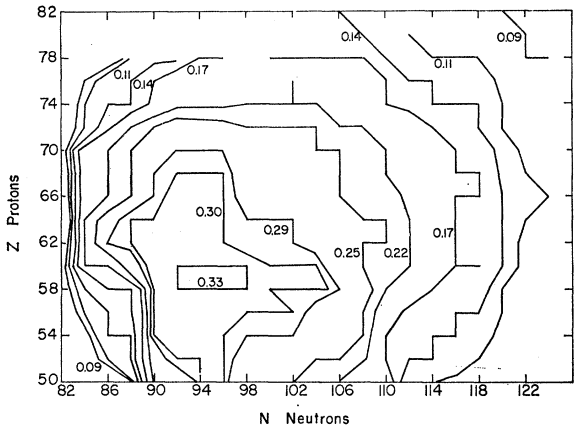

FIG. 6. A contour map of the deformation vs neutron and proton number in the region where the neutrons go from 82 to 126 and the protons go from 50 to 82. The numbers on the contour lines are values for the deformation, $\epsilon \approx 0.95\beta$.

of the approximations involved in the calculations. However, calculated deformations for those few nuclei whose level structure is most amenable to experimental observation are presented in Table II.

TABLE II. Calculated deformations for even-even nuclei whose level structures are amenable to experimental observation.

Nucleus	Deformation calculated without pairing	Nucleus	Deformation calculated without pairing
$_{56}Ba^{124}$	0.30	$_{88}Ra^{206}$	0.11
Ba^{126}	0.30	Ra^{208}	0.10
Ba^{128}	0.27	Ra^{210}	0.09
Ba^{130}	0.17	$_{90}Th^{208}$	0.11
$_{58}Ce^{126}$	0.31	Th^{210}	0.10
Ce^{128}	0.30	Th^{212}	0.10
Ce^{130}	0.29	$_{92}U^{210}$	0.14
Ce^{132}	0.21	U^{212}	0.11
$_{60}Nd^{132}$	0.26	U^{214}	0.10
Nd^{134}	0.22		

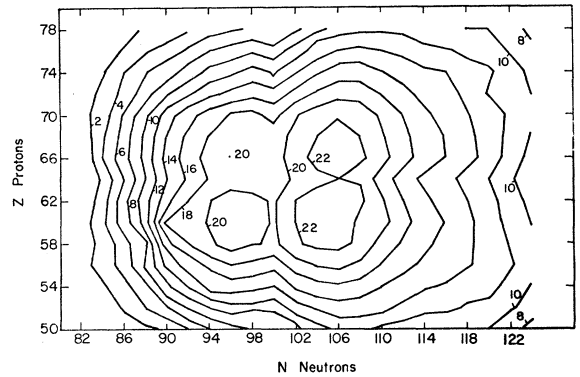

FIG. 7. A contour map of the energy of deformation vs neutron and proton number in the region where the neutrons go from 82 to 126 and the protons go from 50 to 82. The numbers on the contour lines are the values for the energy differences between spherical and deformed nuclei in MeV.

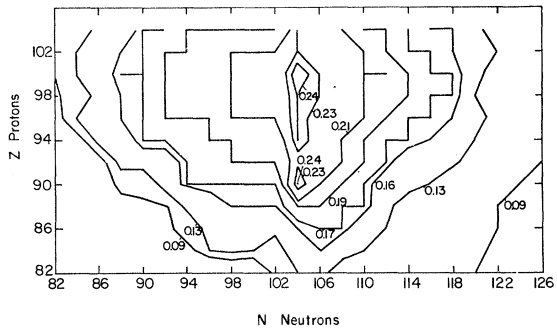

FIG. 8. A contour map of the deformation vs neutron and proton number in the region where the neutrons go from 82 to 126 and the protons go from 82 to 106. The numbers on the contour lines are values for the deformation, $\epsilon \approx 0.95\beta$. The calculations have not been carried out for the entire region because nuclei with proton numbers above 106 are not likely to be studied thoroughly enough for these considerations.

IV. DISCUSSION

A. Caution about the Exactness of the Values of Deformation and Energies of Deformation

In our calculation of the deformations and deformation energies, kinds of scalloped curves similar to those indicated in Fig. 1 were obtained. In general, several minima were observed, with the differences in energy among the minima usually being very small in comparison with the total energy difference between spherical and deformed nuclei. In general, we used the absolute minimum, but on occasion when the absolute minimum did not follow the trend of absolute minima of neighboring nuclei, we chose a relative minimum that did follow this trend. When this happened, the effect on the energy was very small, whereas the effect on the deformation was relatively larger. By using Fig. 12, we can make a

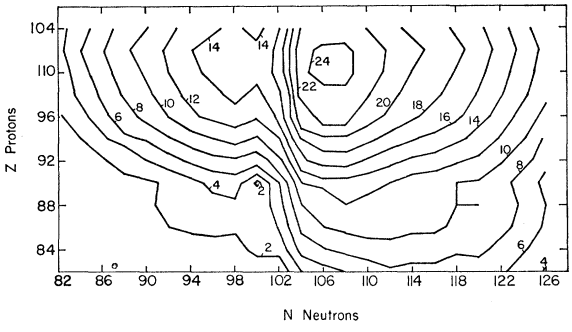

FIG. 9. A contour map of the energy of deformation vs neutron and proton number in the region where the neutrons go from 82 to 126 and the protons go from 82 to 106. The numbers on the contour lines are the values for the energy differences between spherical and deformed nuclei in MeV. The calculations have not been carried out for the entire region because nuclei with proton numbers above 106 are not likely to be studied thoroughly enough for these considerations.

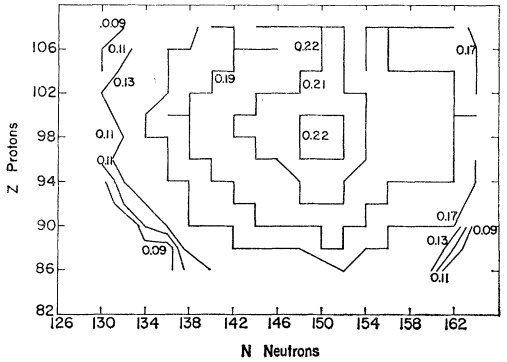

FIG. 10. A contour map of the deformation vs neutron and proton number in the region where the neutrons go from 126 to 166 and the protons go from 82 to 110. The numbers on the contour lines are values for the deformation, $\epsilon \approx 0.95\beta$. The calculations have not been carried out for the entire region because nuclei with proton numbers above 110 are not likely to be studied thoroughly enough for these considerations.

comparison between a contour map of the first excited states in the rare-earth deformed nuclei and the calculated deformation of Fig. 6.

The absolute value, both for the deformation and for the energy of deformation, is to some extent at the mercy of the approximations involved in these calculations. The nature and effect of these approximations have been carefully considered in Sec. II, but trends rather than absolute values have meaning in these calculations.

B. Comparisons between the Different Regions of Deformation

In general, the contour of energy of deformation follows approximately the contour of deformation.

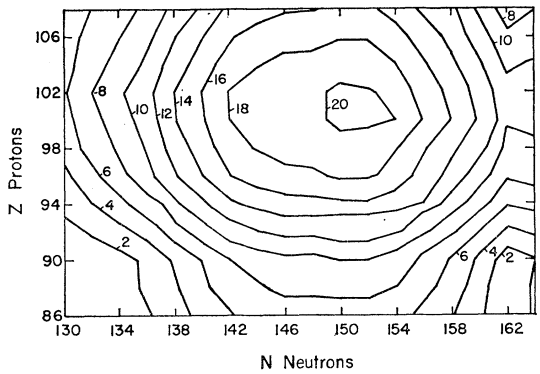

FIG. 11. A contour map of the energy of deformation vs neutron and proton number in the region where the neutrons go from 130 to 166 and the protons go from 86 to 110. The numbers on the contour lines are the values for the energy differences between spherical and deformed nuclei in MeV. The calculations have not been carried out for the entire region because nuclei with proton numbers above 110 are not likely to be studied thoroughly enough for these considerations.

(See Figs. 4 through 11.) However, some important differences are immediately evident. Thus, while the deformation rises steeply to a maximum early in the region of deformation and then decreases gradually, the energy of deformation is much more symmetrical. Particularly, in region 1 the contour of deformation rises most steeply in the proton–neutron particle configurations (i.e., in the lower left-hand corner) as compared to the proton–neutron hole configurations (the upper right-hand region). This observation seems most striking in region 1, but is to some extent true in other regions also. The deformation generally decreases slowly from region 1 to region 2 and from region 3 to region 4. However, there is a large decrease in deformation, by about 1/3, in going from regions 1 and 2 to regions 3 and 4. Furthermore, the contours

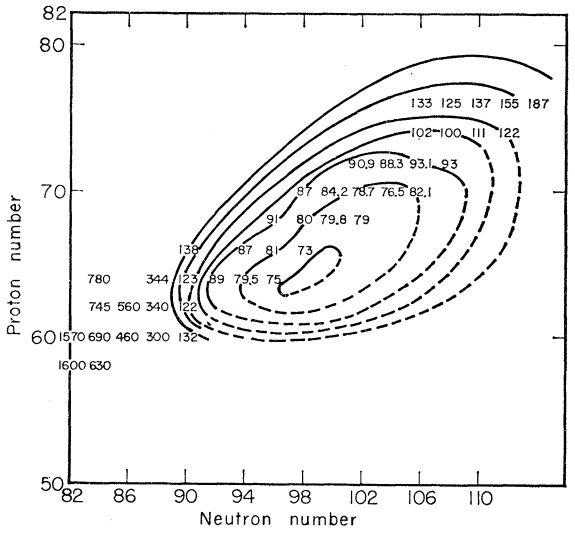

FIG. 12. Contour plot of the first-excited-state energies of even–even nuclei against neutron and proton number in the rare-earth region. A comparison with the calculated deformation in Fig. 6 reveals similarities.

indicate that the deformation is considerably flatter in regions 3 and 4 than in regions 1 and 2. The experimental values for deformation for the heavy elements (region 4) are about 2/3 the values of the rare-earth region (region 2) and are in fair agreement with the calculated deformations. One of the most striking differences between the contour maps for deformation and those for energy of deformation is the existence of four-leaf-clover contours in the energy of deformation maps not present in the deformation maps. This is most probably because the energy of deformation is determined as $\delta E = E_{\text{def}} - E_{\text{spher}}$. Since the E_{spher} has a prominent closed subshell structure, this is mirrored in the δE of defor-

mation. One expects closed subshells at

$$Z = N = 64 = (g_{7/2})^8 + (d_{5/2})^6 + 50 = 64\,,$$

$$N = 100 = (h_{9/2})^{10} + (f_{7/2})^8 + 82 = 100$$

and

$$N = 164 = (g_{9/2})^{10} + (i_{11/2})^{12} + (j_{15/2})^{16} + 126 = 164\,.$$

These subshells probably give rise to the waists of the cloverleaf regions. The waists seem to occur at $N = 162$ instead of the expected $N = 164$.

C. Interpretation of the Contours within a Region

We have used the simple deformed harmonic oscillator for calculating the deformation δ, which is proportional to the quadrupole moments (see Lemmer and Weisskopf[13]), and

$$\delta = \tfrac{3}{2}\frac{\sum_i (2n_3^i - n_1^i)}{\sum_i (N^i + \tfrac{3}{2})}\,,$$

where the summation is over the nucleon numbers, the N are the total oscillator quantum numbers, the n_3 are the z components of the total oscillator quantum numbers, and n_1 is the component perpendicular to the z-axis.

Figure 13, where the deformation δ is plotted against A for a fixed value of Z in the $N = 4$ oscillator shell, shows that δ rises more rapidly at the beginning of the shell than it falls at the end of the shell.

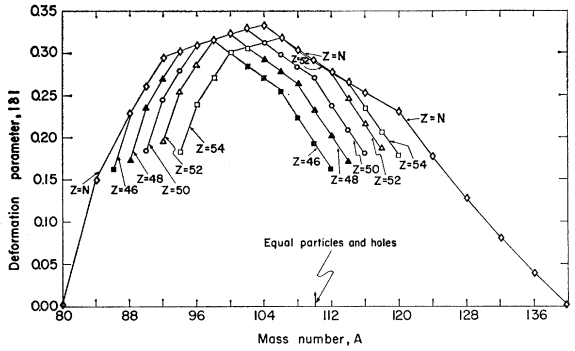

FIG. 13. The diagram of the deformation parameter plotted as a function of the neutron and proton numbers in the $N = 4$ simple-harmonic-oscillator shell.

At $A = 110$, where the numbers of particles and holes are equal, the deformation increases as the number of particles and holes increase. This indicates that in our contour plots of deformation the contour lines should be closer in spacing at the beginning of the shell than at the end of the shell.

[13] R. H. Lemmer and V. F. Weisskopf, Nucl. Phys. 25, 624 (1961).

D. Comparison between Even, Odd A, and Odd–Odd Nuclei

Calculations have also been made for odd-A and odd-odd nuclei in the limited region $50 \leqslant Z \leqslant 60$, and $50 \leqslant N \leqslant 82$. The calculated deformations indicate only small differences between even, odd-A and odd-odd nuclei in this limited region. This is to be expected, since the pairing interaction has been neglected in our calculation. In view of this fact, calculations are not presented for odd-A and odd-odd nuclei.

V. CONCLUSION

Calculations have been made over large regions of the nuclear periodic table to determine the deformation of and the energy difference between spherical and deformed nuclei as functions of neutron and proton numbers. These calculations lead to the following conclusions:

1. There are extended regions of deformation for relatively neutron-deficient nuclei in the region where both proton and neutron numbers go from 50 to 82 and in the region where both proton and neutron numbers go from 82 to 126.

2. Contour maps of the deformation and the energy difference between spherical and deformed nuclei do not mirror each other. Indeed, the differences are quite striking.

3. The deformation contour maps reach a maximum fairly soon after nuclear deformation is achieved. The deformation then decreases slowly as additional neutrons and protons are added. Such a trend has been noted experimentally in the rare earths.

4. Energy differences between spherical and deformed nuclei are much more symmetrical and tend to show a "four-leaf clover" kind of contour. The waists in the "four-leaf clovers" seem to be connected with closed subshell structures in the spherical nuclei.

5. A number of experiments are suggested by these calculations. Among them is the possibility of observing another region of nuclear deformation, the region labeled 1 in Fig. 1. These calculations were started simultaneously with an experimental attempt to observe this new region of deformation. The experimental observation of the new region of deformation came slightly before its confirmation by these theoretical calculations.

a. In the experiment, In and Sb were bombarded with heavy-ion beams of O^{16} and C^{12} to produce a number of new nuclides. The new observed activities, their half lives, and the nuclear reactions involved

are shown in Table III. The decay of the odd-odd nuclear species La^{126}, La^{128}, and La^{130} into even-even Ba^{126}, Ba^{128}, and Ba^{130} allows observation of first excited states in each of these species. The energies of these excited states indicate strongly that the

TABLE III. Nuclear reactions showing the cross bombardments used to produce the new nuclides La^{126}, La^{128}, and La^{130} whose decay suggests a new region of nuclear deformation.

New nuclide	Half-life (min)	Nuclear reactions	Heavy-ion bombarding energy (MeV)
La^{126}	1.0 ± 0.3	$In^{115}(O^{16},5n)La^{126}$	94
		$Sb^{121}(C^{12},7n)La^{126}$	117
La^{128}	6.5 ± 1.0	$In^{115}(O^{16},3n)La^{128}$	65
		$Sb^{121}(C^{12},5n)La^{128}$	84
		$Sb^{123}(C^{12},7n)La^{128}$	117
La^{130}	9.0 ± 1.0	$Sb^{121}(C^{12},3n)La^{130}$	53
		$Sb^{123}(C^{12},5n)La^{130}$	84

even-even Ba isotopes are deformed and that, in fact, one can expect a large region of deformation involving several hundred neutron-deficient species.[14] It will be important not only to find convincing evidence for rotational bands in even-even nuclei, but also to assign Nilsson levels in odd-A nuclei, to observe vibrational bands, vibration-rotation interaction, Coriolis coupling, and all the rest of the rich experimental possibilities, which result from the additional degrees of freedom in deformed nuclei.

b. A similar region of deformation (labeled 3 in Fig. 1) is certainly also amenable to observation. Perhaps the easiest way to search for this region of deformation is to observe the alpha-particle fine structure resulting from the decay of even-even nuclei that is produced from nuclear reactions of the types Hg^{196} $(Ne^{20},8n)$ Th^{208} and Hg^{198} $(Ne^{20},8n)$ Th^{210}. The Th^{210} and Th^{208} should have extremely short half-lives, in the millisecond-to-microsecond range. However, the background involved in an alpha-particle experiment is so small that good results are possible.[15] These isotopes should have deformations (β) of the order of 0.10 to 0.13 similar to those of the Ra isotopes and Th^{226}.

c. In addition to those regions that have been considered in these calculations, there are certain other limited regions where deformation might be expected (e.g., Fe isotopes other than Fe^{54}, and As and Se isotopes). These regions have already been discussed to some extent in the literature. In general, the possibilities of observing the deformations are marginal. In particular, however, we might mention the region of the neutron-excess Ru and Pd nuclides. Published information on Pd^{110} and Ru^{104}, both on the energy of the first excited state and on the ratio

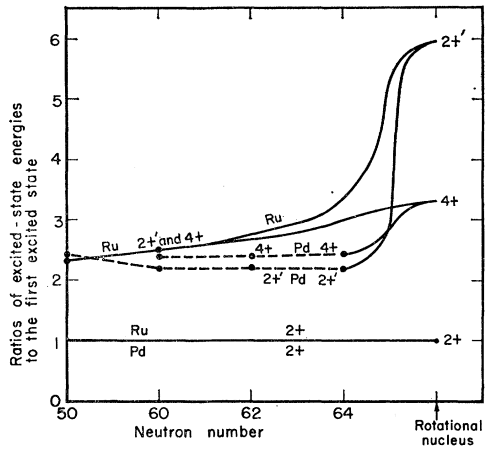

FIG. 14. Data on the nuclear levels in even-even Ru and Pd isotopes taken from K. Way, N. B. Gove, C. L. McGinnis, and R. Nakasima *Nuclear Data Sheets* (National Research Council, Washington 25, D. C.). The dotted lines are Pd isotopes; the solid lines are Ru isotopes. Extrapolations toward rotational-level systematics (solid lines) seem to indicate that one can reach only an intermediate situation between vibrational and rotational-level systematics. The best possibility for a study of an even-even nucleus amenable to present experimental procedures would probably be Ru^{106} by the reaction $Ru^{104}(t,p)$ Ru^{106}.

of the second excited state to the first excited state, indicate that we are approaching a new very limited region of nuclear deformation (see Fig. 14). Of particular interest, therefore, in this case, would be the observation of the nuclear levels in Ru^{106} and Pd^{112}. We can study the levels in these nuclei by investigating such reactions as $Ru^{104}(t,p)$ Ru^{106} and $Pd^{110}(t,p)$ Pd^{112}. It should also be of interest to look at odd-A nuclei such as Ru^{105}, Pd^{111}, Ag^{111}, etc., to see if the intrinsic states correspond at all to those suggested by the Nilsson levels. It seems probable from the information already available that one will approach only a transitional region where the nuclear characteristics are intermediate, between those of spherical and deformed nuclei.

Finally, these calculations do seem to indicate that we understand where to expect deformed nuclei. The simple empirical considerations presented in Fig. 2 are quite satisfactory in this regard. Thus, for example, it is not hard to predict (using Fig. 2) that if the calculations were made, we would probably pre-

[14] Raymond K. Sheline, Torbjørn Sikkeland, and Richard N. Chanda, Phys. Rev. Letters **7**, 446 (1961).
[15] R. D. MacFarlane and R. D. Griffioen, Bull. Am. Phys. Soc. **6**, 451 (1961).

dict a region of deformation where the proton number goes from 50 to 82 and the neutron number from 126 to 184 (i.e., in the highly neutron-excess rare earths). However, at present the possibility of the experimental observation of deformation in these nuclei seems remote.

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The Two-Nucleon Interaction

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1. INTRODUCTION

IN recent years a considerable amount of progress has been achieved in the study of the two-nucleon interaction. The experimental investigation of polarization and the triple scattering parameters of p-p scattering at high energies has made it possible to look into several features of the two-nucleon interaction which did not exhibit themselves in data on cross section at lower energies. The first set of triple scattering experiments was performed by Chamberlain, Segrè, Tripp, Wiegand, and Ypsilantis[1] who measured five scattering parameters; cross section σ, polarization P, depolarization D, rotation parameters R, and A for p-p scattering at 310 MeV. Similar experiments have since been performed at 150 MeV,[2,3] 210 MeV[4] and at other energies. The theoretical investigation of the experimental data has followed two main procedures, viz., (i) phase shift analysis of the experimental data, and (ii) phenomenological potential models. In the first approach one obtains several sets of phase-shift solutions which fit the experimental data and then the problem is that of discriminating among the various sets of solutions on the basis of other experimental evidence. Stapp, Ypsilantis, and Metropolis[5] have carried out the phase shift analysis of the experiment of Chamberlain *et al.*, obtaining several phase-shift solutions out of which finally only four were acceptable. A later modified analysis of the p-p scattering data at 310 MeV was carried out by Cziffra, MacGregor, Moravcsik, and Stapp[6] in which the partial waves G and higher were calculated from the one-pion-exchange pole in the scattering amplitude. This procedure left only two distinguishable solutions. In the second approach followed by Signell and Marshak,[7] Gammel and Thaler,[8] Otsuki,[9] Watari,[10] and Tamagaki,[11] one starts with a phenomenological potential and calculates a set of phase shifts which finally enable one to calculate the experimental scattering parameters. Both Signell and Marshak (SM)[7] and Gammel and Thaler (GT)[8] found it necessary to add to the central and tensor potentials a strong spin-

[1] P. Chamberlain, E. Segrè, R. D. Tripp, C. Wiegand, and T. J. Ypsilantis, Phys. Rev. **105**, 288 (1957).
[2] J. N. Palmieri, A. M. Cormack, N. F. Ramsey, and R. Wilson, Ann. Phys. (N. Y.) **5**, 299 (1958); T. R. Ophel, E. H. Thorndike, R. Wilson, and N. F. Ramsey, Phys. Rev. Letters **2**, 310 (1959); C. F. Hwang, T. R. Ophel, E. H. Thorndike, and R. Wilson, Phys. Rev. **119**, 325 (1960); E. H. Thorndike, J. Lefrancois, and R. Wilson, *ibid.* **120**, 1819 (1960).
[3] A. E. Taylor and E. Wood, *Proceedings of the Sixth and Seventh Annual International Conference on High Energy Physics*, 1956 and 1957 (Interscience Publishers, Inc., New York).
[4] K. Gotow and E. Heer, Phys. Rev. Letters **5**, 111 (1960); A. England, W. Gibson, K. Gotow, E. Heer, and J. Tinlot, Phys. Rev. **124**, 561 (1961), J. Tinlot and R. Warner, *ibid.* **124**, 890 (1961).
[5] H. P. Stapp, T. J. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957).
[6] P. Cziffra, M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Phys. Rev. **114**, 880 (1959); M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, *ibid.* **116**, 1248 (1959).
[7] P. S. Signell and R. E. Marshak, Phys. Rev. **106**, 832 (1957); **109**, 1229 (1958).
[8] J. Gammel and R. Thaler, Phys. Rev. **107**, 291 (1957).
[9] S. Otsuki, Progr. Theoret. Phys. (Kyoto) **20**, 171 (1958).
[10] W. Watari, Progr. Theoret. Phys. (Kyoto) **20**, 181 (1958).
[11] R. Tamagaki, Progr. Theoret. Phys. (Kyoto) **20**, 505 1958).